Stochastic Calculus and Derivatives Pricing in the Nigerian Stock Market

URAMA T. C. Ph.D. 2018

Stochastic Calculus and Derivatives Pricing in the Nigerian Stock Market

URAMA Thomas Chinwe

A Thesis submitted in partial fulfilment of the requirement of

Sheffield Hallam University

For the Degree of Doctor of Philosophy

March 2018

DECLARATION

I certify that the substance of this thesis has not been already submitted for any degree and is not currently being submitted for any other degree. I also certify that to the best of my knowledge any assistance received in preparing this thesis, and all sources used, have been duly acknowledged and referenced in this thesis.

Abstract

Led by the Central Bank of Nigeria (CBN) and the Nigerian Stock Exchange (NSE), policy makers, investors and other stakeholders in the Nigerian Stock Market consider the introduction of derivative products in Nigerian capital markets essential for their investment and risk management needs. This research foregrounds these interests through detailed theoretical and empirical review of derivative pricing models. The specific objectives of the research include: 1) To explore the key stochastic calculus models used in pricing and trading financial derivatives (e.g. the Black-Scholes model and its extensions); 2) To examine the investment objectives fulfilled by derivatives; 3) To investigate the links between the stylized facts in the Nigerian Stock Market (NSM), the risk management techniques to be adopted, and the workings of the pricing models*;* and 4) To apply the research results to the NSM, by comparing the investment performance of selected derivative pricing models under different market scenarios, represented by the stylized facts of the underlying assets and market characteristics of the NSM.

The foundational concepts that underpin the research include: stochastic calculus models of derivative pricing, especially the Black-Scholes (1973) model; its extensions; the practitioners' Ad-Hoc Black Scholes models, which directly support proposed derivative products in the NSM; and Random Matric Theory (RMT). RMT correlates market data from the NSM and Johannesburg Stock Exchange (JSE) and facilitates possible simulation of non-existing derivative prices in the NSM, from those in the JSE. Furthermore, the research explores in detail the workings of different derivative pricing models, for example various structures for the Ad-Hoc Black Scholes models, using selected underlying asset prices, to determine the applicability of the models in the NSM.

The key research findings include: 1) ways to estimate the parameters of the stochastic calculus models; 2) exploring the benefits of introducing pioneer derivative products in the NSM, including risk hedging, arbitrage, and price speculation; 3) using NSM stylized facts to calibrate selected derivative pricing models; and 4) explaining how the results could be used in future experimental modelling to compare the investment performance of selected models.

By way of contributions to knowledge, this is the first study known to the researcher that provides in-depth review of the theoretical and empirical underpinnings of derivative pricing possible in the NSM. This forms the basis for the Black Scholes approach to asset pricing of European option contract, which is the kind of call/put option contract that is being adopted in the NSM. The research provides the initial foundations for effective derivatives trading in the NSM. By explaining the heuristics for developing derivative products in the NSM from JSE information, the research will support future work in this important area of study.

ACKNOWLEDGEMENTS

I would like to express my profound gratitude and deep appreciation to my supervisory team which includes my former and present Directors of Studies Dr. Patrick Oseloka Ezepue and Dr. Alboul Lyuba respectively for their esteemed guidance and encouragement throughout the duration of this research. My gratitude also goes to my Supervisor, Professor Jacques Penders, whose professorial suggestions and advice helped immensely in shaping the thesis to its final form for presentation and subsequently examination. The staffs and students of Material and Engineering Research Institute, Sheffield Hallam University, UK have been very wonderful throughout my stay with them in the University, as they made my sojourn in the United Kingdom, a home away from home. I thank you all for your individual and collective assistance to me here in the UK.

My heartfelt gratitude also goes to my mother for her care, love, prayers and encouragement throughout my programme here in Sheffield. Furthermore, I am also indebted to members of my research group and friends including Maruf, Chimezie, Nonso, Cajetan, Rabia, Kun, Nana, and Adiza Sadik, whose critical contributions have been of much benefit to me in the development of the research ideas. It is also worthy to mention here the contributions of my dear wife, Mercy and Children among whom are Chinweike, Kamsiyochukwu, Udochukwu, Christiantus and Olivia, without whose love, fervent prayers and support this research would not have been completed.

I will not fail to mention the sponsors of this programme - Federal Republic of Nigeria, under the auspices of Tertiary Education Trust Fund (TETFund), Maitama Abuja and Institute of Management and Technology (IMT) Enugu - Nigeria for their financial assistance towards the accomplishment of my research degree here in Sheffield Hallam University, (SHU), Sheffield, United Kingdom.

Above all, I am very grateful to the Almighty and awesome God whose protection and Devine guidance made it possible for me to accomplish this onerous task.

Thomas Chinwe URAMA

DEDICATION

This dissertation is dedicated to

My beloved mother Madam Mary O. Urama

My late father Mr Ugwu Attama Urama

My kind and benign wife Mercy URAMA

My lovely children:

Franklyn Chinweike URAMA

Stephanie Kamsiyochukwu URAMA

Philemon Udochukwu URAMA

Christiantus Chimdindu URAMA

and

Olivia Ifechimeremasoka URAMA

Table of Contents

List of Tables

List of Figures

ABBREVIATIONS

NSM	Nigerian Stock Market
JSE	Johannesburg Stock Exchange
IPO	Initial Public Offer
CBN	Central Bank of Nigeria
NASD	Nigerian Association of Securities Dealers
RMT	Random Matrix Theory
CsTK	Contributions to Knowledge
SDEs	Stochastic Differential Equations
OTC	Over The Counter
FX	Foreign Exchange
AfDB	African Development Bank
SPDEs	Stochastic Partial Differential Equations
RQ	Research Questions
SRO	Self-Regulatory Organisation
MERI	Materials and Engineering Research Institute
SIMFIM	Statistics, Information Modelling and Financial Mathematics Research Group
SHU	Sheffield Hallam University, UK
LSR	Listings, Sales and Retention
CSCS	Central Securities Clearing System
SEC	Security and Exchange Commission
ERM	Enterprise Risk Management
CSD	Central Security Depository
DMO	Debt Management Office
FMDQ	Financial markets Dealers Quotation
MD	Managing Director
CEO	Chief Executive Officer
EMs	Emerging Markets
RBI	Reserve Bank of India
BESA	Bond Exchange of South Africa
FSB	Financial Services Board of South Africa
FRA	Forward rate Agreement
FCD	Foreign Currency Derivatives
FDD	Foreign Dominated Debt
PPDE	Parabolic Partial Differential Equations
GBM	Geometric Brownian Motion
PCP	Put-Call Parity
NDF	Non-deliverable Forwards

LIBOR	London Inter-Bank Offered Rate
Z_d	Domestic Interest Rate
BIS	Bank for International Settlements
DVF	Deterministic Volatility Function
AHBS	Ad-Hoc Black-Scholes
Z_f	Foreign Interest Rate
DVF_R	Relative Smile Models/ Deterministic Volatility Function
DVF_A	Absolute (Smile) Model/ Deterministic Volatility Function
BS	Black-Scholes
PDE	Partial differential Equations
CRR	Cox, Ross and Rubinstein
ARCH	Auto-Regressive Conditional Heteroscedasticity
GARCH	Generalized Auto-Regressive Conditional Heteroscedasticity
GED	Generalized Error Distribution
GJR-GARCH	Gosten, Jagannathan and Runkle-Generalized Auto-Regressive Conditional Heteroscedasticity
N-GARCH	Non-Linear-Generalized Auto-Regressive Conditional Heteroscedasticity
CEV	Constant Elasticity of Variance
IV	Implied Volatility
ISD	Implied Standard Deviation
P/E	Per-Earnings Ratios
EMH	Efficient Market Hypothesis
MATLAB	Matrix Laboratory
SPSS	Statistical Package for Social Sciences
Excel VBA	Excel Visual Basic for Applications
MVT	Mean Value Theorem
IPR	Inverse Participation Ratios
ODEs	Ordinary Differential Equations
EM	Euler-Maruyama
ANOVA	Analysis of Variance
UBA	United Bank for Africa
FCMB	First City Monument Bank
FBN	First Bank of Nigeria
CIX	Correlation Index
WACM	Weighted Average Correlation Matrix.

CHAPTER 1

INTRODUCTION

1.0 Basics of the Study

This research investigates the market conditions and appropriate models for introducing financial derivatives in the Nigerian Stock Market (NSM), to deepen the market. The research uses mainly techniques in Stochastic Calculus, particularly stochastic differential equations (SDEs) which have become the standard models for pricing financial derivatives.

The research investigates the financial market characteristics (stylized facts) of developing and emerging markets which will provide the theoretical research findings that will encourage the derivatives trading in the Nigerian Stock Market. Of special interest is the Johannesburg Stock Exchange (JSE), South Africa, which the research seeks to use in the experimental trial of derivative and allied products aimed at developing the trade in derivatives in Nigerian Stock Market. Based on a 2014 scientific visit embarked by our research group to the headquarters of the Nigeria Stock Exchange in Lagos, the researcher discovered that Nigerian policy makers are looking at the products and models that work in South Africa towards the development of derivative products in the NSM, since according to them, JSE is the most robust emerging market in Sub-Saharan Africa nearest to the NSM, that trades on derivative products.

Financial derivatives enable market participants to trade specific financial risks, such as interest rate risk, currency, equity and commodity price, and credit risk, thereby transferring the risk on their investment to other entities more willing or better suited to bear those risks. The risk embodied in a derivative contract can be mitigated either by trading the contract itself or by creating a new (reverse) contract which offsets the risks of the existing contract. An industrialist who produces beer, for instance, needs some cereal crops for his brewing industry and therefore may need to enter into some futures or forwards contract to guard against an unanticipated rise in the prices of cereals which they need to produce beer. If, however, they realise that the cost of most of those cereals has some futures contracts which are on the downward trend, they may decide to take a reverse contact by writing some put option contract on them, to reduce the losses they will incur in the futures contracts.

Derivatives can be traded through an organised exchange or over-the-counter (OTC). Exchange-traded derivatives include options and interest rate futures, while OTC-traded derivatives include forwards, foreign exchange (FX) currency swaps, and options as well. (The diagram below represents derivative tree for over-the-counter and exchange traded derivatives).

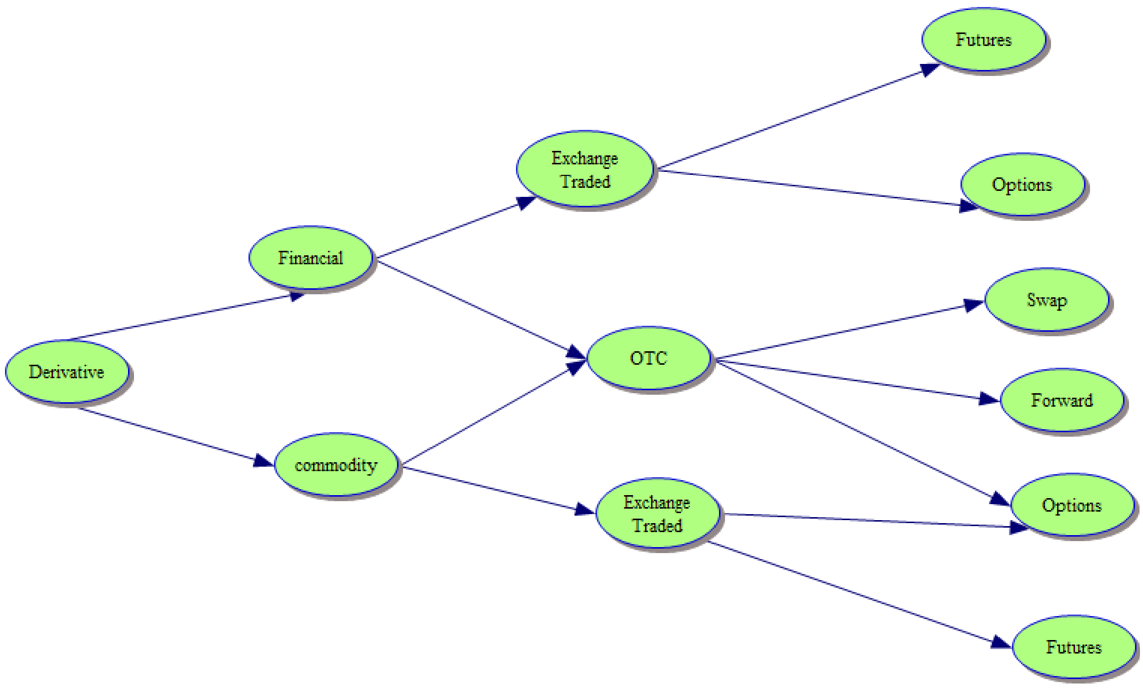


Figure 1.0 A tree diagram showing types and links with OTC and Exchange traded derivatives.

1.1 Derivatives trading in Nigeria

Derivatives play significant roles in the development and growth of an economy through risk management, speculation and or price discovery. They also promote market completeness and efficiency which are associated with low transaction costs, greater market liquidity and leverage to investors, thereby enabling them to go short easily.

Recently in 2014, the Central Bank of Nigeria (CBN) granted approval for stakeholders in the Nigerian capital markets to formally kick-start derivatives trading. This policy interest in derivatives is also evidenced by the establishment of the Nigerian Association of Securities Dealers (NASD) in 2012, and the fact that the CBN provided a N40 million grant to this body towards the development of the OTC derivatives trading platform.

This study, therefore, aims to provide theoretical findings from research-based evidence, with a special focus on the types of derivative products which are seen by the Nigerian Stock Exchange as more promising for early adoption in the Nigerian Stock Market (NSM). The thesis refers to these products below as approved pioneer products.

The approved pioneer derivative products for take-off in the NSM include: Foreign Exchange (FX) options, Forwards (Outright and Non-deliverables), FX Swaps and Cross-Currency Interest Rate Swaps, most of them being OTC derivatives products.

We note here that traces of trade in derivatives products in Nigeria have been in existence for quite some time among market participants in an informal capacity. For example, the African Development Bank, AfDB (2010) asserts that FX forwards has been informally traded in Nigeria and that it is usually subject to a maximum of three years, allowing dealers to engage in foreign exchange swap transactions among themselves and with retail/wholesale customers.

1.2 Rationale for the study

The need to adapt derivative products to the NSM cannot be overemphasized since the products are useful for hedging interest-rate and currency risks, (Neftci, 1996), and these risks are among the most prominent challenges confronting investors in the NSM. Hence, this research examines the structure, functioning and pricing of these derivative products.

In March 2011, the Federal Government of Nigeria through the Central Bank issued policy guidelines for derivative trading in Nigerian Financial markets, CBN (2011). The introduction of derivatives on the foreign exchange markets, according to the CBN, will enable operators and end-users hedge against losses arising from exchange rate fluctuations.

Also, the Nigerian Stock Exchange (NSE) is currently conducting feasibility studies on the introduction of derivative products in the Nigerian Stock Market (NSM). For this purpose, understanding how derivative trading works in other financial markets, especially similar emerging markets to Nigeria, including the Johannesburg Stock Exchange, and other developed markets, is important to this work.

1.3 Related works in some emerging markets

In Brazil, a study by Mullins and Murphy (2009) observes that the growth in derivatives and other financial instruments has afforded the Brazilian stock market more autonomy. Aysun and Guldi (2011), using Brazil, Chile, Israel, Korea, Mexico and Turkey as case studies, show that interest rate exposure is negatively related to derivative usage. Shiu, Moles and Shin (2010), who investigate what motivates banks to use derivatives, find that the propensity to use derivatives is positively related to bank size, currency exposure and issuance of preferred stock, while it is negatively related to leverage and diversification of long term liabilities.

In the Malaysian stock market, Ameer (2009) discovers that there is a significant positive correlation between total earning and the use of derivatives. According to Randall Dodd and Griffith Jones (2007), there is a substantial use of derivative products in Chile and Brazil financial markets. Of special importance in their finding is the over the counter (OTC) market in foreign exchange, which has become an established market with dealers and with high liquidity and low bid-ask spreads. These derivatives markets have helped firms lower their risks and their borrowing costs.

The above facts show the typical contexts in which derivative are useful, such as exchange rate exposures and interest rate risks, asset and liability management, and stock options.

1.4 Brief notes on derivative pricing models

In practical terms, most derivatives pricing uses the Black and Scholes result for theoretical and applied work by researchers and practitioners alike. Black and Scholes (1973) in their paper entitled 'The pricing of options and corporate liabilities' propose a formula for computing call and put option prices. Although the formula has some criticisms on the underlying assumption of constant volatility throughout the option life span, it still constitutes a robust mechanism for derivatives pricing in the financial markets.

This research therefore examines the model and its assumptions in line with the stylized facts of emerging stock markets, for example leptokurtic behaviour of the empirical distributions underpinning stock returns, abnormal skewness and thick tails associated with such non-normal distributions, lack of depth (thinness) in the markets, and asymmetry or leverage effects. It is known in empirical finance that these stylized facts are related to some extent with the differing profiles of six market characteristics among developed and emerging markets, namely Efficiency, Anomalies, Bubbles, Volatility, Predictability, and Valuation (Ezepue and Omar, 2012; Omar, 2012; Islam and Watanapalachaikul, 2005).

Hence, a research direction pursued in this study is to examine the extent to which some of these stylized facts and market characteristics in emerging markets such as the NSM influence the nature of derivative products which are suitable for such markets. In pursuit of this objective we will look at the Random Matrix Theory (RMT) in both Nigerian Stock Market and that of the Johannesburg Stock Exchange, for a comparison of the two most dominant exchanges. This will yield necessary information towards developing the approved pioneer derivative products with the NSM, taking clues from the working on those products and their pricing models in JSE vis-à-vis the empirical results emanating from RMT in both exchanges.

1.5 Aims and objectives of the study

Aims of the research

The research explores some stylized facts and financial market characteristics of developing and emerging markets that will encourage derivatives trading in the Nigeria

Stock market (NSM), compares these market features with those in (developed) markets with successful derivative trading, in order to develop the theoretical underpinnings and some practical results useful for trading in such derivatives in the NSM.

Remarks:

The researcher notes that the theoretical background for the study will focus mainly on topics in Stochastic Calculus for example Stochastic Partial Differential equations (SPDEs), which underpin the Black-Scholes model and its extensions and other derivatives pricing models.

1.6 Key research objectives and questions

The key research questions (RQs) which guide the research on the various work objectives include:

For Objective 1: *To explore the key stochastic calculus models used in pricing and trading financial derivatives (e.g. the Black-Scholes model and its extensions)*

RQ1: What are the differentiating characteristics, performance trade-offs, assumptions, equations, and parameters, among stochastic calculus models used in derivative pricing, and how are the model parameters typically determined from market data?

For Objective 2: *To examine the investment objectives fulfilled by derivatives*

RQ2: What are the links between the model features/derivative products and key investment objectives fulfilled by the products in financial markets, for instance risk hedging, arbitrage and speculation?

For Objective 3: *To investigate the links between the stylized facts and the stock market characteristics (of the NSM), including the empirical correlation matrix properties and derivative pricing models, for example how changes in the stylized facts and market characteristics influence the risk management techniques to be adopted and the workings of the pricing models*

RQ3: Which stylized facts of stock markets are particularly associated with derivative pricing models, and how do they inform adaptations of these and related derivatives to the NSM?

For Objective 4: *To apply the research results to the NSM, by comparing the investment performance of selected derivative pricing models under different market scenarios represented by the stylized facts of the underlying assets and market characterisation of the NSM*

RQ4: How do the research ideas including findings from Random Matrix Theory apply to the NSM? For example, how can the ideas be used to implement relevant experimental modelling for comparing the investment performance of selected derivative pricing models under different market scenarios in the NSM?

1.7 Benefits of this research to market participants in the Nigerian financial system

Derivatives help fund managers and investors to manage risk in their portfolio through hedging. The research will, therefore, provide the necessary theoretical support for trading derivatives in NSM. The study will propose suitable derivative products based on market characteristics and stylized facts, like the Forwards and Options that will assist market participants to speculate and hedge against the risk(s) associated with their portfolio.

The work will also provide research-based theoretical evidence that will support the policy thrust of the NSM towards the development of suitable derivatives models. These models will be adapted in the NSM through a comparison of existing models and results in developed and some similar emerging markets, notably South Africa. This will be achieved by comparing the stylised facts of market data from the NSM and South Africa, and simulating the behaviour of plausible, albeit not yet existing, derivatives in the NSM.

The work will also provide researchers with the fundamental derivatives asset pricing models for the NSM, which will help to deepen the NSM. The results can be applied to other emerging markets with similar market characteristics to the NSM, especially in Africa.

1.7.1 Contributions of the research to knowledge and why it is suitable for PhD work

Results from work on the various objectives of the research including investment goals realizable in adopting the proposed pioneer derivatives products in NSM and the contributions to knowledge both theoretical and practical are as shown in chapter 10 of the thesis. Meanwhile the following publications have been realised from the Thesis as at the time of viva:

Related research publications

Chapters 5, 7 & 8 of the Thesis have been published, accepted for publication or are being reviewed for publication as shown below.

Results from chapter 5 of the Thesis have been accepted for publication in Central Bank of Nigeria Journal of Applied Statistics as

1. Urama, T.C. & Ezepue, P.O. Stochastic Ito-Calculus and Numerical Approximations for Asset Price Forecasting in the Nigerian Stock Market (NSM), Central Bank of Nigeria Journal of Applied Statistics: Accepted to Appear, March 2018.

2. The above article was also published in the Proceedings of International Symposium on Mathematical and Statistical Finance held on 1-3 September, 2015, Mathematics and Statistics Complex, University of Ibadan, Oyo State Nigeria, Publications of the SIMFIM-3E-ICMCS Research Consortium, ISBN: 978-37246-5-7.

3. Aspects of the research was earlier presented as

   Urama, T. C. (2015), Stochastic Calculus and Derivative Pricing in the Nigerian Stock Market

   during the Materials and Engineering Research Institute Symposium, Sheffield Hallam University, SHU, UK held at Cantor Building, 19-20th May 2015.

Chapter 7 of the Thesis was published as

1. Urama, T. C., Ezepue, P. O. & Nnanwa, P. C. (2017) Analysis of Cross-Correlations in Emerging Markets Using Random Matrix Theory, Journal of Mathematical Finance, 2017; 291-307; http://www.scrip.org/journal/jmf.

2. A conference version of the above paper was presented during the 6th Annual International Conference on Computational Geometry and Statistics (CMCGS) 2017 and the 5th Annual International Conference on Operations Research and Statistics (ORS), 2017 held 6-7 March 2017 in Singapore, under the auspices of Global Science and Technology Forum. The paper as titled was accepted and published in the Conference Proceedings (Testimonial interview for excellent presentation is on YouTube with link: https://www.youtube.com/Thomas Urama)

Chapters 7 & 8 of Thesis have been published as:

1. Urama, T.C. Nnanwa, C.P. & Ezepue, P.O. (2017) Application of Random Matrix Theory in Estimating Realistic Implied Correlation Matrix from Option Prices, Proceedings of 6th Annual International Conference on Computational Geometry and Statistics (CMCGS) 2017 and the 5th Annual International Conference on Operations

Research and Statistics (ORS), 2017 held March 2017 in Singapore, Global Science and Technology Forum.

2. Nnanwa, C. P., Urama, T. C. & Ezepue, P. O. (2016) Portfolio Optimization of Financial Services in the Nigerian Stock Exchange. American Review of Mathematics and Statistics, December, 2016, Volume 4 Number 2, pp.1-9.

3. Still under Review with Physica A Journal under the title:

   Urama, T.C. Nnanwa, C.P. & Ezepue, P.O. Application of Random Matrix Theory in Estimating Realistic Implied Correlations from Option Prices; Submitted to Physica A Journal.

4. Another Article was sent out and accepted to appear under the title:

   Nnanwa, C.P; Urama, T.C. & Ezepue, P.O. Random Matrix Theory Analysis of Cross-Correlations in the Nigerian Stock Exchange. Accepted to appear in Proceedings of International Scientific Forum (2017).

1.8 Indicative structure of the thesis

Guided by the key ideas in the research objectives, the thesis is structured as follows:

Chapter 1: Introduction
Chapter 2: Background to the Nigerian Stock Market and Financial System
Chapter 3: Literature Review
Chapter 4: Concept Chapter
Chapter 5: Stochastic Calculus Models for Derivative Pricing (Objective 1)
Chapter 6: Implied Volatility Analysis and Its Application in Risk Management Using AHBS or Practitioners Black Scholes
Chapter 7: Random Matrix Theory (RMT) and Statistical Distribution of the Study Data (Objective 2)
Chapter 8: Application of RMT in Estimating Realistic Correlation Matrix in Option Prices
Chapter 9: Interpretation of Results, Discussions and Findings
Chapter 10: Summary, Contribution to Knowledge and Recommendations

References
Appendices

CHAPTER 2

BACKGROUND TO THE STUDY

2.0 Introduction

This chapter discusses the Nigerian Stock Market (NSM) operations and the extent of development in the Nigerian capital markets, towards the introduction and pricing of derivative products in the NSM. The specific objectives of the chapter are to: present background information on the NSM, obtained from the operations of the Nigerian Stock Exchange (NSE) which manages the NSM; describe the importance of derivatives trading in the NSM; discuss current plans for introducing derivatives trading in the NSM; and explore the need for trading derivatives in Nigeria, especially the proposed pioneer products earmarked for introduction of the trade in the NSM.

This research investigates suitable option pricing models based on the stock market characteristics and stylized facts of the NSM, in line with the recommended pioneer and other derivative products that may be introduced in the NSM. In carrying out the theoretical research, the efficacy of some of these models in other developed markets where trade in derivative products are practised are explored, which will inform the NSM applications. The possibility of adopting the Black-Scholes model as a standard model for derivatives products pricing will be considered, and the alternative of using implied volatility estimates to address the assumption of constant volatility in the Black-Scholes model examined.

2.1 Stock Market

A stock market or an equity market is that financial outlet where shares of publicly held companies are issued and traded either through the exchanges or over-the-counter (OTC) markets. The equity market is known to encourage a free-market economy as it provides company management access to capital, in exchange for offering investors some measure of ownership in the company through the sale of some units of stocks in the company to investors. As some money is required for the purchase of some company shares (to enrol into the ownership of the company), the stock market provides shareholders and investors with the opportunity to grow the initial sum of money invested in the purchase of shares into large sums. This enables investors to get wealthy without necessarily taking the risk of starting their own business, which in most cases requires high capital and a lot of sacrifices for an effective take-off of the businesses of their choice.

The Nigerian Stock Exchange (NSE) was formed in 1960, the same year Nigeria got her independence from Great Britain, and it was referred to then as Lagos Stock Exchange.

This was the name it bore until December 1977, when the name was changed to the present one - Nigerian Stock Exchange.

The NSE is governed by a National Council with its Head office in Lagos, 13 branches across Nigeria, and a total number of 191 companies listed in the Exchange, Gamde (2014). NSE currently has three types of markets in its operation, namely: Equity market, bond market and Exchange Traded Funds (ETFs), and two types of market structure, referred to as Quote-driven market and Order-driven market. For the quote driven market, the exchange allows market makers to provide two-way quotes and licenced broker/dealers of the Exchange to submit orders, whereas in the order-driven market making, all the orders of prospective buyers and sellers are displayed, showing the quantity and the price at which, a buyer/seller is willing to trade. Omotosho (2014) asserts that for stock brokers to attain the status of market makers with the Nigerian Stock Market, they should among other requirements provide evidence of a net income to the tune of five hundred million naira (₦500,000,000).

The NSE is known to operate a 'hybrid' market in its operation with the market makers in the exchange. Hybrid market operation of the NSE is a system that allows market makers to provide two-way quotes for the licenced dealers of the exchange to submit orders. These quotes and orders are allowed to interact in the order book within the Exchange, in order to discover the best price for a security. The NSE, like all other stock exchanges around the globe, is simply a market system that provides a fair, efficient and transparent securities market to investors. It has 5 main objectives which include:

1. Trading Business

NSE provides trading floors/market opportunities for the buying and selling of securities within the market and controls the activities of market participants by ensuring that disciplinary actions are taken against people that flout the rules of the trade.

2. Listing Business

The NSE has the mandate to conduct initial listing of securities through Initial Public Offers (IPOs) for companies that satisfy the prescribed requirements for listing. It further ensures that companies that fall short of the minimum standard for remaining in operation with the NSM are delisted.

3. Index Business

The NSE carries out index definition and maintenance including index data distribution and index licencing.

4. Data Business

NSE maintains a data management process through the dissemination of all references, market corporate activities, and participants in the NSM. It also carries out data vendor enrolment for stakeholders who are interested in the dissemination of market data for trade in the NSM. The discharge of this corporate responsibility of the NSE that made it possible for the researcher to obtain the study data used in this research. The NSE also protects the interest of investors by ensuring that they derive maximum satisfaction from their investment with the NSM.

5. Other duties of the NSE include

Acting as a self-regulatory organisation (SRO), providing information technology outfit and telecommunications infrastructure to investors and stakeholders in the NSM, and offering the needed education, certification and research initiatives to staff members and other interested partners in the activities of the NSM. It was in pursuance of this objective of the NSE that a study visits of the Statistics, Information Modelling & Financial Mathematics Research Group (SIMFIM), Materials and Engineering Research Institute (MERI), Sheffield Hallam University, (SHU), UK, was undertaken in 3-5 May 2014 to the NSE. The visit aimed to explore areas of collaboration between the SHU and Nigerian researchers from the University of Ibadan Mathematics and Statistics departments with the exchange, especially about the intention of the Exchange to introduce derivatives products in the Nigerian market. The visit aimed to facilitate research collaboration with policy makers in the Nigerian Capital Market, to support the planned trade in the derivatives earlier earmarked to take off in 2014.

The department in charge of carrying out the product research in the NSE is known as Product and Business Development and Management. This department also manages Listings, Sales and Retention (LSR), attracts and retains companies on the boards of the NSE, and is responsible for the development and sale of existing and new products on the NSE. Dipo Omotosho (2014), Head Product Management, NSE, in a paper he presented during the study visit of SIMFIM research group, asserts that the product management:

- Oversees all activities that pertain to asset classes in the NSE to ensure that all products are appropriately positioned, promoted and supported to enhance increased order flow;
- Ensures close coordination and maintenance of mutual co-operations among all stakeholders in the market both local and international;
- Enhances market micro-structure in the areas such as transaction fees, to increase product profitability;
- Liaises with all divisions in the Exchange, for instance Market Operations and technology division to ensure the system compatibility with market structure enhancements in support of all product lines;
- Coordinates and operates investor education in partnership with other stakeholders on exchange-based products.

Product management department drives the initiative of the Exchange to increase its products offering (asset classes), a goal fulfilled partly by introducing derivatives in the NSM. It was given the mandate to have created five products by the year 2016 and prominent among them are four derivative products which is the aim of this research. This research, therefore, seeks to provide theoretical and empirical research findings that will support the trade on derivatives products in the NSM.

2.1.1 Listing requirement in the NSE

Taba Peterside (2014), General Manager Listing, Sales and Retention (LSR), asserts that the listing requirements for companies in the NSE are categorized into three options:

Option 1: The Company seeking permission to register should possess a cumulative profit of ₦300million naira minimum for at least 3 years with a pre-tax profit of a ₦100 million naira minimum in 2 out of the 3 years and at least another ₦3 billion naira in shareholders' equity

Option 2: Possess a cumulative consolidated pre-tax profit of at least ₦600million naira within 1 or 2 years and at least ₦3billion in shareholders' equity

Option 3: Possess at least ₦4billion naira in market capitalisation at the time of listing, based on issue price and issued share capital.

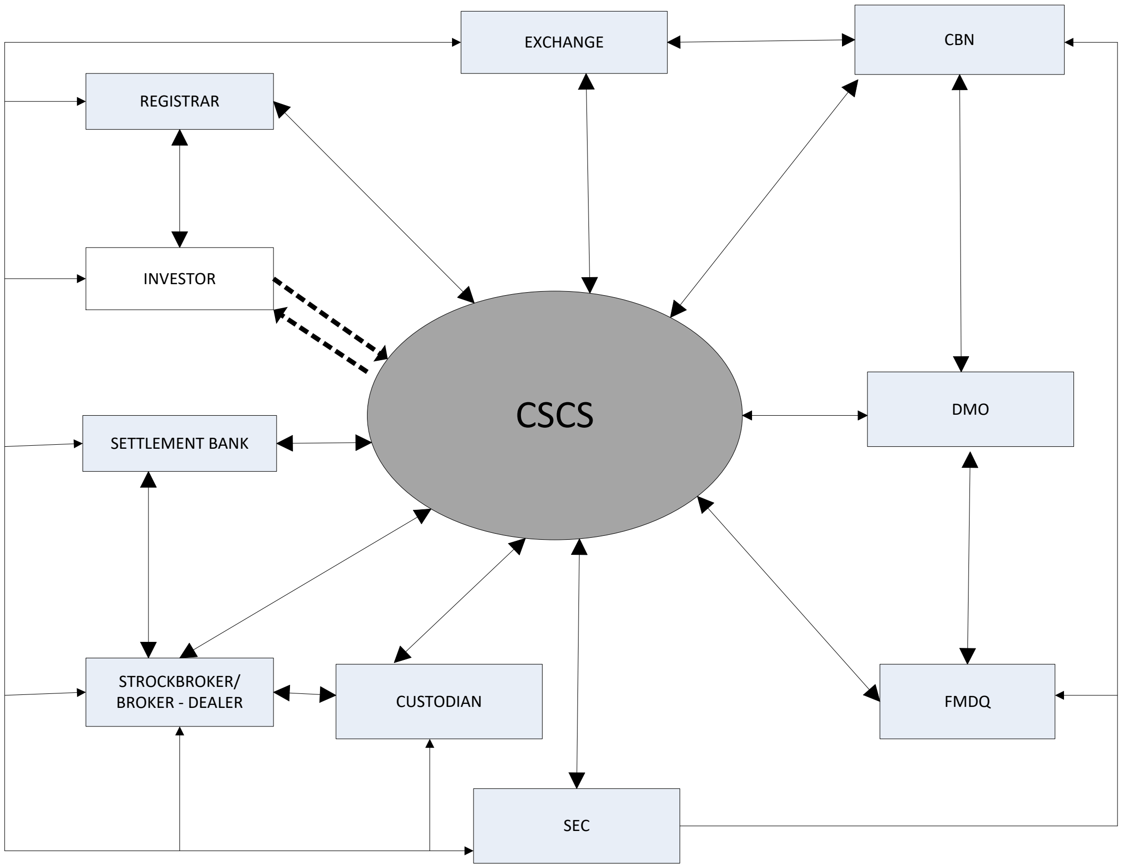
In addition to satisfying any of the options (1) - (3), the Company seeking registration/enlisting with the NSE should have:

3 years' operating track record of Company and/or core investor; 20% of share capital must be offered as public float; and the Company must at the time of applying for enlisting have a minimum of 300 shareholders that have subscribed to it.

2.1.2 Clearing, delivery and settlement

Clearing, settlement and delivery of transactions on the NSE are done electronically by the Central Securities Clearing System (CSCS) Limited. The CSCS is a subsidiary of the stock exchange under the supervision of the Security and Exchange Commission (SEC) that was established under the Company and Allied Matters Act of 1990 and was later incorporated in 1992 as part of the effort to make NSM more efficient and investor friendly. Apart from clearing settlement and delivery, the CSCS offers custodian services and it became a public liability company (plc) on May 16, 2012. According to Ayo Adaralegbe (2014), of the Enterprise Risk Management (ERM) department in CSCS, CSCS is a Central Security Depository (CSD) established to hold securities in a dematerialised or electronic form. CSCS was established to promote efficient clearing and settlement of securities traded in the Nigerian Capital Market.

CSCS provides post-trade services to the capital market and also eliminates delays and risks previously associated with trading of securities in the market, thereby enhancing investors' confidence in the market through process automation, settlement and risk management. He further illustrated in the figure below the relationship between CSCS and other stakeholders (participants) in the exchange.



Source: CSCS post trade services May 7 (2014)

Figure 2.0: participants' relationship

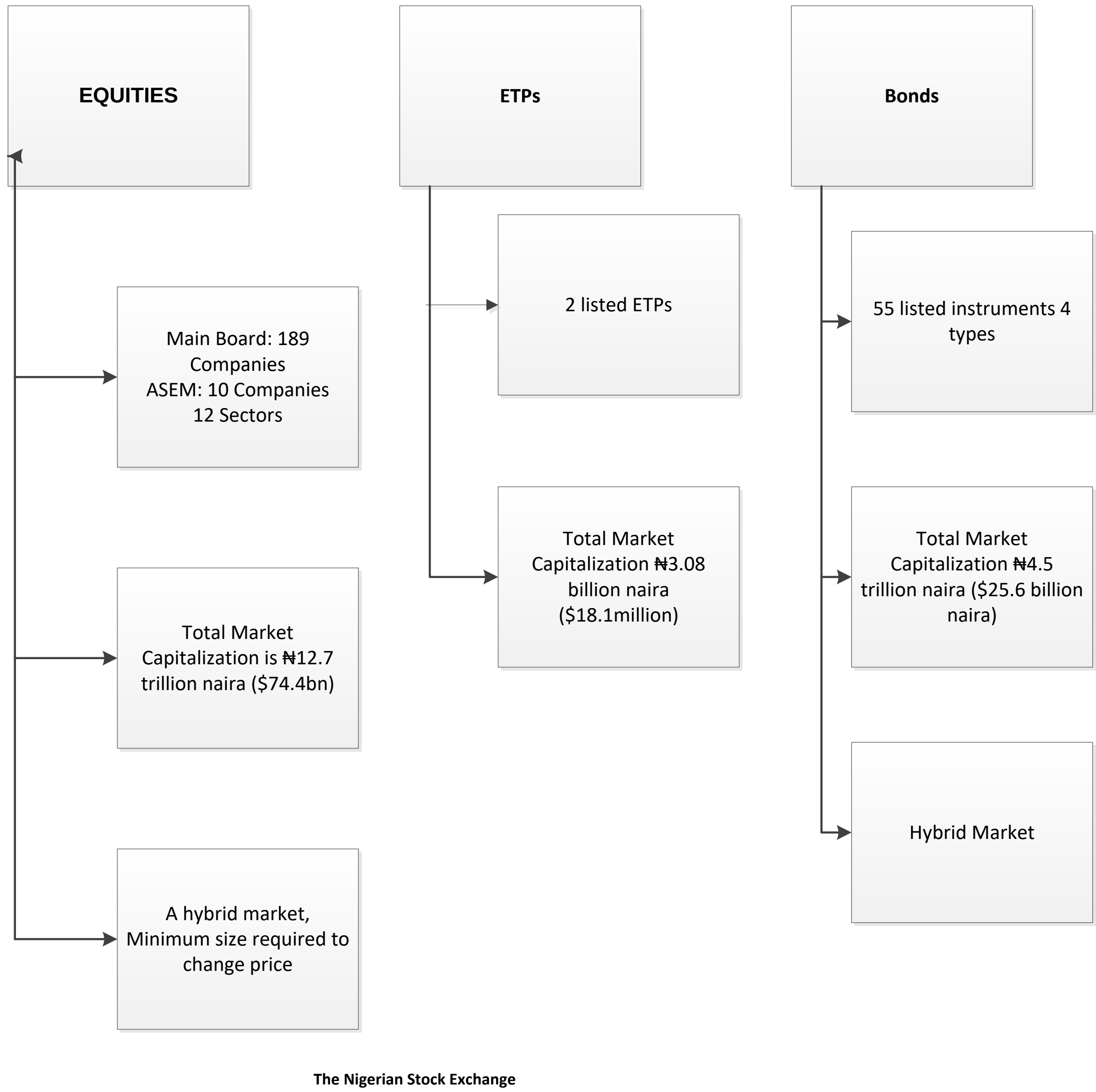


Figure 2.1: Products traded in the NSM

Figure 2.1 illustrates the products being traded in the NSM with the trade in the derivatives products excluded as it is still in the formative stage with few OTC trades.

The OTC markets according to financial market infrastructure of the Nigerian Stock Market is being organised by the Financial Markets Dealers Quotations (FMDQ) OTC plc. FMDQ, in its capacity as a market organiser of the Nigerian OTC markets, receives trade data from its dealing members on a weekly basis, and in line with its duties of providing transparency to the market (OTC) publishes monthly turnover figures across all products traded on the FMDQ OTC platform.

FMDQ OTC plc is a Nigerian Securities and Exchange Commission (SEC) licenced (OTC) market securities exchange and self-regulatory Organisation (SRO), with a target of becoming the most liquid, efficient, secure and technology-driven OTC platform in

Africa by 2018. Its mission is to empower the OTC financial markets to be innovative and credible, in support of the Nigerian economy.

Onadele (2014), who is the MD/CEO FMDQ asserts that prior to the development of FMDQ, the governance over the Nigerian inter-bank OTC market was fragmented, thereby limiting its development, credibility, operational processes, liquidity and capacity development. He stated further that the need to address these challenges necessitated the formation of FMDQ and that through its function as a market organiser and self-regulated organisation, FMDQ drives liquidity and enhances the efficiency of the price discovery mechanism, which is one of the characteristics/properties of derivatives products.

With an effective and efficient collaboration with key financial market regulators, FMDQ is deepening the OTC financial markets, thus complementing other securities exchanges and providing local and international stakeholders with much-needed market governance in capital transfers. Furthermore, through its function as a market organiser and self-regulated organisation, FMDQ will also drive liquidity and efficient price discovery by disseminating information through a centralised platform which would serve the interest of market operators, investors and regulators.

The FMDQ is owned by 25 Nigerian commercial Banks, the Central Bank of Nigeria (CBN), the Finance Dealers Association, and the Nigerian Stock Exchange (NSE).

2.2 Derivatives trading

A derivative asset is a financial security whose value is derived from an underlying financial variable such as a commodity price, a stock price, an exchange rate, an interest rate, an index level or sometimes the price of another derivative security. The three most common types of derivative securities are forward/futures, swaps and options.

Sundaram (2013) infers that the danger of trading in derivatives comes from the interaction of three factors that form a potentially lethal cocktail if the risks associated with investing on the derivative products are not properly understood and managed. The three factors include leverage, volatility and (il)liquidity.

Derivatives are highly leveraged financial instruments since in Futures contract, for instance, only a margin of 10% (or less) of the total value of the contract is required for one to be engaged in Future derivative trade, thus encouraging excessive risk taking from participants in the market. Market volatility is also known to compound the problems emanating from the leverage effects of derivatives trade. As volatility in the price of the underlying increases and unexpectedly large price movements occur, the impact of leverage increases, leading to potentially larger losses on the downside.
With respect to the liquidity and or illiquidity factor, periods of market turmoil are always accompanied by not just higher volatility but also liquidity drying up selectively. This

makes it difficult to exit unprofitable market strategies, thereby increasing the risk of the derivatives position.

Derivatives, however, when properly managed, will foster financial innovations and market developments, thereby increasing the market resilience to shocks. This could be achieved by taking opposite positions with the underlying assets to that of options through call/put option. In other words, when an investor perceives that his/her underlying asset security has every possibility of going down in price, he/she is expected to take a put option position on the same underlying to cushion the effect of the anticipated fall in price of the underlying asset. Therefore, a put option 'increases in value' when the underlying stock it is attached to 'declines in price', and 'decreases in value' when the stock 'goes up in price'.

Similarly, when the stock price 'increases' in value, a call option premium will also increase thereby providing the opportunity for investors who want to diversify their portfolio to also take an equal position by investing in a call option. It, therefore, behoves on the policy makers to ensure that derivative transactions are properly tracked and prudently supervised. This entails designing rules and regulations that are aimed at preventing excessive risk-taking by market participants, and at the same time maintaining the financial innovations in the industry.

2.2.1 Derivatives trading in emerging markets

Mihaljek et al. (2010) assert that about $1.2 trillion a day is the size of derivatives trade executed in emerging markets (EMs) with a 50-50 split overall on both the OTC and exchange-traded derivatives products, although it is of varying degrees across countries. In their findings, of the four largest centres for EM derivatives (Hong Kong, Singapore, Brazil and Korea), exchange-traded derivatives dominate in Brazil and Korea, while OTC derivatives dominate overwhelmingly in Singapore and Hong Kong. For the risk traded in both derivatives, they discovered that 50% of the total derivatives turnover is in Currency derivatives and 30% in equity derivatives, showing that exchange-rate risk is of utmost concern in emerging market economies. Policy makers in the NSM that designed the proposed pioneer derivative products seem to agree totally with the findings of Mihaljek et al. (2010) as most of the derivative products earmarked for introduction into the NSM are aimed at hedging the risks associated with exchange rate.

To support the use of derivatives products in emerging markets and indeed African financial system, there is the need for further education of bank staffers in the field of derivatives trading and its potentials as a tool for risk management. Emira et al. (2012) infer that the main reason for the low level of derivatives supply and demand, especially in emerging markets, is the lack of information and education of banking personnel in derivatives contracting and banks' caution following the global financial crisis. In Brazil, Mullins and Murphy (2009) observe that the growth in derivatives and other financial instruments have afforded the Brazilian stock market more autonomy. In India, the principal regulatory authority for OTC derivatives market is the Reserve Bank of India (RBI) and RBI places restrictions on participation to discourage excessive speculation by users as they are expected to have an existing market exposure that they want to hedge via the derivatives before taking those contracts, (Sundaram, 2013).

Aysun and Guldi (2011), using Brazil, Chile, Israel, Korea, Mexico and Turkey as case studies, show that risk exposure is negatively related to derivative usage. The finding of Shiu and Moles (2010) who investigated what motivates banks to use derivatives discover that the propensity to use derivatives is positively related to bank size, currency exposure and issuance of preferred stock, while negatively related to leverage and diversification of long term liabilities. Nigerian banks are indeed well capitalized in terms of bank size and the naira has a very good international exposure and since Nigerian economy is import-driven through massive importation of consumer goods, there is the need for foreign exchange derivatives in (NSM).

Also, since petroleum products and crude oil prices fluctuate regularly and these products are the mainstay of the Nigerian economy, there is the need for policy makers in the nation's financial industry to devise ways of stabilizing the foreign exchange earnings, through the use of some derivative products like forwards and options in trading petroleum products. In the Malaysian market, Ameer (2009) finds that there is a significant positive correlation between total earning and the use of derivatives and those derivatives have value relevance.

For Dodd and Griffith (2007), derivatives market in Chile and Brazil play a significant role in their financial market and overall economic activity. They infer that of special importance is the OTC market in foreign exchange which has become an established market with dealers, and that it possesses high liquidity and low bid-ask spreads. These derivatives markets have helped firms lower their risks and their borrowing costs.

The trade in derivatives products in global financial markets, including emerging markets is known to provide the following services to market participants: hedging, arbitraging and speculation. Hedgers enter into a derivative contract to protect themselves against adverse changes in the value of their assets and liabilities. In particular, hedgers enter into the contract with the aim that a fall in the value of their assets or security will be compensated by an increase in the value of the corresponding derivatives assets and vice-versa.

One of the implications of efficient risk hedging (shifting) in the derivatives market is the ability to raise capital cheaply in capital markets (leverage). The development of Chile's cross-currency swaps market has enabled some large corporations and banks to lower their cost of borrowing without increasing their exchange rate risk. Dodd et al. (2007) infer that they borrow abroad in hard currency at interest rates lower than in Chile, and then use derivatives to shift out of foreign currency exposure and back into Peso liabilities at an effective Peso interest rate that is lower than borrowing directly in the Chilean capital market. This characteristic will, no doubt, aid the Nigerian banks in mitigating their risk exposure to foreign exchange, while trading in FX swaps as one of the pioneer derivatives products approved for trade in the NSM.

It is worthy of note that the current instability in Nigerian currency (naira) which worsened from 2015 could be reduced not only by CBN directly selling FX to banks and Bureau de Change (BDC) operators from the nation's foreign exchange reserves that depletes same, but by encouraging banks and other stakeholders in the FX businesses to adopt the newly approved foreign exchange swaps and or foreign exchange options in sourcing their needs for foreign exchange. This, however, will help in stabilizing the naira and at the same time strengthening the nation's foreign reserve, thus maintaining a positive outlook for the economy in general.

Keith (1997) refers to arbitrageurs as the market participants that look for opportunities to earn riskless profits by simultaneously taking positions in two or more markets. Speculators attempt to profit from anticipating changes in market prices or rates and credit instruments by entering a derivative contract. The role of hedging and speculation in derivative contracts are said to go together, since according to Jarrow et al (1999), hedging aims at risk reduction, whereas speculation is geared towards risk augmentation, thereby making them flip sides of the same coin.

2.2.2 Derivatives market in Africa

The literature on derivatives trading in emerging markets (EMs), including few African countries that engage in the trade, is believed to be highly fragmented, and mostly limited to individual countries due to lack of unified data base. This shortcoming notwithstanding, turnover of derivatives contracts has grown more rapidly in emerging markets (EMs) than in developed economies and mostly on foreign exchange derivatives contracts (Mihaljek et al., 2010).

Derivatives contract in most African economies is still at the formative stage except for South Africa, where the trade on derivatives products is acclaimed to have grown to an emerging market status. Virtually all the products of derivatives securities are noticeable in South African Stock Market and the derivatives trade has grown rapidly in recent years, supporting capital flows and helping market participants to price, unbundle and transfer risk associated with the portfolio of investments from risk-averse clients to those who are willing and able to take them.

Adelegan (2009) asserts that South Africa's derivatives market was established to further develop the financial system, enhance liquidity, manage risk, and meet the challenges of globalization. Hence, the South African derivatives market, just like other emerging derivatives markets, was introduced primarily because of the need to "self- insure" against volatile capital flows and manage financial risks associated with the high volatility of asset prices. The Johannesburg Stock Exchange (JSE) and Bond Exchange of South Africa (BESA) are the licensed exchanges for derivatives trading in South Africa under the supervision of Financial Services Board of South Africa (FSB).

The African Development Bank Group, AfDB (2010) infers that it is only in South Africa that contracts on derivatives product are well-developed within Sub-Saharan Africa. Other African countries where there are traces of derivatives market products, according to AfDB, include Algeria, Botswana, Egypt, Kenya, Mauritius, Morocco, Namibia, Nigeria, Tanzania and Tunisia. The derivative products evident in most of these countries are foreign exchange forwards, currency swaps, currency forwards, interest rate swaps, with only Morocco and Tunisia operating interest rate Forwards, otherwise called forward rate agreement (FRA) as obtains in South Africa. The FRA may be used by investors to lock-in an interest rate for borrowing or lending over a specified period in the future. We now look at efforts made towards deepening the markets through the introduction of commodity exchanges and financial derivative trade by various regional economic blocks in Africa.

Market information on derivative trade in Africa is generally not available in the research literature as the trade in the region are fragmented and most economies in the Sub-region are still at the formative stage in the process of trading on derivatives with exception of South Africa. For this reason, comprehensive information on the trade in Africa can mostly be found from World Bank assisted activities and sponsored research. In particular, from the proceedings of United Nations Conference on trade and Development (UNCTAD 14) held in Nairobi Kenya July 17-22, 2016, Chakri Selloua (2016) highlight African commodity markets with an insight into various efforts being made by countries in the respective economic blocks of Africa towards kick-starting full derivative trade in the continent. It is therefore necessary to list some of the economic blocks in Africa and their country memberships which are based on interest, geographical affiliation and economic goals of the respective member nations as follows:

(i) Economic Community of West African States (**ECOWAS**) Benin Republic, Burkina Faso, Cape Verde, Cote d'Ivoire, The Gambia, Ghana, Guinea, Guinea Bissau, Liberia, Mali, Niger Republic, Nigeria, Senegal, Sierra Leone and Togo.

(ii) Common Market for Eastern and Southern Africa (**COMESA**) comprising of Burundi, Comoros, Democratic Republic of Congo, Djibouti, Egypt, Eritrea, Ethiopia, Kenya, Libya, Madagascar, Malawi, Mauritius, Rwanda, Seychelles, Sudan, Swaziland, Uganda, Zambia and Zimbabwe.

(iii) Southern Africa Development Community (**SADC**) with members as Angola, Botswana, Democratic Republic of Congo, Lesotho, Madagascar, Malawi, Mauritius, Mozambique, Namibia, Seychelles, South Africa, Swaziland, Tanzania, Zambia and Zimbabwe.

(iv) Arab Maghreb Union (**AMU**) made up of five nations namely: Algeria, Libya, Mauritania, Morocco and Tunisia.

(v) East Africa Community (**ECA**) also comprised of five nations namely: Burundi, Kenya, Rwanda, Uganda and Tanzania.

(vi) Economic Community of Central African States (**ECCAS**) with membership drawn from Angola, Burundi, Cameroon, Central Africa Republic, Chad, Congo Brazzaville, Democratic Republic of Congo, Equatorial Guinea, Gabon, Sao-Tome and Principe.

The above six are the major economic blocks in Africa although there are a few others with membership comprising of almost the same as above. Some of them are:

(vii) Economic and Monetary Community of Central Africa (**CEMAC**) with member states as Cameroon, Central Africa Republic, Chad, Democratic Republic of Congo, Equatorial Guinea and Gabon.

(viii) Community of Sahel-Sahara States (**CEN-SAD**) comprised of Benin Republic, Burkina Faso, Central Africa Republic, Chad, Comoros, Djibouti, Egypt, Eritrea, The Gambia, Ghana, Guinea Bissau, Ivory Coast, Kenya, Liberia, Libya, Mali, Morocco, Niger Republic, Nigeria, Sao-Tome & Principe, Senegal, Sierra Leone, Somalia, Sudan, Togo and Tunisia.

We now look at the derivative markets trade (both commodity and financial) derivatives where they exist in some countries for the above listed six most dominant active economic blocks in Africa.

Ghana (ECOWAS): The commodity derivative trade in Ghana is still at the formative stage as the trade was scheduled to take off in 2016 and till date no concrete evidence is available in the literature on the actual commencement of the trade. The policy makers have only succeeded in initiating a project called Commodity Clearing House (CCH) aimed at introducing an exchange-oriented trade on derivatives at banks that will offer the trade in commodity-backed warrants (warehouse receipts), (Mbeng Mezui et. al.2013). The idea was that the operations of CCH will be regulated by Ghana Commodity Exchange (GCX) and in 2012, CCH arranged for credits towards the development of trade on grains, coffee and sheanuts from local banks. The CCH is also pioneering the trading of repos in the money market in Ghana.

Burkina Faso (ECOWAS): In Burkina Faso, trade on derivatives is still very scanty as it is some Non-Governmental Organisation (NGO) called Afrique Verte that organised 'bourses cerealieves' which can be translated to mean cereal exchanges or cereal fairs in December 1991 in Burkina Faso, to enable direct meetings of the first farmers' association aimed at facilitating cereals trading among regions in Africa. This effort yielded some positive results as few organised trades were carried out on cereal commodities for the first time in Burkina Faso.

Mali (ECOWAS): The quest for trade in derivatives in Mali dates back to 1995 when an NGO called AMASSA- Afrique Verte Mali started organising cereal fairs/exchanges in the country to facilitate intra and inter regional trade, and since then the awareness and interest of market participants in derivatives market in Mali has been on the increase. In December 2012 they were able to, through the National exchange of Mali, bring together 300 market participants from several countries where 129,000 tons of cereals was on the offer from 272,000 trade demands available out of which 44 contracts totalling 55,000 tons of cereal worth 6.6 million Euros was signed, (Mbeng Mezui et. al. 2013).

Niger Republic (ECOWAS): The Afrique Verte, an NGO that set physical spot exchanges in Burkina Faso and Mali, created similar fairs in Niger Republic with two organised exchanges in 2010. The two established exchanges were aimed at bringing together market participants from surplus and deficit regions of the country, although derivatives trade has some presence in the country the volumes were very small as only 1,000 tons were traded.

Nigeria (ECOWAS): Development of derivatives trade is always being encouraged by Nigerian government through enactment of extant laws to regulate and promote the trade, but the non-commencement of full derivatives trade in the NSM has been a source of concern to investors and other stakeholder in the Nigerian market. In 1999 an Investment and Securities Act was passed by the government mandating the Security and Exchange Commission to register and regulate futures, option, derivatives and commodity exchanges (Mbeng Mezui et al. 2013). Nigeria got its own commodity exchange in 2001, when Abuja Securities Exchange was converted into the Abuja Securities and Commodity Exchange (ASCE). Due to administrative bottleneck in the system, the commodity exchange could not record any serious progress in market development as ASCE got depleted which led to it becoming bankrupt and the 40% government equity were taken over by the Ministry of Finance. In 2006, ASCE was revived with intensive effort to get commodity trading back by setting up of some institutional support for effective and efficient trading. Further developments of trade on derivatives in Nigeria are as shown in other sectors of the thesis.

Egypt (COMESA): The Alexandria Cotton Exchange was established in 1861 about 10 years before the New York Stock Exchange (NYSE) came on board and was recorded as the world's oldest futures market. It was adjudged the world's leading exchange for about 90 years of trade on spot and futures contract in not just cotton but also in cotton seed and cereals. The Alexandria Cotton Exchange was closed temporarily for 3years in 1952 and was finally disbanded in 1961 exactly 100 years after establishment, Mbeng Mezui et al. (2013). It was in mid 2000s that the United States Agency for International Development, USAID rescued Egyptian commodity market by commissioning a report towards the establishment of new commodity exchange referred to as reformed commodity exchange in Egypt with a more comprehensive derivatives market that would trade both commodities and financial instruments. Although this effort towards reinventing commodity trade in Egypt has been put in place, evidence of actual trade data on derivatives in Egypt is still scanty.

Ethiopia (COMESA): In 2016, Ethiopian government established the Ethiopian Commodity Exchange (ECX) which received some support from other development partners including United Nations Development Programme (UNDP), World Bank and USAID for its development. In April 2008, ECX started trading on agricultural commodities like coffee, sesame, pea beans, maize and wheat. The Ethiopian Commodity

Exchange had a rapid growth and was seen as a model for other African countries as coffee exports in the country increased from $529 million worth in 2007/2008 to $797 million in 2011/2012 (Whitehead, 2013). ECX is known to be Africa's largest functional exchange after that of South African SAFEX.

Kenya (COMESA/EAC): The recorded first attempt at official trading in commodity exchange in Kenya was 1997 when a private entrepreneur established the Kenya Agricultural Commodities Exchange, KACE, which consisted of two main components, namely the Physical Delivery Platform and a Regional Commodity Trade and Information System. Due to paucity of funds in its early years of existence, the KACE could not afford to develop a functional trading platform; hence, it decided to focus on the provision of market information which the development partners are interested, in in-lieu of operating a commodity exchange. In 1998, the coffee board of Kenya set up the Nairobi Coffee Exchange with an electronic auctioning system with the ambition of becoming a regional hub for coffee trading. At this time of the history of African commodity markets, Kenya was regarded as the site for Africa's first internet-based commodity exchange called 'Africanlion' meaning (where Africa trades).

Malawi (COMESA, SADC): Member of COMESA and SADC regional economic blocks in Africa, Malawi has been active in terms of commodity exchanges as it has three exchange initiatives. The Agricultural Commodity Exchange for Africa (ACE) started in 2004 but commenced operation in 2006 under the USAID assisted project to the National Smallholder Farmer's Association of Malawi (NASFAM). The duties of the ACE border on collection and dissemination of market information, trade facilitation, and implementation of warehouse receipt system, and financing of goods under warehouse receipts. Also, in 2004, another exchange that was modelled after the Kenya's KACE was established and called Malawi Agricultural Commodity Exchange (MACE) with main focus on the provision of exchange information to market participants. Finally, in 2012, the third commodity exchange was established in Malawi and was named AHL Commodity Exchange (AHCX) which was driven by Auction Holdings Limited, a leading tobacco company in Malawi. It was hoped that the exchange will trade on grains like maize, rice, soybeans, pigeon pea, and other commodities like groundnuts and cotton.

Uganda (COMESA, EAC): With support from USAID, the Bank of Uganda established a commodity exchange to trade on coffee, maize, beans, rice, sesame, soybeans, and wheat in December 1998, called the Uganda Commodity Exchange (UCE). Members of the exchange are made up of Uganda Corporative Alliance, The Uganda Coffee Trade Federation, The National Famers' Association, The Commercial Farmers' Association and representatives of two private trading firms (Mbeng Mezui et al. 2013). Although the commodity exchange was established in 1998, the follow up trade was slow as actual trade commenced in 2002 and between March 2002 and 2004 only eleven contracts were traded.

Zimbabwe (SADC, COMESA): As a way of contributing to the development of derivative trade in Zimbabwe, a private sector launched in March 1994 Zimbabwe Agricultural Commodity (ZIMACE). The exchange provided a platform for negotiating contracts which were based on standardized ZIMACE warehouse receipts and commodities they were actually trading on included maize, wheat, and soybeans with trade on these products reaching a volume of 550 million United States of American dollars in 2001. ZIMACE was suspended later in 2001 when the government gave the state-owned grain marketing company board a monopoly for the trading of maize and wheat in the country, and consequently in addition to successive government interventions and unprecedented increases in the prices of commodities, ZIMACE collapsed in 2010, (Rashid et al. 2010). Later in 2010, the government of Zimbabwe declared that it was reintroducing a commodity exchange, the Commodity Exchange of Zimbabwe (COMEZ), but this time handing over the leadership of the exchange to the Ministry of Finance under the public-private partnership.

Zambia (COMESA, SADC): The successful commodity exchange after previous efforts in Zambia to establish a commodity market was the Zambia Agricultural Commodity Exchange (ZAMACE) which was established in May 2007 by a group of 15 grain traders and brokers as a non-profit open outcry exchange. However, the trading on the exchange stopped in 2011 as a result of government undue interference in the activities of the exchange and this necessitated USAID to withdraw its funding to the commodity exchange. The ZAMACE had to undergo structural transformations which resulted in the adoption of a new less interventionist agricultural marketing act of 2012 after which ZAMACE started trading again in 2013, (Mbeng Mezui et. al. 2013).

In a bid to deepen further their trade using derivatives products, ZAMACE signed an agreement with the South Africa Futures Exchange (SAFEX) which is a subsidiary of Johannesburg Stock Exchange (JSE) for SAFEX to start trading Zambian maize, wheat and soybeans in the United States of America dollars, which would provide arbitrage opportunities to traders on ZAMACE thereby increasing the volume of trade in the exchange.

There is always commitment on the part of government in Zambia towards the development of derivative trade in the country and in 2012 the Zambian government licensed a new exchange called the "Bond and Derivatives Exchange" (BDE) which was owned by local banks, pension funds and securities brokers and was designed to use the South Africa trading system for its operations. Products earmarked to be traded in the BDE include corporate bonds, municipal bonds, currency futures and options, interest rate derivatives which includes swaps, equity derivatives and commodity derivatives using copper, cobalt, gold, oil, wheat as the underlying assets, spot and currency derivatives market, commodity derivatives and commodity spot markets, agricultural derivatives, energy derivatives and precious metals derivatives market. Although the BDE have these lofty ideas of potential products to be traded on derivatives in Zambia, evidence of effective commencement of actual trading on those lines of products are still being expected.

South Africa (SADC): Mbeng Mezui et al. (2013) declare that Uganda, Zimbabwe, Kenya, Zambia and South Africa were pioneers in the launching of commodity exchanges; and that the only successful one was the South Africa Futures Exchange, SAFEX, a subsidiary of the Johannesburg Stock Exchange (JSE). Supporting this view, Adelagan (2009) asserts that South Africa's derivative market which is comprised of two categories - options and futures - is the only functional derivatives market in Africa and that it was established to further develop the financial system, enhance liquidity, manage risk and meet the challenges emanating from globalization of economy. The SAFEX was birthed from an informal financial market introduced in 1987 by Rand Merchant Bank and subsequently option contracts were introduced in 1992, followed by agricultural commodity futures in 1995.

In contrast, equity derivatives division of the JSE was introduced in 1990 whose responsibility were the coordination of trading activities in warrants, single stock futures (SSF), equity indices and interest rate futures and options.

The deregulation of agricultural products market in 1995 paved way for the establishment of an agricultural commodity market in South Africa otherwise known as the South Africa Futures Exchange with about 52 companies listed on the exchange. Initially, the exchange started with trading on physical settled beef contract and potato contract which were later delisted and replaced with contracts on white and yellow maize in 1996, wheat in 1997, and sunflower seeds contracts in 1999. Options contract were, however, introduced on all the above grain commodities which resulted in advanced price risk management for market participants and by 2002 SAFEX could boast of over a hundred thousand contracts monthly.

A licensing agreement was signed in 2009 between SAFEX and the world's largest exchange group, Chicago Mercantile Exchange (CME), which permits the former to introduce contracts denominated in local currency that were indexed to CME contracts on maize, gold, crude oil. This agreement permits proxy access to the international market to South Africa investors. In 2013, new lines of products were introduced for derivative contracts in SAFEX, which include heating oil, gasoline, natural gas, palladium, sugar, cotton, cocoa and coffee.

2.2.3 Derivatives trading in Nigeria

The Management of the Nigerian Stock Exchange (NSE) indicated during the Scientific visit of our Research group - SIMFIM, SHU, UK - with them in Nigeria in 2014 their intention to introduce derivative products in the NSM. In the policy statement, the NSM was interested in using derivative products to deepen the markets, and at the same time see how they could use the products to enable market participants to perform the traditional roles of derivatives in risk hedging, speculation and arbitrage.

The NSE management also noted that the NSM is benchmarking their plan to trade on the products based on the performance of the Johannesburg Stock Exchange (JSE), in those products that they are interested in, given the relatively more advanced status of the JSE as the only exchange in Sub-Saharan Africa where there is some good evidence of trade on derivative products. It is in these regards that the researcher would, in addition to studying the underlying stock market price behaviour of the products traded in the NSM, take a closer look and compare the stock market characteristics of the NSM with those of the JSE, for the banks and other underlying stocks in this research.

Derivatives trading plays significant role in the development and growth of an economy through risk management, speculation or price discovery. Trade in derivative products also promote market completeness and efficiency which includes low transaction costs, greater market liquidity and leverage to investors, enabling them to go short very easily. Derivatives will also, apart from hedging ability mentioned earlier, provide market participants with the price discovery of the underlying asset(s) like the exchange rate of the Naira over time, (Dodd et al., 2007). Derivatives markets can serve to determine the spot price and future prices, and in case of options the price of the risk is determined in the form of premiums paid by the option holders.

Nigeria's quest to join the league of nations in the derivative trade which is targeted towards deepening the financial markets, received a boost through an approval granted by the Central Bank of Nigeria (CBN) to formally kick start the trade with the necessary legal and logistic backing required for its take-off in 2014, and through the establishment of Nigerian Association of Securities Dealers (NASD) Ltd in 2012, with a grant of N40 million for the (NASD) OTC platform. NASD, established in 2012, is to boost the market and it is a formal OTC platform for the trading of unlisted equities, bonds and money market instruments. With most of the pioneer products slated for the formal take of derivatives trade meant to be over-the-counter traded derivatives, there is, therefore, the need for transparency in the industry so that deepening the market through the derivatives trading would attract foreign capital inflows and strengthen the economy.

2.2.4 Approved Derivatives Products for Nigerian Capital Market and their Features

The strong emergence of derivatives in last few decades as the most cost-effective way to manage risks, has triggered considerable interest among financial market participants; the Nigerian financial market therefore cannot be an exception. The contemporary finance discipline is also becoming more and more focused on hedging activities and risk management practices of corporations, Nguyen et al. (2003). Most of the derivatives instruments earmarked for initial trading in the Nigerian financial market are mostly Foreign Currency Derivatives (FCD). Elliot et al. (2003) asserts that Foreign Denominated Debt (FDD) is used as a hedge and substitutes for the use of FCD in reducing currency risk.

Foreign currency derivative is any financial instrument that locks in a future foreign exchange rate. The foreign currency derivative can be used by currency or forex traders, as well as large multinational corporations. The latter often use these products when they expect to receive large amounts of money in the future but want to hedge their exposure to currency exchange risk. Financial instruments that fall into this category include currency option contract, currency swaps, forward contracts and futures. These products, except for futures, are essentially the derivatives products earmarked for trade in Nigeria. The option contract could be both over-the-counter products and traded on organised exchanges like the NSE. Currency options can be priced using Black-Scholes option pricing model with some little modifications, by replacing the risk-free rate with domestic and foreign currency interest rates respectively.

We now look at the attributes of each of the derivatives in the following order: Foreign Exchange options, Forwards (Outright and Non-deliverables), Foreign Exchange Swaps, and Cross-Currency Interest Rate Swaps.

2.3 Features of the approved derivatives products in the NSM

2.3.1 Options

An option is a derivative security that offers its owner a right, not an obligation, to trade in a fixed number of shares of a specified common stock at a fixed price at any time on or before a pre-determined future date. The power to exercise this right only on the exact given or pre-determined date (expiry date) is referred to as European option, while that which confers on the option holder the leverage or authority to exercise this right on or before the expiry date is termed American option. Thus, the name European or American has nothing to do with the geographical location but rather on the type of exercise right conferred on a given option contract.

Option trading is where the action is in the security markets and virtually every financial contract has option features or can be decomposed into options. Black (1975) attributes the growing popularity in options trading to the fact that the brokerage charge for taking a position in options can sometimes be lower than the charge for taking an equivalent position directly in the underlying stock. He further stated that an option on a stock that is expected to go up has the same value, in terms of the stock, as an option on a stock that is expected to go down. The rules for an option buyer are the same as the rules for an option writer. If the option is under-priced, buy it and sell when it is overpriced. The writer's gains are the buyer’s losses, and the writer's losses are the buyer's gains. Hence, for an overpriced option the writer is likely to gain while in the under-priced he is likely to lose giving the buyer the opportunity to gain.

Option listing, however, has some significant impact on the underlying stock prices. Ma and Rao (1988) argue that there is a differential market impact of options on underlying stocks, with volatile stocks becoming more stable after listing because of hedging behaviour by uninformed traders, and stable stocks becoming more volatile after listing due to increased speculation in the options markets on the part of informed traders. DeTemple and P. Jorion (1988) assert that options listing and subsequently trading on them provide significant welfare benefits to investors with greater risk assessments, compared to those who move into stocks market.

Determining the economic value of the contract obligations is in many cases a matter of valuing the underlying options. Hulle (1998) asserts that option literature knows how to price options on shares, bonds options, foreign exchange, futures, options, commodities, derivatives asset, and multi-asset options like options to exchange assets, multicurrency bond options, options on the minimum and maximum of assets, with remarkable evidence of progress recorded in using options for real assets valuation.

The indispensability property of option in derivatives trade was confirmed by Boyle (1976), who notes that option valuation models are very important in the theory of finance, since many corporate liabilities can be expressed in terms of options or combination of two or more options.

Option pricing theory however has a long and illustrious history, but it also underwent a revolutionary change in 1973 (Cox et al., 1979). They assert that it was Black and Scholes who presented the first completely satisfactory equilibrium option pricing model, which was extended the same year by Robert Merton with several other extensions following afterwards.

The Black-Scholes parabolic partial differential equation (PPDE) is one of the most important mathematical models of financial markets commonly used in option pricing. It is pertinent to mention here that, in my own view, there is the need for appropriate pricing of financial instrument as correct pricing would prevent pure arbitrage opportunities, thereby ensuring that the trades in derivative instruments are based entirely on the perceived true value of those derivative instruments under consideration.

For a European option, which Nigeria is adopting, the Black-Scholes (PPDE), BS (1973) is given by:

$$\frac{\partial V}{\partial t} + \frac{\delta^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} - \mathrm{rv} + \mathrm{rS}\frac{\partial V}{\partial S} = 0 \qquad (2.1)$$

where S is the underlying asset price at time t, V is the value of the option at time t, defined as a function of S and t, and r is the risk-free interest rate.

The price of the underlying stock follows a geometric Brownian motion (GBM) process with μ and δ constant. Furthermore, the (GBM) satisfies the following stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{2.2}$$

which can also be termed as stock price model, where S(t) is the underlying stock price at time t, μ is the rate of return on riskless asset (or drift), δ captures the volatility of the stock price, and W(t) represents the Brownian motion or the white noise in the trade.

As Black Scholes option pricing model is typical for European options, which incidentally is the type of option pricing formula Nigerian Stock Exchange is interested in, we will therefore prioritize the Black-Scholes (1973) seminal work on option pricing and its extension in this research. Other derivatives pricing option that were not considered in detail in this work but worthy of mention include, for example, American options, Russian option, Israeli options and the Asian options. The American option is the same with the European option, the only difference being the possibility of exercising the option before expiry for American option, unlike the European option that can only be exercised on the expiry date. Duisttermaat et al. (2005) assert that there are other different types of options which are American in nature, for example, the Russian option where given a risky asset whose price dynamics is represented by

$$S_t = sexp[\sigma W_t + \mu t], \tag{2.2a}$$

($s > 0$ $where$ $\sigma =$ Volatility, $\mu =$drift and W_t is a Wieners process), the pay-out on the Russian option contract is of the form

$$Y_t = e^{-\alpha t}[\max\{m, \sup_{u\in[0,t]} S_u\}] \tag{2.2b}$$

for $\alpha \geq 0, m > 0$ and $t \in [0, T]$.

Furthermore, Kyprianou (2004), Baurdoux and Kyprianou (2004) describe another kind of American option which they refer to as Israeli ∂-penalty Russian options where the writer can annul the contract at will but his punishment for the early annulment of the contract attracts a penalty equivalent to $e^{-\alpha t}\partial$. However, when the contract is allowed to mature, the claim for an option holder with $\partial > 0$ is given by

$$Y_t = e^{-\alpha t}[\max\{m, \sup_{u\in[0,t]} S_u\} + \partial S_t] \tag{2.2c}$$

Vecer and Xu (2004) describe Asian options as securities whose payoff depend on the average of the underlying stock price S over a certain time interval. They declare that for λ representing the average factor of the option, one can write the general Asian option payoff as

$$Y_t = [\xi\{S_t d\lambda(t) - K_1 S_T - K_2\}]^+ \quad (2.2d)$$

and that for K_1= 0 we obtain a fixed strike option whereas for K_2= 0 we will have a floating strike option. It is the value of the parameter ξ = ±1 that determines if the option will be a call or put option.

The Black-Scholes model for European call option is given by:

$$c = SN(d_1) - Xe^{r\tau} N(d_2) \quad (2.3)$$

with $$d_1 = \frac{\ln\left(\frac{S}{X}\right)+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}.$$

where c is the market value of the European call option (In American type of option we represent C in capital); S is the price of the underlying security; X is the exercise price; τ is the time to expiration; r is the short-term interest rate which is continuous and constant through time; σ^2 is the variance rate of return for the underlying security; $N(d_i)$ is the cumulative normal density function evaluated at d_i, i = 1 and 2.

The main advantage of Black-Scholes (1973) option pricing model is that all the parameters are observable, except the underlying stock volatility, thereby making it possible for the model to provide a closed form solution to option prices. It, however, has some shortcomings from the assumptions mostly from the underlying stock for the derivatives products, which include the assumption that volatility of the underlying stock is constant throughout the duration of option contracts.

The need to extend the Black-Scholes formula is indisputable following the underlying assumptions that characterize the use of Black-Scholes option pricing formula not holding in some contexts. These include no dividend payment throughout the life time of the option on the underlying stock, fixed interest rate with volatility of the underlying stock known and constant. For the model to be applicable, it is necessary that the option must be European meaning that it can only be exercised on the expiry date, markets are efficient (market movement cannot be predicted), no commission is charged for buying and or selling of the option (no arbitrage) and that returns on the underlying stocks are lognormally distributed.

New developments in option pricing formula have been on before the major breakthroughs of 1973 with the put-call-parity (PCP) relationship that was originally developed by Stoll (1969), and later on extended and modified by Merton (1973) to account for European stock options with continuous dividend stream, and this was further modified by Hoque et al. (2008) with the spot market bid-ask-spread as a measure of transaction cost which was overlooked by Black-Scholes.

For the Put-Call parity relation, Stoll (1969) states that If C(t) and P(t) are prices of European call and put options on the same asset with the same maturity T and strike price K, then the put-call parity relation will be

$$P(t) + S(t)e^{-q(T-t)} - C(t) = Ke^{-r(T-t)} \quad (2.4)$$

where S(t) is the stock price at time t, q = continuous asset dividend rate, r = interest rate and T = option maturity time.

2.3.2 Foreign Exchange Options

Foreign exchange options (hereafter referred to as FX option or currency option) are a recent financial derivative market innovation. The standard Black-Scholes option pricing model does not apply well directly to Foreign Exchange Options as noted by Garman et al. (1983), since multiple interest rates are involved in ways differing from the Black-Scholes assumptions. In the standard Black-Scholes (1973) option pricing model, the underlying deliverable instrument is a non-dividend paying stock. A FX option is a derivative financial instrument that gives the owner the right but not an obligation to exchange money denominated in one currency into another currency at a pre-determined exchange rate on a specified date.

Foreign currency options arise in international finance in three principal contexts, Grabbe (1983). The three uses of FX options are: in organised trading on an exchange;
FX options are also used in the banking industry where money-centre banks write FX options directly to their corporate customers. However, the bank transactions in the FX options market are largely invisible as banks are reluctant to make public any data regarding their activities in this sphere with their customers; and FX options feature on bond contracts in the international bond market.

The FX component of bond market is witnessed when the repayments or redemption of the bond is at the owner's discretion on if it is with the local currency or foreign currency. Consider, for instance, a bond of $2,000 was sold to an investor at the coupon rate of 10% per annum. At maturity the bond is redeemed at the owner’s discretion in Naira or dollars at an exchange rate of say N160 per dollar or 0.00625$/N. The bond owner will opt for repayment of principal in naira if the spot price of naira is greater than 0.00625$/N. If the spot rate is 0.00725$/N, the owner would redeem it for 2000/.00625 = 320000 naira and then sell the naira for (320000) (0.00725) = $2320. Thus the value of this bond can be viewed as the sum of the value of an ordinary $2000 bond with a 10% coupon, plus the value of European call option on N320000 with an exercise price of 0.00625$/N at the expiry date.

The FX option market is the deepest, largest and most liquid option market and is mostly traded over-the-counter but is highly regulated. One of the biggest innovations in financial markets industry has been the introduction of options on currencies. Hoque et al. (2008) assert that FX options were designed not as substitute to forwards or futures contracts, but as an additional and potentially more versatile financial vehicle that can offer significant opportunities and advantages to those seeking protection on their investment against unanticipated changes in exchange rates.

Nigeria, like most emerging African markets, are faced with the fluctuations in the value of its local currency - the Naira - when compared with the rate at which the naira exchanges with United States of American Dollars, for instance, with reference to the rate at which the other major currencies like Pounds Sterling and Euro exchange with each other. Hence, there is need to trade on derivative options like Foreign Currency to guard against these erratic fluctuations in the value of the naira. This, no doubt, will encourage more trading partners in the Nigerian market and help the market participants reduce the risks associated with their portfolio of investment in the Nigerian economy, which will, therefore, increase the growth and development of the economy.

The difference between FX options and the traditional options is that in traditional options, the option buyer is to give an amount of money and receive the right to buy or sell a commodity, stock or other non-money asset, whereas in FX options the underlying asset is also money denominated in another currency. Corporations primarily use FX options to hedge uncertain future cash flows with forward contracts. In general, foreign exchange derivative is a financial derivative whose payoff depends on the foreign exchange rate(s) of two (or more) currencies. These derivative products are used for speculation, arbitrage and for hedging foreign exchange risk. The instruments used in foreign exchange derivatives are: Binary Option-Foreign exchange, Currency Future, Currency Swap, Foreign exchange forward, Foreign exchange option, Foreign exchange swap, and foreign exchange rate.

2.3.3 Forwards (Outright and Non-deliverables)

As an illustration, consider an American importer that expects to receive £100 million in three months with the current price of pound sterling as $1.40. Suppose the price of pound falls by 10 percent over the next three months, the exporter losses $14 million. By selling pound sterling forward, the exporter locks in the current forward rate (if the forward rate is $1.16, the exporter receives $116 million at maturity)

A forward derivatives contract obligates one party to buy the underlying asset at a fixed price at a certain time in the future, called 'maturity', from a counterparty who is obligated to sell the underlying at that fixed price, Stulz, (2004). Forward contracting is very valuable in hedging and speculation. The classic hedging application would be that of a rice farmer forward selling his harvest at a known price in order to eliminate price risk. If a speculator has an information or analysis which forecasts an upturn in a price, then he can go long on the forward market instead of the cash market. The speculator would go long on forward and wait for the price to rise after which he would then carry out a reverse transaction, thereby making profit.

A foreign exchange outright forward is a contract to exchange two currencies at a future date at an agreed upon exchange rate. Forward contracts (both deliverables and outright) represent agreements for delayed delivery of financial instruments or commodities in which the buyer agrees to purchase and the seller agrees to deliver, at a specified future date, a specified instrument or commodity at a specified price or yield. Forward contracts are generally not traded on organized exchanges and their contractual terms are not standardized. This type of derivative also includes transactions where only the difference between the contracted forward outright rate and the prevailing spot rate is settled at maturity, such as non-deliverable forwards (i.e. forwards which do not require physical delivery of a non-convertible currency).

The pricing of most forward foreign exchange contracts is primarily based on the interest rate parity formula which determines equivalent returns over a set time-period based on two currencies' interest rates and the current spot exchange rate. In addition to interest rate parity calculations, many other factors can affect pricing of forward contracts such as trading flows, liquidity, and counterparty risk.

An outright forward is a forward foreign exchange contract (normally contract between the market making bank and the client), in which a bank undertakes to deliver a currency or purchase a currency on a specified date in the future, other than the spot date, at an exchange rate agreed upfront. The formula is:

$$Outright\ forward = SP * \frac{[1 + (ir_{vc} * t)]}{[1 + (ir_{bc} * t)]} \tag{2.5}$$

with
SP = spot price / exchange rate
ir_{vc} = interest rate on variable currency
ir_{bc} = interest rate on base currency
t = term, expressed as number of days / 365.

The above formula is referred to as the standard formula, since the vast majority of forwards are contracted for standard periods of less than a year (like 30-days, 60-days, 90-days, 180-days, and so on).

When the period is longer than a year, the formula becomes:

$$Outright\ forward = SP * \frac{n[1 + ir_{vc}]}{[1 + ir_{bc}]^n} \quad (2.6)$$

n = number of years (but when the period is broken years, like 430 days, then n = 430 / 365).

The non-deliverable forwards (hereafter referred to as NDF) markets are used for currencies that have convertibility restrictions and this is peculiar to currency of emerging financial markets. These restrictions emanate from control imposed by local financial regulators and consequently the non-existence of a natural forward market for non-domestic players, which forces the private companies and investors in these economies to look for alternative avenues to hedge their exposure to such currencies.

The reason for the restrictions and or perceived non-liberalization of the onshore trade is not farfetched. Local monetary authorities fear that easy access to onshore local currency loans and deposits, and the ability to easily transfer local currencies to non-residents, encourages speculative financial movements, greater exchange rate volatility, and ultimately some loss of monetary control (Higgins and Humpage, 2005). NDF therefore is a popular derivative instrument that takes care of offshore investors' hedging need and is also a derivative trading in non-convertible or restricted currencies without delivery of the underlying currency whose trade normally takes place in offshore centres, Sangita et al. (2006).

In NDF transactions, no exchange takes place in the two currencies' principal sums and the only cash flow is the payment of the difference between the NDF rate and the prevailing spot market rate, but this amount is however settled on the expiry date of the contract in a convertible currency, usually the US dollars in an offshore financial centre. The other currency, usually the emerging market currency that has the capital control is non-deliverable. The NDF prices usually depend on the anticipated changes in foreign exchange regime, speculative positioning, prevailing local onshore interest rate markets, the relationship existing between the offshore and onshore currency forward markets and Central bank monetary policy.

To reiterate earlier notes, NDF contracts are used to hedge or speculate against currencies when exchange controls make it difficult for foreigners to trade in the spot market directly. The idea is the same as a regular foreign exchange forward where an investor or company wants to lock in an exchange rate for a certain period in the future. The contracts are called 'no-deliverable' since no exchange of the underlying currency takes place but instead the whole deal is settled in a widely traded currency, normally in United States of America dollars.

We note that it is not only speculations or hedging roles that NDF are known to play in derivatives market. The offshore markets also form an important part of the global and

Asian foreign exchange markets, equilibrating market demand and supply in the presence of capital controls (Ishii et al. (2001), Watanabe et al. (2002)). Ma et al. (2004) claim that while the NDF markets have at times presented challenges to policymakers, the rise of NDF trading could nevertheless prove beneficial to the development of local currency bond markets in Asia. Consequently, NDF markets could potentially facilitate foreign investment in Asia's expanding local currency bond markets, thereby diversifying and adding liquidity to them, (Jiang and McCauley, 2004).

The difference between onshore currency forward prices, where they are available, and NDFs can increase in periods of heightened investor caution or concern over potential change in the exchange rate regime or a perceived increase in onshore country risk, Lipscomb (2005). Prices in the NDF market can be a useful informational tool for authorities and investors to gauge market expectations of potential pressures on an exchange rate regime going forward.

2.3.4 Foreign Exchange Swaps

A currency swap or a Foreign exchange swap (which is not the same as a cross-currency swap) is a derivative contract that simultaneously agrees to buy (sell) a specified amount of currency at an agreed rate, on the one hand, and to resell (repurchase) the same amount of currency for a later value date to (from) the same counterparty, also at an agreed rate, on the other hand. In a FX swap two parties exchange specific amount of two distinct currencies and repay the resulting amount on the exchange at a future date through a predetermined rule that reflects both interest payments and amortization of the principal. Swaps can take place both in the domestic and international markets and are used by a variety of market participants which include banks, corporations, and insurance companies, international agencies like the World Bank, and sovereign states. They are of FOUR types:

1. Parallel or back-to-back loans
2. Swap transactions, comprising of
    - i credit swaps,
    - ii currency swaps,
    - iii currency coupon swaps,
    - iv interest rate swaps,
    - v basis rate swaps,
    - vi commodity swaps,
    - vii swaps with timing mismatches,
    - viii swaps with option-like payoffs (swaptions),
    - ix amortizing swaps,
    - x zero swaps, long-dated or long-term foreign exchange contracts

3. Forward rate agreements (FRAs)
4. Caps, collars and floors.

Hooyman (1994) notes that Currency swaps, like the interest rate swaps and cross-currency interest rate swaps, are used: a) to exploit the differences in credit rating and differential access to markets, thereby obtaining low-cost financing or high-yield assets; b) to hedge interest rate or currency exposure; c) to manage short-term assets and liabilities; d) to speculate; e) Central banks are also known to use currency swaps for hedging asset-liabilities although this is not common features of currency swaps; and f) swaps are also used by developing countries as a tool for the management and acquisition of foreign exchange reserves.

Wall et al. (1989) assert that swaps could be a more efficient alternative method for risk management in that they allow a firm to reduce the agency costs of long-term debt without exposure to changes in interest rate. They further note that a firm that wishes to lock in a long-term rate but is unwilling to pay the premium required to compensate for the problems of underinvestment and asset substitution when issuing long-term debt, can issue short-term debt and enter a swap as a fixed rate payer.

Nance et al. (1993) investigate the determinants of firm hedging and the attributes, including the use of forwards, futures, swaps and options. They discover that firms that hedge are larger, face more convex tax functions, lower interest coverage, and have more growth opportunities. In a related development, Geczy et al. (1997) find that firms using currency derivatives to hedge have greater growth opportunities and tighter financial constraints.

2.3.5 Cross-Currency Interest Rate Swaps

Foreign exchange (FX) and their hybrid derivatives markets like Foreign exchange swaps and cross-currency interest rate swaps are some of the most liquid markets in the world, and the growth of interest rate and FX or currency swaps is often cited as a factor promoting the further integration of global financial markets.

A cross-currency basis swap agreement is a contract in which one party borrows one currency from another party and simultaneously lends the same value, at current spot rates, of a second currency to that party. The market participants involved in basis swaps are usually financial institutions, either acting on their own or as agents for non-financial corporations.

Balsam et al. (2001) find that firms engaging in swaps subsequently have lower cash flow variance than non-swapping firms, a finding consistent with firms engaging in swaps for risk reduction/hedging purposes. They further suggest that firms that have decided to reduce their total risk may adopt a package of measures to reduce risk, for example, currency swaps to reduce exchange rate risk, interest rate swaps to reduce interest rate risk, and changes in investment policy to reduce operating risk.

Cross-currency basis swaps have been employed to fund foreign currency investments, both by financial institutions and their customers, including multinational corporations engaged in foreign direct investment. They have also been used as a tool for converting currency liabilities, particularly by issuers of bonds denominated in foreign currencies. Mirroring the tenor of the transactions they are meant to fund, most cross-currency basis swaps are long-term, generally ranging between one and 30 years in maturity, Baba et al. (2008). It is also worthy of mention here that in Cross-Currency Swaps, the two interest rates being swapped are in different currencies, one local or domestic, Z_d and the other foreign currency Z_f respectively.

Interest rate swaps are agreements between two institutions in which each commits to make periodic payments to the other based on a predetermined amount of notional principal for a predetermined life, called the maturity. The periodic payments may either be fixed or floating rate with an agreed-upon floating index such as the six-month London interbank offered rate (LIBOR), Sun et al. (1993).

Dempster et al. (1996) assert that one might also use a Cross-Currency model to price currency swaps, since cross-currency model also incorporates two additional explanatory variables that affect the domestic term structure through correlation. They further state that the most common (vanilla) cross-currency swap is the exchange of floating or fixed rate interest payments on principals Z_d and Z_f. In interest rate swaps, no principal amounts exchange hands, and for the so-called generic interest rate swaps, one counterparty pays a fixed rate while the other a floating rate, with the payment frequency coinciding with the term of the floating index.

Cross-Currency Interest Rate swaps are indispensable in a developing economy like Nigeria, as it is known to provide the following services. Cross-Currency Interest-Rate Swaps allows the firm to switch its loan from one currency to another. As Nigerian naira is known to have been unstable with regards to its value compared with the major currencies of the world including United States of America dollars, European Euro and British Pounds, it is imperative that banks and other investors in Nigeria should subscribe to currency swaps to reduce risks associated with their investments in the Nigerian Market. These investors in the currency swaps are also known to be allowed to choose between fixed- or floating-rate interests, and this measure provides the required insurance they may require in protecting themselves against unanticipated fluctuations in the prices of currencies that they may be interested in swapping.

The swap as a derivative instrument allows the firm to borrow in the currency which will give it the best terms. The firm can use Cross-Currency Interest-Rate Swaps to switch the loan back into any currency it chooses. Cross-Currency Interest-Rate Swaps can reduce foreign currency exposures. The firm can use money it receives in foreign currency to pay off its loans when it switches them, and the firm can protect itself against changes in interest rates by creating fixed-rate loans.

According to AfDB (2010), foreign exchange forwards exist in Nigeria which is usually subject to a maximum of three years, allowing dealers to engage in swaps transactions among themselves or with retail/wholesale customers. These transactions are deliverable forwards and swaps; the AfDB report declares that the undeliverable forward market in Nigeria is underdeveloped and has very poor liquidity with a tenor of up to six months.

2.4 summary and conclusion

The need to fully adopt derivative trade in the NSM cannot be overemphasized since, according to Neftci (1996), it is mainly the need to hedge interest-rate and currency risks that brought about the prolific increase in markets for derivative products, and the Nigerian financial market is currently confronted with problems of interest rate and currency risk. This chapter reviewed the background to this research mainly in form of the structure of the NSE and the NSM which it oversees, the policy to introduce some selected derivatives in Nigerian capital markets, the different types of products, and related literature on where and how these products are traded, with a focus on emerging markets with similar characteristics as the NSM, particularly the benchmark JSE.

With the new derivatives products being developed for the Nigerian financial market, the conceptual understanding of the structure, functioning and pricing of these derivatives and other derivatives and financial engineering products are of prime importance to the stakeholders in the Nigerian financial system, hence this research. As Nigeria is now ready to formally introduce derivatives trading in its capital market, it is pertinent to explore the characteristics of some derivatives products in developed and emerging markets, to be able to compare features of these derivatives products in such economies and seek ways of adapting the derivatives pricing models to the NSM. Chapter 3 of the thesis will review the stochastic calculus foundations of the research and the key derivative pricing models.

CHAPTER 3

LITERATURE REVIEW

3.0 Introduction

In this chapter, we look at some models that will constitute the fundamental concepts of interest in the research work. Prominent among them are the stochastic calculus models, including the Black-Scholes option pricing models for call and put options. The literature review is organised in accordance with the various objectives of the research as follows: stochastic calculus models, extensions of Black-Scholes (1973) option pricing model, stylized facts of asset returns, and the concept of Random Matrix Theory.

3.1 Stochastic calculus models for financial derivative pricing and trading

Stochastic calculus in the research is a mathematical method used in modelling and analysing the behaviour of economic and financial phenomena under uncertainty, by means of Ito lemma, stochastic differential/integral equations, stochastic stability and control. Stochastic partial differential equations (SPDEs) will be of utmost concern, including their use in asset pricing or portfolio modelling.

Malliaris and Brook (1982) assert that stochastic calculus is useful in such areas as determining the solution of Black-Scholes option pricing, and market risk adjustment in project valuation by method of Constatinides (1978). In light of this, we formally state the Black-Scholes (BS 1973) partial differential equation, which is a stochastic calculus model given by equation (2.1). The BS equation uses the geometric Brownian motion as the governing relation for the underlying asset stock price. This asserts that in a financial market the value of an underlying asset S(t), t ϵ R^+ satisfies the stochastic differential equation (2.2) as shown in chapter 2.

The Black-Scholes model assumes constant volatility on the Geometric Brownian Motion (GBM) for the underlying asset price. Since volatilities are not necessarily constant over typical life spans of derivative products, for example options, other models for asset pricing have emerged, some of which are variants of Black-Scholes that are adjudged to perform better than the original Black-Scholes model, [Amin & Ng (1993), Heston (1993), Jiang and Sluis (2000), Scott (1997)]. In view of this assertion, this research seeks to look at the Black-Scholes method and other derivatives pricing models with regards to the stylized facts and market characteristics of the NSM, to provide the desired theoretical result which will give the expected research support for the proposed introduction of some (pioneer) derivative products to the NSM.

3.2 Ito calculus

Ito calculus, named after Kiyoshi Ito, a Japanese that worked with the Japanese Bureau of Statistics around 1942, is an extension of classical calculus to stochastic processes such as Brownian motion. This type of calculus has numerous applications in mathematical finance and stochastic differential equations as the future prices of stocks are not known in advance and as such is stochastic. Fekete et al. (2017) in their research on continuous state branching process adopted the principle of Ito formula for non-negative twice differentiable and compact supported function to illustrate some theorem in a conditional probability of a Poisson distribution under a prescribed intensity for a time in-homogenous process. We review here the concept of Ito Calculus which is required for the underlying stock price dynamics in financial derivative asset pricing.
We note the following relations in Ito's calculus:

	dW	dt
dW	dt	0
dt	0	0

Table 3.1

Ito calculus is indispensable in the theory of asset derivatives pricing especially for the underlying asset price in the BS option pricing formula for call and put options. If the underlying stock return in a derivatives asset is driven by the Wiener's process as stated in equation (2.2) and for f = f(s, t), a function of stock price at time t, from Ito's lemma, (which is used in deriving the Black-Scholes option pricing formula), we shall have:

$$\boldsymbol{df = f_s ds + f_t dt + \frac{1}{2}\{f_{ss}(dS)^2 + 2f_{st}dSdt + f_{tt}(dt)^2\}},$$

that is:

$$\mathrm{df} = \frac{\delta f}{\delta S}\mathrm{dS} + \frac{\delta f}{\delta t}\mathrm{dt} + \frac{1}{2}\left\{\frac{\delta^2 f}{\delta S^2}(\mathrm{dS})^2 + 2\frac{\delta^2 f}{\delta S \delta t}\mathrm{dSdt} + \frac{\delta^2 f}{\delta t^2}(\mathrm{dt})^2\right\} \qquad (3.1)$$

From equation (2.2) and table (2.1), we shall have:

$$= \left\{\mu S\frac{\delta f}{\delta S} + \frac{\delta f}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 f}{\delta S^2}\right\}dt + \sigma S\frac{\delta f}{\delta S}dW \qquad (3.2)$$

where, μ, σ have their usual meanings with dW, S as Geometric Brownian motion and underlying stock price, respectively.

We intend to look at other stochastic calculus models that try to address the shortcomings from the underlying assumptions of the Black-Scholes (BS) models which results in the mispricing of security asset especially for deep-in (out) of-the money options and near deep-in-(out)-of-the money options.

Remarks:

The most attractive feature of the BS model is that all the parameters in the model, except the volatility, that is, the time to maturity, the risk-free interest rate, the strike price, the current underlying asset price, are observable. This is because in option pricing theory, the risk-neutrality assumption allows us to replace the expected rate of return by the risk-free rate of interest. That is, the only unobservable value in the stock price process of the Brownian motion in equation (2.2) and the associated option pricing formula is **σ.** The unobservable parameter **σ** can be estimated from the history of stock prices, that is using the sample standard deviation of the return rate, Hull (2002).

3.2.1 Main features of competing pricing models

We now highlight the main features of some pricing models associated with derivative products. Most of the models are theoretically robust, but lack the practicability of the BS model, as stakeholders see the models as burdensome and near impracticable for use in the valuation and pricing of the products. We will for trials of some models that could be useful in Nigerian Stock Market demonstrate the use of some of the models including Black-Scholes and Practitioners Black-Scholes, otherwise called Ad-Hoc Black-Scholes for the pricing of derivative products. To take care of the constant volatility assumed by Black-Scholes, which is widely adjudged to be untrue, the implied volatility procedure is the focus of practitioners Black-Scholes and is known to work well in derivative pricing.

To this end, we will look at various aspects of the Practitioners model which include the 'relative smile' and 'absolute smile' models, depending on the emphasis in the model. Some models emphasise time to maturity and the exercise price as the main determinants of implied volatility, whereas others like ‘relative smile’ put more emphasis on the moneyness and time to maturity as the key factors in implied volatility.

3.2.2 Black Fisher and Myron Scholes (1973) call option pricing model is given by

$$C_{BS73} = SN(d_1) - Xe^{-rT}N(d_2) \tag{3.3}$$

S = Stock price, X = exercise or strike price, r = risk-free interest rate, T = time to expiration and σ = standard deviation of log return (volatility), which is assumed constant throughout the life span of the call (put) option;

$$d_1 = \frac{\log(\frac{S}{X}) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \text{ and } d_2 = \frac{\log(\frac{S}{X}) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{3.4}$$

Similarly, for put option, the Black-Scholes option pricing formula will then be:

$$P_{BS73} = Xe^{-rT}[1 - N(d_2)] - S[1 - N(d_2)] \tag{3.5}$$

3.3 Extensions of the Black-Scholes model

Here we look at the following extensions of the seminal paper popularly known as the Black-Scholes (1973) model for pricing the European call and put options of derivative products.

3.3.1 Hull John and Allan White model (1987)

The derivative pricing model of Hull and White (1987) addresses the issue of constant volatility by relaxing the assumption on the underlying stock property of constant volatility in the BS (1973) option pricing model. In this context, unlike the Black-Scholes model where the volatility of the underlying stock is assumed constant throughout the option life span, the underlying stock volatility varies as the time to expiration of the option, and is therefore, stochastic. We therefore seek to obtain the call option pricing formula in a stochastic volatility setting. As $\log(\frac{S_T}{S_0})$ conditioned on $\tilde{V}$ is normally distributed with variance $\tilde{V}T$ when S and $\tilde{V}$ are instantaneously uncorrelated, the BS option price $C(\tilde{V})$ for stochastic volatility according to Hull and White (1987) is given by

$$C_{HW87}(\tilde{V}) = S_t N(d_1) - Xe^{-r(T-t)}N(d_2) \qquad (3.6)$$

where, $d_1 = \frac{\log\left(\frac{S_t}{X}\right)+(r+\frac{\tilde{V}}{2})(T-t)}{\sqrt{\tilde{V}(T-t)}}$, $d_2 = d_1 - \sqrt{\tilde{V}(T-t)}$

with the option value given by

$$f(S_t,\delta_t^2) = \int C_{HW}(\tilde{V})\, h(\tilde{V}|\delta_t^2)d\tilde{V},\ \tilde{V} = \frac{1}{T-t}\int_t^T \sigma_t^2 d\tau$$

given that T = time at which the option matures, S_t = security (underlying stock) price at time t; σ_t = instantaneous standard deviation at time t.

3.3.2 The Merton (1973) model

As the BS (1973) proposes that there are no dividend pay-outs in the option priced with the model, Merton (1973) option pricing formula is a generalisation of BS (1973) formula with the capacity of pricing European options on stocks or stock indices that is paying some dividend accrued to shareholders/investors in the stock. The yield is expressed as an annual continuously compounded rate **q.** The values for a call/put option price in the Merton's model which we refer to as $C_{M73}\ and\ P_{M73}$ for the Merton's (1973) call and put options respectively are:

$$C_{M73} = Se^{-qT}\emptyset(d_1) - Xe^{-rT}\emptyset(d_2) \qquad (3.7)$$

$$P_{M73} = Xe^{-rT}\emptyset(-d_2) - Se^{-qT}\emptyset(-d_1)$$

with $d_1 = \frac{\log(\frac{S}{X})+\left(r-q+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$,

where, log connotes the natural logarithm, S = the underlying stock price, X = the strike price, r = the continuously compounded risk-free interest rate, q = the continuously compounded annual dividend yield, T = the time in years until the expiration of the option contract, σ = the implied volatility for the underlying stock, $\emptyset$ = the standard normal cumulative distribution function.

3.3.3 Foreign Currency option

Merton (1973) as stated above extended the Black-Scholes (1973) model to include stocks that pay continuous dividend during the life span of the option contract. Similarly, Jorion et al. (1996) assert that as foreign currency derivative options pay continuous rate of interest which can be interpreted to mean dividend yield, one can therefore extend the Black-Scholes (1973) option pricing model to cover currency options as shown by Garman and Kohlhagen (1983). They derived the solution to second order partial differential equation

$$\frac{\sigma^2}{2}S^2\frac{\delta^2 C}{\delta S^2} - r_d C + (r_d - r_f)S\frac{\delta C}{\delta S} = \frac{\delta C}{\delta T} \qquad (3.7a)$$

as the formula for a call option given by

$$C(S,T) = e^{-r_f T}S\emptyset(x + \sigma\sqrt{T}) - e^{-r_d T}K\emptyset(x) \qquad (3.7b)$$

$$x = \frac{\ln(S/K) + \left\{r_d - r_f - \left(\frac{\sigma^2}{2}\right)\right\}T}{\sigma\sqrt{T}}$$

and similarly, for a European put option, we shall have

$$P(S,T) = e^{-r_f T}S[\emptyset(x + \sigma\sqrt{T}) - 1] - e^{-r_d T}K[\emptyset(x) - 1] \qquad (3.7c)$$

where,

S = the spot price of the deliverable currency (domestic unit per foreign unit)

F = forward price of the currency to be delivered at option maturity

K = strike/exercise price of option (domestic unit per foreign unit

T = time remaining before option maturity (in days per annum)

r_d = domestic (riskless) interest rate

r_f = foreign (riskless) interest rate

σ = volatility of the spot currency price

$\emptyset(.)$ = cumulative normal distribution

3.3.3.1 Relationship between call and put option prices to a contemporaneous forward price

From Keynes (1923)'s findings on interest rate parity, the forward price of currency with respect to the spot price of currency deliverable contemporaneously within the maturity period of an option is given by

$$F = e^{(r_d - r_f)T} S$$

When we substitute the above forward price into call and put option in equations (3.7b) and (3.7c) we obtain:

$$C(F,T) = \{F\emptyset\big(x + \sigma\sqrt{T}\big) - K\emptyset(x)\}e^{-r_d T} \tag{3.7d}$$

and

$$P(F,T) = \{F[\emptyset\big(x + \sigma\sqrt{T}\big) - 1] - K[\emptyset(x) - 1]\}e^{-r_d T} \tag{3.7e}$$

where $$x = \frac{\ln(F/K) - (\frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$

thus, changing the call and put price for a European type of option to be a function of the forward price and domestic interest rate, r_d.

3.3.4 Practitioners' or Ad-hoc Black-Scholes model

The practitioners' Black-Scholes version of the original Black-Scholes model is an extension of the later that addresses its constant volatility assumption for pricing European call and put options. There have been many empirical studies investigating the efficacy of Black-Scholes equation (2.1) on option pricing. This constant volatility assumption not being generally true leads to volatility smile, which shows that the implied volatility option depends to a large extent on the strike price, time to maturity and moneyness of the option. It is the smile and smirk shapes of implied volatility that have motivated researchers to model implied volatility as a quadratic function of moneyness and time to maturity, which thereafter, will be substituted into the Black-Scholes model for the actual pricing of the options.

To this end, Dumas, Fleming & Whaley (1998) introduce an ad-hoc/practitioners' Black-Scholes model that uses a deterministic volatility function (DVF) method to model implied volatility. It is a known fact that despite the pricing and hedging biases of the Black-Scholes model, it is still widely used by market practitioners, Kim (2009). He observes that when practitioners apply the Black-Scholes model, they usually allow the only unobservable parameter of the model (volatility) to vary across strike prices and maturities of options, in order to fit the volatility to the observe smile pattern. Dumas et al. (1998) declare that this procedure will circumvent some of the model biases associated with the constant volatility assumption of the Black-Scholes model. The Ad-Hoc Black Scholes (AHBS) is an extension of Black-Scholes model where each option has its own implied volatility depending on a strike and time to maturity.

There are two approaches to AHBS models which we are going to consider in this thesis, namely 'relative smile' and the 'absolute smile' AHBS for the implied volatilities. For the relative smile approach, implied volatility is treated as function of moneyness whereas in absolute smile, implied volatility is treated a fixed function of the strike price, K, but independent of the value of the underlying stock. We will in the analysis examine and compare what constitutes an efficient combination of the independent variables (moneyness, strike price and time to maturity), by considering what happens when the number of independent variables increases, using the resulting p-values to assess the relative significance of the variables.

It is known from empirical research findings that implied volatility varies for an option with the same strike price but different maturity dates. Options whose maturity dates are closer are known to have higher implied volatility, hence the time to maturity and its interaction with the strike price, KT constitute significant factor in the value of implied volatilities. The models generated and considered in this research that have practical applications for the proposed derivative asset pricing, in the Nigerian Stock Market, which I called extensions of Dumas, Fleming and Whaley (1998) model or Deterministic Volatility Functions, are as follows:

$$DVF_{R1}: \sigma_{iv} = a_0 + a_1(S/K) + a_2T + a_3(S/K)T$$

$$DVF_{R2}: \sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T$$

$$DVF_{R3}: \sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T + a_4(S/K)T$$

$$DVF_{R4}: \sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T + a_4T^2$$

$$DVF_{A1}: \sigma_{iv} = a_0 + a_1K + a_2T + a_3KT$$

$$DVF_{A2}: \sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T$$

$$DVF_{A3}: \sigma_{iv} = a_0 + a_1K + a_2T + a_3T^2 + a_4KT$$

$$DVF_{A4}: \sigma_{iv} = a_0 + a_1K + a_2T + a_3K^2 + a_4KT$$

$$DVF_{A5}: \sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T + a_4T^2 + a_5KT$$

3.3.5 Merton (1976) Option Pricing Model

This model addresses the constant volatility assumption of the BS (1973) model. Merton (1976) asserts that BS (1973) option pricing model is not valid when the stock price dynamics cannot be represented by a stochastic process with a continuous sample path. To this end, the validity of the BS formula depends largely on whether or not stock price changes satisfy a kind of 'local' Markov property. By this he refers to the ability of the stock price to change by a small amount in a short interval of time. The discontinuous path of the stochastic process is called **"jump"** stochastic process defined in a continuous time that permits a positive probability of a stock price change of an extraordinary magnitude in a short interval of time.

This process results in negative skewness and excess kurtosis of the underlying stock price density and hence fat tails, which necessitate the inclusion of Poisson jump component in the generation of the underlying stock returns. Thus, from Merton, R. C. (1976)'s model, a stock price that follows a geometric Brownian motion (BS) with an additional jump component in a European call option price C_{M76} is given by:

$$C_{M76} = \sum_{n=0}^{\infty} \frac{e^{-\bar{\gamma}T}(\bar{\gamma}T)^n}{n!} C_{BS73}(S, X, T, \sigma_i, r_i) \qquad (3.8),$$

where σ_i = total variance without jumps, r_i = the adjusted risk-free rate;

with $C_{BS73} = S\emptyset(d_1) - Xe^{-rT}\emptyset(d_2)$ as the Black-Scholes option pricing formula.

$$d_1 = \frac{\log\left(\frac{S}{X}\right) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad , \quad d_2 = d_1 - \sigma\sqrt{T}$$

And when we assume, as in Cheang and Chiarella (2011), that the jump sizes are normally distributed with mean α, variance ∂ and jump intensity γ under a Martingale measure,

$$\bar{\gamma} = \gamma e^{\alpha + \frac{\partial^2}{2}} \; and \;\; \delta_i^2 = \partial^2 + \frac{i\partial^2}{T}; \; r_i = r - \gamma(e^{-\bar{\gamma}T} - 1) + \frac{i(\alpha + \frac{\partial^2}{2})}{T}.$$

Hence, the parameters we need to estimate here include the volatility of the underlying, δ, the three jump parameters given by: $\gamma, \alpha \; and \; \partial$.

(3.3.6) Heston S.L (1993) Model (a variant of Black-Scholes)

Heston (1993) addresses the constant volatility assumption by using a new technique to derive a closed-form solution, not based on the BS model, for the price of a European call/put option on an asset with stochastic volatility which permits arbitrary correlation between volatility and spot returns and the call option pricing model is given by

$$C_{H93}(S, v, t) = SP_1 - KP(t, T)P_2 \equiv SP_1 - e^{-rT}KP_2 \qquad (3.9);$$

Thus, $P \equiv e^{-rT}$, with the first term as the present value of the spot asset upon optimal exercise, and the second term is the present value of the strike-price payment, P_1, P_2 satisfying the desired PDE. P_1 is the option delta and P_2 is the risk-neutral probability of exercise.

Using $X = \log S$, the characteristic function is:

$$f_j(X, v, t, \emptyset) = e^{C_H(T-t;\emptyset)+D(T-t;\emptyset)v+i\emptyset X}$$

$$\text{where } C_H(r; \emptyset) = r\emptyset i\tau + \frac{a}{\delta^2}\left\{\left(b_j - \rho\delta\emptyset i + d\right)\tau - 2\log[\frac{1-ge^{d\tau}}{1-g}]\right\}$$

$$D(\tau; \emptyset) = \frac{b_j - \rho\delta\emptyset i + d}{\delta^2}\left[\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right] and\ g = \frac{b_j - \rho\delta\emptyset i + d}{b_j - \rho\delta\emptyset i - d}$$

$$d = \sqrt{(\rho\delta\emptyset_i - b_j)^2 - \delta^2(2u_j\emptyset_i - \emptyset^2)}$$

and on inverting the characteristic functions to obtain the desired probabilities we shall have:

$$P_j\{X, v, T;\ \log[K]\} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\emptyset\log[K]}f_j(X, v, T; \emptyset)}{i\emptyset}\right]d\emptyset$$

$$for\ j = 1{,}2\ with\ u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, a = k\theta, b_1 = k + \gamma - \rho\delta, b_2 = k + \gamma.$$

From the standard arbitrage argument of the BS (1973) model and Merton (1973) the value of any asset, $\cup(S, v, T)$, including accruing payments must satisfy the partial differential equation (PDE) given by:

$$\frac{1}{2}vS^2\frac{\partial^2\cup}{\partial S^2} + \rho\delta vS\frac{\partial^2\cup}{\partial S\partial v} + \frac{1}{2}\partial^2 v\frac{\partial^2\cup}{\partial v^2} + rS\frac{\partial\cup}{\partial S} + \{k[\theta - v(t)] - \gamma(S, v, t)\}\frac{\partial\cup}{\partial v} - r\cup + \frac{\partial\cup}{\partial t} = 0$$

This model can also be adapted for stochastic interest rate, Bakshi et al. (1997). The parameters that we need to estimate in the Heston (1993) model include:

$\gamma = market\ price\ of\ volatility\ risk$

$\delta = the\ variance\ of\ the\ underlying$

$v = the\ volatility\ of\ the\ variance$, which is the volatility of the volatility referred to here as simply variance

$$k = mean\ reversion\ rate$$

$$\emptyset = the\ long\ run\ variance$$

$$S = underlying\ stock\ price,\ T = Time\ to\ maturity\ of\ the\ option$$

$$\rho = correlation\ between\ the\ logreturns\ and\ the\ volatility\ of\ asset.$$

3.3.7 Corrado C. J. and Su, Tie (1996) - An Extension of the Merton (1973) model

This model is known to account for the biases associated with Merton (1973) and therefore also takes care of the shortcomings of the Black-Scholes model, since Merton's model itself is an extension of the Black-Scholes model. The BS model (1973) is known to misprice deep in(out) of the money options and these strike price biases could be referred to as volatility smiles.

Corrado et al. (1996)'s model addresses the biases induced by non-normal skewness and kurtosis in stock return distributions, by using Gram-Charlier series expansion of the normal density function, which adjusts the skewness and kurtosis in the BS formula. The model particularly addresses the underlying assumption of the BS that trading in the underlying stock return is log normally distributed, with no dividend payments during the life span of the option contract. This method of extending the BS method to address the skewness and kurtosis adopted by Corrado et al. (1996) is analogous to that of Jarrow and Rudd (1982).

While Jarrow and Rudd method accounts for the skewness and kurtosis deviations from log normality for stock returns, the method of Corrado et al. (1996) accounts for skewness and kurtosis for normality of stock returns. Both methods are equally good for option price adjustment, but the underlying difference is that skewness and kurtosis from normality of stock returns are known constants 0 and 3 respectively, Stuart and Ord (1987), while skewness and kurtosis coefficients for log normal distributions vary across different normal distributions, Aitchison and Brown (1963).

The Gram-Charlier series expansion of the density function f(x) is defined as

$$f(x) = \sum_{n=0}^{\infty} C_n H_n(x)\varphi(x),$$

$where\ \varphi(x)$ is a normal density function, $H_n(x)$ are Hermite polynomials derived from successively higher derivatives of $\varphi(x)$ and the coefficients c_n are determined by moments of the distribution $F(x)$. The series $F(x)$ when standardized will be:

$$g(z) = n(z)\left\{1 + \frac{\mu_3}{3!}(z^3 - 3z) + \frac{\mu_4 - 3}{4!}(z^4 - 6z^2 + 3)\right\}$$

$$\text{where } n(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad z = \left[\log\left(\frac{S_t}{S_0}\right) - \left(r - \frac{\delta^2}{s}\right)t\right] / (\delta\sqrt{t})$$

$$S_0 = current\ stock\ price, \qquad S_t = random\ stock\ price\ at\ time\ t,$$

$$r = risk - free\ interet\ rate.,$$

$$\delta = standard\ deviation\ of\ returns\ for\ the\ underlying\ stock,$$

$$t = time\ remaining\ until\ option\ maturity.$$

The formula therefore for the European call option obtained by Corrado and Su (1996) and represented as C_{COSU73} is given by:

$$C_{COSU73} = C_{M73} + \mu_3 Q_3 + (\mu_4 - 3) Q_4 \qquad (3.10)$$

with, $Q_3 = \frac{1}{3!} S_t e^{-\delta T} \delta\sqrt{T}[(2\delta\sqrt{T} - d_1)h(d_1) + \delta^2 T N(d_1)]$

$$Q_4 = \frac{1}{4!} S_t e^{-\delta T} \delta\sqrt{T}[(d_1)^2 - 1 - 3\delta\sqrt{T}(d_1 - \delta\sqrt{T})]h(d_1) + \delta^3 T^{\frac{3}{2}} N(d_1)$$

$$h(z) = \frac{1}{\sqrt{2\pi}} exp\left(\frac{-z^2}{2}\right) = n(z).$$

$\mu_3\ and\ \mu_4$ are the standardized coefficients of skewness and kurtosis of the returns respectively, which are unobserved just like the variance δ, and they are the parameters that will be estimated.

We note here that when the skewness is zero [(i.e. $\mu_3 = 0)\ and\ kurtosis\ \mu_4 = 3,$] the Corrado and Su (1996) model is equivalent to the Merton (1973) model for option pricing.

(3.3.8) Jarrow and Rudd (1982)

This model takes care of the lognormal assumption of the Black-Scholes model on the underlying stocks. This is an option valuation formula where the underlying security distribution, if not lognormal can be approximated by a lognormally distributed random variable, by deriving a series expansion of a given distribution in terms of an unspecified approximating function, A(s) using Edgeworth series expansion.

The resulting true option price will be expressed as a sum of the BS option price plus adjustment terms that will depend on the second and higher order moments of the underlying stochastic process for the security. That is, the approximate (true) option price will be the BS price plus three adjustments which depend respectively on the difference between the variance, skewness and kurtosis (2nd, 3rd & 4th order moments) of the underlying and the normal distribution. This was carried out through a method of finding the relationship between cumulants and moments (mean, variance, skewness and kurtosis), (Kendall and Stuart, 1977). The first cumulant is the mean, the second cumulant is the variance, third cumulant is the skewness and finally the fourth cumulant stands for the measure of kurtosis. The first four cumulants are:

$$K_1 = \alpha_1(F),\ K_2(F) = \mu_2(F),\ K_3(F) = \mu_3(F)$$

$$K_4 = \mu_4(F) - 3\mu_2(F)^2$$

with,

$$\alpha_j(F) = \int_{-\infty}^{\infty} S^j f(s)ds$$

$$\mu_j(F) = \int_{-\infty}^{\infty} [S - \alpha_1(F)]^j f(s)ds$$

$$\emptyset(F,t) = \int_{-\infty}^{\infty} e^{itS} f(s)ds,$$

$where\ i^2 = -1$, $\alpha_j(F)$ = the jth moment of distribution F, $\mu_j(F)$ is the jth central moment distribution F, and $\emptyset(F,t)$ is the characteristic function of F.

The approximate option price of Jarrow and Rudd (1982) represented as C_{JR82} in terms of Black-Scholes (1973), [written as C(A)], and the moments is given by:

$$C_{JR82} = C(A) + \frac{e^{-rt}K_2(F) - K_2(A)}{2!}a(K) - e^{-rt}\frac{[K_3(F) - K_3(A)]}{3!}\frac{da(K)}{dS_t} +$$

$$e^{-rt}[\{K_4(F) - K_4(A)\} + 3\{K_2(F) - K_2(A)\}^2]\frac{d^2a(K)}{dS_t^2} + \varepsilon(K) \quad (3.11)$$

where,

$$C(A) = S_0N(d) - Xe^{-rt}N(d - \delta\sqrt{t})$$

$$d = \log(\frac{S_0}{Ke^{-rt}}) + \frac{\delta^2 t}{2}$$

N(.) is the cumulative standard normal distribution.

$\alpha_1(A) = S_0e^{rt}$ $,and\ defining\ q^2 = e^{\delta^2 t} - 1$, the cumulants are written as follows, according to Mitchell (1968);

$$K_1(A) = \alpha_1(A)$$

$$K_2(A) = \mu_2(A) = K_1(A)^2q^2$$

$$K_3(A) = K_1(A)^3(3q + q^3)q^3$$

$$K_4(A) = K_1(A)^4q^4(16q^2 + 15q^4 + 6q^6 + q^8)$$

and finally

$$\delta^2 t = \int_{-\infty}^{\infty}(\log S_t)^2 dF(S_t) - [\int_{-\infty}^{\infty}\log S_t dF(S_t)]^2.$$

For in the money option, $S_0 > Ke^{-rt}, at\ the\ money\ option\ is\ S_0 = Ke^{-rt}$and finally out of the money option arises when $S_0 < Ke^{-rt}$. However, since the mean of the distribution is $S_0 e^{rt}$, one can classify in/at/out of the money options as:

$$K > \alpha_1(A)[\,out\ of\ the\ money]$$

$$K = \alpha_1(A)\ [at\ the\ money]$$

$$K < \alpha_1(A)\ [in\ the\ money]$$

Remarks:

Skewness and Kurtosis will be defined formally later when we will treat volatility modelling in our study of volatility as one of the stylized facts of the stock market characterisation.

Cox, Ross and Rubinstein, CRR (1979), is a simple binomial option price model that derives the BS pricing formula for a geometric Brownian motion as a limiting case of the binomial option pricing formula.

The binomial option pricing model of Cox, Ross and Rubinstein (1979) was originally proposed by Cox and Ross (1976) and was later extended by Cox and Rubinstein. It is a simple discrete-time formula for valuing options. It supports the economic principles of option pricing by arbitrage principle and gives rise to a simple and efficient numerical procedure for valuing options for which premature exercise may be optimal. It also takes into consideration the pricing of option contracts that have dividend payments on their underlying assets, by proposing a numerical procedure for the valuation of such option contracts.

CRR (1979) is a discrete binomial pricing model for the option price of an underlying stock in a given time interval [0, T], divided into n steps such that T = nh. In each step, the price S (of the underlying) moves up to uS with a probability q or downwards to dS with probability 1- q. These upward and downward movements with interest rate are regarded as constants with $d < r < u$ and an appropriate choice of the parameters u, d, q, n leads in the limit of the process to a lognormal model.

The model relaxes the BS assumption of continuous evolution of the share price through the introduction of some jumps in the pricing process. As stated earlier the rate on the stock over each period can have two values:

$$u - 1\ with\ probability\ q$$

$$d - 1\, with\, probability\, 1 - q,$$

meaning, as stated earlier, that the stock price can either move up or down. Let r denote one plus the riskless interest rate over one period and we require (as before) that $u > r > d$ with

$$p = \frac{r-d}{u-d} \quad , 1 - p = \frac{u-r}{u-d}$$

The call option pricing formula of Cox, Ross and Rubinstein model (1979) written as C_{CRR79}, is given by:

$$C_{CRR79} = \frac{1}{r^n} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} Max[0, u^j d^{n-j} S - X] \qquad (3.12)$$

where n= the number of periods remaining until expiration. It suffices to note here that we can modify the binomial option pricing model above by restricting the value of "a" and carrying out some algebraic manipulations of the parameters. Now for "a" representing minimum moves upwards the loop for n-periods to finish in-the-money for "a" a nonnegative integer (i.e. $''a'' \in Z^+$), [$such\, that\, u^a d^{n-a} S > X$], we shall have the call option pricing formula above could now be written as:

$$C_{CRR79} = \frac{1}{r^n} \sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} [u^j d^{n-j} S - X] \qquad (3.12a)$$

For, $a > n$, the call option will finish out-of-the-money, so that we will have upon separating the terms in S and X, respectively, we should have:

$$C_{CRR79} = \frac{S}{r^n} \sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} u^j d^{n-j} - \frac{X}{r^n} \sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \qquad (3.12b)$$

In summary, binomial option pricing formula is:

$$C_{CRR79} = S\emptyset(a; n, p') - Xr^{-n}\emptyset(a; n, p) \qquad (3.12c)$$

$$where\ p \equiv \frac{r-d}{u-d}\ and\ p' = \frac{u}{r}p$$

$$a \equiv the\ smallest\ nonnegative\ integer\ greater\ than\ \log(\frac{X}{Sd^n})/\log(\frac{u}{d})$$

$$\text{If, } a > n, \quad C = 0.$$

C_{CRR79} in their formula, however, discover that if we re-state the Black-Scholes (1973) option pricing model as:

$$C_{BS} = SN(x) - Xr^{-t}N(x - \delta\sqrt{t})$$

$$\text{where } x \equiv \frac{\log\left(\frac{S}{Ke^{-t}}\right)}{\delta\sqrt{t}} + \frac{1}{2}\delta\sqrt{t}.$$

From C_{CRR79} it is easy to confirm that the binomial formula converges to the BS formula if 't' is divided into more subintervals with appropriate choices of $r, u, d, and\ q$. The underlying similarity between BS and CRR model is that both assume continuous trading and lognormal distribution, although CRR model is known to be a combination of BS option pricing formula and for perceived cases of a jump process formula for option contract.

In order to capture the jump process, we invoke the findings of Cox-Ross (1975) model, with u, d, and q instead of the values as before, are now given by $u = u, d = e^{\varepsilon(\frac{t}{n})}\ and\ q = \gamma(\frac{t}{n})$ with the underlying assumption that the stock price dynamics will no longer be explained through the initial conditions of central limit theorem of the lognormal process, but rather will converge to a log-Poisson distribution given by:

$$\varphi\ [x; y] \equiv \sum_{i=x}^{\infty} \frac{e^{-y} y^{i}}{i!}$$

The jump process option pricing formula is given by:

$$C_{CRR79}(with\ jumps) = S\varphi[x; y] - Xr^{-t}\varphi\left[x; \frac{y}{n}\right], \qquad (3.12d)$$

$where, y = \frac{(\log r - \varepsilon)ut}{u-1}$ and $x =$ the smallest non-negative integer greater that $\frac{[\log\left(\frac{X}{S}\right) - \varepsilon t]}{\log u}$.

3.3.9 Bakshi et al. (1997) - Empirical Performance of Alternative Option Pricing Models

Bakshi et al. (1997) developed an option pricing model that improves on the restrictive BS (1973), by relaxing some of the assumptions. Their model allows volatility, interest rates and jumps in the process to be stochastic. While the stochastic volatility and jumps in the process are important for pricing and internal consistency, hedging requires stochastic volatility to obtain optimum performance. Their model is so robust that virtually all the known closed-form option pricing formulas are special cases of the Bakshi et al model.

The motivation for their research, just like many others in the literature, is that the benchmark BS formula exhibits strong pricing biases across both moneyness and maturity (i.e. the "smile"), and the BS especially underprices deep out-of- the-money calls and puts. This shortcoming according to Bakshi et al. (1997) was as a result of wrong distributional assumption and therefore necessitates the need to find the right distributional structure for the pricing process. The stochastic volatility model, for instance, offers a flexible distributional structure in which the correlation between volatility shocks and underlying stock returns serves in controlling the level of skewness and the volatility variation coefficient to control the kurtosis.

However, Bakshi et al. (1997) through the diffusion model assert that it is the occasional, discontinuous jumps and crashes that cause negative implicit skewness and high implicit kurtosis to exist in option prices. They propose that the random-jump and the stochastic-volatility features can in principle improve the pricing and hedging of short term and relatively long-term options, respectively.

In their view, the inclusion of stochastic interest rate term structure model in an option pricing framework is required for the valuing and discounting of future payoffs, instead of enhancing the flexibility of permissible distributions.

The European call option written on the stock with strike price X and time-to-expiration τ is given by:

$$C_{BCC97}(t,\tau) = S(t)\prod_1(t,\tau;S,R,V) - XB(t,\tau)\prod_2(t,\tau;S,R,V), \quad (3.13)$$

where, $\prod_1(.)$ and $\prod_2(.)$ are recovered from inverting the respective characteristic functions.

$$\prod_j[t,\tau;S(t),R(t),V(t)] = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re[\frac{e^{-i\emptyset \log X} f_j[t,\tau,S(t),R(t),V(t);\emptyset]}{i\emptyset}]d\emptyset$$

$for\ j = 1,2$. The characteristic functions f_j respectively are given in the equations below.

$$f_1(t,\tau) = exp\left\{-\frac{\theta_R}{\sigma_R^2}\left[2\log(1-\frac{[\varepsilon_R - X_R](1-e^{-\varepsilon_R\tau})}{2\varepsilon_R}\right) + [\varepsilon_R - X_R]\tau] - \frac{\theta_v}{\sigma_v^2}[2\log(1-\frac{[\varepsilon_v - X_v + (1+i\emptyset)\rho\sigma_v](1-e^{-\varepsilon_v\tau})}{2\varepsilon_v})] - \frac{\theta_v}{\sigma_v^2}[\varepsilon_v - X_v + (1+i\emptyset)\rho\sigma_v]\tau + i\emptyset\log[S(t)] + \frac{2i\emptyset(1-e^{-i\omega t})}{2\varepsilon_R - [\varepsilon_R - X_R](1-e^{-\varepsilon_R\tau})}R(t) + \gamma(1+\mu_J)\tau[\left(1+\mu_J\right)^{i\emptyset}e^{(i\emptyset/2(1+i\emptyset)\sigma_J^2} - 1\Big] - \gamma i\emptyset\mu_J\tau + \frac{i\emptyset(i\emptyset+1)(1-e^{-\varepsilon_v\tau})}{2\varepsilon_v - [\varepsilon_v - X_v + (1+i\emptyset)\rho\sigma_v](1-e^{-i\omega t})}V(t)\right\}$$

$$f_2(t,\tau) = exp\left\{-\frac{\theta_R}{\sigma_R^2}\left[2\log(1-\frac{[\varepsilon_R^* - X_R]\left(1-e^{-\varepsilon_R^*\tau}\right)}{2\varepsilon_R}\right) + [\varepsilon_R^* - X_R]\tau] - \frac{\theta_v}{\sigma_v^2}[2\log(1-\frac{[\varepsilon_v^* - X_v + (1+i\emptyset)\rho\sigma_v]\left(1-e^{-\varepsilon_v^*\tau}\right)}{2\varepsilon_v^*})] - \frac{\theta_v}{\sigma_v^2}[\varepsilon_v^* - X_v + (1+i\emptyset)\rho\sigma_v]\tau + i\emptyset\log[S(t)] -$$

$$\ln[B(t,\tau) + \frac{2i\emptyset\left(1-e^{-i\omega t}\right)}{2\varepsilon_R^* - [\varepsilon_R^* - X_R]\left(1-e^{-\varepsilon_R^*\tau}\right)} R(t) + \gamma(1+\mu_J)\tau[\left(1+\mu_J\right)^{i\emptyset} e^{(i\emptyset/2(1+i\emptyset)\sigma_J^2} - 1] -$$

$$\gamma i\emptyset\mu_J\tau + \frac{i\emptyset(i\emptyset+1)\left(1-e^{-\varepsilon_v^*\tau}\right)}{2\varepsilon_v^* - [\varepsilon_v^* - X_v + (1+i\emptyset)\rho\sigma_v](1-e^{-}\varepsilon_v^*)} V(t) \Big\}$$

$C_{BCC97}(t,\tau)$ asserts that their model must solve the following second order stochastic partial differential equation:

$$\frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + [R - \gamma\mu_J]S\frac{\partial C}{\partial S} + \rho\sigma_v VS\frac{\partial^2 C}{\partial S\partial V} + \frac{1}{2}\sigma_v^2 V\frac{\partial^2 C}{\partial V^2} + [\theta_v - X_v V]\frac{\partial C}{\partial V}$$
$$+ \frac{1}{2}\sigma_R^2 R\frac{\partial^2 C}{\partial R^2} + [\theta_R - X_R R]\frac{\partial C}{\partial R} - \frac{\partial C}{\partial \tau} - RC$$
$$+ \gamma E\{C(t,\tau,S(1+J),R,V) - C(t,\tau;S,R,V)\} = 0$$

$$subject\ to\ C(t+\tau,0) = max\{S(t+\tau) - X, 0\}$$

3.3.10 Other variants of BS Model: [G(ARCH) and Stochastic Volatility Models]

Campbell and Mackinlay (1997) argue that it is not only logically inconsistent but also statistically inefficient to use volatility measures that are based on the assumption of constant volatility over some period when the resulting series moves through time. In the case of financial data for instance, large and small errors tend to occur in clusters. In other words, it is known that large returns are followed by more large returns, and small returns by more small returns which suggest that returns are serially correlated. It is therefore imperative to use the G(ARCH) family of models in analysing financial data bearing in mind that such models will address the issue of varying volatility across the life span of derivatives and other financial asset contract.

ARCH and GARCH models are the most popular time series tools for modelling volatility and the ARCH models are usually estimated using maximum likelihood estimators, although there are other known estimators we may encounter in the course of this research. Bollerslev (1986) asserts that while conventional time series and economic models operate under an assumption of constant variance, for example the Black-Scholes (1973) model, the ARCH process of Engle (1982) allows the conditional variance to change over time thus addressing the past errors of leaving unconditional variance constant. Similarly, the GARCH model of Bollerslev allows the volatility to change over time and provides a longer memory and a more flexible lag structure. That is, the generalised autoregressive conditional heteroscedasticity GARCH models are unlike the ARCH models, where the next period's variance only depends on last period's squared residuals.

The ARCH model has a volatility equation written as:

ARCH model, $$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q}\alpha_i r_{t-i}^2 \quad (3.14)$$

For GARCH model, $$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i r_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 \quad (3.15),$$

So that when $p = 0$, GARCH process reduces to ARCH(q) process.

In the ARCH (q) process, the conditional variance is specified as a linear function of past sample variances only, whereas the GARCH (p, q) allows lagged conditional variances to enter as well. GARCH models are designed to capture the volatility clustering effects in the returns. GARCH (1, 1) can, for instance, model the dependence in the squared returns (or squared residuals) and they can also capture some of the unconditional leptokurtosis.

It is a known result among researchers of statistical economics and mathematical finance that financial asset returns/stock returns exhibit volatility clustering, asymmetry and leptokurtosis. These characteristics of asset return indicates rise in financial risk which can affect investors adversely. Volatility clustering refers to the situation when large stock price changes are followed by large price change, of either sign, and similarly, small changes are followed by periods of small changes.

Asymmetry, otherwise known as leverage effect, means that a fall in asset return is followed by an increase in volatility greater than the volatility induced by an increase in returns. In other words, the impact of bad news on volatility is always greater that the corresponding impact of good news on volatility.

Leptokurtosis on the other hand refers to market condition where the distribution of stock return is not normal but rather exhibits fat tails. Leptokurtosis means that there exists the higher propensity for extreme values to occur more regularly than the normal law predicts.

These three financial asset characteristics mentioned above expose investors to pay higher risk premium, to insure against the increased uncertainty in their portfolio of investments. Volatility clustering, for instance, makes the investors to be more averse to holding stocks due to high stock price uncertainty.

Emenike (2010) used GARCH (1,1) model to capture the nature of volatility, the Generalised Error Distribution (GED) to capture fat tails, and GJR-GARCH (1993) a modification of GARCH (1, 1) to capture leverage (asymmetry) effects.

3.3.11 GARCH in Option Pricing

Hsieh et al (2005) assert that recent empirical studies have shown that GARCH models can be successfully used to describe option prices and that pricing such contracts requires knowledge of the risk neutral cumulative return distribution. Duan et al. (1999) use Edgeworth expansions to provide analytical approximation for European options where the underlying asset is driven by N-GARCH process. Duan et al. (2006) have extended the Duan et al. (1999) approach to approximate option pricing under GARCH specifications of Glosten, Jagannathan and Runkle (1993) and the exponential GARCH specification of Nelson (1991). Finally, Heston and Nandi (2000) model developed a `closed form' solution for European options under a very specific GARCH-like volatility updating scheme.

3.3.12 Constant elasticity of variance (CEV)

Some of the models formulated to take care of the shortcomings from equation (2.1) include Hull and White (1987), Black (1976), MacBeth and Merville (1979) and Heston (1993) researches that have contrary results on the assumption of constant volatility of the underlying stock in a derivative option. It is also this unsuitable assumption of constant volatility that prompted Cox (1975) and Cox and Ross (1976) to propose the constant elasticity of variance (CEV) diffusion process which takes the form

$$dS = \mu S dt + \delta S^{\frac{\beta}{2}} dW, \quad S_o = x \tag{3.16}$$

as the option pricing model with β = elasticity of the underlying stock price $\beta, \delta\ are\ constants$ with $\mu\ and\ dW$ having their usual meanings.

The model proposes the following deterministic relationship between stock price, S and volatility, σ:

$$\sigma(S,t) = \delta S^{(\beta-2)/2} \tag{3.17}$$

The elasticity of variance with respect to price equals β - 2, and if β < 2, the volatility and the stock price are inversely related. It suffices to state here that volatility is an increasing (decreasing) function of S when β > 2 (β < 2). That is, for:

β = 2, the CEV option pricing formula reduces to the usual Black-Scholes model,

β < 2, volatility falls as stock price rises (and hence, generates a fatter left tail)

β > 2, Volatility rises as stock price rises.

Under the model (3.16) above, and some assumptions of BS (1973) framework, Cox (1975) derived the equilibrium price of a call option for β < 2 while Emmanuel and MacBeth (1982) extended the pricing formula to the case when β > 2.

The CEV diffusion model with stochastic volatility is a natural extension of geometric Brownian motion (GBM) BS (1973) model, Jianwu et al. (2007). Cox (1975, 1996), Cox and Ross (1976) proposed extension of the (GBM) model that allows volatility to change over time without introducing a new source of uncertainty, called the constant elasticity of variance model. Some other popular CEV models are Beckers (1980), Emmanuel and MacBeth (1982), Davydor and Linetsky (2001), Basu and Samanta (2001).

The Cox (1975) CEV formula was extended by Schroder (1989) in terms of non-central chi-square distribution. Schroder (1989) states that empirical and theoretical arguments support the hypothesis that there is an association between stock price and volatility. In order to account for this relationship, Cox (1996) introduced the CEV model which nests the constant volatility diffusion process of Black-Scholes. The Cox (1975) and the extension Cox and Ross (1976) model is an SDE given by (3.16)

There are lots of financial applications of the CEV model which include pricing of financial derivatives and portfolio selection, Sang-Hyeon et al. (2011). Its advantage over BS model is that it captures implied volatility smile or skew phenomena which the BS model does not. It has, however, the shortcoming that the transition density function of the CEV diffusion of the underlying stock consists of an infinite sum of the Bessel's functions, Davydor et al. (2001) and Schroder (1989). Hence, one has to rely heavily on numerical methods under many circumstances.

On the other hand, it is also known that the empirically observed negative relationship between a stock price and its return volatility can be captured by the CEV option pricing model, Thakoor et al. (2013). For elasticity factors close to 1, the analytical formula for CEV models is known to be computationally expensive as it yields slow convergence rate. Although there are few numerical methods like Wong and Zhao (2008) who propose a Crank-Nicolson scheme for pricing the European and American options, for pricing techniques of CEV models, numerical solution remains a better alternative especially when elasticity is close to 1.

In Cox (1975) formula for CEV option pricing, the underlying stock price dynamics is described by the process:

$$\frac{dS_t}{S_t} = \mu_t dt + \delta_o S_t^{\gamma-1} dW_t \tag{3.18}$$

where, $\gamma - 1$, is the elasticity of the volatility function $\delta_t(S_t) = \delta_0 S_t^{\gamma-1}$, with respect to the underlying stock price. For γ less than unity (1), there exists an inverse relation between the stock and the instantaneous volatility sometimes referred to as "leverage effect".

Cox (1975) shows that, for $\gamma \in (0,1)$, the price of the European call option would be obtained from:

$$C_{C75}(S_t, T-t, \delta, K) = S_t e^{-\delta(T-t)} \sum_{k=1}^{\infty} g(\varepsilon_t'; K) G(\theta_t' K^{2-2\gamma}; K + \frac{1}{2-2\gamma}) - Ke^{-r(T-t)} \sum_{k=1}^{\infty} g(\varepsilon_t'; K + \frac{1}{2-2\gamma} G(\theta_t' K^{2-2\gamma}; K), \tag{3.19},$$

$$where,\ \theta_t' = \frac{r-\delta}{\delta_o^2(1-\gamma)[e^{2(1-\gamma)(r-\delta)(T-t)}-1]},\ \varepsilon_t' = S_t^{2-2\gamma}\theta_t' e^{2(1-\gamma)(r-\delta)(T-t)}$$

$g(x;\alpha)$ = the gamma probability density function (p.d.f) with shape parameter α, $and\ G(x;\alpha)$is the complementary gamma cumulative density function (c.d.f).

Shroder (1989) states that this formula is applicable for the cases when $\gamma < 1$, while Emmanuel and MacBeth (1982) extend the formula to the case when $\gamma > 1$. Verchenko (2011) asserts that this model produces thick tails in the distribution of asset returns, and can accommodate the smirk pattern of implied volatilities but it cannot account for the other side of the volatility smile, and fails to produce the term structure of implied volatilities.

3.4 Objectives Fulfilled by Derivatives Trading

Ezepue and Solarin (2009) argue that Nigeria and other Sub-Saharan African countries need a systemic study of the characteristics of the financial systems and markets, to strengthen market knowledge and deepen technical development of the markets in such areas as Financial Engineering of appropriate products, sophisticated risk management, and diversified portfolio management.

Osuoha (2010) identifies reasons for derivatives trading in Nigerian financial markets to include: need to deepen the financial markets; presently Nigerian capital and money markets do not have a hedging mechanism that will protect investors (derivatives are known to play this role); foreign investment funds managers have a preference for more sophisticated investments like derivatives products that will provide the mechanism for hedging the price fluctuations in oil and gas and other natural resources that are abundantly available in Nigeria; need to stabilize other market segments for example real estate; need to increase participants in the capital market such as banks, insurance companies, oil companies and pension funds; Nigerian investors deserve numerous benefits associated with derivative trade; and finally the need to enhance price discovery, market completeness and efficiency in Nigerian market.

As stated earlier, foreign investors in their risk management strategy prefer more sophisticated investments like derivatives, and as such introduction of derivatives trade in the NSM will obviously provide more foreign direct investments in the Nigerian oil and gas, and agricultural products, for example. This will improve Nigerian export trade, thereby increasing her foreign exchange earnings.

Derivatives trading plays significant role in the development and growth of an economy through risk management, speculation or price discovery. Risk management is concerned with the understanding of risks inherent in a portfolio of securities and managing them through speculations and hedging. Speculators take long or short positions in derivatives to increase their exposure to the market. The stock market players in this category usually bet that the underlying asset will go up or down through speculation.

Arbitrageurs find mispriced securities and instantaneously lock in a profit by adapting certain trading strategies. Hedgers are players who take positions in derivative securities opposite those taken in the underlying security assets to help them manage risks associated with their portfolio better. In other words, short position in underlying stock security is equivalent to long position in derivative security (option).

The major motivation for entering into a forward (an OTC platform contract, which is one of the pioneer derivative products of the NSM) or futures (exchange counterpart of the derivative forward), and in fact any derivatives contract, is to speculate and or hedge an existing market exposure to reduce cash flow uncertainties resulting from the market exposure. The forward contact will enable market participants with the NSM to insure themselves against fluctuating values of the Naira in relation to other major currency of trade within the NSM, for instance, American dollars and British pound sterling. While the forward or futures contract is mainly for hedging, an option contract, also one of the new products earmarked for introduction in the NSM, provides financial insurance to their holders. Thus, holding a call/put option provides the investor with the protection (insurance) against an increase/decrease in the price above/below the prevailing contract price. The writer of the call/put option who takes the reverse side of the contract is referred to as the provider of the insurance.

Another very important use of derivative products is to speculate over the price(s) of securities by investors. Theorists generally define a speculator as someone who purchases an asset with the intent of quickly reselling it, or sells an asset with the intent of quickly repurchasing it, Stout (1999). Therefore, introduction of derivative products into the NSM will enable market participants bet on prices of security assets with the hope of making some profits from these transactions.

As the price of derivative product depends on the underlying assets, it is therefore a market strategy to substitute one for the other. Arnold et al. (2006) in their test for a substitution effect where options are purchased in lieu of the underlying stock found some reasons that necessitate the substitution of options for stocks as follows.

A call option is a limited-life security with value derived from the price of an underlying stock and provides a larger potential return than investing in the underlying stocks. There is usually a higher expected payoff from trading in options contracts since from their findings, average return of options is about 12 times that of common stock.

The risk-averse investors pay to avoid taking risk (like through the insurance policies) while investors with greater tolerance for risk reap some profit through accepting the risk rejected by the risk-averse investors. In the risk hedging model, speculators are relatively risk-neutral traders. For instance, a risk-averse rice farmer in Abakiliki, Ebonyi State, Nigeria, whose crops will soon be ready for harvest may be more worried about the fall in price of rice during the harvest as many farmers are likely to flood the market with their own products, than the possible rise in price of rice. For this fear in the possible fall in the price of rice during the harvest period, the risk-averse farmer might prefer to sell his crops well ahead of harvesting period at some discount (forward derivative) to deliver it in, say forty days' time. On the contrary a more risk-neutral rice speculator might purchase the contract since the price discount creates for him a "risk-premium" that compensates him for accepting the possible changes of future price of rice within the forty days the goods will be delivered.

This risk hedging behaviour implies that speculative traders generally involve "hedgers" on the one side of the transaction, and "speculators" on the other side. The risk-averse (hedgers) like the rice farmer is therefore happy to pay in order to avoid the price variation (presumably downwards) inherent in holding the asset(s), for example, the rice product, while a more risk-neutral speculator is happy to be paid a premium to assume the risk. Risk management that reduces return volatility is frequently termed hedging while risk management that increases the return volatility is called speculation.

Trade in derivatives also promotes market completeness and efficiency which includes low transaction costs, greater market liquidity and leverage to investors enabling them to go short very easily. Derivatives will also, apart from hedging ability mentioned earlier, provide market participants with the price discovery of the underlying asset(s) like the exchange rate of the Naira over time, Dodd et al. (2007).

Derivatives markets can serve to determine not just the spot price but also future prices (and in case of options the price of the risk is determined) in the form of premium paid by the option holders. This research will, based on the market characteristics of the NSM, indicate how suitable investment derivatives products that will best suit the Nigerian market can be developed from the stylised facts of a benchmark market, the South African market, Johannesburg Stock Exchange.

The parameters that will help investigate the extent to which derivatives products fulfil the investment objectives include the stock volatility, σ, the underlying stock return, μ (which in Black-Scholes model will be replaced by risk-free interest rate), r, the stock price S, the dividends for stock that are assumed to be paying dividend, and duration of the contract.

3.5 Stylized facts of the NSM as an emerging market and the development of suitable derivative products in the NSM

Bekaert et al. (1998) identify some distinct features in the characteristics of stock market returns in emerging markets to include: high volatility, little or no correlation with developed and emerging markets, long-term high yields in returns, high predictability potentials than could be recorded with the developed markets, exposure to the influence of external shocks like political instability, changing economic and fiscal policies or exchange rate.

Furthermore, Bekaert and Harvey (1997) examine the cause of varying volatility across emerging markets, particularly regarding the timing of reforms in the capital market and discover that capital market liberalization which is usually responsible for high correlation between local market returns and the developed market, has been unable to trigger local market volatility.

To exploit knowledge of the stylised facts in the NSM in developing suitable derivatives for the market, we will compare market features, similarities and differences in the two most dominant markets in Sub-Saharan Africa, NSM and the Johannesburg Stock Exchange (JSE), using the concept of Random Matrix Theory. We will study the type of correlations among stocks, volatility indices and inverse participation ratios, to determine the sectors that drive the entire markets for the two markets under consideration. Further research applying the results to constructing and pricing the said the derivatives will be based on this study.

3.6 Stylized Facts of Asset Returns

Thompson (2011) asserts that the main purpose of modelling stock market data is to approximate the behaviour of the unobservable data generating process that determine observed stock prices and that the process of examining how fit this approximation is to the data leads to identifying the stylized facts of the stock returns. In the same vein, for derivative products we have the underlying stock price volatility, σ which is the only variable that is unobserved that we seek to determine. Taylor (2011) and Cont (2001) opine that stylized fact is a statistical property that is expected to exist in any series of observed stock market returns. Cont (2001) further maintains that these stylized facts are evident in many financial assets and are found in various markets.

Research findings from various studies investigating the dynamic nature of major stock markets for developed and emerging markets discover the following stylized facts:

- Asymmetry [Brock et al. (1992); Campbell et al (1993); Sentana and Wadhwani (1992)]
- Volume or volatility correlation, Cont, (2001)
- Absence of autocorrelations in returns [Pagan, (1996); Taylor, (2005); Ding et al. (1993); Cont, (2001)]
- Volatility clustering [Scruggs and Glabanidis, (2003); Bollerslev and Zhou, (2002); Mandelbrot, (1963), P.418; Engle, (1982); Bollerslev et al., (1992); Koutmos and Knif, (2002); Moschini and Myers, (2002)]
- High probabilities for extreme events (or thick tails of the distribution - 'heavy tails'), hence non-normality [De Santis and Imrohoroglu, (1997); Pagan, (1996); Taylor, (2011); Cont, (2001)]
- Positive autocorrelation in squared returns and variance [Ding et al., (1993) and finally
- Slow decay of autocorrelation in absolute returns [Ding and Granger, (1996); Taylor, (2005); Cont, (2001); Pagan, (1996)].

We will adopt some of the results on stylized facts that may be useful for derivative pricing obtained by other researchers in the Statistics and Information Modelling Research Group of MERI, Sheffield Hallam University, in carrying out the empirical study of derivative products pricing in the NSM. We now look at some key stock market characteristics that affect derivatives trading:

3.6.1 Volatility

Volatility is a measure of the spread of positive and negative outcomes, unlike risk which is a measure of uncertainty of the negative outcome of some event/process like the stock market returns. A good forecast of asset price volatility over the investment period is a good process towards the assessment of investment risk. There are two general classes of volatility models, namely:

Volatility models that formulate the conditional variance directly as a function of observables (including historical and implied volatility) and others like the ARCH and GARCH models that are not functions of purely observable parameters like the stochastic volatility models. The stochastic volatility model is very popular in option pricing where semi-closed form solution exists.

Hyung et al. (2008) assert that stochastic volatility models are less common as time series model when compared with GARCH models, since the estimation of stochastic volatility model using time series data is a non-trivial task. This is because maximum likelihood function cannot be written straightforwardly when the volatility itself is stochastic. Stochastic models are usually approximated through Markov Chain Monte Carlo methods. These stochastic volatility models are usually simulated, and they are difficult to estimate.

A good volatility model should be able to forecast volatility, which is the central requirement in almost all financial applications. In modelling volatility of a financial system, one should take into cognizance the stylized facts of volatility which include: pronounced persistence and mean reversion, asymmetry such that the sign of an innovation also affects volatility, and the possibility of exogenous or pre-determined variables affecting volatility, Engle and Patton (2001). Essentially, all the financial uses of volatility models entail forecasting aspects of future returns and a typical volatility model used to forecast the absolute magnitude of returns can also be used to predict quartiles or the entire density.

The forecasts of volatility for absolute magnitude of returns are therefore applied by the stakeholders in financial industry in risk management, derivatives pricing and hedging, market making, market timing, portfolio selection, and a host of other financial activities. Volatility is the most important variable in the pricing of derivative securities, the volume of which in the world trade has increased tremendously in recent years. To price an option, one needs to know the volatility of the underlying asset from the time of entering into the contract to expiration date of the contract.

Poon and Granger (2003) assert that nowadays it is possible to buy derivative written on volatility itself, in which case the definition and measurement of volatility will be clearly specified in the derivative contracts. In such case, volatility forecast and a second prediction on the volatility of volatility over the defined period is needed to price such derivative contracts.

A risk manager should know as at today the likelihood that his portfolio will rise or decline in future just like a stakeholder in option contract would wish to know the expected volatility over the entire life span of his contract. A farmer on his own side may wish to write a forward contract to sell his agricultural product, to hedge against fall in price of his produce at the time of harvesting and so on. Dynamic risk management uses the correct estimate of historical volatility and short-term forecast in risk management process. Volatility (historical) is, therefore, from Poon and Granger (2003) given by

$$\sigma = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(r_t - \mu)^2}\ , \tag{3.20}$$

where $r_t = \log\left(\frac{S_t}{S_{t-1}}\right)$, $\mu = the\ expected\ return$ is a quantified measure of market risk.

The main characteristic of any financial asset is its return which is considered as a random variable. The spread of this random variable is known as asset volatility which plays pivotal role in numerous financial applications. The primary role is to estimate the market risk and serve as a key parameter for pricing financial derivatives like the option pricing as seen earlier. It is also used for risk assessment and management and to a larger extent in portfolio management.

3.6.1.1 Market risk

Market risk is one of the main sources of uncertainties for any financial establishment that has a stake in given risky asset(s). This market risk refers to the possibility that an asset value will decrease owing to changes in interest rates, currency rates, and the price of securities.

The method of estimating a financial institution's exposure to market risk is the value-at-risk methodology. The value at risk methodology adopts a system of dynamic risk management whereby the market risk is monitored on daily basis.

GARCH models, as stated above, are also referred to as volatility models and are usually formulated in terms of the conditional moments. GARCH (p, q) lags denoted by GARCH (p, q) has a volatility equation written as:

$$\boldsymbol{\sigma_t^2 = \vartheta_0 + \vartheta_1\varepsilon_{t-1}^2 + \cdots + \vartheta_p\varepsilon_{t-p}^2 + \lambda_1\sigma_{t-1}^2 + \cdots + \lambda_p\sigma_{t-p}^2} \tag{3.21}$$

When the coefficient of the term σ_{t-1}^2 is insignificant in GARCH (1, 1) model, the implication is that ARCH (1) model is likely to be good enough for the volatility data estimation.

As stated earlier in the stylized facts, financial asset returns (stock returns) exhibit volatility clustering, leptokurtosis and asymmetry. These characteristics of asset return indicate increase in financial risk which can affect investors adversely. Volatility clustering refers to the situation when large stock price changes are followed by large price change, of either sign, and similarly small changes are followed by periods of small changes. Leptokurtosis refers to the market condition where the distribution of stock return is not normal but rather exhibits fat tails. In other words, leptokurtosis means that there are higher propensities for extreme values to occur more regularly than the normal law predicts in a series.

Asymmetry, otherwise known as leverage effect, means that a fall in asset return is followed by an increase in volatility greater than the volatility induced by increase in returns. These three characteristics mentioned above make investors to pay higher risk premium to insure against the increased uncertainty in the portfolio of investments. Volatility clustering for instance makes investors to be more averse to holding stocks due to high stock price uncertainty. Emenike (2010) advocates for the use of GARCH (1,1) model to capture the nature of volatility, the Generalised Error Distribution (GED) to capture fat tails, Glosten, Jagannathan and Runkle, GDR-GARCH (1, 1) model (1993) which is a modification of GARCH (1, 1) to capture the leverage (asymmetry) effects of stock return.

Higher moments of a returns distribution include the unconditional skewness and kurtosis defined as:

$$\varepsilon = \frac{E[(r_t - \mu)^3]}{\delta^3} \; and \; \vartheta = \frac{E[(r_t - \mu)^4]}{\delta^4}, \text{ respectively.}$$

The conditional skewness and kurtosis are similarly defined respectively as:

$$\boldsymbol{S_t} = \frac{\boldsymbol{E_{t-1}[(r_t - M_t)^3]}}{\boldsymbol{h_{t-1}^{3/2}}}, \quad \boldsymbol{K_t} = \frac{\boldsymbol{E_{t-1}[(r_t - M_t)^4]}}{\boldsymbol{h_{t-1}^{4}}} \qquad (\mathbf{3.22})$$

$r_t = \log(p_t) - \log(p_{t-1})$ is the asset return and $p_t,\ p_{t-1}$ are asset prices at t and $t-1$, respectively.

$$M_t = E_{t-1}(r_t) is\ the\ condional\ mean$$

$$h_t = E_{t-1}[(r_t - M_t)^2] is\ the\ condional\ variance$$

Conditional volatility is made up of: Historical volatility like the Exponential weighted moving average; implied volatility as in the Black-Scholes model for option prices; and ARCH models like the GARCH family of models.

3.6.2 Implied volatility (IV)

The market's assessment of the underlying assets volatility as reflected in an option is known as implied volatility (IV) of the option. This is obtained through an observation of the market price of the option, and through an inversion of BS (1973) option pricing formula we can determine the volatility implied by the market, Mayhew (1995). In other words, given the Geometric Brownian motion, with some other assumptions, Black-Scholes (1973) obtained exact formula for pricing European call and put options.

Usually, options are traded on volatility with implied volatility serving as an efficient and effective price of the option and therefore implied volatility is important in financial assets risk management. To this end, investors can adjust their portfolios in order to reduce their exposure to those instruments whose volatilities are predicted to be on the increase, thereby managing effectively their exposure to risk in investment. For instance, applying implied volatility to the Black-Scholes (1973) model in equation (3.3) we shall have

$$C_{BS73} = f(S, X, r, t, \sigma) \tag{3.23}$$

So that using equation (3.23) above, the implied standard deviation is denoted by $\sigma_{imp}[X, t]$ which for prescribed values of strike price X, underlying stock price S, risk-free interest rate r and time to expiration t, satisfies equation (3.24) below

$$_{BS73} = f(S, X, r, t, \sigma_{imp}[X, t]\,) \tag{3.24}$$

This equation has the desired positive solution for $\sigma_{imp}[X, t]$ if and only if the option is rationally priced (Manaster et al. 1982) so that

$$Max(0, S - Xe^{-rt}) \leq C_{BS73} \tag{3.25}$$

since according to Hull (1997), prior to maturity, at any given time t, the option price will have a value not less than zero (negative payoff in option pricing is not allowed). Also, the option price should not be less than the current share price less the present value of the exercise price discounted at the risk-free rate, that is $S - Xe^{-rt}$.

However, from the Black-Scholes formula and other derivative option pricing formulas like Heston, Rubinstein, or stochastic volatility option pricing formulas, with the observed option price in the market we can also find the implied option value of **σ** the implied volatility.

Traditionally, due to their robustness, implied volatility (IV) has been calculated using either the BS formula or the Cross-Ross-Rubinstein binomial model for option pricing, and from the underlying stock price assumption of the BS model, IV could be interpreted as the option market's estimate of the constant volatility parameter.

The BS assumption of constant variance does not hold exactly in the markets due to jumps in the underlying asset prices, movement of volatility over time, transaction cost on the assets, and non-synchronous trading which will therefore cause the observed implied volatility to differ across options.

If the underlying asset volatility, as opposed to the assumptions of the BS mode, is allowed to vary deterministically over time, IV is interpreted as the market's assessment of the average volatility over the remaining life of the option. However, when the options pricing formula cannot be inverted analytically as is usually the case, IV is calculated through numerical approximations.

Many options with varying strike price and time to expiration could be written on the same underlying asset and by the BS model (with constant variance) these options should be priced so that they all have exactly the same IV which of course is not true. This systemic deviation from the predictions of the BS constant variance model is referred to as "volatility smile". Volatility smile refers to the use of different values of implied volatility by practitioners in the derivatives contract for different strike prices. As IV are not necessarily the same across the life span of the option, some literature suggested calculating implied volatilities for each option and then using a weighted average of these implied volatilities as a point estimate of future volatilities. Many subscribed to placing more weights on options with higher Vegas (higher sensitivities to volatility), like the Latane and Rendleman (1976) model given by:

$$\hat{\sigma} = \frac{1}{\sum_{i=1}^{N} w_i} \sqrt{\sum_{i=1}^{N} w_i^2 \sigma_i^2} \qquad (3.26)$$

where the weights, w_i are the BS Vega of the options. This method however is bedevilled with the criticism that the weights do not sum to 1. In another development, Becker (1981) found that using the IV of the option with the highest Vega outperforms all other techniques. Garman and Kohlhagen (1983) also state that several other option pricing formulas could be used to calculate IV, and that the currency option pricing formula can also be inverted to calculate the implied volatilities. This model will however, be of much interest in the NSM as currency option is among the derivative products being considered for introduction in the Nigerian Capital Market.

3.6.2.1 Methods of estimating implied volatility

There are two principal ways of estimating implied volatility, namely: Analytical method or closed form solution and Numerical solution which include Newton-Raphson and Bisection methods. Analytical method is applied only for special cases of calculating the implied volatility for at-the -money options. Brenner and Subrahmanyam (1988) demonstrate that we can use Black-Scholes option pricing model to obtain the implied volatility using the relation that for an at-the-money option,

$$S = Xe^{-rt} = ke^{-rt} \tag{3.27}$$

In this regard, we approximate cumulative normal distribution $N(d_1)$ as the integral of normal density function $N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ between the bounds $(-\infty, d_1)$, that is:

$$N(d_1) = \int_{-\infty}^{d_1} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_0^{d_1} e^{-\frac{x^2}{2}}dx \tag{3.28}$$

Evaluation of the integrand is through a Taylor series expansion of $e^{-\frac{x^2}{2}}$ and integrating term by term. Thus,

$$N(d_1) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_0^{d_1}\left[1 - \frac{x^2}{2} + \frac{x^4}{2^2 2!} - \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} - \cdots + \right]dx \tag{3.29}$$

$$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}[d_1 - \frac{d_1^3}{2.3} + \frac{d_1^5}{2^2 2!5} - \frac{d_1^7}{2^3 3!7} + \frac{d_1^9}{2^4 4!9} - \cdots + \cdots -]$$

Similarly, for d_2.

For small values of d_1 ($|\ d_1\ | \leq 0.2$)terms beyond d_1 or order ≥ 3 are ignored for a better approximation of $N(d)$.

Thus, $N(d) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}d\ \forall\ d. \qquad d_1 = \frac{1}{2}\sigma\sqrt{t}, \qquad d_2 = -\frac{1}{2}\sigma\sqrt{t}$

Therefore, $N(d_1) \cong \frac{1}{2} + 0.398d_1 \quad = 0.5 + 0.199\sigma\sqrt{t}$

$$N(d_2) = 1 - N(d_1) = 0.5 - 0.199\sigma\sqrt{t}$$

so that the value of at-the-money option from Black-Scholes option pricing formula will be $C_{BS73} = 0.398S\sigma\sqrt{t}$.

From equation (3.27) we shall then have:

$$C_{BS73} = 0.398S\sigma\sqrt{t} = 0.398ke^{-rt}\sigma\sqrt{t}$$

$$\sigma = \frac{C_{BS73}}{S}\ x\ \frac{1}{0.398\sqrt{t}} \tag{3.30}$$

Corrado and Miller (1996) modified this implied volatility formula in (3.30) above as:

$$\sigma = \frac{C_{BS73}}{S}\sqrt{\frac{2\pi}{t}} \tag{3.31}$$

since, $0.398 = \frac{1}{\sqrt{2\pi}}$

3.6.2.2 Various weighting schemes for implied volatility

To this end, we must observe that after calculating the various standard deviations for various options written on each stock by the method of Newton Raphson or bisection methods, we have to combine them into a single weighted average standard deviation. We look at the quadratic approximation method for implied standard deviation when the option is not at the money.

3.6.2.3 Quadratic approximation of implied volatility

Using the method of Brenner and Subrahmanyam (1988), we can obtain a simple, accurate formula for the computation of implied volatility (standard deviation) using a quadratic approximation. Recall equation (3.28) from where we can state:

$$N(d) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(d - \frac{d^3}{6} + \frac{d^5}{40} + \cdots\right) \qquad (3.32)$$

and from the expansions of the normal probabilities $N(d)\, and\ N(d - \sigma\sqrt{T})$ in the C_{BS73} we shall obtain

$$C_{BS73} = S\left(\frac{1}{2} + \frac{d}{\sqrt{2\pi}}\right) - X\left(\frac{1}{2} + \frac{d - \sigma\sqrt{T}}{\sqrt{2\pi}}\right) \qquad (3.33)$$

Corrado and Miller (1996) assert that (3.33) can be manipulated to yield the following quadratic equation in $\sigma\sqrt{T}$

$$\sigma^2 T(S + X) - \sigma\sqrt{T}\,\sqrt{8\pi}\left(C - \frac{S - X}{2}\right) + 2(S - X)\ln\left(\frac{S}{X}\right) = 0 \qquad (3.34)$$

having non-negative real roots with the largest roots as

$$\sigma\sqrt{T} = \sqrt{2\pi}\left\{\frac{C - \frac{S-X}{2}}{S+X}\right\} + \sqrt{2\pi\left\{\frac{C - \frac{S-X}{2}}{S+X}\right\}^2 - \frac{2(S-X)\ln(\frac{S}{X})}{S+X}} \qquad (3.35)$$

which can further be reduced to

$$\sigma\sqrt{T} = \frac{\sqrt{2\pi}}{S+X}\left\{C - \frac{S-X}{2} + \sqrt{\left(C - \frac{S-X}{2}\right)^2 - \frac{(S-X)^2}{\pi}}\right\} \qquad (3.36)$$

Observe that whenever S = X we will obtain the Brenner and Subrahmanyam (1988) model and it is written by Corrado and Miller (1996) as

$$\sigma = \frac{C_{BS73}}{S}\sqrt{\frac{2\pi}{T}}\ as\ in\ equation\ (3.27)$$

Chambers and Nawalkha (2001) assert that because of the shortcoming inherent in equation (3.31) and consequently that of equation (3.32), a solution to the expressions can give a negative square root (which means no real solution to the implied volatility) and that this could be obtained for short term options that are very substantially away from the money. As a remedy to this shortcoming, Bharadia et al. (1996) derived a very simplified implied volatility model given by

$$\sigma = \sqrt{\frac{2\pi}{T}}\ \frac{C - \frac{S-X}{2}}{S - \frac{S-X}{2}} \qquad (3.37)$$

Moneyness

Moneyness in a security asset is the ratio between its strike price K and the price of the underlying asset, S. That is, moneyness is how far from the strike price is the current underlying price. For a European call option, Moneyness can be defined as a ratio of the underlying stock price S with that of the exercise (strike) price K, i.e.:

$$X = \left(\frac{K}{S}\right) \qquad (3.38),$$

A European call option is said to be at the money if S = K; if S > K the option is said to be in the money; whereas, if S < K, the option is said to be out of the money. The converse is true for a European put option

From the second expression of equation of moneyness, we can calculate implied volatility as:

$$\sigma_{imp} = a_0 + a_1X + a_2X^2 + a_3\tau + a_4\tau^2 + D(a_5 + a_6X + a_7X^2 + a_8\tau + a_9\tau^2) \qquad (3.39)$$

$$where\ D = \begin{cases} 0, if\ X < 0 \\ 1, if\ X \geq 0 \end{cases}$$

τ = time to expiration given by $\tau = T - t$, T is the expiration date of a given option and t = the current date.

3.6.3 Bubbles

Over the years, a substantial number of market inefficiencies or 'anomalies' have been of concern to financial managers and researchers in financial markets. Similarly, bubbles in financial markets are expressions of market inefficiencies that cause damage to the real economy, Stefan Palan (2009). It is then pertinent to ask if the derivatives markets improve the informational efficiency of spot markets and if in the affirmative, can the prediction markets which are just another form of a market place for trading of derivatives contracts reduce or prevent the formation of price bubbles at financial exchanges?

The standard model of asset prices values the assets based on the present value of the stream of dividends that the owner expects to receive. When the prices of assets conform to this expectation, the rational expectation is said to be driven by the fundamentals. Any other price expectation not based on this fundamental dividend stream is called "bubble".

Bubbles and option prices

A bubble in the derivative sense is defined as a price process which when discounted is a local Martingale under the risk-neutral measure, but not a Martingale. In a market with bubbles, many standard results from the folklore become false. For example, the put-call parity fails, the price of an American option call exceeds that of European, and call prices are no longer increasing in maturity (for a fixed strike), Cox et al. (1985). For instance, if S is a discounted price of a given financial security, and S is continuous, the no arbitrage theory tells us that S is a local martingale under the pricing measure and defines bubbles.

There are two types of bubbles: deterministic bubbles and rational stochastic bubbles. Diba and Grossman (1987) show that conditions that rule out certain deterministic bubbles also rule out all rational stochastic bubbles of the form suggested by Blachard and Watson (1982). The bubbles could be speculative, and the speculative bubbles are characterised by a long run-up in price followed by crash.

The most important feature of rational speculative bubbles is that stock prices may deviate from their fundamental value without assuming or having irrational investors, Chan et al. (1998). They assert that investors realise that prices exceed fundamental values, but they believe that, with high probability, the bubble will continue to expand and yield a high return which compensates them for the probability of a crash, thus justifying the rationality of staying in the market despite the overvaluation.

3.6.4 Speculation

As noted, the major motivation for entering into a forward or futures and in fact any derivatives contract is to speculate and or hedge an existing market exposure so as to reduce cash flow uncertainties resulting from the market exposure. While the forward or futures contract is mainly for hedging, an option contract provides a form of financial insurance to their holders. Thus, holding a call/put option provides the investor with the protection (insurance) against an increase/decrease in the price above/below the contract's price. The writer of the call/put option who takes the reverse side of the contract is referred to as the provider of the insurance. Theorists generally define a speculator as someone who purchases an asset with the intent of quickly reselling it or sells an asset with the intent of quickly repurchasing it, Stout Lynn (1999). Speculative trading behaviour incorporates two motives in the activity; risk hedging and information arbitrage.

3.6.4.1 Risk-hedging

The risk-averse investors pay to avoid taking risk (like through insurance policies), while investors with greater tolerance to risk reap some profit through accepting the risk rejected by the risk-averse investors. In the risk-hedging model of speculation, speculators are relatively risk-neutral traders. For instance, a risk-averse rice farmer in Abakiliki, Ebonyi State, Nigeria, whose crops will soon be ready for harvest, and as a risk averse farmer, is more worried about the fall in price of rice during the harvest than the possible rise in price, might prefer to sell his crops now at a slight discount (forward derivative) to deliver it in, say forty days' time. On the contrary a more risk-neutral rice speculator might purchase the contract since the price discount creates for him a "risk-premium" that compensates him for accepting the changes of future price of rice within the forty days.

This risk hedging model implies that speculative traders generally involve "hedgers" on the one side of the transaction, and "speculators" on the other side. The risk-averse (hedgers) like the rice farmer is therefore happy to pay, to avoid the price variation (presumably downwards) inherent in holding the asset(s) (rice product), while a more risk-neutral speculator is happy to be paid a premium to assume the risk. Risk management that reduces return volatility is frequently termed hedging, while risk management that increases the return volatility is called speculation.

Information arbitrage

The other model of speculative trading different from risk hedging is the information arbitrage model. The information arbitrage approach describes speculators as traders who through financial research are able to predict future changes in prices of assets and liabilities. They are equipped with superior knowledge of market information that permits them to trade on favourable terms with less-informed buyers and sellers who are trading for other reasons. As an illustration, a major dealer in Nigerian rice who collects data about other rice farmers in several regions like Lafia, Gboko, Nassarawa, Ugbawka and Kano, all in different rice producing areas of Nigeria that might show a low harvest yield in the regions which will necessitate price increase, may profit form the strategy of buying and storing rice from less well-informed farmers and stakeholders in the rice industry.

On a larger spectrum, Smith and Stulz (1985) demonstrate that when a risk-averse manager owns a large number of firm's shares, his expected utility of wealth is significantly affected by the variance of the firms expected profits. The Manager will direct the firm to hedge when he believes that it is less costly for the firm to hedge the share price risk than it is for him to hedge the risk on his own account. Consequently, Smith and Stulz predict a positive relation between managerial wealth invested in the firm and the use of derivatives. Thus, for speculation to be a profit-making activity in rational markets, either a firm must have an information advantage related to the prices of the instruments underlying the derivatives, or it must have economies of scale in transactions costs allowing for profitable arbitrage opportunities.

However, Hentschel et al. (2001) state that public discussion regarding corporate use of derivatives focuses on whether firms use derivatives to reduce or increase firm risk. They opine that in contrast, empirical academic studies of corporate derivatives usually take it for granted that firms hedge with derivatives. Their findings are consistent with Stulz's (1984) argument that firms primarily use derivatives to reduce the risks associated with short-term contracts.

[Stulz (1984), Smith and Stulz (1985), and Froot et al. (1993)] construct models of corporate hedging that could be useful to investors in Nigeria when the derivative products take off fully in Nigeria. These models predict that firms attempt to reduce the risks they face if they have poorly diversified and risk-averse investors face progressive taxes, suffer large costs from potential bankruptcy or have some funding needs for future investment projects in the face of strongly asymmetric information.

3.6.5 Market efficiency

The fact that a market is efficient or not and where the inefficiencies lie is a vital tool in investment valuation. For efficient market, the market price of assets gives the best estimate of value and the associated process of asset valuation becomes the one that justifies the actual market price. For markets that are not efficient, the asset market price could deviate from the actual value and the process of valuation is directed towards realising a reasonable estimate of this value. The market inefficiency increases the possibility of having under or overvalued stocks.

A market is said to be efficient when the market price is unbiased estimate of the true value of the investment. However, market efficiency does not necessarily mean that the market price is equal to the true value at every point in time but rather it emphasizes that errors in the asset market price is unbiased. That is to say, asset market price can be greater than or less than the true value. So long as these deviations are random, the market is said to be efficient. Randomness in the price deviation here means that there is an equal probability that stock prices are undervalued or overvalued at any given time, and these deviations are uncorrelated with any observable parameter.

Also, in an efficient market where the deviations from true values are random, no investor or group of investors should be able to consistently find under or overvalued stocks or any other investment assets using any known investment strategy.

There are three categories of efficiency in the efficient market hypothesis:

The weak form efficient: The weak-form of the efficient market hypothesis claims that prices fully reflect the information implicit in the sequence of past prices.
Semi-strong efficient: Semi-strong type asserts that prices reflect all relevant information that is publicly available.
The strong-form efficient market: The strong-form efficient market asserts that information known to any participant is reflected in market prices.

Basu (1977) notes that the case where lower (P/E) securities perform better than the higher P/E counterparts is indicative of market inefficiency. P/E represents the price to earnings ratios of the securities being considered. In his finding, "securities trading at different multiples of earnings, on average, seem to have been inappropriately priced vis-à-vis one another and opportunities for 'abnormal' earnings were afforded to investors". This contradicts the fact that in an efficient market, stock with lower P/E ratios should be no more or less likely to be undervalued than the stocks with higher P/E ratios.

Tests of market efficiency are aimed at checking whether a given investment strategy earns excess returns. In all cases, tests of market efficiency are a combined test of market efficiency and the efficacy of the model used for expected returns. In other words, the BS (1973) model that under-prices and overprices deep in-the-money and deep out-of-the-money options does not show model efficacy in the market.

For efficiency in the derivatives market, insider knowledge trading should be discouraged to ensure market efficiency in the transactions. This could be achieved by using news reflected in the stock market as a benchmark for public information, and banks, for instance, must not use private knowledge of corporate clients to trade instruments like the credit default swaps. It is a public knowledge that many financial institutions are fond of trading credit default swaps in the same companies they finance, probably to reduce the risk on their own balance sheets, Acharya et al. (2007). Modest regulatory framework would address this problem to ensure transparency in derivatives trade.

Baxter (1995) identifies three major problems with market efficiency tests:
He asserts that the major problem with market efficiency test is that they are extremely vulnerable to selection bias. Imperfect synchronization with the underlying asset price and bid-ask spread (on options or on the underlying asset) can generate large percentage error in option prices, especially for low priced out-of-the money options.
The second and statistical reason is that the distribution of profits from option trading strategies is typically extremely skewed and leptokurtic. This is evidently true for unhedged options positions, since buying options involves limited liability but unlimited profit. Merton (1976), however, points out that this is also the case with delta-hedged positions and specification error.
Finally, the problem with 'market efficiency' studies is that they give no clue about which options are mispriced and that the typical approach pools options of different strike prices, maturities and even options on different stocks together.

3.6.6 Predictability

Mathematical modelling can assist in the establishment of the relationship between current values of the financial indicators and their future expected values. Model based quantitative forecasts can provide the stakeholders in financial markets with a valuable estimate of a future market trend. Some schools of thought, however, hold the view that future events are unpredictable, while others have the contrary opinion. That is why financial assets volatility has the tendency to cluster (large moves follow large moves and small moves also align with small moves), and thus exhibits considerable autocorrelation signalling the dependency of future values on past values. This attribute justifies the concept of volatility forecasting as a mathematical technique in financial asset pricing.

The efficient market hypothesis (EMH) which disagrees with asset return predictability evolved in 1960's from the random walk theory of asset prices, as was proposed by Samuelson (1965). He shows that in an informationally efficient market, price changes must be unpredictable. As a result of individual errors and irrationality of market participants, some departure from market efficiency could be observed resulting in the occurrence of bubbles and crashes in the financial market operations.

However, it is often argued that if the stock market returns are efficient, then it should not be possible to predict stock returns, namely that none of the variables in the stock market regression should be statistically significant, Paseran (2010). He declares that market efficiency needs to be defined separately from predictability, since stock market returns will be non-predictable only if market efficiency is combined with risk-neutrality.

A risk-neutral investor, as seen from speculative property of asset returns segment of the stylized facts of NSM, is an indifferent investor in whose belief a position in a risk-free asset like bond makes no different with that in a risky asset like the underlying stock. In other words, the risk-neutral investor will be indifferent between the certainty of return and the expectation of the pay-out from risky asset investments.

Lo and Wang (1995) argue that predictability of an asset's return could affect the prices of options written on that asset, even though predictability is induced by the drift which does not enter the option pricing formula. Similarly, Leon and Enrique (1997), analyse the effect of predictability of an asset's return on the prices of options on that asset for a class of stochastic processes for prices, and obtained predictable, yet serially uncorrelated returns.

3.6.7 Valuation

To excel in options and derivatives trading in general, one is required to have a fair understanding of the characteristics of these market instruments, especially the options valuation, in order not to lose a great deal of money. All modern option pricing techniques rely heavily on the volatility parameter for price valuation. However, in evaluating the cost/price of the options one is expected to take into consideration the following factors in addition to the volatility parameter:
the current market price of the stock;
the interest rate;
underlying stock dividend;
the strike price of the option (particularly with reference to the stock market price);

Remaining life of the option (time left before expiration);
Taxation; and
the Greeks.

We note that in all these factors, the investor(s) have control in only two factors, namely time to expiration and the strike price of the option.

The current market price of the stock

It is a known fact from research literature that when the stock price increases or decreases, a call option premium (price) will increase or decrease, respectively, whereas for put option the reverse is the case. In other words, underlying stock price is directly proportional to the call option premium, while on the contrary stock price is inversely related to the put option premium. The rewarding market strategy is therefore to buy call option when you think (from your market strategy analysis) that the underlying stock price is going up and puts when you forecast otherwise.

3.6.8 Anomalies

Schwert William G. (2002) asserts that anomalies are empirical results that seem to be inconsistent with existing theories of asset price behaviour. They indicate either market inefficiency (profit opportunities) or inadequacies in the underlying asset pricing model.

Causes of anomalies in the financial system

Stambaugh et al. (2012) assert that financial distress is often attributed to the cause of anomalous patterns in the cross section of stock returns. However, Campbell et al. (2008) find that firms with high failure probability have lower, not higher, subsequent returns anomaly.

Small firms outperform: The first stock market anomaly is that smaller firms (that is firms with smaller market capitalization) tend to outperform larger companies. Banz (1981) and Reinganum (1981) show that small-capitalization firms on the NYSE earned higher average returns, just as Basu (1977) in a study of 1400 firms including both small and big firms observe that low P/E securities outperformed their high P/E counterparts by over 7% per annum.

Net stock issues and composite equity issues: The stock issuing market has been viewed as producing an anomaly arising from sentiment-driven mispricing. It is known that smart managers issue shares when sentiment driven traders push prices to an overvalued level.

Seasonal effect: Seasonality or calendar anomalies such as month of the year, day of the week (weekend effect), are also known to have effects on stock market anomaly. Despite strong evidence that stock market is highly efficient, there have been scores of studies that have documented long term historical anomalies in the stock market that seem to contradict the efficient market hypothesis (EMH), Kuria Allan et al. (2013). In their finding, seasonal anomalies are persistent in the markets of both advanced and emerging economies (which probably extends to the Nigerian security market), thus showing the inefficiency in the stock market. Research has shown that anomalies tend to disappear, reverse or alternate when they are documented and analysed in academic literature, hence the need to take into consideration stock market anomalies and relate same to the NSM.

Keim (1983) and Reiganum (1983) discover that most abnormal returns for small firms measured relative to the Capital Asset Pricing Model (CAPM) are prevalent within the first two weeks in January. Islam and Watanapalachaikul (2005) proffer a model for testing the daily seasonality in stock market adjusted returns, by estimating the following regression equation:

$$W_t = \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3 + \alpha_4 d_4 + \alpha_5 d_5 + \varepsilon_t \qquad (3.40)$$

$where, \alpha_1, \alpha_2, \ldots \alpha_5$ $are\ parameters, d_1, d_2, \ldots, d_5$ are days of the week (Monday - Friday) with:

$$\begin{cases} d_1 = 1, if\ t\ is\ monday \\ \quad 0, \qquad elsewhere \end{cases}, \ and\ \varepsilon_t\ is\ the\ error\ term.$$

Similarly, for monthly seasonality, (month of the year effect) we adopt the model:

$$M_t = \beta_1 m_1 + \beta_2 m_2 + \cdots + \beta_{12} m_{12} + \varepsilon_t$$

where as usual, $m_1, m_2, \ldots, m_{12}$ represent January to December, with$m_i = \begin{cases} 1, i = 1 \\ 0, i \neq 1. \end{cases}$

$$\beta_1, \beta_2, \ldots., \beta_{12}\ are\ parameters, and\ \varepsilon_t\ the\ error\ term.$$

Total accruals: Sloan (1996) shows that firms with high accruals (aligned estimates of revenue and cost in a given period) earn abnormal lower returns, on average, than firms with low accruals, which suggests that investors overestimate the persistence of the accruals component of earnings when forming earnings expectations.

3.6.9 Momentum

The momentum effect as observed by Jegadeesh and Titman (1993) is seen as one of the most robust anomalies associated with asset pricing. The momentum effect stipulates that high past returns forecast high future returns.

3.6.9.1 The value effect

Basu (1977, 1983) observes that firms with high earning-to-price ratios earn positive abnormal returns with regards to Capital Asset Pricing Model (CAPM). Other researchers also infer that positive abnormal returns seem to accrue to portfolios of stocks that have high dividend yields or to stocks that have high book-to-market values. The measure for abnormal return α_i is called Jensen's (1968) alpha from the model:

$$(R_{it} - R_{ft}) = \alpha_i + \beta_i(R_{mt} - R_{ft}) + \varepsilon_{it} \tag{3.37}$$

where R_{it} is the return on US Dimensional Fund Advisors (DFA), equivalent to say the Pension Fund in Nigeria, R_{ft} = the yield on a one-month treasury bill, and R_{mt} = the return on the US Centre for Research in Security Prices (CRSP) mutual Fund database.

3.7 Random Matrix Theory (RMT)

Random Matrix Theory (RMT) is used for the study and analysis of cross-correlations between price fluctuations of different stocks in a given financial market, Plerou et al. (2002). As Nigeria policy makers in the NSM are modelling the trade on derivative products after that of the Johannesburg Stock Exchange (JSE), it is pertinent for this research to look at the nature and characteristics of correlations that exist among stocks in the two exchanges. This procedure will provide the necessary hints on appropriate pricing and evaluation of derivative products earmarked for introduction into the NSM. Furthermore, there was a current adjustment to Basel 11 market risk framework on banks carried out by Basel Committee on Banking Supervision, 2011 which recommends a continuation of focus on the risk related to correlation trading portfolios.

Numpacharoen (2013) asserts that financial institutions or investment fund managers usually hold multiple assets and asset classes in their portfolios, which include basket of derivatives, credit derivatives or other correlation trading products, and these portfolios of assets depend heavily on correlation coefficients among underlying assets. Naturally, the linear relationships among assets in a given financial system are encapsulated in an empirical correlation matrix derived from time series of historical returns of the respective assets of interest in that financial market. Numpacharoen et al. (2013) declare that correlation is useful in portfolio management as it can be applied in reducing the risk associated with the investments.

The empirical cross-correlation matrix represented by C is constructed from the returns of various stocks considered in a given stock exchange for a specified period of time, usually in years. The empirical correlation matrix obtained is compared with a random Wishart matrix of an equivalent dimension with that of C for the analysis of nature of correlations that exist between the component stocks in the financial market being considered. Usually, we test the statistics of the eigenvalues λ_i of C (the empirical correlation matrix) against a 'null hypothesis' - the random correlation Wishart matrix constructed from mutually uncorrelated time series.

The length of the historical period, T, for the time series data on stock price returns is expected to be large enough with respect to the number of stocks under consideration to prevent noise from dominating the data analysis. For an appropriate period T and a given number of stocks, N, if all the eigenvalues of the empirical correlation matrix and that of the Wishart matrix lie in the same region without any significant deviations, then the stocks are said to be uncorrelated. In this case, no information could be obtained, or deductions made about the nature of the market since it is the deviations of the eigenvalues of the empirical correlation matrix from that of the Wishart matrix that carries information about the entire market. If, however, the analysis is not dominated by noise but rather there exists at least one eigenvalue that lies outside the theoretical bounds of the eigenvalues in the empirical correlation matrix obtained from the historical price returns, then the deviating eigenvalue(s) is (are) known to carry information about the market under consideration.

It is, therefore, by comparing the eigenvalue spectrum of the empirical correlation matrix and that of the Wishart matrix to the analytical result obtained for random matrix ensembles that we can deduce the significant deviations from the RMT eigenvalue predictions which will in turn provide the required genuine information about the correlation structure of the system, Conlon et al. (2007). It is the analysis of information on deviations of the eigenvalues that is used to reduce the difference between predicted and realised risks associated with various stocks in the investment portfolio in a given market.

The effect of noise on RMT applications has different impact on the analysis depending on whether we want to optimize the portfolio or merely wish to measure the risk of a given portfolio. For the case of portfolio optimization, the effect of noise is more significant compared to when we are measuring the risk in a given portfolio for an acceptable ratio of $N:T$ with N representing the number of stocks and T is the period considered in the time series analysis, Pafka and Kondor (2003).

The quantification of correlation between various stocks in a financial market is of much interest, not just for scientific reasons of understanding the economy in question as a complex dynamical system, but also for practical reasons which includes asset allocation and the estimation of risks associated with the portfolio in such financial system, (Farmer and Lo (1999), Mantegna et al. (2000), Bouchaud et al. (2000), J. Campbell et al. (1997)) using the of RMT. In particular, RMT has most often been applied in filtering the desired market information from statistical fluctuations that are associated with the empirical cross-correlation matrices obtained from the financial time series of stock price returns, (Laloux et al. (1999), Plerou et al. (1999), Bouchaud et al. (2004), Conlon et al. (2007)).

Furthermore, we note from the demonstrations of (Plerou et al. (1999), Plerou et al. (2000), Plerou et al. (2001), Laloux et al. (2000)), that filtering techniques based on RMT are of immense benefit in portfolio optimization both for reduction of the realised risk associated with an optimised portfolio and improving the forecast of the realised risk.

To the best of my knowledge, no such work on the comparison of stock market correlations has been carried out on African emerging markets, especially JSE and NSM which are major emerging markets in the Sub-Saharan Africa. Most of the work on such comparison has been carried out for developed markets or developed versus emerging markets, see, for instance, Shen, and Zheng (2009), Podobnik et al. (2010), Kumar and Sinha (2007), Sensoy et al. (2013) and Fenn et al. (2011). Also, for some comparison of different stock exchanges within the same market environment the reader is recommended to see some article from Kumar and Sinha (2007).

3.8 Summary and conclusion

This chapter summarized the key stochastic calculus models that are required for pricing and trading in derivative products in the Nigerian stock Market. It also looked at the Black-Scholes, BS (1973) model which underpins pricing of derivatives products and the various extensions of this model necessitated by the underlying assumptions of the BS. In our bid to overcome the perceived shortcoming of the Black-Scholes (1973) model we will examine the practitioners' ad-hoc Black-Scholes model, which could be recommended in the interim for pricing of pioneer derivative products for the NSM.

The literature review took cognizance of the fact that trade in derivatives products in Nigeria is still at the formative stage, and thus demands research and theoretical background that will provide support to policy makers, market participants and researchers in the Nigerian financial services sector. In doing this, cautious efforts were made to examine the pioneer products earmarked for the commencement of derivatives products in the NSM, in line with other similar derivatives products that may have some affinity with the inherent markets characteristics and stylized facts of the NSM.

CHAPTER 4

Analytical Approaches and Concepts

4.0 Data presentation and coverage

The data for this research is from the Nigerian Stock Market, a benchmark emerging market, Johannesburg Stock Exchange (JSE), and some developed markets. As a result of the mathematical equations involved in the research, we used MATLAB, Monte-Carlo Simulation, SPSS and Excel VBA to analyse the data.

The data from the NSM are meant to represent the fundamental properties of the underlying stocks upon which the derivative products will be built. The data also provide the fundamental underlying stock features that are necessary for the formal commencement of the trade in the derivatives products in the NSM.

In Chapter 5, we used data from the Nigerian Stock Market (NSM) for some daily market prices of Access Bank of Nigeria, Plc in 2016 to demonstrate the application of Euler-Maruyama approximation in the estimation and or forecast in the prices of stocks taking Access Bank as a case study.

In Chapter 6, to demonstrate the workings of Black-Scholes model, the shortcoming of the model as regards the type of implied volatility surface obtained (no-flat surfaces contrary to the expectation from the model assumption of constant volatility), and the application of Ad-Hoc Black-Scholes, from some given call options, data on Apple stock from a developed economy (USA) for the year 2016 to 2017 were used since Nigeria has no data yet on derivative trade in her capital market.

To estimate the nature of correlations of stocks in the NSM and JSE, as discussed in Chapters 7 and 8, the data set consists of the daily closing prices of 82 stocks listed in the Nigerian Stock Market, NSM from 3rd August 2009 to 26th August 2013, giving a total of 1019 daily closing returns after removing

(a) assets that were delisted,

(b) those that did not trade at all or

(c) are partially in business for the period under review.

The stocks considered for NSM are drawn from the Agriculture, Oil and Gas, Real Estates/Construction, Consumer Goods and Services, Health care, ICT, Financial Services, Conglomerates, Industrial Goods, and Natural Resources. For the JSE, we have a total of 35 stocks selected from Top 40 shares in the Industrial Metals and Mining, Banking, Insurance, Health care, Mobil Telecommunications, Oil and Gas, Financial services, Food and Drugs, Tobacco, Forestry and Paper, Real Estate, Media, Personal Goods and Beverages, covering the period 2nd January 2009 to 01st August 2013 covering a similar period as that of NSM (This period was chosen for the research because that was the period when we could get the complete market information for the two stock exchanges being considered).

For the banks stocks in the NSM, the Data set is made up of the daily closing prices of 15 bank stocks listed in the Nigerian Stock Market, NSM from 3rd August 2009 to 26th August 2013, giving a total of 1019 daily closing returns after removing assets that were delisted, that did not trade at all or are partially traded in the period under review. The bank stocks considered are Access, Diamond, Equatorial Trust, First Bank of Nigeria, First City Monument, Fidelity, Guaranty Trust bank, Skye bank, Stanbic, Sterling, United Bank for Africa, Union Bank, Unity Bank, WEMA and Zenith Bank.

We remark that for the daily asset prices to be continuous and to minimize the effect of thin trading, it is expedient to remove the public holidays in the period under consideration. Furthermore, to reduce noise in the analysis, market data for the present day is assumed to be the same with that of the previous day in the cases where there is no information on trade for any particular asset on a given date.

4.1 Research Design

The research was carried out quantitatively using the Causal-Comparative research method. For the causal-comparative type of quantitative design we looked at the features of the South African market especially the underlying stocks and compare same with that of the Nigerian Stock market for similarity and differences for the two most dominant markets in the Sub-Saharan Africa. As stated earlier, from the interaction we had during the scientific research visit to Nigeria, we were informed that the NSE is trying to adopt some derivative products from the JSE into the NSM. Hence, the features and characteristics of the two markets are needed for appropriate pricing and evaluation of the proposed derivative products to be adopted into the NSM.

We looked at the pricing of some of the derivative models including the Black-Scholes and some of its variants especially the practitioners Black-Scholes in some organised markets, to be able to propose appropriate models for an emerging market like the NSM.

We also considered some concepts of stochastic models and financial engineering tools necessary for understanding the trade in derivatives and financial engineering products. These concepts include Brownian motion, Ito processes, Ito Integral and Differential equations, Stratonovich and Stochastic Integrals and their relationships, Euler-Maruyama methods.

As trade on derivative products are still in the formative stage, the extensions of Black-Scholes considered in this work include Merton (1973), Ad-Hoc Black-Scholes (AHBS) models in the form of "Relative smiles" and "Absolute smiles".

Stylized facts of the NSM, which are relevant in derivative trading, were critically examined through the concept of Random Matrix Theory. To achieve this, we looked at the Random Matrix Theory (RMT) for Nigerian Stock Market (NSM) and The Johannesburg Stock Exchange (JSE). The study provided the desired empirical evidence for NSM policy makers towards effective pricing and evaluation of derivative products slated for introduction into NSM. Through this study we examined the type of correlation, volatility and the leading stock(s) that drive the markets for both exchanges.

4.2 Estimation of stock price using historical volatility

For stock price data obtained for some periods (usually in days), the estimate of historical volatility according to Poon and Granger (2003) is given by

$$\widehat{\sigma} = \frac{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i-\bar{u})^2}}{\sqrt{\tau}}, \qquad where\ u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \tag{4.1}$$

$\bar{u}$ is the sample average $of\ u_i$, S_i is the stock price in the period i and τ is the total length of each period in years. The annualized estimate of this standard deviation could also be written as

$$\hat{\sigma} = \frac{S}{\sqrt{\tau}} \tag{4.2}$$

where, $S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n}u_i\right)^2}$

4.3 Stochastic Calculus

Stochastic calculus in this research context is a mathematical method used in modelling and analysing the behaviour of economic and financial phenomena under uncertainty, by means of Ito's lemma, stochastic differential equations, stochastic stability and control. This necessitates the presentation of various mathematical concepts and results such as the notion of stochastic integral, the properties and solutions of stochastic differential equations, some approaches to stochastic stability and control.

Stochastic partial differential equations (SPDEs) were studied and their use in asset pricing or portfolio modelling and multi-species asset pricing models. In light of this, Black-Scholes partial differential equations and the concept of Brownian motion are of great importance in asset pricing for call or put options of the derivative products.

Malliaris and Brock (1982) assert that stochastic calculus is useful in determining: stochastic inflationary rates experienced in the use of Ito's lemma for examining the solution and the behaviour of prices including real return of an asset when inflation is described by an Ito process; in the process of finding the solution of Black-Scholes option pricing model; for term structure analysis in an efficient market for interest rate by Vasicek model (1977); and in market risk adjustment in project valuation by method of Constantinides (1978).

A Stochastic process $S(t)$ is called a geometric Brownian motion (GBM) with parameters μ and δ if its logarithm forms a Brownian motion with mean μ and variance δ^2. The price of a stock follows a (GBM) process with mean μ and variance δ^2 as constants. Furthermore, the (GBM) for a stock price satisfies the following stochastic differential equation of equation (2.2), where $S(t)$ is the stock price at a time t, μ is the rate of return on riskless asset (or drift), δ is the volatility index of the stock, and $w(t)$ is the white noise or the Wieners process. The solution of (2.2) as shown is of the form

$$S(t) = S(0)\exp\left[\left(\mu - \frac{1}{2}\delta^2\right)t + \delta w(t)\right]$$
$$= S(0)\exp\left[\left(\mu - \frac{1}{2}\delta^2\right)t + \delta\int_0^t dw\right]$$

Brownian motion

A real valued continuous time stochastic process B_t(ω), t ε [0, T] is called a Brownian motion or Wieners process if

B_t(ω) is a Gaussian process
E(B_t) = 0 for all t.
E[$B_t B_s$] = $min(t, s)$

It is denoted by B_t or W_t and if $\delta^2 = 1$, then the process is called standard Brownian motion.

Some properties of the Wieners process

$$E[dW(t)] = 0$$
$$E[dw(t)dt] = E[dw(t)]dt = 0$$
$$E[dW(t)^2] = E(dt) = dt$$
$$E\{[dW(t)dt]^2\} = E[dw(t]^2]dt^2 = 0$$
$$Var[dw(t)dt] = E[(dW(t)dt)^2] - E^2[dW(t)dt] = 0.$$

The variables for constructing option pricing Strategies

Delta

A by-product of application of the BS model is the calculation of the delta. Delta is the degree to which an option price will move given a small change in the underlying stock price. For instance, an option with a delta of 0.5 will move half a naira (50 kobo) for every full one (1) naira movement in the underlying stock price. A deeply out-of-the money call will have a delta very close to zero while a deeply in-the-money call will have a delta very close to 1. The formula for a 'delta' in a European call on a non-dividend paying stock is Delta = $N(d_1)$ where d_1 is as defined in the BS call option pricing formula. Call deltas are positive whereas 'put delta' are negative thus reflecting the fact that the put option price and the underlying stock price are inversely related. In fact, the put delta equals the call delta minus 1.

Delta as a hedge ratio

The delta is often called the hedge ratio. As an illustration, when you have a portfolio short n options (example when you write n calls), then n multiplied by the delta gives you the number of shares (i.e. the units of the underlying stock) you would need in order to create a riskless position. By this we mean a portfolio which is worth the same whether the stock price rose by a very small amount or fell by a very small amount. In such a 'delta-neutral' portfolio any gain in the value of the shares would be exactly offset by a loss on the value of the calls written and vice-versa.

It is noteworthy here that delta changes with the stock price and time to expiration, and the number of shares would need to be continually adjusted to maintain the hedge. The formula for estimating the delta is represented as:

$$\Delta = \frac{\partial C}{\partial S} = e^{-(T-t)}\left[\emptyset(d_+) + S\frac{\partial}{\partial S}\emptyset(d_+)\right] - Ke^{-r(T-t)}\frac{\partial}{\partial S}[\emptyset(d_-)],$$

$\emptyset(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{x^2}{2}}$, $d_+ = \frac{[\log(\frac{S}{K})+(r-q+\frac{\delta^2}{2})(T-t)]}{\delta\sqrt{T-t}}$, $d_- = d_+ - \delta\sqrt{T-t} = \log(\frac{S}{K}) + (r - q - \frac{\delta^2}{2})(T-t)$, S = asset price, K = strike price, T = maturity, t = time (current), δ = volatility, q = (continuous) asset dividend rate.

Gamma

Gamma measures how fast the delta changes for small changes in the underlying stock price i.e. delta of the delta. When one is hedging a portfolio by 'delta hedge' technique, one needs to keep gamma as small as possible since the smaller it is the less often one needs to adjust the hedge, to maintain a delta neutral position. When gamma is too large, a small change in stock price could wreck the whole hedge.

Adjusting gamma, however, can be tricky and is generally done using options. Unlike delta, it can be done by buying or selling the underlying asset as the gamma of the underlying asset is by definition always zero, so more or less of it will not affect the gamma of the portfolio.

$$Gamma\ (\Gamma)\ is\ the\ rate\ of\ change\ of\ \Delta\ wrt\ S\ or\ \Gamma = \frac{\partial\Delta}{\partial S} = \frac{\partial^2 C}{\partial S^2}$$

$$\Gamma = \frac{\partial}{\partial S}[e^{-q(T-t)}\emptyset(d_+) + \frac{e^{-q(T-t)}\emptyset(d_+)}{\delta\sqrt{T-t}} - \frac{Ke^{-r(T-t)}\emptyset(d_-)}{S\delta\sqrt{T-t}}$$

$$= \frac{e^{-q(T-t)}}{S\delta\sqrt{T-t}}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_+^2}{2}}$$

Vega

The change in option price given a one percentage point change in volatility is called the option's Vega. Like delta and gamma, Vega is also used in hedging of asset securities. Vega = the rate of change of option price, C, with respect to volatility, δ.

$$\text{Vega} = \frac{\partial C}{\partial\delta} = Se^{-q(T-t)}\emptyset(d_+)\frac{\partial(d_+)}{\partial\delta} - Ke^{-r(T-t)}\emptyset(d_-)\frac{\partial(d_-)}{\partial\delta}$$

$$= Se^{-q(T-t)}\emptyset(d_+)\frac{\partial\,(d_+-d_-)}{\partial\delta}$$

Theta

Theta is the change in option price given a one day decrease in time to expiration. Basically, theta is a measure of time decay. Unless, however, you and your portfolio are travelling at close to the speed of light, the passage of time is constant and inexorable, thus hedging a portfolio against time decay, the effects of which are completely predictable, would be pointless.

$$Theta\ (\theta) = rate\ of\ change\ of\ option\ price, C\ with\ time, t.$$

$$\boldsymbol{\theta} = \frac{\boldsymbol{\partial C}}{\boldsymbol{\partial t}} = \frac{-\delta Se^{-q(T-t)}}{2\sqrt{T-t}}\emptyset(d_+) + qSe^{-q(T-t)}\emptyset(d_+) - rKe^{-r(T-t)}\emptyset(d_-)$$

Rho

Rho is the change in option price given a one percentage point change in the risk-free interest rate.

$$\rho = \frac{\partial}{\partial r}[Se^{-q(T-t)}\emptyset(d_+) - Ke^{-r(T-t)}\emptyset(d_-)$$

$$= K(T-t)e^{-r(T-t)}\emptyset(d_-) + Se^{-q(T-t)}\emptyset(d_+)\frac{\sqrt{T-t}}{\delta} + Ke^{-r(T-t)}\emptyset(d_-)\frac{\sqrt{T-t}}{\delta}.$$

4.4 Ito Calculus

Ito Calculus is indispensable in the theory of derivative asset pricing, especially for the underlying asset price in the Black-Scholes option pricing formula for call and put options. Suppose the underlying stock returns in a derivative asset is driven by the Wiener process as stated in equation (2.2), that is

$$dS = \mu S dt + \delta S dW,$$

μ = the return of the underlying stock in a financial market, δ^2 = the variance and $f = f(s,t)$ is a function of stock of the asset price at time t. Now, from Ito's lemma we shall have:

$$df = f_s ds + f_t dt + \frac{1}{2}\{f_{ss}(dS)^2 + 2f_{st} dSdt + f_{tt}(dt)^2\},$$

that is:

$$df = \frac{\delta f}{\delta S} dS + \frac{\delta f}{\delta t} dt + \frac{1}{2}\left\{\frac{\delta^2 f}{\delta S^2}(dS)^2 + 2\frac{\delta^2 f}{\delta S \delta t} + \frac{\delta^2 f}{\delta t^2}(dt)^2\right\} \quad (4.3)$$

We recall the following relations in Ito's calculus from table (3.1):

	dW	dt
dW	dt	0
dt	0	0

Consequently, $(dt)^3 = 0$, $(dt)^2 dW = 0, dt(dW)^2 = 0, (dW)^3 = (dt)^2 dt = 0(dt) = 0$.

so that, $(dS)^2 = \{\mu S dt + \delta S dW\}^2 = \mu^2 S^2 (dt)^2 + 2\mu\delta S^2 dt + \delta^2 S^2 (dW)^2$

$$= \mu^2 S^2(0) + 2\mu\delta S^2(0) + \delta^2 S^2 dt$$

$$= \delta^2 S^2 dt.$$

Similarly, from the multiplication table above,

$$dSdt = (\mu S dt + \delta S dW)dt = 0.$$

From equation (3.1), we shall have:

$$df = \frac{\delta f}{\delta S} dS + \frac{\delta f}{\delta t} dt + \frac{1}{2}\left\{\frac{\delta^2 f}{\delta S^2}(S^2\delta^2 dt) + 0 + 0\right\}$$

$$= \frac{\delta f}{\delta s}(\mu S dt + \delta S dW) + \frac{\delta f}{\delta t} dt + \frac{1}{2}\delta^2 S^2 \frac{\delta^2 f}{\delta S^2} dt.$$

$$= \left\{\mu S \frac{\delta f}{\delta S} + \frac{\delta f}{\delta t} + \frac{1}{2}\delta^2 S^2 \frac{\delta^2 f}{\delta S^2}\right\} dt + \delta S \frac{\delta f}{\delta S} dW.$$

In general, if $X_t = f(X_t)dt + \delta(X_t)dW_t$, and F a smooth function, then the Ito's formula is given by:

$$dF(t, X_t) = \frac{\delta F}{\delta t} dt + \frac{\delta F}{\delta X} dX_t + \frac{1}{2}\frac{\delta^2 F}{\delta X^2} dX_t dX_t$$

$$= \left\{\frac{\delta F}{\delta t} + f(X_t)\frac{\delta F}{\delta x} + \frac{1}{2}\delta^2(X_t)\frac{\delta^2 F}{\delta x^2}\right\} dt + \delta(X_t)\frac{\boldsymbol{\delta F}}{\boldsymbol{\delta x}} dW_t$$

Ito formula in Brownian motion: Theorem 1

Let $X(t)$ be an Ito process represented by the SDE

$$dX(t) = \alpha\big(t, X(t)\big)dt + \beta\big(t, X(t)\big)dW(t)$$

Let $g(t,x)$ be twice differentiable function defined on $[0,\infty)x\mathbb{R}, or\ g(t,x) \in C^2[0,\infty)x\mathbb{R}$, then $Y(t) = g(t, X(t))$ is also an Ito process and

$$dY(t) = \frac{\partial g}{\partial t}\big(t, X(t)\big)dt + \frac{\partial g}{\partial x}\big(t, x(t)\big)dX(t) + \frac{1}{2}\frac{\partial^2 g(t, X(t))}{\partial x^2}[dX(t)]^2$$

(proof omitted)

4.5 Forecasting solutions to stochastic calculus (derivative) models

The solutions to some stochastic calculus models are proposed by method of simulation. The underlying principle is to generate some system of random numbers, using some codes in the forms:

```
Brownian path simulation
randn('state',400) % set the state of randn represent a collection of
random numbers.
T = 1; N = 500; dt = T/N; This process discretises the derivative
function dt
dW = zeros (1, N); % preallocate arrays for efficiency, represents the
weiners process
W = zeros (1, N);
dW(1) = sqrt(dt)*randn;  % first approximation outside the loop ....
This is obtained by using the property John, C. Hull (2012) of Wieners
process which states that the change Δz during a small interval of time
Δt is represented by Δz= ϵ√Δt where ϵ has a standard normal
distribution ∅(0,1).
W(1) = dW(1)              % since W(0) = 0 is not allowed
for j = 2: N
    dW(j) = sqrt(dt)*randn; % general increment
    W(j) = W(j-1) +dW(j);
```

4.6 Estimation of implied volatility from a set of option prices

To calculate the implied volatility, we used Excel goal seek method for single option prices. However, in this work, when we have a set of option prices, we estimate implied volatility by bisection method. The set of option prices with their respective strike prices and times to maturation will lead to construction of implied volatility surfaces for the given set of option prices for fixed underlying asset price.

4.7 Method of Bisection

Step 1: From equation (3.24) to obtain the implied volatility of an option, conceptually we are trying to find the root of the equation given below:

$$f\left(\sigma_{imp}\right) = f\left[S, X, r, t, \sigma_{imp}(X.t)\right] - C_{BS73} \qquad (4.4)$$

In other words, we need the value of σ for which $f\left(\sigma_{imp}\right) = 0$. To do this, we begin by picking an upper and lower bound of the volatility (σ_{lower} *and* σ_{upper}) such that the value of $f(\sigma_L)$ and $f(\sigma_U)$ have different (opposite) signs. This relation from mean value theorem (MVT)/Rolle's Theorem means that the root of equation (4.4) or the value of implied volatility lies between the lower and upper volatility so picked. The lower estimate of volatility corresponds to a low option value and a high estimate for volatility corresponds to a high option value.

Step 2: We then calculate a volatility that lies half way between the upper and lower volatilities. That is, $Vol_{mid} = \frac{\sigma_L + \sigma_U}{2}$, If we set $Vol_{mid} = \sigma_M, and\ if\ for\ C(\sigma_M) > C$ (observed) then the new mid-point σ_N will be $\sigma_N = \frac{\sigma_L + \sigma_M}{2}$ or else we have $\sigma_N = \frac{\sigma_U + \sigma_M}{2}$. This method is continued in this fashion until a reasonable approximation of implied volatility is obtained. In other words, when the option value corresponding to our interpolated estimate for volatility is below the actual (observed) option price, we replace our low volatility estimate with the interpolated estimate and repeat the calculation, Kritzman (1991). However, if the estimated option value is above the actual option price, we replace the high volatility estimate with the interpolated estimate and continue in this way until the reasonable implied volatility approximation is achieved.

Step 3: When the option value corresponding to the volatility estimate is equal to the actual price of option, we have thus arrived at the required implied volatility of the option. In other words, if $f(vol_{mid}) = 0$ or less than a given ε, we have found the required implied volatility and that terminates the iterations.

Step 4 Summary: If $f(vol_{lower})$ multiplied by $f(vol_{mid}) < 0$ then the root lies between vol_{lower} and vol_{mid}. If, however, the value of $f(vol_{lower})$ multiplied by $f(vol_{mid}) > 0$, then the root lies between vol_{mid} and $vol_{upper.}$ In other words when $f(vol_{lower}) * f(vol_{mid}) < 0$, then allow $vol_{upper.}$ to be vol_{mid} and apply step 2 again. But when $f(vol_{lower}) * f(vol_{mid}) > 0$ then allow $vol_{upper.} = vol_{mid}$ and proceed by going back to step 2.

4.8 Computation of implied volatility from option prices

The Black Scholes implied volatility for a given set of call option prices is calculated using the Bisection method in an Excel VBA with a code given by

Black-Scholes Implied volatility = $BScImVol(S, K, r, q, T, callmktprice)$

where, $S =$ underlying stock price, $K =$ exercise or strike price of individual option contract on the same underlying; $r =$ risk-free rate

$q =$ dividend paid out on the underlying stock

$callmktprice$ =average of bid/ask prices for the respective options under consideration

```
Function BSC (S, K, r, q, sigma, T)

Dim dOne, dTwo, Nd1, Nd2

    dOne = (Log(S / K) + (r - q + 0.5 * sigma) * T) / (sigma
* Sqr(T))

    dTwo = dOne - sigma * Sqr(T)

    Nd1 = Application.NormSDist(dOne)

    Nd2 = Application.NormSDist(dTwo)

    BSC = Exp (-q * T) * S * Nd1 - Exp(-r * T) * K * Nd2

End Function

Function BSCImVol(S, K, r, q, T, callmktprice)

    H = 5

    L = 0

Do While (H - L) > 0.00000001

If BSC (S, K, r, q, (H + L) / 2, T) > callmktprice Then

    H = (H + L) / 2

    Else: L = (H + L) / 2

End If

Loop
```

```
    BSCImVol = (H + L) / 2

End Function
```

4.9 Estimation of parameters of some implied volatility models

We also estimate the parameters of implied volatility, fit the implied volatility and plot the corresponding surface of a typical implied volatility. For example, we will parameterize the constants by using the linest function in Excel to determine the values of $a_0, a_1, \ldots, a_5$ in an absolute smile type of Ad-Hoc Black-Scholes model, and thereafter construct the implied volatility surface using the Dumas, Fleming and Whaley model:

$$\sigma_{iv} = a_0 + a_1 K + a_2 K^2 + a_3 T + a_4 T^2 + a_5 KT$$

For the various models of Practitioners Black-Scholes/Ad-Hoc Black-Scholes both "Absolute and Relative smiles" models, we estimate the parameters and determine from the p-values so obtained the most appropriate model for the option prices under consideration. By hypothesis, p-values are usually less than 0.05 for the multiple regressions to confirm the significance of estimated parameters in a model. The implied volatility model above, according to Dumas et al. (1998), could be used to estimate the Black-Scholes option pricing model for call and or put which will take care of the volatility smirk and smile associated with the Black-Scholes model. Thus, for a given array of strike price and time to maturity in years, an implied volatility surface could therefore, be plotted.

4.10 Correlation Matrix

4.10.1 Normalization: In Random Matrix Theory (RMT) we calculate the price changes over a time scale, Δt, which is equivalent to one day and this represents the corresponding price change or logarithmic returns $G_i(t)$ over the time scale Δt by

$$G_i(t) = \ln[S_i(t + \Delta t)] - \ln[S_i(t)] \quad (4.5)$$

It suffices to note here that as different stocks vary on different scales, we are expected to normalize the return using

$$M_i(t) = \frac{G_i(t) - \langle G_i(t) \rangle}{\sigma_i} \quad (4.6)$$

where $\sigma_i = \sqrt{\langle G_i(t)^2 \rangle - \langle G_i(t) \rangle^2}$ *and* $\langle \ldots \rangle$ represents the average in the period studied.

Remarks

The observed stylized facts will be compared with known results for other emerging and developed financial markets, in order to provide insight into the behaviour of the underlying stock returns of the NSM.

From real time series return data, we can calculate the element of N x N correlation matrix C as follows

$$C_{ij} = \langle g_i(t) g_j(t) \rangle = \frac{\langle [G_i(t) - \langle G_i \rangle][G_j(t) - \langle G_j \rangle] \rangle}{\sqrt{[\langle G_i^2 \rangle - \langle G_i \rangle^2][\langle G_j^2 \rangle - \langle G_j \rangle^2]}} \quad (4.7)$$

C_{ij} lies in the range of the closed interval $-1 \leq C_{ij} \leq 1$, with $C_{ij} = 0$ means there is no correlation, $C_{ij} = -1$ implies anti-correlation and $C_{ij} = 1$ means perfect correlation for the empirical correlation matrix.

It can be shown from Sharifi (2004) that the empirical correlation matrix C can be expressed as

$$C = \frac{1}{L} G G^T \quad (4.8)$$

where G is the normalized N x L matrix and G^T is the transpose of G. This empirical correlation will be compared with a random Wishart matrix (random matrix) R given by:

$$R = \frac{1}{L} A A^T \quad (4.9)$$

to classify the information and noise in the system, Conlon et al. (2007) and Gopikrishnan et al. (2001), where A is an N x L matrix whose entries are independent identically distributed random variables that are normally distributed and have zero mean and unit variance. Edelman (1988) assert that the statistical properties of R are known and that in particular for the limit as $N \to \infty, and\ L \to \infty$ we have that $Q = \frac{L}{N} (\geq 1)$ is fixed and that the probability function $P_{rm}(\lambda)$ of eigenvalues λ of the random correlation matrix R is given by

$$P(\lambda) = \frac{Q}{2\pi\sigma^2} \frac{\sqrt{(\lambda_{max} - \lambda)(\lambda - \lambda_{min})}}{\lambda} \quad (4.10)$$

for λ such that $\lambda_{min} \leq \lambda \leq \lambda_{max}$, where σ^2 is the variance of the elements of A. Here $\sigma^2 = 1$ and $\lambda_{min}\ and\ \lambda_{max}$ satisfy

$$\lambda_{max/min} = \sigma^2 (1 + \frac{1}{Q} \mp 2\sqrt{1/Q}) \quad (4.11)$$

The values of lambda from equation (4.9) that satisfy (4.10) and (4.11) are called the Wishart distribution of eigenvalues from the correlation matrix. These values of lambda, as stated before, determine the bounds of theoretical eigenvalue distribution. When the eigenvalues of empirical correlation matrix C are beyond these bounds, they are said to deviate from the random matrix bounds and are therefore supposed to carry some useful information about the market, Cukur (2007).

4.11 Distribution of eigenvector component

The concept that low-lying eigenvalues are really random can also be verified by studying the statistical structure of the corresponding eigenvectors. The jth component of the eigenvector corresponding to each eigenvalue λ_α will be denoted by $v_{\alpha,j}$ and then normalized such that $\sum_{j=1}^{N} v_{\alpha,j}^2 = N$. Plerou et al. (1999) assert that if there is no information contained in the eigenvector $v_{\alpha,j}$, one expects that for a fixed α, the distribution of $u = v_{\alpha,j}(as\ j\ is\ varied)$ is a maximum entropy distribution. This, therefore, leads to what is called Porter-Thomas distribution in the theory of random matrices written as

$$p(u) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) \tag{4.12}$$

4.12 Inverse participation ratio

Guhr, et al. (1998) assert that to quantify the number of components that participates significantly in each eigenvector, we use inverse participation ratio. Inverse participation ratio (IPR) shows the degree of deviation of the distribution of eigenvectors from RMT results and distinguishes one eigenvector with approximately equal components with another that has a small number of large components. For each eigenvector V_α, Plerou et al. [2002] defined the inverse participation ratio as

$$I_\alpha = \sum_{l=1}^{N} [V_\alpha(l)]^4 \tag{4.13}$$

where N is the number of the time series (the number of implied volatilities considered) and hence the number of eigenvalue components and $V_\alpha(l)$ is the $l-$ th component of the eigenvector V_α. There are two limiting cases of I_α (i) If an eigenvector V_α has an identical component, $V_\alpha(l) = \frac{1}{\sqrt{N}}, then\ I_\alpha = \frac{1}{N}$ and (ii) For the case when V_α has one element with $V_\alpha(l) = 1$ and the remaining components zero, then $I_\alpha = 1$.

Therefore, the IPR can be illustrated as the inverse of the number of elements of an eigenvector that are different from zero that contribute significantly to the value of the eigenvector. Utsugi et al. [2004] in their study of the RMT assert that the expectation of the IPR is given by

$$\langle I_\alpha \rangle = N \int_{-\infty}^{\infty} [V_\alpha(l)]^4 \frac{1}{\sqrt{2\pi N}} \exp\left(-\frac{[V_\alpha(l)]^2}{2N}\right) dV_\alpha(l) = \frac{3}{N} \tag{4.14}$$

since the kurtosis for the distribution of eigenvector components s 3.

4.13 Some notes on the rationale for RMT analysis

RMT is a theoretical suite of techniques which originated from modern Physics (Astrophysics and theoretical particle Physics) and is widely applied in Statistical Physics and Econophysics. It relates to using correlation measures among clusters of measurements, based on eigenvalue and eigenvector analyses, among other techniques in multivariate statistics, underpinned by assumptions about the likely types of probability distributions which generate the data clusters, to explore the relationships among the data clusters.

In this section of the thesis we simply note that these RMT techniques are used as baseline tools for initially studying the closeness or otherwise among the selected data clusters from sectors and sets of asset prices in the JSM and NSM. The results will then be combined with further knowledge of: a) the statistical distributions which govern the respective data cluster, b) the extent to which the data behaviours support the assumptions of different derivative pricing models, with the BS model as a reference point, hence the plausible validity of the models in deciphering derivative prices, in order to simulate supposed NSM data that fit the distributions and models, and thereby produce plausible derivative prices for the NSM.

It is expected that the research will serve as a point of departure in further modelling of derivative prices in NSM, post-introduction of such products in the market. Importantly, the results provide theoretical knowledge of the limitations of different derivative pricing models reviewed in the literature presented in Chapter 3 of this thesis, which will be useful for further theoretical research on the models and their applications in Nigeria and similar emerging markets, with particular emphasis on markets in some African regional economic blocks such as ECOWAS, COMESA, EAC, AMU, SADC and ECCAS.

We will present the crucial touch-points of RMT and the JSM-NSM characterisation results in Chapter 7 of the thesis. We will follow this up with the remaining steps in modelling selected JSM-NSM data and derivative prices for risk hedging, speculation and arbitrage investment goals, in the subsequent chapters to the RMT chapter.

4.14 Summary and Conclusion

We have explianed in this concept chapter the source of the data needed for the research, and the detailed basic terms that underpin the entire work. In the subsequent chapters, we are going to look at the dynamics of some stochastic calculus models of interest, using Monte Carlo method of simulation.

We also showed how to estimate implied volatilities for some given sets of option prices, which will be obtained from yahoo finance in developed economies for comparative empirical data analysis, including the construction of implied volatility surfaces. Also, the stock market returns characteristics for both markets - Nigerian Stock Market (NSM) and the Johannesburg Stock Exchange (JSE) - were studied using the concept of Random Matrix Theory (RMT). We also considered the valid and empirical correlation matrices and their relations with empirical implied volatility of option prices.

CHAPTER FIVE

STOCHASTIC CALCULUS MODELS FOR DERIVATIVE ASSETS

5.0 Introduction

Numerical Solutions to SDEs: The use of stochastic Ito and Stratonovic integrals in deriving security asset pricing

Ito stochastic and Stratonovich integrals are good numerical approximations of solution dynamics to SDEs of the stock price for an underlying asset in a European call option. Ito and Stratonovich integral representations to SDEs are known to provide useful approximate solutions from which we can predict the stock market prices, Panzar et al. (2004).

As mentioned earlier, price of the underlying asset could be represented by the geometric Brownian motion: $dS = \mu S dt + \delta S dW$ and the concern in this research is how to represent the above differential equation in an integral form and thus be able to use the resulting stochastic integral to estimate the price of the underlying asset security.

5.1 Euler's Method

Euler's method is one of the most fundamental methods for numerical approximations of integrals. Euler's approximation has been very useful in the solutions of stochastic differential equations especially as it concerns the discretisation scheme in finding the solutions to SDEs of interest. While in this research we applied the approach of discretising the stochastic component (Wieners process component) of the price dynamics of asset prices, Castilla et al. (2016) in their work on Levy-driven SDEs modified the standard Euler's scheme by replacing equally spaced time steps by exponentially distributed ones in order to ensure that the grid points are equivalent to arrival times of a Poisson process. To evaluate the integral given by $I(x) = \int_a^b x(t)dt,$ Euler proposes the partition of the interval into $a = t_o < t_1 < \cdots < t_{n-1} < t_n = b;$ so that
$I_n(x) = \sum_{i=1}^{n} x(t_{i-1})(t_i - t_{i-1})$, so that $I_n(x) \rightarrow I(x) \;\; if \; x$ is well defined.

Similarly, for $I(x) = \int_a^b x(t)dw, where \; w$ is also a function of t. Here we need the derivative of w given by w' so that the integrand will then be $x(t)w'$ and then carry out the usual operation as we did with I(x). Thus, from the Euler's approach, the equivalent approximating sum will now be

$\int_a^b x(t)dw = \sum_{i=1}^{n} x(t_{i-1})\, w'(t_{i-1})(t_i - t_{i-1})$, so that for well-defined w, we shall have

$$\int_a^b x(t)dw = \sum_{i=1}^{n} x(t_{i-1})[w(t_i) - w(t_{i-1})] \quad (5.1)$$

Remarks:

For Wieners process, that is when the functions $X(t)$ and $W(t)$ are random, we add limit to the stochastic integral for the region where the limit exist. That is,

$$\int_a^b X(t)dW = \lim_{n\to\infty} \sum_{i=1}^{n} X(t_{i-1})[W(t_i) - W(t_{i-1})]$$

To get rid of the limiting process we need some fundamental definitions:

5.1.1 Ito integral in an Elementary Process

If X is an elementary, progressive, non-anticipative process and square integrable from a to b, then the Ito integral from a to b is given by

$$\int_a^b X(t)dW = \sum_{i=0}^{n} X(t_i)[W(t_{i+1}) - W(t_i)] \quad (5.2)$$

This is basically the Riemann-Stieltjes integral.

5.2 Stochastic Integrals

Higham, D.J (2001) states that for Riemann-Stieltjes integrals we have

$$\int_0^T h(t)\, dt \cong \sum_{j=0}^{n} h(t_j)(t_{j+1} - t_j) \quad (5.3)$$

by using triangle rule, or

$$\int_0^T h(t)\, dt \cong \sum_{j=0}^{n} h\left(\frac{t_j + t_{j+1}}{2}\right)(t_{j+1} - t_j) \quad (5.4),$$

by mid-point rule.

The relations (5.3) and (5.4) can be extended to stochastic integrals with respect to a Brownian motion W(t) so that we have

$$\int_0^T h(t)dW(t) \quad (5.5)$$

We seek to apply the above quadrature ideas in obtaining equivalent formula in an entirely stochastic setting through replacing $h(t)\ in$ (5.5) $by\ W(t)$.

Hence, the entire stochastic formula is given from (5.3) by:

$$\int_0^T W(t)dW(t) \cong \sum_{j=0}^{n} W(\tau_j)[W(t_{j+1}) - W(t_j)]$$

$$= \lim_{n\to\infty} \sum_{j=0}^{n} W(\tau_j)[W(t_{j+1}) - W(t_j)] \quad (5.6)$$

Observation:

The limit that defines the integral in (5.6) above depends largely on where τ_j lies in the closed interval $[\tau_j, t_{j+1}]$. Different choices on the value of τ_j lead to distinct stochastic calculi:

if $\tau_j = t_j$ we obtain the Ito Stochastic calculus and for, and if

$\tau_j = \frac{t_j + t_{j+1}}{2}$ we have the Stratonovich calculus.

So, in applying the triangle quadrature rule in equation (5.3) with $h(t) = W(t)$ gives the Ito integral:

$$\int_0^T W(t)dW(t) \cong \sum_{j=o}^{n} W(t_j)[W(t_{j+1}) - W(t_j)]$$

$$= \lim_{n\to\infty} \sum_{j=0}^{n} W(t_j)[W(t_{j+1}) - W(t_j)] \qquad (5.7)$$

Similarly, applying the mid-point quadrature as in (5.4) gives the Stratonovich integral written as

$$\int_0^T W(t)^\circ \, dW(t) \cong \sum_{j=0}^{n} W\left(\frac{t_j+t_{j+1}}{2}\right)[W(t_{j+1}) - W(t_j)]$$

$$= \lim_{n\to\infty} \sum_{j=0}^{n} W\left(\frac{t_j+t_{j+1}}{2}\right)[W(t_{j+1}) - W(t_j)] \qquad (5.8)$$

Note that algebraically, $\frac{1}{2}[W(t_j) + W(t_{j+1})] = W(t_j) + \frac{1}{2}[W(t_{j+1}) - W(t_j)]$ and with this equation (5.8) can be re-written as

$$\lim_{n\to\infty} \sum_{j=0}^{n} W\left(\frac{t_j+t_{j+1}}{2}\right)[W(t_{j+1})\, W(t_j)] = \lim_{n\to\infty} \sum_{j=0}^{n} W(t_j)[W(t_{j+1}) - W(t_j)] + \frac{1}{2}\lim_{n\to\infty} \sum_{j=0}^{n} [W(t_{j+1}) - W(t_j)][W(t_{j+1}) - W(t_j)] \qquad (5.9)$$

⤇ Stratonovich integral = Ito integral + the last term; we can show this:

For Ito integral; $\int_0^T W(t)dW(t) = \sum_{j=0}^{n} W(t_j)[W(t_{j+1}) - W(t_j)]$

$$= \frac{1}{2}\sum_{j=0}^{n} W(t_{j+1})^2 - W(t_j)^2 - [W(t_{j+1}) - W(t_j)]^2$$

since $a(b-a) = ab - a^2 = \frac{1}{2}[b^2 - a^2 - (b-a)^2$

We observe that $W(t_{j+1})^2 - W(t_j)^2 - [W(t_{j+1}) - W(t_j)]^2 = W(t_{j+1})^2 - W(t_j)^2 - W(t_{j+1})^2 + 2W(t_{j+1})W(t_j) - W(t_j)^2$

$$= 2[W(t_{j+1})W(t_j) - W(t_j)^2]$$

Therefore $\sum_{j=0}^{n} W(t_j)[W(t_{j+1}) - W(t_j)] = \frac{1}{2}\{W(t_{j+1})^2 - W(t_j)^2 - [W(t_{j+1}) - W(t_j)]^2\}$

$$= \frac{1}{2}\{W(T)^2 - W(0)^2 - \sum_{j=0}^{n}[W(t_{j+1}) - W(t_j)]^2$$

$$\rightarrow \quad \int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T$$

For Stratonovich integrals,

$$\int_0^T W(t)dW(t) = \lim_{n\to\infty}\sum_{j=0}^{n} W\left(\frac{t_j + t_{j+1}}{2}\right)[W(t_{j+1}) - W(t_j)]$$

$$= \lim_{n\to\infty} \sum_{j=0}^{n}\{\tfrac{1}{2}[W(t_j) + W(t_{j+1})] + \Delta z_j\}[W(t_{j+1}) - W(t_j)]$$

since, $W\left(\frac{t_j+t_{j+1}}{2}\right) \cong \frac{1}{2}[W(t_j) + W(t_{j+1})] + \Delta z_j \; by\ definition$

$$= \lim_{n\to\infty} \sum_{j=0}^{n} \tfrac{1}{2}\{[W(t_j) + W(t_{j+1})][W(t_{j+1}) - W(t_j)] + \Delta z_j[W(t_{j+1}) - W(t_j)]$$

$$= \lim_{n\to\infty} \sum_{j=0}^{n}\{\tfrac{1}{2}W(t_{j+1})^2 - W(t_j)^2 + \Delta z_j[W(t_{j+1}) - W(t_j)]\}$$

The term on the RHS tends to zero thus reducing the expression to:

$$= \tfrac{1}{2}\{W(T)^2 - W(0)^2\} + 0 = -W(T)^2$$

⤇ Stratonovich integral = Ito integral + the last term as required.

The fact that Ito integrations do not always conform to the traditional rules of the Riemann-integrals posits Ito integrals as an abnormal procedure for approximating solutions to SDEs. It is the quest to circumvent this shortcoming that necessitated the introduction of Stratonovich integrals. Stratonovich integrals are known to obey the traditional rules of integration although the use of Stratonovich integrals has its own demerit, which is the loss of Martingale property which Ito processes (and indeed integrals) are known to possess. The reason for this is that Stratonovich integrals are sub-Martingale.

5.3 Generalisation of the relationship between Ito and Stratonovich Integrals

We can use Ito's formula to obtain a "translation" between Ito and Stratonovich stochastic differential equations in the expressions below:
For Ito SDE

$$\frac{dx}{dt} = a(t,x)dt + b(t,x)dW(t) \tag{5.10}$$

The equivalent Stratonovich integral to the SDE will be of the form;

$$\frac{dx}{dt} = \left\{a(t,x) - \frac{1}{2}b(t,x)\frac{\delta[b(t,x)]}{\delta x}\right\} + b(t,x)^{\circ}dW(t) \tag{5.11}$$

In other words, Ito integral $f(t, W(t))$ and its equivalent Stratonovich integral are connected by the identity:

$$\int_a^b [f(t,W(t))] \circ dW(t) = \int_a^b f(t,W(t))dW(t) + \frac{1}{2}\int_a^b \frac{\delta f(t,W(t))}{\delta W(t)}\,dt \tag{5.12}$$

Whenever,

$$\int_a^b E[f(t,W(t))]^2 dt < \infty, \; and \; \int_a^b E[\frac{\delta f(t,\ W(t))}{\delta W(t)}]^2 dt < \infty \tag{5.13}$$

The expressions in equation (5.13) on expectation are necessary for the convergence of Ito and Stratonovich integrals. We can also infer from equation (5.12) that every Ito integral has an equivalent Stratonovich integral representation.

Mark Richardson (2009) asserts that implementing the quadrature method in MATLAB is very easy and straightforward, if we notice that $dW(t_{j+1}) = W(t_{j+1}) - W(t_j)$ in the construction of discretised Brownian motion.

The use of MATLAB and other numerical approximation methods for the evaluation of Ito and Stratonovich integrals will be studied later in the thesis. However, we note that in Ito integral as in equation (5.7) above, we will use the vector inner (dot) product to compute the finite sum of the right-hand side of (5.7) by using the following command:

$$Ito = [0, W(1{:}n-1)] * dW'$$

Note that we discard the last entry of W and shift the values along 1, adding a zero as the first entry Richardson Mark (2009).

For the Stratonovich integral, we must rearrange the term $W\left(\frac{t_j+t_{j+1}}{2}\right)$ using

$$W\left(\frac{t_j+t_{j+1}}{2}\right) \cong \frac{1}{2}\left[W(t_j) + W(t_{j+1})\right] + \Delta z_j \text{ , where, } \Delta z_j \sim N(0, \tfrac{\Delta t}{4}),$$

The MATLAB implementation for this is therefore given by:

$$Strat = [0.5 * \{[0, W(1{:}n-1)] + W\} + 0.5 * sqrt(dt) * rand(1,n)] * dW' \quad (5.14)$$

5.4 Monte Carlo approximation

Monte Carlo approximations are adopted for solutions to differential equations where the analytic approach is difficult or in some cases seem to be infeasible. In ordinary differential equations (ODEs), Euler's method provides the desired approximation to solutions of ODEs where the analytic solutions fail. However, for stochastic differential equations (SDEs), which are often more difficult to approximate when compared with the ODEs, the two main numerical schemes also called Monte-Carlo approximations are the Euler-Maruyama and Milstein's approximation.

Monte-Carlo approximations are very useful for option pricing especially in estimating the price dynamics of the underlying stock prices to derivative products. In this regard, Ballota and Kyprianou (2001) adopted the Monte-Carlo simulation for the solution to α-quantile option where the analytic pricing formulas are difficult to compute. Their approach for quantile option is that in which the α-quantile of the Brownian motion is generated directly as the sum of two independent samples of the extremes of X_t where X_t is an arithmetic Brownian motion and is defined by $X_t = \mu t + \sigma W_t$, with μ = drift, σ = volatility, $t \geq 0$.

5.4.1 Euler-Maruyama approximation

Considering $dx(t) = \mu(t, x_t) + \delta(t, x_t)dW_t$, the integral solution of the differential equation could be expressed as

$$x(t) = x(s) + \int_s^t \mu(r, x_r)dr + \int_s^t \delta(r, x_r)dW_r \qquad (5.15),$$

by integrating both sides with respect to the respective arguments from s to t. We consider the computation of the right-hand side of equation (5.15), over the closed interval [t, t+h], where h is infinitesimally small.

If x_t is a continuous random function of t while $\mu(t,x) and\, \delta(t,x)$ are continuous functions of (t,x) then $\mu(t,x_r)$ and $\delta(t,x_r)$ in (5.15), can be approximated by $\mu(t, x_t)\, and\ \ \delta(t, x_t)$ respectively giving

$$x(t+h) \cong x(t) + \mu(t, x_t)\int_t^{t+h} dr + \delta(t, x_t)\int_t^{t+h} dW_r \qquad (5.17)$$

$$= x(t) + \mu(t, x_t)h + \delta(t, x_t)[W(t+h) - W(t)]$$

We observe here that $\delta(t, x_t)[W(t+h) - W(t)] = 0$, hence the stochastic solution in non-anticipative. This agree with the statement that no information is known about the future solution to the process (which is true about the stock market asset prices) and that the best estimate of the future solution is the current state plus a drift {i.e. $[\mu(t, x_t)]$} which of course is deterministic. This method of approximation is known as Euler-Maruyama approximation.

5.4.2 Milstein's higher order method

We seek here to construct a method that guarantees a higher rate of convergence than the Euler-Maruyama method. The Milstein's method named after Grigori N. Milstein, a Russian Mathematician, is a technique for approximating numerical solutions of stochastic differential equations using Ito's lemma, by means of the stochastic Taylor series expansion. The Milstein's (1974) higher-order method of approximating the numerical solution of a discretised stochastic differential equation is given by:

$$X_j = X_{j-1} + \delta t f(X_{j-1}) + g(X_{j-1})[W(\tau_j) - W(\tau_{j-1})] + \frac{1}{2}g(X_{j-1})g'(X_{j-1}) \qquad (5.18)$$

$$for\ j = 1, 2, \dots., L;\ with\, X_0 = X(0)$$

The Milstein method has order 1 and that of Euler-Maruyama has order $\frac{1}{2}$. It suffices to mention here that Milstein method is identical to the Euler-Maruyama method if there is no X term in the diffusion part $b(X, t)$ of the equation (5.10), Sauer (2008). Both methods are known to be useful for numerical approximation of solutions of the Black-Scholes Stochastic differential equation (Brownian motion for stock asset prices), given by

$$dX(t) = \mu\, X dt + \delta\, X dW(t),$$

where X is the stock price of the underlying asset and other quantities have their usual meanings.

Lemma 1: In any plain vanilla option, the evolution of a firm's stock price for the underlying stock is given by a geometric Brownian motion

$$dS_t = \mu(S_t,t)dt + \sigma(S_t,t)dW_t, \; S(o) = S_o$$

if and only if the exact solution of the Brownian motion is given by

$$S_t = S_0 exp\left\{\left[\mu - \frac{\sigma^2}{2}\right]t + \sigma W_t\right\}$$

where, μ, σ, W_t denote respectively drift of the asset return, its volatility and the Wieners process which is the random perturbation affecting the evolution of the process.

Proof: By B-S (1973) seminal paper, the underlying stock price for derivative option pricing follows a geometric Brownian, is lognormally distributed, and could be represented by equation (2.2) and satisfies a certain second order differential equation represented by equation (3.1).

For this we set $F(S,t) = logS$, and by Ito's lemma and Taylor series,

$$dF(S,t) = d(logS)$$
$$= \frac{\partial F}{\partial S}dS + \frac{\partial F}{\partial t}\,dt + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(dS)^2 + \frac{\partial^2 F}{\partial S \partial t}dSdt + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}(dt)^2 \quad (5.19)$$

(since all other higher powers of the derivatives vanish by Ito formula after the second order)

$$d(logS) = \frac{\partial F}{\partial S}(\mu Sdt + \sigma SdW) + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(\mu Sdt + \sigma SdW)^2 + \frac{\partial^2 F}{\partial S \partial t}dSdt + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}(dt)^2$$

But $\frac{\partial F}{\partial S} = \frac{1}{S}, \; \frac{\partial^2 F}{\partial S^2} = -\frac{1}{S^2}, \; \frac{\partial F}{\partial t} = 0 \; and \; (dt^2) = dtdt = 0, dtdW = 0 = dWdt, \; dW^2 = dWdW = dt; so \; that$

$$d(logS) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW$$

Integrating both sides with respect to t, t ϵ [0,T] yield

$$\log S(t) - \log S(0) = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

$$\Rightarrow \; S_t = S_0 exp\left\{\left[\mu - \frac{\sigma^2}{2}\right]t + \sigma W_t\right\} \quad (5.20)$$

For the converse, suppose that the solution of the geometric Brownian motion is given by

$$S_t = S_0 exp\left\{\left[\mu - \frac{\sigma^2}{2}\right]t + \sigma W_t\right\} \equiv S_t = S_0 exp\left\{\left[\mu - \frac{\sigma^2}{2}\right]t + \sigma\int_0^t dW\right\}$$

(since W(0) = 0 in a Brownian motion)

We will use Ito's lemma to establish that $S(t)$ as above satisfies equation (5.19). For this purpose, we set

$$F(t, W) = S_0\exp[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\int_0^t dW] \quad (E.1)$$

$$\frac{\partial F}{\partial t} = S_0\exp[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\int_0^t dW]\left(\mu - \frac{\sigma^2}{2}\right) = S(t)(\mu - \frac{\sigma^2}{2}) \quad (E.2)$$

$$\frac{\partial F}{\partial W} = S_0\exp[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\int_0^t dW](\sigma) = \sigma S(t) \quad (E.3)$$

$$\frac{\partial^2 F}{\partial W^2} = S_0\exp[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\int_0^t dW](\sigma^2) = (\sigma^2)S(t) \quad (E.4)$$

Invoking the Ito formula and equations $(E.1) - (E.4)$, we have

$$dS = dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial W}dW + \frac{1}{2}\frac{\partial^2 F}{dW^2}dW^2$$

$$= S(t)\left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma S(t)dW + \frac{\sigma^2}{2}S(t)(dW)^2$$

$$= \mu S(t)dt - S(t)\frac{\sigma^2}{2}dt + \sigma S(t)dW + \frac{\sigma^2}{2}S(t)dt$$

$$= \mu S(t) + \sigma S(t)dW$$

$$\Rightarrow dS = \mu s(t)dt + \sigma S(t)dW \; as \; required$$

In general equation (5.19) can be written as

$$\Delta f = df(s,t) = \frac{\partial f}{\partial S}\Delta S + \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial t}\Delta\sigma_t + \frac{\partial f}{\partial t}\Delta r_t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\Delta S_t)^2 + \cdots \quad (5.21)$$

and this last equation represents the various risks that an option is exposed to. The terms on the right-hand side represent: for the first term, the risk associated with the underlying stock, delta given by ΔS, that of the change in time, theta or Δt, volatility change vega given by $\Delta\sigma$, interest rate rho is represented by Δr, and finally is gamma the second derivative of delta represented by $(\Delta S_t)^2$.

5.5 The Greeks

Consequently, we can now verify some Greeks of call and put options

Recall equation (2.3) that for the price of a call option we have:

$$C = SN(d_1) - Xe^{-r\tau}N(d_2)$$

where $d_1 = \frac{\ln(\left(\frac{S}{X}\right) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$ and $d_2 = \frac{\ln(\left(\frac{S}{X}\right) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau}$ with $\tau = T - t$

with $N(.)$ as the cumulative density function of normal distribution defined by:

$$N(d_1) = \int_{-\infty}^{d_1} f(x)dx = \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

It follows then that

$$N'(d_1) = \frac{\partial N(d_1)}{\partial d_1} = \frac{1}{\sqrt{2\pi}} e^{-\frac{{d_1}^2}{2}} \qquad (5.21A)$$

$$N'(d_2) = \frac{\partial N(d_2)}{\partial d_2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{{d_2}^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(d_1 - \sigma\sqrt{\tau})^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{{d_1}^2}{2}} . e^{d_1 \sigma\sqrt{\tau}} . e^{-\frac{\sigma^2 \tau}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{{d_1}^2}{2}} . e^{\ln\left(\frac{S}{X}\right) + (r + \frac{\sigma^2}{2})\tau} . e^{-\frac{\sigma^2 \tau}{2}}$$

$$\rightarrow N'(d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{{d_1}^2}{2}} . \frac{S}{X} . e^{r\tau} \qquad (5.21B)$$

Therefore, delta which represents change in call option price with respect to the underlying stock price is given by

$$\Delta = \frac{\partial C}{\partial S} = N(d_1) + S \frac{\partial N(d_1)}{\partial S} - Xe^{-r\tau} \frac{\partial N(d_2)}{\partial S}$$

$$= N(d_1) + S \frac{\partial N(d_1)}{\partial d_1} \frac{\partial d_1}{\partial S} - Xe^{-r\tau} \frac{\partial N(d_2)}{\partial d_2} \frac{\partial d_2}{\partial S}$$

$$= N(d_1) + S \frac{1}{\sqrt{2\pi}} e^{-\frac{{d_1}^2}{2}} . \frac{1}{S\sigma\sqrt{\tau}} - Xe^{-r\tau} \frac{1}{\sqrt{2\pi}} . e^{-\frac{{d_1}^2}{2}} . \frac{S}{X} . e^{r\tau} . \frac{1}{S\sigma\sqrt{\tau}}$$

obtained by using the respective derivatives of $N(d_1)$& $N(d_2)$ with respect to S and also taking cognizance of the fact that if $y = \ln\left(\frac{x}{a}\right) = lnx - lna, then\ \frac{dy}{dx} = \frac{1}{x}$

$$= N(d_1) + S \frac{1}{\sqrt{2\pi}} e^{-\frac{{d_1}^2}{2}} . \frac{1}{S\sigma\sqrt{\tau}} - \frac{1}{\sqrt{2\pi}} . e^{-\frac{{d_1}^2}{2}} . S . \frac{1}{S\sigma\sqrt{\tau}}$$

$$= N(d_1) \tag{5.22}$$

For a call option, delta is strictly positive, whereas in a put option we have the opposite meaning that delta is strictly negative. In call option the range of values of delta is in the closed interval [0, 1]. The value of delta determines the type of correlation between the option price and the underlying stock. A delta value of magnitude 1 implies perfect correlation between the underlying stock and the option, but a correlation of magnitude 0 means no correlation between the option and the underlying stock.

For a put option, delta could be similarly derived to be:

$$\Delta = N(d_1) - 1 \tag{5.22A}$$

other Greeks could be derived similarly for non-dividend paying stocks.

For dividend paying stocks, derivative of the cumulative density function with respect to d_1 will be given by as before:

$$N'(d_1) = \frac{\partial N(d_1)}{\partial d_1} = \frac{1}{\sqrt{2\pi}} e^{-\frac{{d_1}^2}{2}} \tag{5.22B}$$

But for the derivative of the cumulative density function with respect to $d_2$we shall have:

$$N'(d_2) \;\; = \;\; \frac{1}{\sqrt{2\pi}} e^{-\frac{{d_1}^2}{2}} . \frac{S}{X} . e^{(r-q)\tau} \tag{5.22C}$$

with q defined as the dividend yield.

We may recall here that the call option formula for dividend paying stocks is given by:

$$C = SN(d_1) - Xe^{-r\tau} N(d_2)$$

with boundary conditions as;

$d_1 = \frac{\ln(\left(\frac{S}{X}\right) + (r-q+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$ and $d_2 = \frac{\ln(\left(\frac{S}{X}\right) + (r-q-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau}$ with $\tau = T - t$.

In in a similar fashion we can derive the delta for call and put options for dividend paying stocks to be respectively as:

$$\Delta = e^{-q\tau} N(d_1) \;\&\; \Delta = e^{-q\tau}[N(d_1) - 1].$$

5.6 Estimation of Stock Price using Historical Volatility

Given a time series of historical stock price data at some fixed intervals for example days, weeks, months, we can estimate the volatility or standard deviation of stock returns that could be used in the Black-Scholes option pricing formula. The values of drift and volatility so obtained can thus be used in evaluating the analytical solutions and or numerical approximations to stochastic differential equation of the underlying stock price in a European call option.

In Black-Scholes model it is assumed that volatility of the underlying stock is constant and one of the ways of estimating this parameter is using historical volatility. In this section, we are interested in estimating the solutions of SDEs analytically by first computing the diffusion coefficient/volatility through historical volatility. For stock price data obtained for some periods (usually in days), the estimate of historical volatility is given by

$$\hat{\sigma} = \frac{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}}{\sqrt{\tau}} \quad , \qquad where\ u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \tag{5.23}$$

$\bar{u}$ is the sample averageu_i , S_i is the stock price in the period i and τ is the total length of each period in years. The annualized estimate of this standard deviation could also be written as

$$\hat{\sigma} = \frac{S}{\sqrt{\tau}} \quad , \quad \text{where} \quad S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} u_i\right)^2} \tag{5.24}$$

Remarks:
For a more natural approach we will adopt the daily prices of any chosen asset from the NSM for this trial estimation, for instance Access Bank Nig. Plc.

In discrete time, the rentability of stock S(t) or the stock price return over an interval $(t_{i-1}, t_i)\ is$

$$R(t_i) = \frac{S(t_i) - S(t_{i-1})}{S(t_{i-1})}, i \geq 1,$$

and in continuous time the stock price return at any given time t is

$$R(t) = \frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t).$$

Evans (2003) asserts that given a differential equation:

$$\frac{dS(t)}{dt} = \alpha(S(t), t) + B(S(t), t)\varepsilon(t),\ S(0) = S_0 \tag{5.25}$$

$with\ S(.) : [0, \infty) \rightarrow \mathbb{R}^n$ random function, $\alpha : \mathbb{R}^n x\ [0\ T] \rightarrow \mathbb{R}^n$,

$B : \mathbb{R}^n x\ [0\ T] \rightarrow \mathcal{M}_{nxm}(\mathbb{R}),\ \varepsilon : \mathbb{R} \rightarrow \mathbb{R}^m$, m-dimensional white noise defines a stochastic differential equation:

$$dS(t) = \alpha(S(t),t)dt + B(S(t),t)dW(t), S(0) = S_0 \qquad (5.26),$$

if the white noise is solution of an m-dimensional Wieners process.

The integral form of equation (5.26) is therefore given by:

$$S(t) = S_0 + \int_0^t \alpha(S(u),u)du + \int_0^t B(S(u),u)dW(u), \forall\, t \geq 0 \qquad (5.27)$$

The major problem therefore is how to calculate the third term in equation (5.27) above hence we need the Monte-Carlo simulation for the estimation of the Brownian process as is shown in the solution below:

mfile1: Brownian path Simulation

```
%BPATH1  Bownian path simulation
randn('state',400) % set the state of randn
T = 1;N = 500; dt = T/N;
dW = zeros(1,N); % preallocate arrays for efficiency
W = zeros(1,N);
dW(1) = sqrt(dt)*randn;  % first approximation outside the loop ...
W(1) = dW(1)             % since W(0) = 0 is not allowed
for j = 2:N
    dW(j) = sqrt(dt)*randn; % general increment
    W(j) = W(j-1)+dW(j);
end
plot([0:dt:T],[0,W],'r-')  % plot W against t
xlabel('t','FontSize',16)
ylabel('W(t)','FontSize',16,'Rotation',0)
```

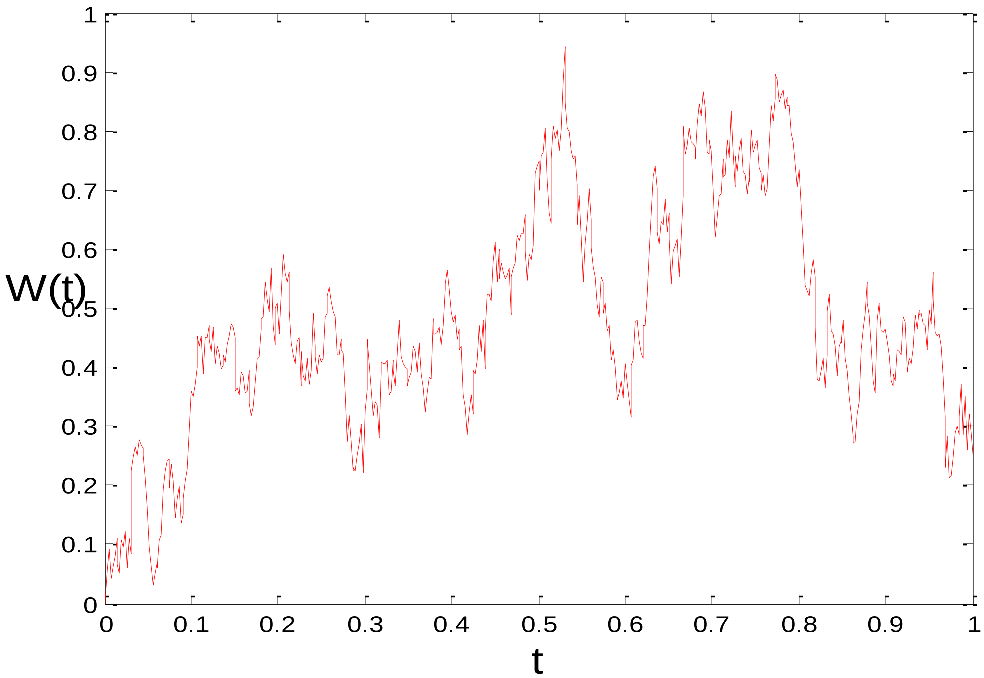


figure 5.0: Brownian path simulation

The diagram above is a representation of typical stock price using Brownian motion which shows the degree of predictability in a given asset price dynamics. It shows that in most cases the underlying stock price dynamics can only be predicted in the short run and that the degree of accuracy in the prediction diminishes as the period under consideration increases.

5.7 Use of Euler-Maruyama Method for Estimating Stock Return

Sauer (2013) asserts that stochastic differential equations (SDEs) provide essential mathematical models that combine deterministic and probabilistic components of dynamic behaviour, and because of this, SDEs have become standard models for diffusion processes in the physical/biological sciences, economics and finance. Consequently, in modern finance the Black-Scholes formula for option pricing and other fundamental asset price models, for example the Langevian equation or Ornstein- Uhlenbeck process, is based on the concept of SDEs where diffusion coefficient represents the volatility.

We recall that given appropriate conditions, an ordinary differential equation has a unique solution for every initial condition, although a numerical solution to ordinary differential equation can be obtained through Euler's method. Similarly, one can numerically obtain the solution to an SDE which is a continuous-time stochastic process by the method of Euler-Maruyama (EM) approximation. This approximation adopts the concept of Ito stochastic calculus, and in modern finance the Black-Scholes formula for option pricing and other fundamental asset price models are based on SDEs.

Since very few SDEs have closed form solution just like the ODEs, it is always necessary to use numerical techniques like the (EM) or Milstein approximation methods in estimating the solutions of SDEs emanating from finance and economics. We usually simulate the solution of nonlinear SDEs when no known analytical solution is available. Riadh et al. (2014) assert that the procedure is by solving the nonlinear SDE and simulating the stochastic (Wieners) process.

Dumbar (2014) defines (EM) method as a numerical method for simulating the solutions of a stochastic differential equation based on the definition of the Ito stochastic integral. As most stochastic processes like the Brownian motion are continuous but not differentiable, Dunbar asserts that given

$$dX(t) = f\big(X(t)\big)dt + g\big(X(t)\big)dW(t), \quad X(t_0) = X_0 \tag{5.28}$$

with step size dt, we can approximate and simulate the given equation (5.28) with the relation

$$X_j = f\big(X_{j-1}\big)dt + g\big(X_{j-1}\big)\big[W\big(t_{j-1} + dt\big) - W\big(t_{j-1}\big)\big] \tag{5.29}$$

or equivalent to:

$$X_j = f\big(X_{j-1}\big)dt + g\big(X_{j-1}\big)\big[W\big(t_j\big) - W\big(t_{j-1}\big)\big] \tag{5.30}$$

5.8 Derivation of Euler-Maruyama method

It could be recalled that a given stochastic differential equation:

$$dX(t) = f\big(X(t)\big)dt + g\big(X(t)\big)dW(t), X(0) = X_0, \qquad 0 \le t \le T$$

in equation (5.28) above, could be transformed into a stochastic integral equation written as

$$X(t) = X_0 + \int_0^t f(X(s)ds + \int_0^t g(X(s))dW(s) \qquad (5.31)$$

where, f and g are scalar functions and the initial condition X_0 is a random variable, and similarly, the solution X(t) is also a random variable for every t.

If however, $g \equiv 0$ and X_0 is a constant, then the problem reduces to deterministic case which is an ordinary differential equation and the solution will therefore be by Euler's approximation.

As a result of the stochastic component of the equation (5.31) above, the Euler-Maruyama method makes use of Ito integral calculus. To apply numerical solution to the SDE, over any prescribed interval [0, T], we have to discretize the interval. For this purpose, we can set $\Delta t = \frac{T}{L}$ for some positive integer L, and $\tau_j = j\delta t \; for \; all \; j = 0,1,2, \dots L$. Suppose we denote $t = \tau_j \; and \; t = \tau_{j-1}$ in equation (5.31) we obtain:

$$X\big(\tau_j\big) = X_0 + \int_0^{\tau_j} f\big(X(s)\big)ds + \int_0^{\tau_j} g\big(X(s)\big)dW(s) \; for \; t = \tau_j$$

$$X\big(\tau_{j-1}\big) = X_0 + \int_0^{\tau_{j-1}} f\big(X(s)\big)ds + \int_0^{\tau_{j-1}} g\big(X(s)\big)dW(s) \; for \; t = \tau_{j-1}$$

and on subtracting we shall have:

$$X\big(\tau_j\big) = X(\tau_{j-1}) + \int_{\tau_{j-1}}^{\tau_j} f\big(X(s)\big)ds + \int_{\tau_{j-1}}^{\tau_j} g\big(X(s)\big)dW(s) \; for \; t = \tau_j \qquad (5.32)$$

by setting $X\big(\tau_j\big) = X_j$ we shall have:

$$X_j = X_{j-1} + \delta t f\big(X_{j-1}\big) + g\big(X_{j-1}\big)[W\big(\tau_j\big) - W\big(\tau_{j-1}\big)] \qquad (5.33)$$

This equation is used to demonstrate how to model stock price dynamics for a security asset in the NSM with mean return µ =2 and volatility (sigma) = 1. To this end we need an actual evolution of a firm's stock prices, to be able to approximate the solution of (5.20) and setting $\mu = 2, \sigma = 1 \; and \; x_0 = 1$ arbitrarily from the initial estimation of the values of expectation and volatility. Thus, for an obtained mean (expected value) of the asset and volatility, we then use the Euler-Maruyama method to simulate the SDE by Monte-Carlo approach as shown below.

mfile2 for exact and Euler-Maruyama approximation

```
%EM Euler-Maruyama method on linear SDE

%EM Euler-Maruyama method on linear SDE
% SDE is dS = mu*Sdt + sigma*SdW, S(0)= Szero,
% where miu = 2, sigma = 1 and Szero = 1.
% Discretized Brownian path over [0,1] has dt = 2^(-8).
% Euler-Maruyama uses timestep R*dt.

randn('state',100)
mu = 2; sigma = 1; Szero = 1;     % problem parameters
T = 1; N = 2^8; dt = 1/N;
dW = sqrt(dt)*randn(1,N);          % Brownian increments
W = cumsum(dW);  % discretized Brownian path
Strue = Szero*exp((mu-0.5*sigma^2)*([dt:dt:T])+sigma*W);
plot([0:dt:T],[Szero,Strue],'m-'),hold on
R = 4;Dt = R*dt; L = N/R;           % L EM steps of size Dt = R*dt
Sem = zeros(1,L);                   % preallocate for efficiency
Stemp = Szero;
for j = 1:L
    Winc = sum(dW(R*(j-1)+1:R*j));
    Stemp = Stemp + Dt*mu*Stemp + sigma*Stemp*Winc;
    Sem(j) = Stemp;
end
plot([0:Dt:T], [Szero,Sem],'r--*)',hold off
xlabel('t','FontSize',12)
ylabel('S','FontSize',16,'Rotation',0,'HorizontalAlignment','right')
emerr = abs(Sem(end)-Strue(end))
```

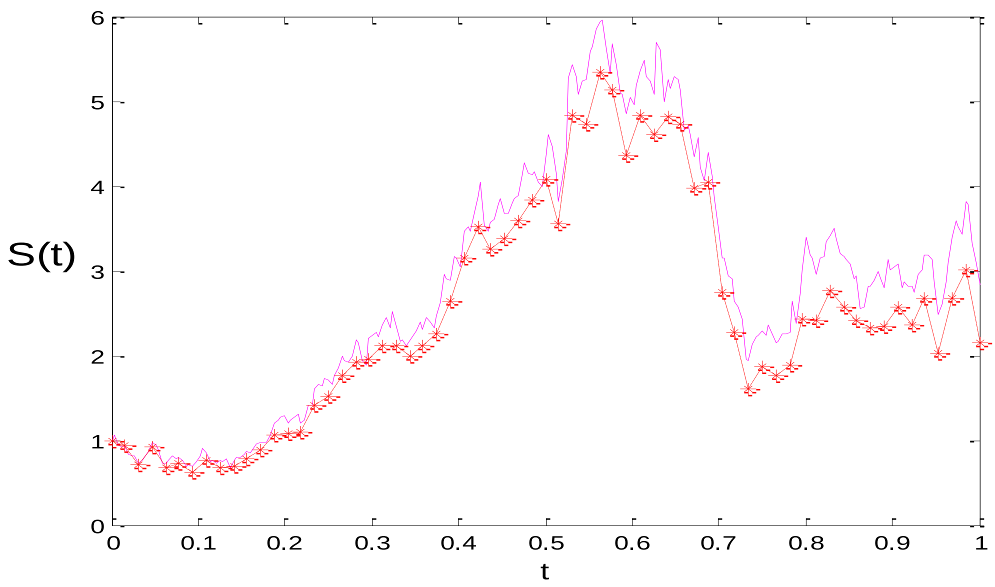


Figure 5.1: Using Euler-Maruyama to approximate stock price dynamics

From figure (5.1), the accuracy in using equations (5.33) to estimate the stock price to that of the analytical solution represented by equation (5.20) depend largely on the value of 'R' in the numerical calculation obtained from MATLAB. Greater accuracy is obtained for smaller values of 'R'. For instance, if we put R = 2 in Euler Maruyama above, we shall obtain the nearer approximation of equation (5.33) to that of the analytical solution.

As practical illustration using some data from Nigerian Stock Market from Access Bank stock as shown below (see appendix), we will obtain the following mfile3 below. We will discover that since the value of drift and variance are negligible the estimated Euler-Maruyama and Exact solution coincides thus giving us essentially the same solution.

We hereby demonstrate the use of the above code in predicting the Access stock price using data from the current price of the Access bank obtained from Cashcraft data base for securities in the Nigerian Stock Market (NSM).

5.9 Exact and Euler-Maruyama approximations using a sample of Access Bank data

(see Table 5.1 of appendix**).**

The mean daily return $\bar{\mu} = \frac{0.295246845}{40} = 0.007381171125$

Standard deviation of daily returns from the sample of access stock is:

$$\sqrt{\frac{0.050823065}{39} - \frac{(0.295246845)^2}{40(39)}} = \sqrt{0.0012472773} = 0.0353168.$$

We annualize this estimate of the standard deviation by assuming that there are about 252 trading days in a year. In this regard,

From equation (5.24)

$$\hat{\sigma} = \left\{\frac{s}{\sqrt{\tau}}\right\} = \left\{\frac{0.0353168}{\sqrt{\frac{1}{252}}}\right\} = 0.0353168\ x\sqrt{252} = 0.560636818$$

Thus, the estimated annualized volatility measure (standard deviation of Access bank stock is 0.560 or 56%), indicates high volatility occasioned by recession in the Nigerian economy, due largely to fall in the oil price and low production output. Therefore, in the Black-Scholes formulation, Access bank returns is assumed to follow a normal distribution with the estimated mean μ = 0.0073812 as the drift and square root variance σ = 0.5606368 expressing the volatility.

```
% EM Euler-Maruyama method for linear SDE
% SDE is dS = miu*S dt + sigma*S dW, S(0) = Szero,
% where sigma = 0.5606, miu = 0.0074 and Szero = 4.28.
% Euler-Maruyama uses timestep R*dt.
randn('state',100)
miu = 0.0074; sigma = 0.5606; Szero = 4.28;    % problem parameters
```

```
T = 1; N = 2^8; dt = 1/N;
dW = sqrt(dt)*randn(1,N);            % Brownian increments
W = cumsum(dW);                      % discretized Brownian path
Strue = Szero*exp((miu - 0.5*sigma^2)*([dt:dt:T]) + sigma*W);
plot([0:dt:T],[Szero,Strue],'m-'),hold on
R = 4; Dt = R*dt; L = N/R;           % L EM steps of size Dt = R*dt
Sem = zeros(1,L);                    % preallocate for efficiency
Stemp = Szero;
for j= 1:L
    Winc = sum(dW(R*(j-1)+1:R*j));
    Stemp = Stemp + Dt*miu*Stemp + sigma*Stemp*Winc;
    Sem(j) = Stemp;
end
plot([0:Dt:T],[Szero,Sem],'r--*'),hold off
xlabel('t','FontSize',12)
ylabel('S','FontSize',16,'Rotation',0,'HorizontalAlignment','right')
emerr = abs(Sem(end)-Strue(end))
```

The error between the Euler-Maruyama and exact solution represented by emerr is given by

$$emerr = abs(sem(end) - Strue(end)) = 0.2413$$

which could be minimized by an appropriate choice of R as stated before.

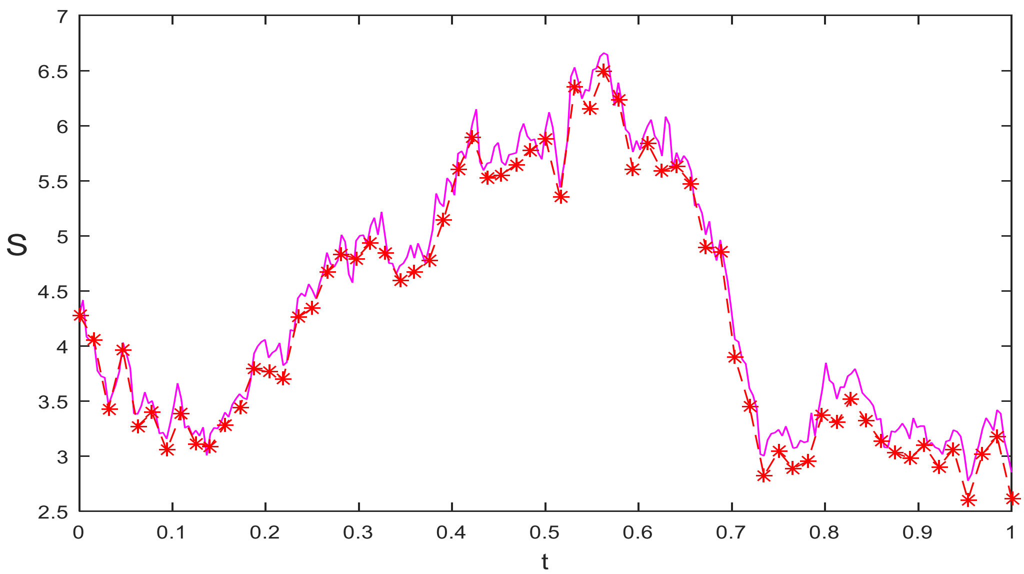


Figure 5.2: Comparison between exact and Euler-Maruyama solutions using NSM stock

Figure (5.2) above shows, as earlier stated, that Euler-Maruyama approximation is a close estimate to solutions of stochastic differential equation of interest, thus making it possible for us to adopt the same approach especially when the analytical solutions are not feasible or difficult to obtain for SDEs of interest.

For further illustrations on the development of appropriate choices of R, For R=2, we shall have:

mfile4 for exact and Euler-Maruyama approximation

```
%EM Euler-Maruyama method on linear SDE

%EM Euler-Maruyama method on linear SDE
% SDE is dS = mu*Sdt + sigma*SdW, S(0)= Szero,
% where miu = 2, sigma = 1 and Szero = 1.
% Discretized Brownian path over [0,1] has dt = 2^(-8).
% Euler-Maruyama uses timestep R*dt.

randn('state',100)
mu = 2; sigma = 1; Szero = 1;     % problem parameters
T = 1; N = 2^8; dt = 1/N;
dW = sqrt(dt)*randn(1,N);          % Brownian increments
W = cumsum(dW);                    % discretized Brownian path
Strue = Szero*exp((mu-0.5*sigma^2)*([dt:dt:T])+sigma*W);
plot([0:dt:T],[Szero,Strue],'m-'),hold on
R = 2; Dt = R*dt; L = N/R;          % L EM steps of size Dt = R*dt
Sem = zeros(1,L);                   % preallocate for efficiency
Stemp = Szero;
for j = 1:L
    Winc = sum(dW(R*(j-1)+1:R*j));
    Stemp = Stemp + Dt*mu*Stemp + sigma*Stemp*Winc;
    Sem(j) = Stemp;
end
plot([0:Dt:T], [Szero,Sem],'r--*)',hold off
xlabel('t','FontSize',12)
ylabel('S','FontSize',16,'Rotation',0,'HorizontalAlignment','right')
emerr = abs(Sem(end)-Strue(end))
```

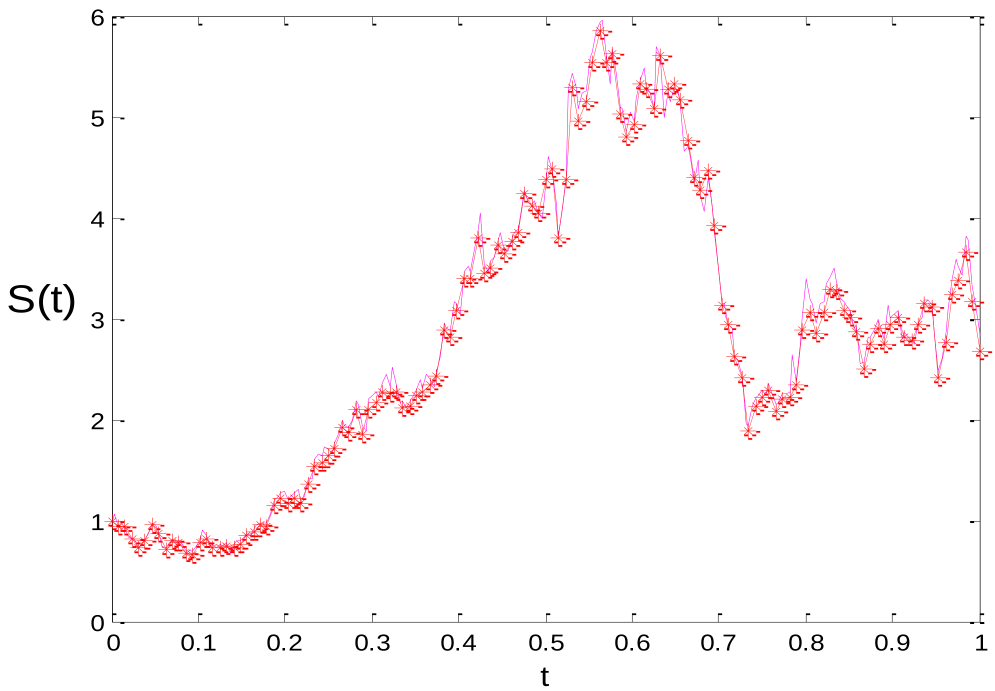


Figure 5.3: Choosing 'R' to obtain Euler-Maruyama approximation near enough to exact solution of the stochastic differential equation

The figure above shows a case where the Euler-Maruyama approximation coincides with the exact solution of the desired stochastic differential equation.

For R = 1 we shall have also:

mfile5 for exact and Euler-Maruyama approximation

```
%EM Euler-Maruyama method on linear SDE

%EM Euler-Maruyama method on linear SDE
% SDE is dS = mu*Sdt + sigma*SdW, S(0)= Szero,
% where miu = 2, sigma = 1 and Szero = 1.
% Discretized Brownian path over [0,1] has dt = 2^(-8).
% Euler-Maruyama uses timestep R*dt.
randn('state',100)
mu = 2; sigma = 1; Szero = 1;    % problem parameters
T = 1; N = 2^8; dt = 1/N;
dW = sqrt(dt)*randn(1,N);         % Brownian increments
W = cumsum(dW);                   % discretized Brownian path
Strue = Szero*exp((mu-0.5*sigma^2)*([dt:dt:T])+sigma*W);
plot([0:dt:T],[Szero,Strue],'m-'),hold on
R = 1;Dt = R*dt; L = N/R;          % L EM steps of size Dt = R*dt
Sem = zeros(1,L);                  % preallocate for efficiency
Stemp = Szero;
for j = 1:L
    Winc = sum(dW(R*(j-1)+1:R*j));
    Stemp = Stemp + Dt*mu*Stemp + sigma*Stemp*Winc;
    Sem(j) = Stemp;
end
plot([0:Dt:T], [Szero,Sem],'r--*)',hold off
xlabel('t','FontSize',12)
ylabel('S','FontSize',16,'Rotation',0,'HorizontalAlignment','right')
emerr = abs(Sem(end)-Strue(end))
```

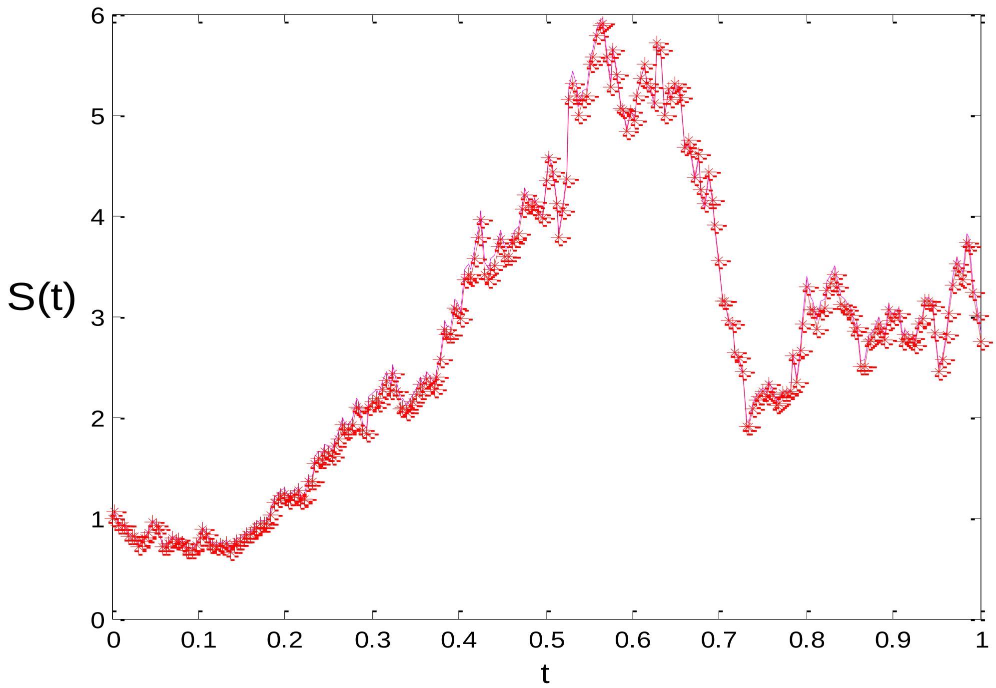


Figure 5.4: Using relevant 'R' to compare exact and Euler-Maruyama approximations

Graph in figure (5.4) goes to illustrate further the earlier statement that smaller 'R' offers an approximate solution very close to the exact (analytical) solution of the stochastic differential equation.

Market makers and investors can therefore explore the method in predicting the asset prices for possible use in pricing of derivative product that has this given primitive security price dynamics as the asset under which the contract (call or put option) is established. This could be carried out using, for instance, the Black-Scholes seminal option pricing formula.

Recall from equations (3.3) - (3.5) that the Black-Scholes formula for a European call option pricing is given by:

$$C(S,T) = SN(d_1) - \text{K}\exp(-rT)\,N(d_2)$$

$$where\ d_1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

For the put option in the European type of option pricing we have:

$$P(T) = Kexp(-rT)N(d_2) - SN(d_1)$$

$$where\ d_1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Or in a more general and compact form we could write it as:

$$C(S,t) = SN(d_1) - \text{Kexp}(-r\tau)\, N(d_2)\ and\ P(S,t) = Kexp(-r\tau)N(d_2) - SN(d_1)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad and \quad d_2 = d_1 - \sigma\sqrt{\tau} = \frac{\ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$C(S,t) =$ Price of the European call option,

$P(S,t) =$ Price of the European put option,

$S =$ Current underlying asset (stock) price,

$K =$ Strike price,

$\tau = T - t$, is the current annualized time-to-expiration, where T is the expiration date,

$r =$The annualized risk-free interest rate,

$\sigma =$ The annualized standard deviation of the underlying asset price,

$N =$ The cumulative distributions function for a standard normal variable.

We present below the MATLAB code for the computation of a European put option using the Black-Scholes Formula:

```
mfile6 for put option parameters

d1 = (log(S/K) + (r+0.5*sigma^2)*T)/(sigma*sqrt(T));

d2 = d1 - sigma*sqrt(T);

N1 = 0.5*(1+erf(-d1/sqrt(2)));

N2 = 0.5*(1+erf(-d2/sqrt(2)));

value = K.*exp(-r*T).*N2 - S.*N1;
```

5.10 Langevin equation (Ornstein-Uhlenbeck process)

This equation is used in modelling mean-reverting processes like the interest rate. Consider an SDE of the form:

$$dX(t) = -\mu X(t)dt + \sigma dW(t) \qquad (5.35)$$

where $\mu, and\ \sigma \in \mathbb{R}^+$ and the solution to this type of equation is called Ornstein-Uhlenbeck process. The Euler-Maruyama and Milstein's approximation methods are identical here, since there are no X(t) terms in diffusion component and it is difficult to obtain an analytic solution to equation (5.35) above through the elementary process as was the case with the geometric Brownian motion seen earlier. To confirm this, we try and solve the SDE (5.35) and examine the nature of the solution.

We recall that $dX(t) + \mu X(t)dt = \sigma dW(t)$, so that multiplying both sides by the integrating factor we shall obtain:

$$d\left(e^{\mu t}X(T)\right) = \mu X(t)e^{\mu t}dt + e^{\mu t}dX(t) = e^{\mu t}\sigma dW(t)$$

Integrating both sides gives;

$$e^{\mu t}X(t)|_0^t = \sigma \int_0^t e^{\mu s}dW(s)$$

$$\Rightarrow e^{\mu t}X(t) - e^0 X(0) = \sigma \int_0^t e^{\mu s}dW(s)$$

$$\Rightarrow e^{\mu t}X(t) - X(0) = \sigma \int_0^t e^{\mu s}dW(s)$$

$$\Rightarrow X(t) = X(0)e^{-\mu t} + \sigma \int_0^t e^{\mu(s-t)}dW(s)$$

The second term on the right-hand side shows that no closed form solution exists and that the only solution is the non-trivial one. Hence, we are left with the numerical simulation to the SDE unlike the Brownian motion where we have the trivial (closed form) solution as well as the numerical approximations.

mfile7

```
%Euler-Maruyama method on Interest rate model (langevian/Ornstein
process
% SDE is dX = - miu*Xdt + sigma*dW, X(0) = Xzero
% Method uses timestep of Delta = 2^(-8) over a single path


clf
randn('state',1)
T = 1; N = 2^8; Delta = T/N;
miu = 0.05; sigma = 0.8; Xzero = 1;

Xem = zeros(1, N+1);
Xem(1) = Xzero;
for j = 1:N
    Winc = sqrt(Delta)*randn;
    Xem(j+1) = Xem(j)- Delta*miu*Xem(j) + sigma*(Xem(j))*Winc;
end

plot([0:Delta:T],Xem,'r--')
xlabel('t','FontSize',16)
ylabel('X','FontSize',
```

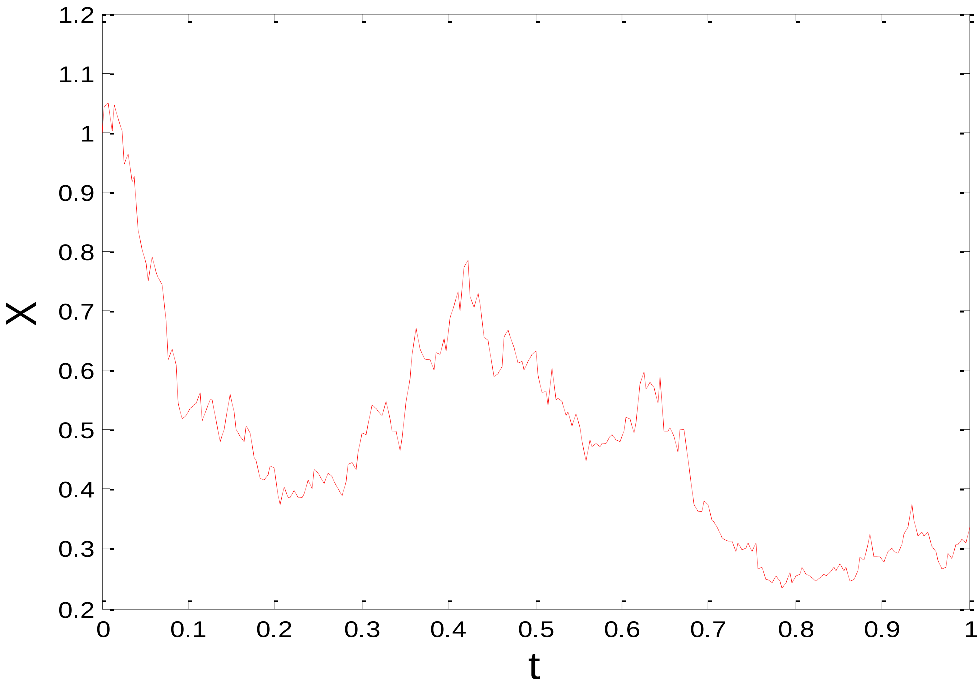


Figure 5.5: Use of Euler-Maruyama approximations for interest rate models

Figure (5.5) above is a typical illustration of numerical approximation to solution of stochastic differential equation model where the analytical solution is difficult to achieve. Such models as interest rate are evaluated through this type of numerical approximation.

Here we want to consider asset price dynamics that is represented by a square root process as in the scalar stochastic differential equation

$$dX(t) = \lambda(t)X(t)dt + \sigma\sqrt{X(t)}dW(t) \qquad 0 \leq t \leq 1$$

The mfile8: square root function using Euler-Maruyama approximation

```
%Euler1 Stochastic Euler method on square root process SDE
%
% SDE is dX = lambda*X dt + sigma*sqrt(X)dW, X(0) = Xzero.
% Method uses timestep of Delta = 2^(-8) over a single path.
clf
randn('state',1)
T = 1; N = 2^8; Delta = T/N;
lambda = 0.05; sigma = 0.8; Xzero = 1;

Xem = zeros(1,N+1);
Xem(1) = Xzero;
for j = 1:N
    Winc = sqrt(Delta)*randn;
    Xem(j+1) = abs(Xem(j) + Delta*lambda*Xem(j) +
sigma*sqrt(Xem(j))*Winc);
end

plot([0:Delta:T],Xem,'r--')
```

```
xlabel('t','FontSize',16), ylabel('X','FontSize',16)
```

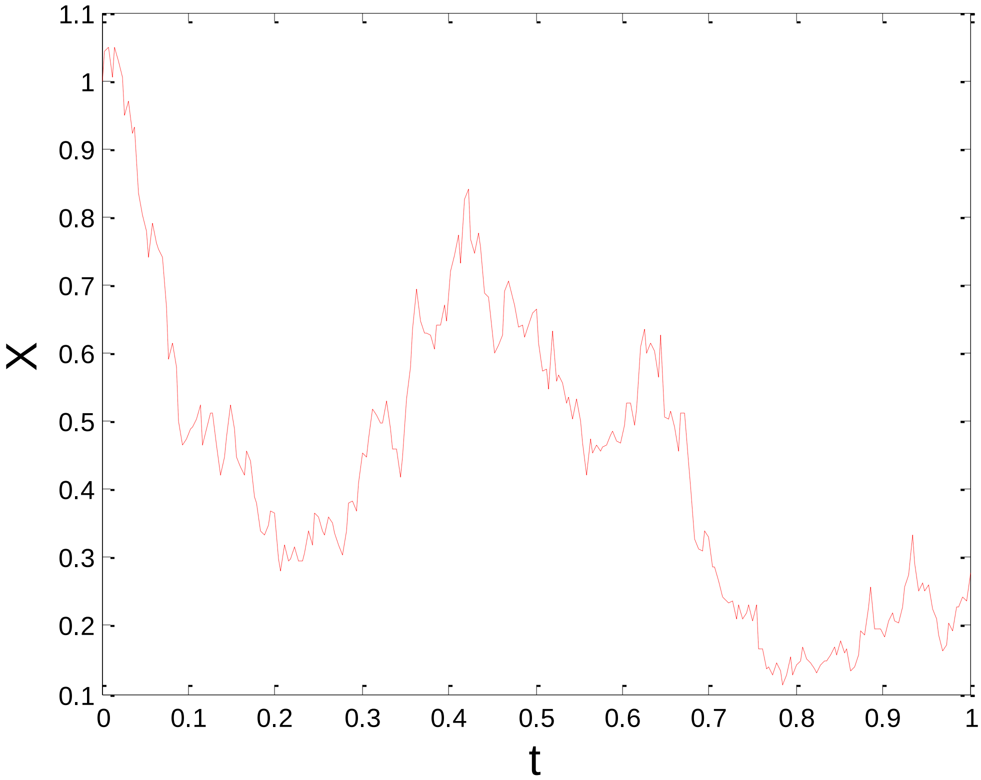


Figure 5.6: Euler-Maruyama approximations for square root functions

The process as shown in figure (5.6) above is used in modelling interest rates or stochastic volatility process for stock prices and it was proposed by Cox et al. (1985) as the prototypical model on interest rates. Thus, figure (5.5) and (5.6) are useful to investors and other stakeholders in the Nigerian Capital Market as interest rate has always been a financial quantity that worries participants in the Nigerian market.

5.11 Errors in Euler-Maruyama and Milstein's approximation

Usually, the analytical solutions of many SDE are not known explicitly and that is why we resort to the method of simulation. However, when the explicit solution to a given SDE is known, then it is realistic to use the absolute error criterion to calculate the error and this was defined by Riadh et al. (2014) as the expectation of the absolute value of the difference between the approximation and the Ito process at time T, written as

$$\varepsilon_\Delta = \boldsymbol{E}(|\boldsymbol{X}_{\boldsymbol{app}}(\boldsymbol{t_i}) - \boldsymbol{X}_{\boldsymbol{true}}(\boldsymbol{t_i})|)$$

where $t_i = i\Delta; i = 1,2,, \ldots, N$ and **E** denotes the mean value.

By repeating N different simulations of sample paths of the Ito Calculus process, and their respective Euler-Maruyama approximations corresponding to the same sample paths of the Wiener process and estimate the absolute error $\boldsymbol{\varepsilon}$, we have

$$\widehat{\varepsilon_\Delta} = \frac{1}{N}\sum_{i=1}^{N} |X_{app}(t_i) - X_{true}(t_i)|$$

5.12 Summary and Conclusion

It has been observed from this work that the difference between the analytical (exact) solution and numerical approximation (Euler-Maruyama) lies on the values of the drift parameter, variance (volatility), and the initial take off price of the asset denoted by S_0. This relation is easily seen from the initial trial simulations of the stock prices using some arbitrary values of drift, volatility and assumed asset price.

As a result of the relatively significant values of theses parameters (drift, volatility and initial asset price), there exists some remarkable difference between the exact or analytical values and that of Euler-Maruyama (EM) approximations for sufficiently large value of R. For data that have infinitesimal values for the parameters, the plotted analytical (exact) solution coincides with that of Euler-Maruyama approximation for sufficiently large values of R. From the Access Bank data therefore, EM solution is essentially the same with the true solution, even for large values of R (Δt). This was demonstrated in Mfile3 and figure 3.

We can therefore infer that EM approximation could be used for an estimation of financial asset price like equity (in Black-Scholes model) and interest rate (in Ornstein Uhlenbeck process model), which in turn is needed in derivative asset pricing.

Market participants can thus use these properties of EM approximations in forecasting the values of assets in their portfolio of investment for appropriate pricing of such securities for use in derivative contracts. That is, the process will facilitate proper pricing of the underlying asset for which the investors and market makers wish to introduce derivative contract thereby ensuring that the assets are properly priced in order to avoid arbitrage opportunities.

Chapter 6

IMPLIED VOLATILITY ANALYSIS AND ITS APPLICATIONS

6.0 Introduction

Implied volatility is a useful tool in financial asset management deployed in monitoring the market opinion regarding the volatility of a given stock. Usually, options are traded on volatility with implied volatility serving as an efficient and effective price mediator of the option. Investors can adjust their portfolio in order to reduce their exposure to those instruments whose volatility are predicted to increase hence implied volatility has some useful implication in risk management. For instance, in applying implied volatility to the seminal Black-Scholes (1973) model we shall have:

$$C\big[(S_t, k, \tau, r, \sigma_{imp}(k,\tau)\big] = C_t^*(k,\tau) \quad (6.1)$$

Where the left-hand side of equation (6.1) is the Black-Scholes call option price, $\sigma_{imp}(k,\tau)$ is the implied volatility and $C_t^*(k,\tau)$ is the market price of a call option at the time instant t; S_t is the price of the underlying stock, k is the exercise price, τ is the time to expiration and r is the interest rate. Similarly, the implied volatility of the European put option with the same maturity and strike can be obtained using put-call parity relation.

The convex shape of the implied volatility (as against flat surface predicted by Black-Scholes option pricing model due to constant volatility and lognormal assumption of the underlying stock price) with respect to moneyness (K/S), is referred to as the smile effect. Jarrow and Rudd (1982) argue that the smile effect can be explained by departures from lognormality in the underlying asset price, especially for out-of-the money options. This smile effect is more noticeable as the option approaches expiration, Hull and White (1987) and is very noticeable in Black-Scholes model as a result of the assumptions on the underlying asset in the Black-Scholes model.

Generally, value of the implied volatility depends on time to expiration τ and strike K. A graphical function:

$$\sigma_{imp}(K,\tau) \rightarrow \sigma_{imp}(K,\tau)$$

is called the implied volatility surface at a date t. In other words, implied volatility surface is the plot of implied volatility across strike and time to maturity.

We recall here that the volatility of an asset/equity is a measure of its return variability. Usually, volatility is measured by using previous prices of the underlying asset to obtain the historical volatility. This method of measuring dispersion in return is not generally acceptable to investors who prefer the market estimate of volatility, thereby advocating for use of implied volatility. Thus, for a correct market price of put and call options, the volatility implied by such market reflects the markets opinion of what volatility should be.

Therefore, due to the shortcomings associated with the constant volatility parameter of the Black-Scholes model, investors have devised an alternative method of estimating and or predicting the volatility parameter, through an observation of the market price of the option, by inverting the option pricing formula to determine the volatility implied by the market price otherwise known as implied volatility.

6.1 Numerical approximation of implied volatility

The numerical approximation of implied volatility could be achieved through Newton - Raphson method or Bisection method.

6.1.1 Newton Raphson method: The most commonly used numerical approximation for implied volatility is the traditional method of solving nonlinear systems of equation, proposed by Adi (1966) which is a root searching algorithm that is used in finding the first few terms of the Taylor series of a function f(x) in the neighbourhood of a suspected root referred to as Newton Raphson method. For equation (3.23) of chapter 3 which is equivalent to (6.1) above we shall obtain from Newton Raphson method

$$\sigma_{n+1} = \sigma - \frac{f(\sigma_n) - C_{BS73}}{f'(\sigma_n)} \tag{6.2}$$

where σ_n is the nth estimate of $\sigma_{imp}(K, t)$, and f' is the first derivative of f(σ), that is first derivative of the option price with respect to volatility, $\frac{\partial C}{\partial \sigma}$ and since $f'(\sigma_n) > 0$ when $t, S, K > 0$, equation (6.2) is well defined over (0,∞).

Mark Kritzman (1991) asserts that the Newton-Raphson method entails starting the iteration with some reasonable estimate of volatility and evaluating the option using this estimate of volatility from equation (6.2). Stewart Mayhew (1995) declares that faster convergence could be achieved if an analytic expression is known for the options 'vega' which as stated before is the derivative of the option price with respect to the volatility parameter. This is readily verifiable for the Black-Scholes formula for which a Newton-Raphson algorithm can usually achieve reasonably accurate estimates of the implied volatility within three iterations. To obtain the initial value for iteration using Newton-Raphson method, Manaster and Koehler (1982) described how to choose this starting value to ensure that the algorithm will converge whenever the solution exists.

Manaster and Koehler (1982) state that a well-known result concerning the Newton-Raphson method iteration is that whenever (6.1) has a solution; there is an open interval (c,d), in the neighbourhood of σ_{imp} such that if $\sigma_n \in (c, d)$ for any n , then $\lfloor\sigma_n\rfloor \to \sigma_{imp}^2$ and when this is the case, we have quadratic convergence. They further stated that whenever Max$(0, S - Xe^{-rt}) < C_{BS73} < S$ then equation (3.23) has a solution.

We are therefore required to find one point in (c, d) to guarantee that (6.2) will usually lead to σ_{imp} and that such point

$$\sigma_1^2 = \left|ln\frac{S}{X} + rt\right| \frac{2}{t} \tag{6.3}$$

It is worthy of mention here that for other options different from the European option (American option), where a significant possibility of early exercise exists or for complex options, the Newton-Raphson method does not work. The preferred method for non-European plain vanilla option is the Bisection method. Here in this work we used the bisection method for the computation of implied volatilities for the given option prices.

6.1.2 Method of Bisection

Step 1: From equation (3.23) to obtain the implied volatility of an option, conceptually we are trying to find the root of the equation given below:

$$f(\sigma_{imp}) = f[S, X, r, t, \sigma_{imp}(X.t)] - C_{BS73} \tag{6.4}$$

In other words, we need the value of σ for which $f(\sigma_{imp}) = 0$. To do this, we begin by picking an upper and lower bound of the volatility ($\sigma_{lower}\ and\ \sigma_{upper}$) such that the value of $f(\sigma_L)$ and $f(\sigma_U)$ have different (opposite) signs. This relation from mean value theorem (MVT) /Rolle's Theorem means that the root of equation (6.4) or the value of implied volatility lies between the lower and upper volatility so picked. The lower estimate of volatility corresponds to a low option value and a high estimate for volatility corresponds to a high option value.

Step 2:

We then calculate a volatility that lies half way between the upper and lower volatilities. That is, $Vol_{mid} = \frac{\sigma_L + \sigma_U}{2}$, If we set $Vol_{mid} = \sigma_M, and\ if\ for\ C(\sigma_M) > C$ (observed) then the new mid-point σ_N will be $\sigma_N = \frac{\sigma_L + \sigma_M}{2}$ or else we have $\sigma_N = \frac{\sigma_U + \sigma_M}{2}$. This method is continued in this fashion until a reasonable approximation of implied volatility is obtained. In other words when the option value corresponding to our interpolated estimate for volatility is below the actual (observed) option price, we replace our low volatility estimate with the interpolated estimate and repeat the calculation, Kritzman Mark (1991). However, if the estimated option value is above the actual option price, we replace the high volatility estimate with the interpolated estimate and continue in this way until the reasonable implied volatility approximation is achieved.

Step 3:

When the option value corresponding to the volatility estimate is equal to the actual price of option, we have thus arrived at the required implied volatility of the option. In other words, if $f(vol_{mid}) = 0$ or less than a given ε, we have therefore found the required implied volatility and that terminates the iterations.

Step 4 Summary: If $f(vol_{lower})$ multiplied by $f(vol_{mid}) < 0$ then the root lies between vol_{lower} and vol_{mid} . If however the value of $f(vol_{lower})$ multiplied by $f(vol_{mid}) > 0$, then the root lies between vol_{mid} and $vol_{upper.}$ In other words when $f(vol_{lower}) * f(vol_{mid}) < 0$, then allow $vol_{upper.}$ to be vol_{mid} and apply step 2 again. But when $f(vol_{lower}) * f(vol_{mid}) > 0$ then allow $vol_{upper.} = vol_{mid}$ and proceed by going back to step 2.

In practice, however, various implied standard deviation obtained are simultaneously from different options on the same stock, and the composite implied standard deviation for any given stock is therefore calculated by taking suitably weighted average of the composite implied standard deviation (implied volatilities). It is indeed necessary that the various weighting schemes to be adopted should reflect the sensitivities of the option prices to volatility as at-the-money (ATM) options are known to be far more sensitive to volatility, than the price of the deep-out-of-the money options.

The main reason, however, for adopting at-the-money options as the best estimate of volatility in the past is that at-the-money options are almost the most actively traded options, and have the smallest measurement errors Brenner and Subrahmanyam (1988).

We note that stocks usually have many options traded on them, thereby providing several different implied standard deviations to be calculated for each stock. In order, therefore, to obtain a single estimate of the implied standard deviation (volatility) associated with each stock will then be combined into a single weighted average standard deviation, Chiras and Manaster (1978).

(see Appendix of this thesis for some detailed computations of implied volatility)

As the policy makers in the NSM are most interested in European type of derivative options and have recommended same for introduction into NSM, we therefore assumed that the option type is European (call) option so that the Black-Scholes formula could be applicable. We estimate the Black-Scholes implied volatility using Excel VBA (Visual Basic for Applications) a programming language in Excel that is very useful in computing both implied volatility for single option price, and for the case when there are series of option prices that we need to calculate their respective implied volatilities. The process is to store the programming codes as written below in modules for use in the various calculations as and when necessary. For the implied volatility computation, we use the bisection method for the estimation which will therefore be inserted into the Black-Scholes option pricing formula.

```
Function BSC (S, K, r, q, sigma, T)

Dim dOne, dTwo, Nd1, Nd2

    dOne = (Log(S / K) + (r - q + 0.5 * sigma) * T) / (sigma
* Sqr(T))

    dTwo = dOne - sigma * Sqr(T)

    Nd1 = Application.NormSDist(dOne)

    Nd2 = Application.NormSDist(dTwo)

    BSC = Exp(-q * T) * S * Nd1 - Exp(-r * T) * K * Nd2

End Function

Function BSCImVol(S, K, r, q, T, callmktprice)

    H = 5

    L = 0

Do While (H - L) > 0.00000001

If BSC(S, K, r, q, (H + L) / 2, T) > callmktprice Then


    H = (H + L) / 2


    Else: L = (H + L) / 2

End If

Loop

    BSCImVol = (H + L) / 2

End Function
```

6.2 Model testing

We now consider the various practitioners Black-Scholes model otherwise called Ad-Hoc Black-Scholes to determine their suitability or otherwise for pricing derivative options. The practitioners Black-Scholes models are categorized into two main groups, namely: The "Relative smile" and "Absolute smile" models for derivative option pricing and we adopt the method of Dumas, Fleming and Whaley (1998) to test these models:

$$DVF_{R1}: \sigma_{iv} = a_0 + a_1\left(S/_K\right) + a_2T + a_3\left(S/_K\right)T$$

$$DVF_{R2}: \sigma_{iv} = a_0 + a_1\left(S/_K\right) + a_2(S/_K)^2 + a_3T$$

$$DVF_{R3}: \sigma_{iv} = a_0 + a_1\left(S/_K\right) + a_2(S/_K)^2 + a_3T + a_4\left(S/_K\right)T$$

$$DVF_{R4}: \sigma_{iv} = a_0 + a_1\left(S/_K\right) + a_2(S/_K)^2 + a_3T + a_4T^2$$

$$DVF_{A1}: \sigma_{iv} = a_0 + a_1K + a_2T + a_3KT$$

$$DVF_{A2}: \sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T$$

$$DVF_{A3}: \sigma_{iv} = a_0 + a_1K + a_2T + a_3T^2 + a_4KT$$

$$DVF_{A4}: \sigma_{iv} = a_0 + a_1K + a_2T + a_3K^2 + a_4KT$$

$$DVF_{A5}: \sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T + a_4T^2 + a_5KT$$

As labelled above, the first four are relative smile models whereas the last five implied volatility models are absolute smile which we considered in this research. We now consider the models one after the other starting with the absolute smile given by

$$DVF_{A5}: \sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T + a_4T^2 + a_5KT.$$

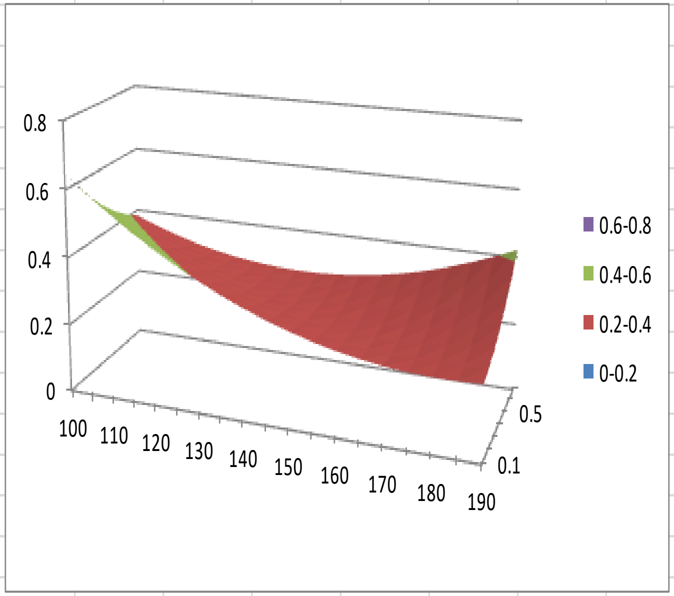
This model may not be recommended for the estimation of implied volatility in the NSM, as the p-value is greater than 0.05, for the coefficient of T^2 which is a_4. Thus, the parameter a_4 does not improve the model estimation significantly (see page 267, Table 6.7 in appendix).

The summary statistics are as seen in the table below. Directly after the summary statistics is the associated implied volatility surface for the option prices for the various time-to-maturity of the given set of option prices. It is obvious that the obtained surface is not flat, supporting the earlier claims from various research results that the constant assumption of volatility throughout the option life span as was proposed by Black and Scholes in their (1973) is not generally true.

(see table 6.3 in the appendix for a detailed computation of implied volatility surface).

Figure 6.0: Implied volatility surfaces

	100	105	110	115	120	125	130	135	140	145	150	155	160	165	170	175	180	185	190
0.1	0.647411	0.59662	0.549015	0.504594	0.463359	0.42531	0.390446	0.358767	0.330273	0.304966	0.282843	0.263906	0.248154	0.235588	0.226207	0.220011	0.217001	0.217176	0.220536
0.2	0.586088	0.53963	0.496357	0.45627	0.419368	0.385651	0.35512	0.327774	0.303614	0.282639	0.26485	0.250245	0.238827	0.230593	0.225545	0.223683	0.225006	0.229514	0.237207
0.3	0.531604	0.489479	0.450539	0.414785	0.382216	0.352833	0.326634	0.303622	0.283794	0.267152	0.253696	0.243425	0.236339	0.232439	0.231724	0.234194	0.23985	0.248691	0.260718
0.4	0.48396	0.446168	0.411561	0.38014	0.351904	0.326853	0.304988	0.286309	0.270814	0.258506	0.249382	0.243444	0.240691	0.241124	0.244742	0.251545	0.261534	0.274709	0.291068
0.5	0.443155	0.409696	0.379423	0.352334	0.328432	0.307714	0.290182	0.275836	0.264674	0.256698	0.251908	0.250303	0.251883	0.256649	0.2646	0.275736	0.290058	0.307566	0.328258
0.6	0.409191	0.380065	0.354124	0.331369	0.311799	0.295415	0.282216	0.272202	0.265374	0.261731	0.261273	0.264001	0.269915	0.279014	0.291298	0.306767	0.325422	0.347262	0.372288
0.7	0.382066	0.357273	0.335665	0.317243	0.302006	0.289955	0.281089	0.275408	0.272913	0.273603	0.277479	0.28454	0.294786	0.308218	0.324835	0.344638	0.367626	0.393799	0.423158



Next, we consider the quadratic implied volatility model of practitioners Black-Scholes given by $\sigma_{iv} = a_0 + a_1K + a_2T + a_3T^2 + a_4KT$. The summary statistics is as shown on Table 6.8 page 268 in the appendix. This model may not be recommended for estimating implied volatilities and consequently in pricing of options in the Nigerian Stock Market as the p-value is greater than 0.05, for T^2 coefficient $(= a_3)$. (see Table 6.4 of Appendix pages for detail and page 268 for summary Statistics).

We now consider the implied volatility model for an Ad-Hoc Black-Scholes model (Absolute smile) where the quadratic terms are $KT \ and \ K^2$. It is given by $\sigma_{iv} = a_0 + a_1K + a_2T + a_3K^2 + a_4KT$. The p-value is within the acceptable range as it is less than 0.05 whereas the R- squared value which measures how close the observed data are to the fitted model has a sufficiently large value (72%). The summary statistics is as shown in Appendix page 269, Table 6.9.

We now consider another type of Ad-Hoc Black-Scholes which is represented by the multiple regression equation given by: $\sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T$. However, it is worthy of mention here that although the model parameters here are fewer than what we had in the preceding model, the former fits the data better than this present one. That is, although the model parameters here are fewer in number than what we had in the preceding model, the former model fits the data better than the latter, hence we can infer from this evidence that increasing the number of explanatory variables do not generally improve the efficiency of the model parameter estimations. Indeed, the R-squared value and adjusted R-squared values are better in $\sigma_{iv} = a_0 + a_1K + a_2T + a_3K^2 + a_4KT$ when compared with what we obtained from$\sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T$. The Summary Statistics for implied volatility model given by $\sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T$ is shown on page 269 Table 6.10.

Finally, on this type of Ad-Hoc Black-Scholes (practitioners Black-Scholes), we look at another absolute smile model where the only quadratic term is the product of exercise price and time to maturity represented by the non-linear equation$\sigma_{iv} = a_0 + a_1K + a_2T + a_3KT$. It is evident from the table below that although the R-squared value is not very high (58.4%), the p-values are within the region ($p < 0.05$),where we can reject the null hypothesis which means that the inclusion of the entire affected predictor variable are necessary for the estimation of the response variable (implied volatility). Summary Statistics for implied volatility model given by $\sigma_{iv} = a_0 + a_1K + a_2T + a_3KT$ is displayed in the appendix section page 270, Table 6.11.

We now consider other types of Ad-Hoc Black-Scholes -" relative smile" types of implied volatility estimation models where the predictor variables are functions of moneyness and time to maturity.

Summary Statistics for implied volatility model given by $\sigma_{iv} = a_0 + a_1 S/K + a_2(S/K)^2 + a_3 T$ is as shown in the appendix specifically on page 271, Table 6.12.

We can see here that the predictor variables are good estimators of the response variable (implied volatility) as shown in the one-way ANOVA table for the parameterization in $\sigma_{iv} = a_0 + a_1 {}^{S}/_{K} + a_2({}^{S}/_{K})^2 + a_3 T,$ hence this type of relative smile model fits the data well for implied volatility estimation.

We now consider other models in the relative smile family of implied volatility estimation but will only show the regression statistics /ANOVA table of the results that are therein. Summary Statistics for implied volatility model given by $\sigma_{iv} = a_0 + a_1 S/K + a_2 T + a_3(S/K)T$ is shown in the appendix on page 272, Table 6.13. We can infer from the table above that the model is a good fit of the data having all the p-values for the predictor variable strictly less than 0.05 and a very nice value of R-squared 68%.

For $\sigma_{iv} = a_0 + a_1({}^{S}/_{K}) + a_2({}^{S}/_{K})^2 + a_3 T + a_4({}^{S}/_{K})T$ we again see that the model p-value lies within the acceptable threshold (less than 0.05) and with a higher R-Squared value of 74% thus showing that the latter is a better model that fits the data obtained from the market.

Summary statistics for model $\sigma_{iv} = a_0 + a_1({}^{S}/_{K}) + a_2({}^{S}/_{K})^2 + a_3 T + a_4({}^{S}/_{K})T$ is on page 273, Table 6.14 of the thesis.

Finally, we consider another similar type of absolute smile family of Ad-Hoc Black-Scholes but in this case instead the last quadratic term as a mixture of moneyness and time to maturity we are going to replace this product with the square of time to maturity. The model is given by: $\sigma_{iv} = a_0 + a_1({}^{S}/_{K}) + a_2({}^{S}/_{K})^2 + a_3 T + a_4 T^2$. We see from the computations that the changes in the predictor variables $(({}^{S}/_{K})^2 T \; and \; T^2 respectively)$ has no relationship with the response variable (implied volatility). This situation is observable from the fact that not only were the values of the parameters for estimation of the explanatory variables T and T^2 not significant, the adjusted R^2 was also seen to diminish in the estimation of the parameters of the new model. Hence the increment on the number of variables in this case does not improve the estimation of the implied volatility for the given relative smile model. Summary Statistics for implied volatility model $\sigma_{iv} = a_0 + a_1({}^{S}/_{K}) + a_2({}^{S}/_{K})^2 + a_3 T + a_4 T^2$ is as shown on page 274 of Table 6.15.

6.3 Summary and Conclusion

We observe that the standard error under (Multiple Regression Statistics) heading determines the usefulness or otherwise of an additional predictor variable introduced into the model. For instance, when we compare the models

$$\sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T \text{ and}$$

$$\sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T + a_4T^2$$

we obtained a standard error value of 0.069441 for the former but when we added another predictor variable in the later we obtained a standard error of 0.0696636 meaning that the additional predictor variable T^2 introduced into the model does not improve the value of the implied volatility so fitted. The standard error is, however, expected to decrease in value whenever a new predictor variable which is added into the model improves the model fitting.

However, for R^2, the value of the model will always increase whenever a new predictor variable is added to the model. Daniel, T. Larose and Champal, D. Larose (2015) assert that while the standard error value decreases when a predictor that improves the model fitting is added, R^2 will always increase in value whenever a new predictor variable is added regardless of its usefulness.

To conclude, if the added explanatory (predictor) variable(s) offer(s) improvement on the response variable, it is necessary that the adjusted R^2value also increases alongside with that of R^2. When this happens with the satisfactory p-values, then the added variable improves the model parameter values for estimating implied volatility compared to the previous model.

The T-test is a measure of the relationship between the response variable (implied volatility) and a particular predictor variable (in this case they are strike price, time to maturity and moneyness). The F-test measures the significance of the regression as a whole. While the t-test could be applied to measure if there is a significant linear relationship between the target (response) variable which in this case is the implied volatility and each of the predictors, F-test considers the linear relationship between implied volatility and the set of predictors as a whole.

CHAPTER SEVEN

RESULTS FROM RANDOM MATRIX THEORY

7.0 Introduction

Most recently, the analysis of equal time cross-correlation matrix for some variety of multivariate data sets including the financial market data have been of much interest to researchers leading to some fundamental properties being examined extensively, [Laloux L.et al. (2000), Sensoy, a. (2013), Plerou, V. et al. (1999), Plerou, V. et al.(2000), Mantegna, R.N. (1999), Utsugi, A. et al. (2004), Conlon, T. et al. (2007), Nobi, A. et al. (2013), Wilcox, D. et al. (2004), Conlon, T. et al (2009)].The dynamics of this equal time cross-correlation matrix of the multivariate times series is studied through an examination of the eigenvalue spectrum over some prescribed time intervals , (Conlon, T. et al., 2009).

It is the need to study the dynamics of stock price returns using the information obtained from the eigenvalue spectrum of the cross-correlation analysis that brought about the concept known as Random Matrix Theory, (RMT) which, several researchers have deployed to filter the relevant information from the statistical properties associated with the empirical cross correlation matrices for various financial times series. The RMT provides the theoretical underpinnings for a possible comparison of the eigenvalue spectrum of the empirical correlation matrix with that of the Wishart matrix generated from a random matrix of equivalent dimension with that of the empirical correlation matrix employed in this study of stock price dynamics of financial assets drawn from the Nigerian Stock Market (NSM) in this work.

Deviations in the eigenvalue spectrum from the eigenvalue predictions (if any) could provide genuine information about the correlation structure of the multivariate time series of stock price return or other analysis required that involves the use of RMT. The analysis of the statistical properties resulting from the information obtained through deviations in the eigenvalue spectrum is necessary for the reduction of risk existing between the predicted and realised risk in different portfolio of investment.

The construction of fund of funds in a hedge fund portfolio requires a correlation matrix that are usually estimated using small samples of monthly returns data that induces noise in the empirical analysis. T. Conlon et al. (2007) assert that a hedge fund is a highly regulated investment strategy which uses a variety of investment instruments that may include short positions, derivatives, leverage and charge incentive-based fees. They are normally structured as a limited partnership or offshore investment companies that pursue positive returns in all markets and are always described as an absolute return strategist.

Empirical correlation matrices are very useful in risk management and asset allocations Laloux, L. et al. (2000). Investors are known to apply the method of asset diversification in the management of risks associated with their portfolio using the knowledge from empirical cross correlation on those assets that have low or even negative correlation coefficient with other assets in their preferred portfolio of investment. This is true since in empirical finance, the probability of large losses for a certain portfolio or option book is usually dominated by correlated moves of its different constituents, T. Conlon et al. (2007).

7.1 Theoretical Backgrounds

For any given set of N different assets, the associated correlation matrix has $\frac{N(N-1)}{2}$ entries that would be determined from N times series of length, T. It suffices to state here that if T is not large enough when compared to the number of assets, N, the obvious implication is that the associated covariance matrix is noisy hence the resulting empirical correlation matrix is therefore said to be random. When this happens, the entire matrix structure is known to be dominated by measurement noise and we cannot, therefore, make any meaningful pronouncement about the properties of the matrix structure so obtained and cannot also use the associated information for risk management and asset allocation. As this research is geared towards risk management of assets by investors in the Nigerian Stock Market using derivatives, and derivative contracts themselves are usually written on some underlying assets, then studying therefore the nature of correlation of stocks in the NSM is of great concern in this work.

Thus, to avoid the entire exercise being dominated by measurement noise, it is necessary for us to choose sufficiently large N and T, i.e the number of stocks and length of period respectively to be able to obtain true information from the matrix correlation gotten from the stock return dynamics. When this is done we can then be assured that we can distinguish real information from noise in the market substructure through a fair and credible analysis of eigenvalues and eigenvectors emanating from the correlation matrices for risk management. This is done by comparing the properties of an empirical correlation matrix C to a null hypothesis of a purely random matrix called the Wishart matrix obtained from a finite times series of strictly independent assets. It is the deviations in the eigenvalue spectrum of the empirical correlation matrix obtained from the times series return of the chosen assets in the financial market being considered with that of an associated Wishart matrix or Laguerre ensemble that suggests the presence of true information in the matrix structure being analysed.

[Laloux, et al. (1999), Sharifi, S. et al. (2004)] state that for any given financial returns written in the context of correlation matrix R, then

$$R = \frac{1}{T} AA^T \qquad (7.1)$$

where A is an NXT matrix whose elements are independent and identically distributed random variables with mean zero and in the limit $N \to \infty, T \to \infty$ such that $Q = {}^{T}/_{N} \geq 1$ is fixed, then the distribution of $P(\lambda)$ of the eigenvalues of R is self-averaging and can be represented by

$$P(\lambda) = \begin{cases} \frac{Q}{2\pi\sigma^2} \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{\lambda}, & \lambda_- \leq \lambda \leq \lambda_+ \\ 0, & elsewhere \end{cases} \quad (7.2)$$

with σ^2 as the variance of the elements of A and $\lambda_{\pm} = \sigma^2(1 + \frac{1}{Q} \pm 2\sqrt{1/Q})$

However, the covariance matrix of returns on the assets under consideration σ_{ij} is represented by $\sigma_{ij} = < G_i(t)G_j(t) > - < G_i(t) >< G_j(t) >$

where $< . >$ refers to the mean of returns over time under consideration (usually in months or years) hence the empirical correlation matrix C is thus given by

$$C_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} \quad (7.3)$$

where $\left\{G_i(t)\right\}_{t=1,2,\ldots T}^{i=1,2,\ldots,N}$ are returns defined as $G_i(t) = ln\left\{\frac{S_i(t)}{S_i(t-1)}\right\}$ and $S_i(t)$ is the spot price of asset i at time t.

In a more practical sense, we will investigate and compare the spectral properties of correlation matrices of price fluctuations in Nigerian and South African Stock Markets, using the Random Matrix Theory (RMT). Alternative research work on dynamics and properties of the correlation matrix could also be studied through other approaches namely: factor and principal components analysis for measuring the extent of correlations as presented in [Cont, R. et al. (2002), Gentle, J. (1998), Jackson, E. (2003), Morrison, D.F. (1990)]. In this research, we use RMT to compare the empirical correlation matrix with Wishart random matrix, which model's normality and departures from which connote the existence of significant market information in the observed price fluctuations Pafka and Kondor (2004).

Pafka and Kondor (2004) assert that correlation matrices of financial returns play crucial role in various aspects of modern finance including investment theory, capital allocation, and risk management. Also, Wang, Gang-Jin et al.(2013) declare that following the introduction of RMT into the financial markets by Laloux et al. (1999) and Plerou et al. (1999), the concept has been used in the study of the statistical properties of cross-correlations in different financial markets, [Shen and Zheng, (2009), Cukur et al. (2007) El-Alaoui, M. (2015), Leonidas and Franca, (2012), Varsha and Nivedita, (2007), Plerou et al. (2002), Chandradew and Banerjee (2015), Wilcox and Gebbie (2007), Kumar and Sinha (2007), Kim Min Jae et al. (2010) Fenn, Daniel et al. (2011), Nobi Ashadun et al. (2013), Laloux et al. (2000), Gopikrishnan, P. et al. (2001), Martin Juan et al. (2015)]. Laurent Laloux et al. (2000) opine that for financial assets, the study of the empirical correlation matrix is very relevant, since, from their finding, it is its estimation in the price movements of different assets that constitutes a significant and indispensable aspect of risk management. They declare that the probability of huge losses for a certain portfolio or option book is dominated by correlated moves of its different components and that a position which is simultaneously long in stocks and short in bonds will be risky as stocks and bonds usually move in opposite directions during crisis periods.

The interesting question that concerned investors need to answer is how (implied) volatility, which is a measure of market fluctuations, and of course market risk, affects the dynamics of the market or vice versa. It is, therefore, expedient to explore the relationship between volatility and the coupling of stocks with one another using the concept of correlation matrix, Varsha and Nivedita, (2007). Thus, correlations amongst the volatility of different assets are very useful, not only for portfolio selection, but also in pricing options and certain multivariate econometric models for price forecasting and volatility estimations. Engle and Figlewski (2014) assert that with regards to Black-Scholes option pricing model the variance of portfolio, ρ, of options exposed to Vega risk only is given by

$$Var(\rho) = \sum_{i,j,k,l} \frac{w_i w_l \Lambda_{ij} \Lambda_{lk} C_{jk}}{v_j v_k \sigma_j \sigma_k} \quad (7.4)$$

where w_i are the weights in the portfolio, C_{ij} is the correlation matrix for the implied volatility for the underlying assets and the Vega matrix Λ_{ij} is defined as

$$\Lambda_{ij} = \frac{\partial p_i}{\partial v_j} \quad (7.5)$$

with p_i as the price of option i, v_j is the implied volatility of asset underlying option j and σ_i is the standard deviation of the implied volatility v_i.

Similarly, for investors using derivatives products as a hedge on the underlying assets and for risk management, it is advisable that such investors should buy call and put options respectively for assets whose returns move in opposing directions, as may be witnessed from the calculated empirical correlation matrix. Furthermore, an accurate quantification of correlations between the returns of various stocks is practically important in quantifying risks of stock portfolios, pricing options, and forecasting. Investors that are interested in diversification of their portfolio may have to choose the assets from stocks that have negative correlation with one another in the empirical correlation matrix obtained or in the alternative investing in the stocks that have very low coefficient with the other assets that they already have in their basket of investment.

Plerou et al. (2000) note that financial correlation matrices are the key input parameters for Markowitz (1952a) fundamental portfolio optimization problem aimed at providing a recipe for the selection of a portfolio of assets, such that the risk associated with the investment is minimized for a given expected return. Edelman Alan (1988) asserts that RMT makes it possible for a comparison between the cross-correlation matrices obtained from a given number of empirical time series data for a period T with an entirely random matrix W, otherwise known as Wishart matrix of the same size with the empirical correlation matrix, to obtain some useful information about the market(s), which is necessary for portfolio optimization and risk management.

RMT predictions represent an average over all possible interactions between the constituents of the assets in a given market under consideration. The deviations from universal predictions of RMT obtained from the Wishart matrix are used in identifying the system specific, non-random properties of the system under consideration and such variations provide information about the underlying interaction of the assets. In other words, we compare the statistics of the cross-correlation coefficients of price fluctuations of stock i and j against a random matrix having the same symmetric properties as that of the empirical matrix. The RMT is known to distinguish the random and non-random parts of the cross-correlation matrix C, the non-random parts of C which deviates from RMT results is known to provide information regarding the genuine collective behaviour of the stocks under consideration and indeed the entire market at large, V. Plerou et al. (2001).

Theoretically, the comparative analyses of asset price fluctuations (hence correlation structures) between the JSE and NSM will enable us to calibrate suitable derivative models to be proposed for adoption in the NSM for portfolio optimization and risk management. This is because from the research visit embarked upon by the researchers to the Nigerian Stock Exchange (NSE) in 2014; policy makers in the NSE are taking a clue from the JSE in their proposed introduction of some pioneer derivative products and subsequently an appropriate pricing and valuation of such products in the NSM. The research into the correlations between price changes of different stocks is not only necessary for quantifying the risk in a given portfolio, but it is also of scientific interest to researchers in Economics and Financial Mathematics [G. Kim and H.M Markowitz (1989), R.G. Palmer et al. (1994)]. Interestingly, interpreting the correlations between individual stocks-price changes in a given financial market can be likened to the difficulties experienced by physicists in the fifties, in interpreting the spectra of complex nuclei. Due to the enormous amounts of spectroscopic data on the energy levels that were available, which were too complicated to be analysed through model calculations, since the nature of the interactions were not known, Random Matrix Theory (RMT) was developed to take care of the Statistics of energy levels of the complex quantum systems [Kondor, I. and Kertesz (1999), Charterjee and B.K. Charkrabarti (2006), Voit, J. (2001)].

Similarly, for financial time series in a stock exchange, the nature of interactions among constituent stock are unknown, hence the need to adopt the RMT approach in exploring these interactions between individual pairs of stocks, for use in portfolio optimization and risk management. The estimation of risk and expected returns based on variance and expected returns in a given portfolio constitutes Markowitz's model (1952b). In this Thesis, we first demonstrate the validity of the general predictions of RMT for the eigenvalue statistics of the correlation matrix and subsequently calculate the deviations, if any, of the empirical data from the Wishart matrix predictions, to identify the nature of the correlations between the individual stocks and distinguish same from those of the deviations due to randomness, in the NSM and JSE. In doing this, the period T under consideration has to be relatively large enough when compared with the number of stocks or assets being considered to minimize the noise in the correlation matrix. The two sources of noise envisaged in the use of RMT in investigating the cross-correlations of stocks in a given financial market include (a) the noise from the period length T considered with respect to the number of stock and; (b) that resulting from the fact that financial time series of historical return itself is finite or bounded thereby introducing inadvertently estimation errors (noise) in the correlation matrix, Szilard Pafka and Imre Kondor (2004).

Szilard and Kondor (2003) also observe that the effect of noise strongly depends on the ratio of stocks to the period considered, given by $r = \frac{N}{T}$, where N is the number of stocks considered and T the length of the available time series. They note that for the ratio $r = 0.6$ and above, there will be a pronounced effect of noise on the empirical analysis as was discovered by [Galluccio, G.

(1998), Plerou V. et al. (1999), Laloux, L. et al. (2000)] and that for a smaller value of r ($r = 0.2$ $or\ less$); the error due to noise drops to tolerable levels. In our case for NSM $r = \frac{82}{1018} = 0.08$ and that of JSE, $r = \frac{35}{1147} = 0.03$ thus both lying in the admissible region for the values of r. When this is done, if the eigenvalues of the empirical correlation matrix and that of the Wishart matrix lie in the same region without any significant deviations, then the stocks are said to be uncorrelated and therefore no information or deduction can be made about the nature of the market, since it is the deviations of the eigenvalues of the correlation matrix from that of the Wishart matrix that carries information about the entire market. However, if there exists at least one eigenvalue lying outside the theoretical predicted bound of the eigenvalues in the empirical correlation matrix obtained from the stock market returns, then the deviating eigenvalue(s) is(are) known to carry information about the market under consideration.

In some sense, the JSE is gradually approaching a developed market whereas the NSM is an ideal African emerging market with no known trades on derivative products currently existing in the market, unlike the JSE where trade on derivatives has been in existence for over two decades. Option contracts were introduced in JSE in October 1992, agricultural commodity futures in 1995 and a fully automated trading system in May 1996, whereas in the NSM trade in derivative products are still at the formative stage, with a recently approved derivative trade on foreign exchange future under the auspices of Financial Market Derivative Quotations (FMDQ) in 2016. As the policy makers in the NSM are benchmarking themselves on the relevant trade on derivatives in JSE towards an effective take off of derivative trade in the NSM, it is pertinent to compare the asset return correlations between the two markets, to understand the similarities and differences in the statistical properties using random matrix theory.

2. Data

The data set consists of the daily closing prices of 82 stocks listed in the Nigerian Stock Market, NSM from 3rd August 2009 to 26th August 2013, giving a total of 1019 daily closing returns after removing (a) assets that were delisted, (b) those that did not trade at all or (c) are partially in business for the period under review. The stocks considered for NSM are drawn from the Agriculture, Oil and Gas, Real Estates/Construction, Consumer Goods and Services, Health care, ICT, Financial Services, Conglomerates, Industrial Goods, and Natural Resources. For the JSE, we have a total in 35 stocks selected from Top 40 shares in the Industrial Metals and Mining, Banking, Insurance, Health care, Mobil Telecommunications, Oil and Gas, Financial services, Food and Drugs, Tobacco, Forestry and Paper, Real Estate, Media, Personal Goods and Beverages, covering the period 2nd January 2009 to 01st August 2013 covering a similar period as that of NSM (This period was chosen for the research because that was the period when we could get the complete market information for the two stock exchanges being considered).

For the values of the daily asset prices to be continuous and to minimize the effect of thin trading, we remove the public holidays in the period under consideration and to reduce noise in the analysis, market data for the present day is assumed to be the same with the previous day for cases where there are no information on trade for any particular asset on a given date. Also, we eliminate stocks that infrequently traded within the period under review. Let $S_i(t)$ be the closing price on a given day t, for stock i and define the natural logarithmic return of the index as

$$G_i(t) = \ln\frac{S_i(t+1)}{S_i(t)} \tag{7.6}$$

where $G_i(t)$ is the logarithmic return of assets in the two stock exchanges, NSM and JSE.

Computing Volatility: We calculate the price changes of assets in the two markets over a time scale Δt which is equivalent to one day and denote the price of asset i at a time t as $s_i(t)$ with the corresponding price change or logarithmic returns $G_i(t)$ over time scale Δt as

$$G_i(t) = \ln[S_i(t+\Delta t)] - \ln[S_i(t)] \tag{7.7}$$

We quantify the volatility in the respective asset return as a local average of the absolute value of daily returns of indices in an appropriate time window of T days as

$$v = \frac{\sum_{t=1}^{T-1}|G_i(t)|}{T-1} \tag{7.8}$$

To standardize the values obtained from equation (7.7) above for all values of i, we normalize $G(t)_i$ as follows

$$g(t)_i = \frac{\prec G(t)_i - \langle G(t)_i\rangle}{\sigma_i} \tag{7.9}$$

where $\sigma_i = \sqrt{\langle G(t)_i^2\rangle - \langle G(t)_i\rangle^2}$ and $\langle\dots\rangle$ represents the average in the period studied.

From real time series data, we can calculate the element of N x N correlation matrix C as follows

$$C_{ij} = \langle g_i(t)g_j(t)\rangle = \frac{\langle[G_i(t)-\langle G_i\rangle][G_j(t)-\langle G_j\rangle]\rangle}{\sqrt{[\langle G_i^2\rangle-\langle G_i\rangle^2][\langle G_j^2\rangle-\langle G_j\rangle^2]}} \tag{7.10}$$

C_{ij} lies in the range of the closed interval $-1 \le C_{ij} \le 1$, with $C_{ij} = 0$ means there is no correlation, $C_{ij} = -1$ implies anti-correlation and $C_{ij} = 1$ means perfect correlation for the empirical correlation matrix.

7.2 Eigenvalue spectrum of the correlation matrix

As stated earlier, our aim is to extract information about the cross-correlation from the empirical correlation matrix C. To this end, we are going to compare the properties of C with those of a random matrix; see [Conlon T. et al. (2007); Laloux, L. et al (2000); Plerou, V. et al. (1999); Gopikrishnan, P. et al. (2001); Plerou, V. et al. (2002)]. It can also be shown from Sharifi, S. (2004) that the empirical correlation matrix C can be expressed as

$$C = \frac{1}{L}GG^T \tag{7.11}$$

where G is the normalized $N\ x\ L$ matrix and G^T is the transpose of G. This empirical matrix will be compared with a random Wishart matrix R given by:

$$R = \frac{1}{L}AA^T \tag{7.1}$$

to classify the information and noise in the system Conlon, T. et al. (2007) and Gopikrishnan, P. et al. (2001), where A is an $N \; x \; L$ matrix whose entries are independent identically distributed random variables that are normally distributed and have zero mean and unit variance.

In our bid to use the random matrix theory in portfolio optimization and (derivative) assets risk management, we should be conversant with the universal properties of random matrices. Wilcox et al. (2007) assert that there are four underlying properties of random matrices which include (a) Wishart distribution eigenvalues from the correlation matrix, (b) Wigner surmise for eigenvalue spacing (c) the distribution of eigenvector components of the corresponding eigenvalues and finally (d) Inverse participation ratio for Eigenvector components of the resulting correlation matrix. Authors like [Dyson, F. (1971); A.M. Sengupta et al. (1999); Bai, Z.D. (1999); Edelman, A. (1988)] assert that the statistical properties of Rare known and that in particular for the limit as $N \to \infty, and \; L \to \infty,$ we have that $Q = \frac{L}{N}(\geq 1)$ is fixed. The probability function $P_{rm}(\lambda)$ of eigenvalues λ of the random correlation matrix R is given by from equation (7.2)

$$P(\lambda) = \frac{Q}{2\pi\sigma^2} \frac{\sqrt{(\lambda_{max}-\lambda)(\lambda-\lambda_{min})}}{\lambda} \tag{7.12}$$

for λ such that $\lambda_{min} \leq \lambda \leq \lambda_{max}$, where σ^2 is the variance of the elements of A. Here $\sigma^2 = 1$ and $\lambda_{min} \; and \; \lambda_{max}$ satisfy

$$\lambda_{max/min} = \sigma^2(1 + \frac{1}{Q} \mp 2\sqrt{1/Q}) \tag{7.13}$$

The values of lambda from equation (7.12) that satisfy (7.13) and (7.14) are called the Wishart distribution of eigenvalues from the correlation matrix. These values of lambda obtained from equation (7.13) as stated before determining the bounds of theoretical eigenvalue distribution. When the eigenvalues of empirical correlation matrix C are beyond these bounds, they are said to deviate from the random matrix bounds and are therefore supposed to carry some useful information about the market, Sadik Cukur et al. (2007).

The distribution of eigenvalue spacing was introduced as the required test for the case when there are not significant deviations of the empirical eigenvalue distribution to that of the random matrix prediction Wilcox et al. (2007). When the eigenvalues so obtained from the correlation matrix do not deviate significantly from the predictions of the RMT we apply the so-called Wigner surmise for eigenvalue spacing otherwise called Gaussian orthogonal ensemble Plerou, V. et al. (2002) and is given by

$$P(s) = \frac{s}{2\pi} \exp\left(-\frac{s\pi^2}{4}\right), \tag{7.14}$$

where $(\lambda_{i+1} - \lambda_i)/d$ and d denotes the average of the differences$\lambda_{i+1} - \lambda_i$ as i varies.

7.3 Distribution of eigenvector component

The concept that low-lying eigenvalues are really random can also be verified by studying the statistical structure of the corresponding eigenvectors. The $l - th$ component of the eigenvector corresponding to each eigenvalue λ_α will be denoted by, $V_\alpha(l)$ and then normalized such that$\sum_{j=1}^{N} V_\alpha^2 \,(l) = N$. Plerou, V. et al. (1999) assert that if there is no information contained in the eigenvector, $v_{\alpha,j}$, one expects that for a fixed α, the distribution of $u = V_\alpha(l)(as\ l\ is\ varied)$ is a maximum entropy. Thus, to compute the set of eigenvectors corresponding to some obtained eigenvalues of a correlation matrix, we adopt the Marcento-Pastur (1967) distribution in the theory of random matrices written as

$$p(u) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) \tag{7.15}$$

In line with the assumption of pure randomness and independence, the distribution of the components, $u_\alpha(l)$ for $l = 1, 2, 3, \ldots, N$ of an eigenvector u_α of a random correlation matrix, R should obey the standard normal distribution with zero mean and unit variance, (Guhr, T et al., 1998). The distribution so obtained from (7.15) above are expected to fit well the histogram of the eigenvector except for those corresponding to the highest eigenvalues which lie beyond the theoretical value of, λ_{max},Plerou, V. et al. (1999)

7.4 Inverse participation ratio

Guhr, T. et al. (1998) assert that to quantify the number of components that participates significantly in each eigenvector, we use inverse participation ratio (IPR). This (IPR) shows the degree of deviation of the distribution of eigenvectors from RMT results and distinguishes one eigenvector with approximately equal components with another that has a small number of huge components. For each eigenvector,v_α, Plerou, V. et al. (2002) defined the inverse participation ratio as

$$I_\alpha = \sum_{l=1}^{N} (V_\alpha(l))^4 \tag{7.16}$$

where N is the number of the time series (or the number of options implied volatility for derivative assets considered) and hence the number of eigenvalue components and $V_\alpha(l)$ is the $l - th$ component of the eigenvector, V_α. There are two limiting cases of I_α (i); If an eigenvector V_α has an identical component, $V_\alpha(l) = \frac{1}{\sqrt{N}}, then\ I_\alpha = \frac{1}{N}$ and (ii) For the case when the eigenvector V_α has one element with $V_\alpha(l) = 1$ and the remaining components zero, then $I_\alpha = 1$. Therefore, the IPR can be illustrated as the inverse of the number of elements of an eigenvector that are different from zero that contribute significantly to the value of the eigenvector. Utsugi, A. et al. (2004) in their study of the RMT assert that the expectation of the IPR is given by

$$\langle I_\alpha \rangle = N \int_{-\infty}^{\infty} [\, V_\alpha(l)]^4 \frac{1}{\sqrt{2\pi N}} \exp\left(-\frac{[V_\alpha(l)]^2}{2N}\right) dV_\alpha(l) = \frac{3}{N} \tag{7.17}$$

since the kurtosis (extreme deviations) for a distribution of eigenvector components s 3.

7.5 Empirical Result and Data Analysis

7.5.1 Eigenvalue and Eigenvector Analysis of Stocks in NSM and JSE

We took a sample study of eighty-two $(N = 82)$ stocks from the Nigerian stock exchange which gave rise to $L = 1019$ daily closing prices. For the Johannesburg stock exchange, JSE we had a sample study of thirty-five $(N' = 35)$ stocks with a total of $L' = 1148$. The theoretical eigenvalue bounds in the NSM are respectively $\lambda_- = 0.51$ and $\lambda_+ = 1.65$ as minimum and maximum values from equation (7.13) with $Q = \frac{L}{N} = 12.41$. Further from the calculation, the market value shows that the largest eigenvalue $\lambda_1 = 4.87$ which is approximately three times larger than the predicted

RMT of value (1.65). Similarly for the JSE, the theoretical eigenvalue bounds of the correlation matrix are $\lambda_{-} = 0.21$ and $\lambda_{+} = 2.37$ as minimum and maximum eigenvalues respectively, with $Q' = \frac{L'}{N'} = 32.77$. A high percentage (54%) of the eigenvalues obtained from the empirical correlation matrix of stock market price returns lie below $\lambda_{min}(\lambda_{-})$, just as obtained by Wilcox and Gebbie (2007) and this is attributable to the fact that many of the liquid stocks behave independently when compared with the rest of the market.

The empirical market value calculations show that the largest eigenvalue $\lambda_1 = 11.86$ which is five times larger than the predicted RMT value of 2.37 above. If there were no correlations between the stocks in NSM and JSE, the eigenvalues derived from the market data would have been bounded between $\lambda_{-} = 0.51$ and $\lambda_{+} = 1.65$ for NSM and $\lambda_{-} = 0.21$ and $\lambda_{+} = 2.37$ for JSE respectively. In NSM 7.3% of the eigenvalue lie outside the theoretical value and therefore contain information about the market whereas in JSE 8.57% of the total eigenvalue carry information about the entire market (see Figures (7.0) and (7.1) respectively). With these significant deviations in the empirical eigenvalue distribution from the RMT predictions, the test for Wigner surmises for eigenvalue spacing are not relevant in this case.

The average $\langle C_{ij} \rangle$ of the elements of the market 82x82 correlation matrix for the NSM is 0.041, and that of the JSE 35x35 is 0.168, showing that even though the two markets are both emerging the JSE is about four times more correlated than that of the NSM. Thus, this shows that the Johannesburg market is much more emerging than the Nigerian market, Shen and Zheng (2009). It, therefore, means that since many assets in JSE are more correlated than that of the NSM, perhaps different macroeconomic forces are driving the two markets, Fenn, D.J. et al. (2011). It is also worthy of mention that the empirical correlation matrices obtained from the two markets are positive definite since all the eigenvalues obtained are all positive.

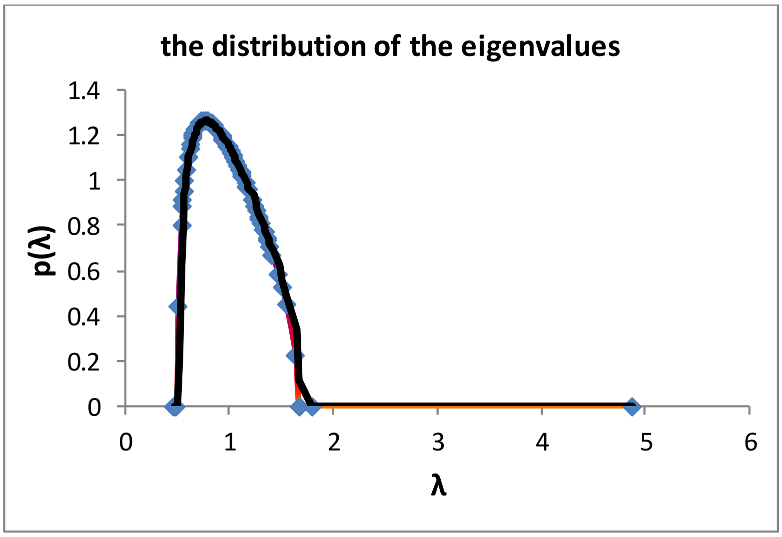


Fig. 7.0: Theoretical (Marcenko-Pastur) empirical eigenvalues for NSM (Source: Nigerian Stock Market price return 2009- 2013).

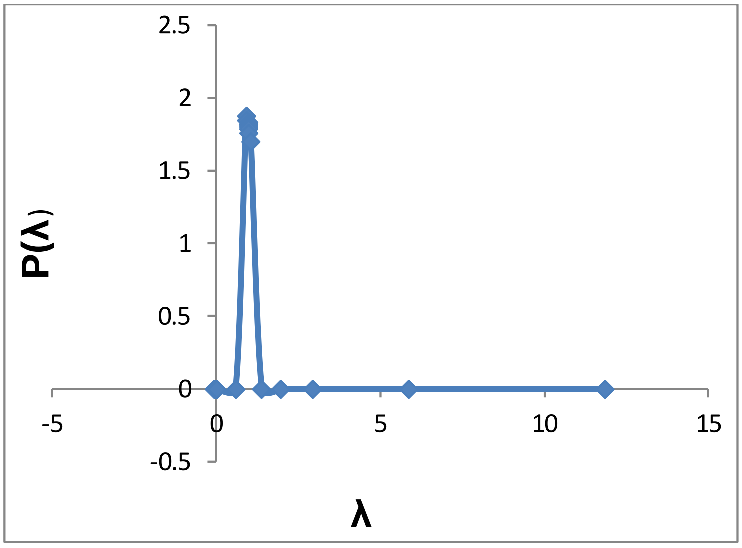


Fig. 7.1: Theoretical (Marcenko-Pastur) empirical eigenvalues for JSE(Source: Johannesburg Stock Exchange price return 2009-2013)

The comparable informative indices (7.3% and 8.6%) for NSM and JSE, respectively, suggest a similarity between the market microstructures in the system.

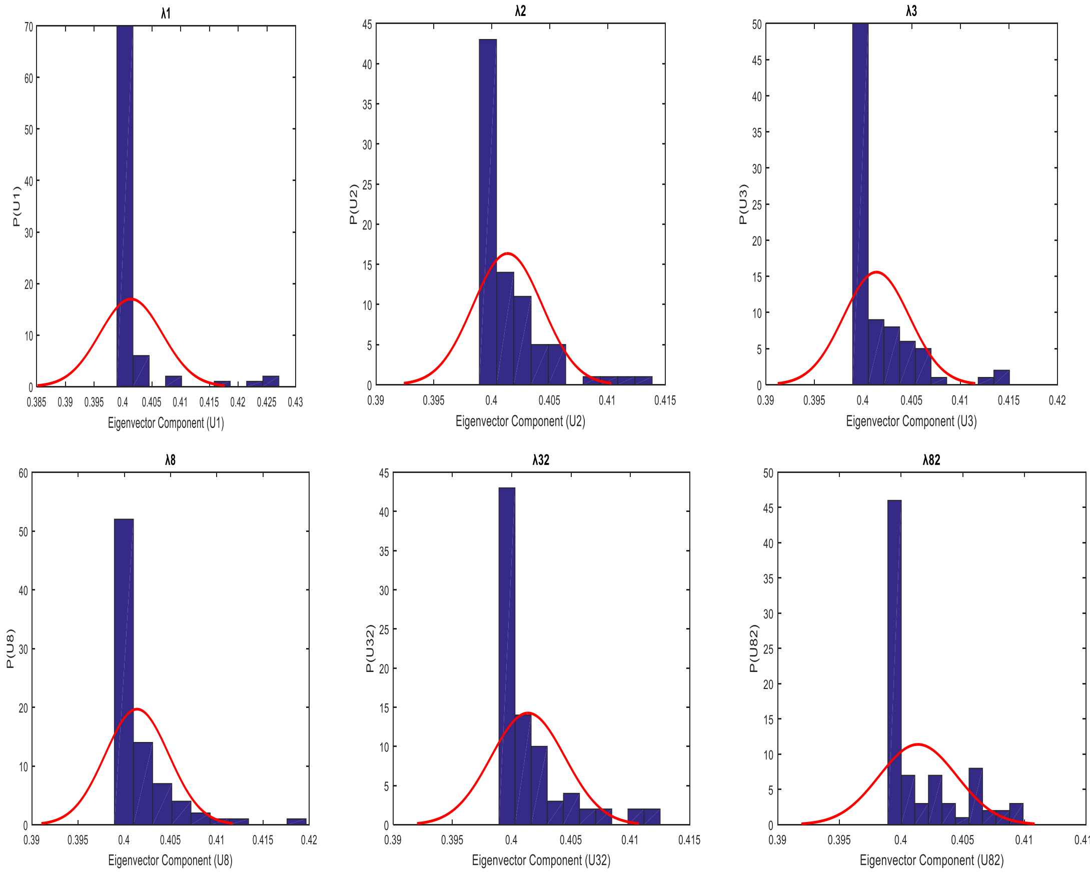


Fig. 7.2: Distribution of eigenvector components of stocks in NSM:

Figures (7.2) above represents the distribution of eigenvectors for the various eigenvalues in the empirical correlation matrix of the NSM. The eigenvector labelled U1 and U82 represents an eigenvector for deviating eigenvalue in the theoretical (hypothetical) region whereas the other 4 diagrams are the eigenvector components of the eigenvalue within the regions predicted from the Random Matrix Theory.

The overwhelming non-informativeness of the remaining 92.7% and 91.43% of the overall markets, further suggests typical random behaviour of the two markets. Typically, the distribution of the first three eigenvectors indicates the key features (mean, standard deviation and kurtosis) of a market. A look at these first three distributions for the NSM shows compared to the normal distribution, they are skewed and leptokurtic in mean and standard deviations, but fairly symmetric in kurtosis. The JSE versions portray similar non-symmetric behaviours, but fairly symmetric in kurtosis. The NSM distributions would seem to follow a beta-gamma family of distribution while the JSE ones are mostly negatively skewed, as opposed to the first two NSM distributions which are positively skewed. In general, higher-order distributions are examined for a more detailed understanding of market-dynamics, for example, market microstructure. These distributions present the same profiles as the first three distributions in the two markets, which suggest persistence of market features and the driving economic forces. Given the fact the distributions reveal the presence of market information outside the noisy RMT range; the results suggest potential market inefficiency and ability to make money from the markets. We cannot, however, say more than this regarding the stylised facts and market features, without a detailed examination of the key financial economics features typically explored in empirical finance, namely market efficiency, volatility, bubbles, anomalies, valuations and predictability.

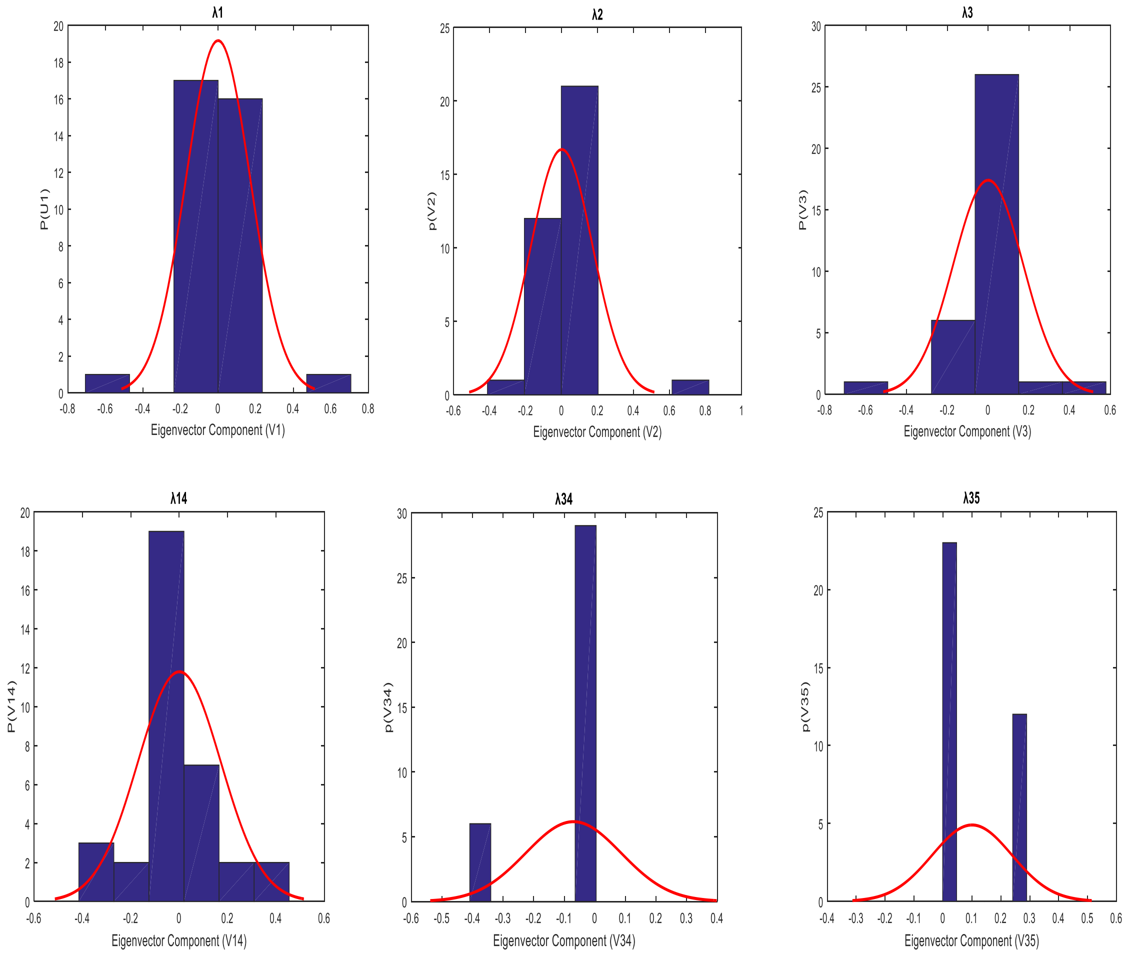


Figure 7.3: Distribution of eigenvector components of stocks in JSE:

Figure (7.3) shows the eigenvector distribution for some eigenvalues within and outside the theoretical region of the Random Matrix Theory. The last diagrams V34 and V35 represent the eigenvectors corresponding to an eigenvalue outside the region predicted by RMT which contain the information about the market. The other eigenvectors correspond to the eigenvalues due to noise as they lie in the region predicted by RMT.

The key interest in this thesis is to assess how similar the NSM and JSE are, to facilitate future modelling of as yet non-existent derivative prices in the NSM using available information on existing derivative prices in the JSE. For this, a comparative look at the two sets of eigenvector distributions suggest a flipping over or reverse dynamics in the JSE in comparison with the NSM. For example, the U2 and U_3(NSM) versus V2 and V_3(JSE) eigenvalue distributions are mirror reflections of each other. The practical implication of this reveals that different market forces seem to drive the NSM and JSE. This result is intuitively meaningful because the NSM is an oil-dependent and erratic in its price dynamics and market microstructure unlike the JSE which is mining dependent and is therefore relatively stable in nature. Consequently, attempts to model, say, non-existent derivative prices in Nigeria using existing prices in the JSE have to be taken cautiously. That said, the flipping-over features suggest that including NSM and JSE stocks in an African Emerging Markets portfolio would achieve reasonable portfolio diversification and corresponding Markowitz-style mean-variance portfolio optimization. These insights reveal the power of statistical physics tools such as RMT in peering through complex market dynamics which may not manifest with traditional mathematical finance techniques.

7.5.2 Inverse Participation Ratios (IPRs) of NSM and JSE Stocks

The inverse participation ratio (IPR) is the multiplicative inverse of the number of eigenvector components that contribute significantly to the eigenmode, Plerou, V. et al. (2002). For the largest eigenvalue deviating from the RMT bounds, almost all the stocks contribute to the corresponding eigenvector thereby justifying treating this eigenvector as the market factor. The eigenvector corresponding to other deviating eigenvalues also exhibits that their corresponding stocks contribute slightly to the overall market features in the two exchanges, NSM and JSE.

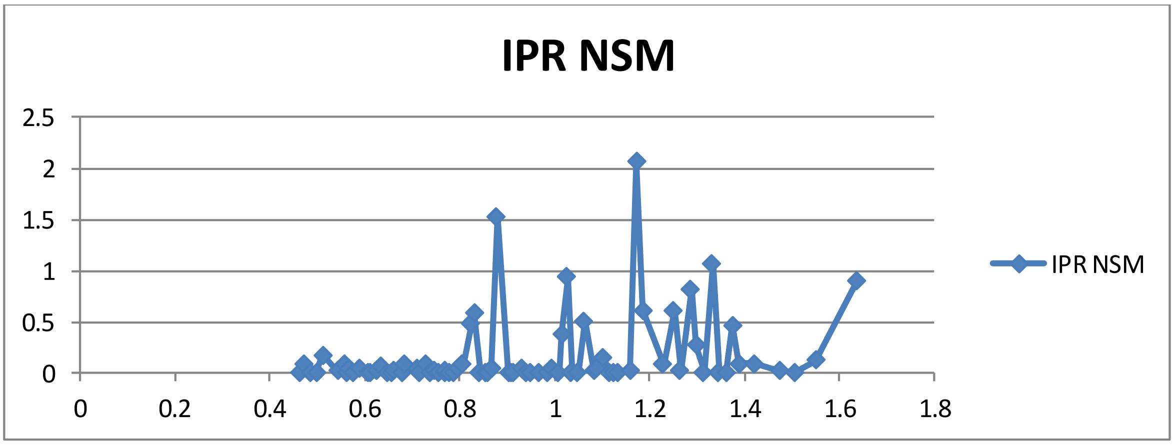


Figure 7.4: Inverse participation ratio and their ranks for NSM,

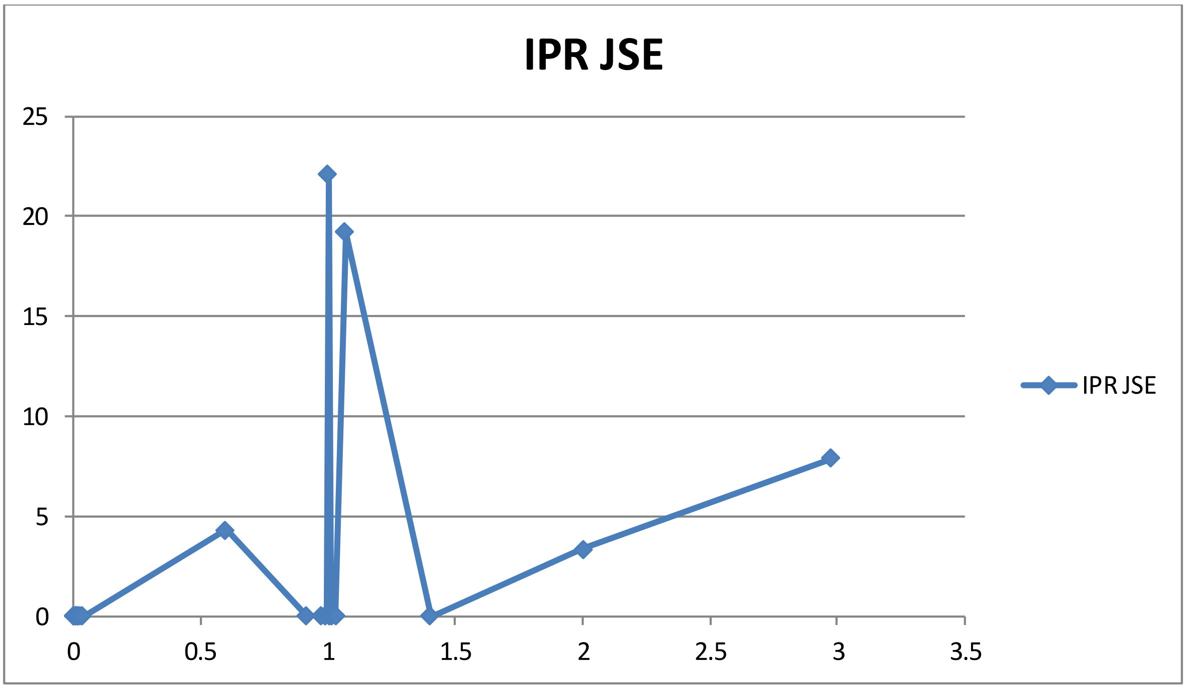


Figure 7.5: Inverse participation ratio and their ranks for JSE

The average IPR value is around $^{3}/_{82}\, for\ NSM\ \&\, ^{3}/_{35}\, for\ JSE$ respectively larger than would be expected $^{1}/_{N} = ^{1}/_{82} = 0.01 for\ NSM \&\, ^{1}/_{35} = 0.03 for\ JSE$, if all components contributed to each eigenvector, Conlon, T. et al. (2007). The remaining eigenvectors appear to be random with some deviations from the predicted value of $^{3}/_{N} = 0.04\ and\ 0.09$ respectively for NSM and JSE possibly because of the existence of fat tails and high kurtosis of the return distributions.

The lower end of JSE and the higher end of the eigenvalues for both exchanges (NSM and JSE) show deviations suggesting the existence of localized modes. It is noticeable from Figures (7.4) and (7.5) that these deviations are fewer in number for NSM than that of the JSE, which implies that distinct groups whose members are mutually correlated in their price movements are witnessed in both markets although they are more noticeable in JSE.

7.6 Limitations of the Study

It would have been preferable to use up to date data (2009-2016) for the two markets to accommodate the recent impact of oil price fluctuation on the market dynamics. This was not possible since for the NSM available data from the Nigerian Stock Exchange when this research was being carried out range from 2009-2013. The authors therefore, used this range that was available for the analysis. Strictly speaking from the point of using the results in derivative pricing, this limitation is not severe as one can forecast parts of the data that are not available or simulate alternative impact scenarios for the revealed price paths of crude oil between 2013 and 2016, for example.

7.7 Random Matrix Theory and Empirical Correlation in the Nigerian Banks

We investigate here the cross-correlation matrix C of stock index returns obtained from Nigerian banking sector for the period 2009 to 2013 using the concept of Random Matrix Theory. The eigenvalues of the empirical correlation matrix gotten from the selected bank stocks in the Nigerian Stock Exchange are tested and their respective eigenvectors used to determine which of the banks that drive the financial sector of the Nigerian Stock Market (NSM) through an analysis of their inverse participation ratios. It was observed from the empirical correlation matrix so obtained, that there are predominantly positive correlation (though not very high) among the respective stocks, meaning that the individual respective stocks although move in the same direction, are not highly positively correlated, hence the diversification method of the portfolio of assets in the banking sector in the NSM is a good investment strategy. There are some few negative coefficients witnessed in some pairs in the empirical correlation matrix involving unity bank and union bank with the rest of the other banks that were considered. From this observation, investment strategy for risk management and optimal portfolio recommendable to the stakeholders in the (NSM) who may not be interested in the only perceived possible diversification method of investment in Union/Unity banks in combination with the rest of the other assets in the banking sector is, therefore, staggering their portfolio in derivative asset in call and put options, which is being introduced into the Nigerian market.

7.7.1 Introduction of RMT to the Banking Sector in the NSM

We investigate the spectral properties of the correlation matrix of the price variations in an emerging market, Nigerian Stock Market (NSM), by scrutinizing the dynamics of bank stocks price movement and trends in the fluctuations, using the Random Matrix Theory (RMT). We examine the correlation matrix using RMT, through a comparison of the empirical correlation matrix with that of the Wishart random matrix. The linear relationships among assets in any given market is usually summarized in a correlation matrix hence the need to study RMT in any financial market(s) of interests.

Szilard Pafka and Imre Kondor (2004) contend that correlation matrices of financial returns play a crucial role in various aspects of modern finance including investment theory, capital allocation and risk management. In their view, for a theoretical perspective, the main interest in examining correlation of price returns is for proper description of the structure and dynamics of correlations whereas for a practitioner, the emphasis is on the ability of the models to provide adequate inputs towards the numerous portfolios and risk management procedures required in the financial industry. Kawee Numpacharoen (2013) observes that financial institutions usually hold multiple assets in their portfolios that may include basket of options/derivatives, credit derivatives or other correlation trading products which depend largely on the correlation coefficients between the underlying assets, hence the need to study RMT.

In this perspective, therefore, good understanding of RMT properties will provide the required theoretical backing that will enable us to propose suitable derivative pricing models to be applied in the NSM, for portfolio optimization, including risk management and appropriate pricing formulae for the proposed pioneer derivative products due for introduction in the NSM. Sensoy et al. (2013)

affirm that high correlation among stocks in any portfolio of assets means that the benefits of portfolio diversification is lowered since from their finding, high correlation is synonymous to high volatility of stock prices. In this situation therefore, the better alternative for investors is thus thinking through the derivative (option) trade as a profitable risk management process in their portfolio of investments. Therefore, it becomes imperative that one should carry out a comprehensive analysis of the nature of correlation among assets in any given financial market and thereafter relate the observed stock price dynamics and the information therein as a useful tool in the hand(s) of investors in such markets.

The corresponding market information from RMT analysis are indispensable in portfolio risk management and could also serves as guide for policy makers in the industry that aims to trade in derivative products, for example Nigeria. It is worthy of mention that following the introduction of RMT into the financial markets by R.N. Mantegna (1999), Laloux et al. (1999) and Plerou et al. (1999), RMT has been used in the study of the statistical properties and stock price dynamics of cross-correlation in different financial markets [Noh, J.D. (2000); Sharifi, S. et al. (2004), Daimov, I.I. et al. (2012); Rosenow, B. et al. (2012); Drozdz, S. et al. (2001)V. Plerou et al. (2001); Gonzalez, M.J et al. (2013); Feng, Ma et al. (2013), Potters, M. et al. (2005); Rosenow, B. et al. (2002); Kim, M. et al (2010); Nobi, A. et al. (2013)].

Laloux, L. et al. (1999) observe that for financial assets, banks inclusive, the study of empirical correlation matrix is very important, since from their investigation, the estimation of the correlations between the price movements of different assets constitutes an important and indispensable aspect of risk management. They proclaim that the likelihood of large losses for a certain portfolio or option book is dominated by correlated moves of its various constituents and that a position which is simultaneously short in bonds and long in stocks will be perilous since bonds and stocks usually move in reverse directions, especially during crisis periods. In view of this, therefore, it is the declared interest of this research to look at the financial service industry in Nigeria, particularly the banking sector through an in-depth study and analyses of correlation among bank assets being the major component in the financial service industry of the NSM and the sector that drives the economy in addition to the oil industry.

When the asset diversification approach for risk management fails as a result of high correlation among stocks, investors in the given financial market are required to use derivatives products as a hedge on the underlying assets and or for risk management and are, consequently encouraged to buy call/put options respectively for those assets whose price returns move in opposite directions as may be inferred from the calculated empirical correlation matrix. Furthermore, V. Plerou et al. Plerou, V. et al. (2000) opine that an accurate quantification of correlations between the returns of various stocks is of practical importance in quantifying the risk of portfolios of stocks, pricing of options and forecasting. They declare that financial correlation matrices are the salient input parameters to Markowitz's fundamental theory of portfolio optimization problem, Markowitz (1952a)` that aims at providing a recipe for the selection of a portfolio of assets so that the risk associated with the investment is minimized for a given expected return.

It is our goal to evaluate the correlation microstructure of the stock price dynamics for all the bank assets enlisted in the Nigerian Stock exchange. This is analogous to the method deployed by Whitehill Sam (2009) in evaluating a pricing model for credit derivatives using a full pair-wise correlation matrix based on historical asset price correlations. Instead of using just a sample of some of the stock enlisted in the NSM as we did in our earlier paper on Urama T.C.et al. (2017a,b.c), here we are interested in considering the entire correlation matrix obtained from all the bank stocks in the NSM.

Edelman Alan (1988) advocates the use of random matrix theory properties as a juxtaposition between the cross-correlation matrices obtained from a given number of empirical time series of underlying stocks data for a period T with an absolutely random matrix W, otherwise known as Wishart matrix of the same size with the empirical correlation matrix, in order to obtain some useful information about the market(s) necessary for portfolio optimization and risk management. RMT predictions represent the mean of all possible interactions between the constituent assets in a given market under consideration. The departure of the eigenvalues from universal predictions of RMT obtained from the Wishart matrix is used in identifying the system specific, non-random properties of the system under consideration and such deviations provide information about the underlying interaction of the assets. The absence of deviating eigenvalues in the region predicted by RMT means that the entire system is engulfed by noise (is random), hence no statistical inference could be drawn from the analysis.

In other words, the process is to compare the statistics of the cross-correlation coefficients of price fluctuations of stock i and j against a random matrix having the same symmetric properties as the empirical matrix. The RMT is known to distinguish the random and non-random parts of the cross-correlation matrix C and the non-random parts of C which deviates from RMT results is known to provide information regarding genuine collective behaviour of the stocks under consideration and indeed the entire market from where the sample stocks were drawn, (Plerou, V., et al. 2012).

The investigation of correlations among price changes of various assets in a given exchange is not only necessary for quantifying the risk in a given portfolio but also of scientific interest to researchers in economics and financial mathematics [Kim, G et al. (1989) and Palmer, R.G. et al. (1994)]. Nonetheless, the problem of interpreting the correlations between individual stocks-price changes in a given financial market can be likened to the difficulties experienced by physicists in the fifties, in interpreting the spectra of complex nuclei. Due to the huge amounts of spectroscopic data on the energy levels that were available which were too complex to be interpreted through model calculations, since the nature of the interactions were not known, the concept of Random Matrix Theory (RMT) was developed to take care of the statistics of energy levels of the complex quantum systems [Kondor, I. et al. (1999); Charterjee, A. et al. (2006); Voit, J. (2001)].

Analogously, for financial time series in a stock exchange, the nature of interactions among constituent stocks are unknown hence the need to adopt the RMT method in explaining the influence each individual asset has with the others within the same market. This, no doubt, will provide the desired market microstructure of stock price dynamics desired for portfolio optimization and risk management. It is, therefore, this estimation of risk and expected returns, based on variance and expected returns in a given portfolio that constitutes Markowitz's model Palmer, R.G. et al. (1994). In carrying out RMT method of portfolio optimization and risk management, the period T, under consideration, has to be relatively large in comparison with the number of stocks being considered in order to minimize the noise in the correlation matrix. The two sources of noise envisaged in the use of RMT in investigating the cross-correlations of stocks in a given financial market include: the noise from the period length T considered with respect to the number of stock and that emanating from the fact that financial time series of historical return itself is finite or bounded, thus introducing, inadvertently estimation errors (noise) in the correlation matrix Pafka, S. and Kondor (2004).

Szilard and Kondor (2003) also discover that the effect of noise strongly depends on the ratio $= \frac{N}{T}$, where N is the number of stocks considered and T the length of the available time series. They remark that for the ratio r = 0.6 and above, there will be a remarkable effect of noise on the empirical analysis, as was discovered by G. Galluccio et al. (1998); V. Plerou et al. (2000); L. Laloux et al. ((2000) and that for smaller value of r ($r = 0.2\ or\ less$); the error due to noise drops to an admissible level.

For this research, we use the empirical data obtained from NSM, with $r = \frac{15}{1018} = 0.01 < 0.2$, thus within the tolerable value of r. The N has to also be relatively large enough for the system not to be dominated by noise. For bank assets in JSE they only have 5 banks stocks times series data that are in operational in JSE, hence the Random Matrix Theory does not apply in JSE bank stocks as it will be dominated by noise hence we could not compare the banks of NSM with that of JSE separately. We therefore rely only on the dynamics and structure of the general stock market behaviour as shown earlier for the two most dominant markets in Africa.

In the following analysis, if the eigenvalues of the empirical correlation matrix and that of the Wishart matrix lie in the same region without any significant deviations, then the stocks are said to be uncorrelated and therefore no information or deduction can be made about the nature of the market. This is because it is the deviations of the eigenvalues of the correlation matrix from that of the Wishart matrix that carries information about the entire market and when there are no such deviating eigenvalues, the RMT method approach to portfolio risk analysis fails and we try another method(s). However, if on the contrary there exists at least one eigenvalue lying outside the theoretical bound of the eigenvalues in the empirical correlation matrix obtained from the stock market returns, then the deviating eigenvalue(s) is (are) known to carry information about the market under consideration, and the asset whose component corresponds with the leading deviating eigenvalue is said to drive the entire market.

7.7.2 Data on Bank Stocks

The Data set is made up of the daily closing prices of 15 bank stocks listed in the Nigerian Stock Market, NSM from 3rd August 2009 to 26th August 2013, giving a total of 1019 daily closing returns after removing assets that were delisted, that did not trade at all or are partially traded in the period under review. The bank stocks considered are Access, Diamond, Equatorial Trust, First Bank of Nigeria, First City Monument, Fidelity, Guaranty Trust bank, Skye bank, Stanbic, Sterling, United Bank for Africa, Union Bank, Unity Bank, WEMA and Zenith Bank.

We remark that for the daily asset prices to be continuous and to minimize the effect of thin trading, it is, therefore, expedient to remove the public holidays in the period under consideration, furthermore to reduce noise in the analysis, market data for the present day is assumed to be the same with that of the previous day in the cases where there are no information on trade for any particular asset on a given date.

From equation (7.1), (7.6) - (7.11) we can obtain the empirical correlation matrix for the Nigerian banks as shown below. For the analysis of the theoretical bounds of the eigenvalue spectrum with that of the empirical correlation matrix so obtained, we use equation (7.12) and (7.13).

7.7.3 Empirical Result and data analysis

	Access	Diamond	ETI	FBN	FCMB	Fidelity	Guaranty	SkyeBank	Stanbic	Sterling	UBA	Union	Unity	WEMA	Zenith
Access	1	0.2463	0.2178	0.2031	0.18	0.2008	0.1409	0.1854	0.1004	0.1383	0.2218	-0.0213	0.0692	0.05	0.2212
Diamond	0.2463	1	0.1127	0.2144	0.1374	0.3052	0.1989	0.2435	0.1506	0.1598	0.2375	-0.0296	0.0533	0.0707	0.1916
ETI	0.2178	0.1127	1	0.1402	0.0973	0.1133	0.1058	0.1405	0.0953	0.1152	0.1107	0.012	0.0625	0.0586	0.1545
FBN	0.2031	0.2144	0.1402	1	0.1566	0.1465	0.3439	0.1768	0.129	0.1363	0.283	-0.037	0.032	0.0962	0.3592
FCMB	0.18	0.1374	0.0973	0.1566	1	0.1616	0.1469	0.2119	0.1137	0.175	0.1484	-0.0476	0.1042	0.0676	0.151
Fidelity	0.2008	0.3052	0.1133	0.1465	0.1616	1	0.1701	0.2422	0.0796	0.1504	0.1909	0.0182	0.1271	0.1313	0.2514
Guaranty	0.1409	0.1989	0.1058	0.3439	0.1469	0.1701	1	0.1571	0.1174	0.1119	0.1829	0.0161	-0.005	0.0514	0.2802
SkyeBank	0.1854	0.2435	0.1405	0.1768	0.2119	0.2422	0.1571	1	0.1209	0.1325	0.2318	-0.0511	0.0553	0.1131	0.1585
Stanbic	0.1004	0.1506	0.0953	0.129	0.1137	0.0796	0.1174	0.1209	1	0.0921	0.1461	-0.0145	0.0159	0.0941	0.131
Sterling	0.1383	0.1598	0.1152	0.1363	0.175	0.1504	0.1119	0.1325	0.0921	1	0.097	-0.0391	0.072	0.0978	0.1555
UBA	0.2218	0.2375	0.1107	0.283	0.1484	0.1909	0.1829	0.2318	0.1461	0.097	1	-0.0235	0.0505	0.0835	0.2293
Union	-0.0213	-0.0296	0.012	-0.037	-0.0476	0.0182	0.0161	-0.0511	-0.0145	-0.0391	-0.0235	1	0.0195	-0.0249	-0.0158
Unity	0.0692	0.0533	0.0625	0.032	0.1042	0.1271	-0.005	0.0553	0.0159	0.072	0.0505	0.0195	1	0.1501	0.0546
WEMA	0.05	0.0707	0.0586	0.0962	0.0676	0.1313	0.0514	0.1131	0.0941	0.0978	0.0835	-0.0249	0.1501	1	0.1333
Zenith	0.2212	0.1916	0.1545	0.3592	0.151	0.2514	0.2802	0.1585	0.131	0.1555	0.2293	-0.0158	0.0546	0.1333	1

Table 7.0: Empirical correlation matrix for bankstocks in the NSM

7.7.4 Eigenvalue and Eigenvector analysis of Bank Stocks in NSM

We took a sample study of 15 (N=15) bank stocks from the Nigerian stock exchange totalling L= 1019 daily closing prices and the theoretical eigenvalue bounds are respectively $\lambda_{-} = 0.7719$ and $\lambda_{+} = 1.2575$ as minimum and maximum values with $Q = \frac{L}{N} = \frac{1018}{15} = 67.87$. Further from the calculation the market value shows that the largest eigenvalue $\lambda_1 = 3.02$ which is approximately two and a half times larger than the predicted RMT of value (1.26). The average value of $C_{i,j}$ the empirical correlation matrix above was found to be 0.18 meaning that there is higher correlation among the bank stocks. Furthermore, most of the banks are positively correlated with one another with exception of Union bank and Unity bank that are mostly negatively correlated with the rest of the bank stocks considered conforming to earlier finding by previous research that assets in the same industry should be more correlated together and that assets with high market capitalisation should be less correlated to assets with low market capitalisation.

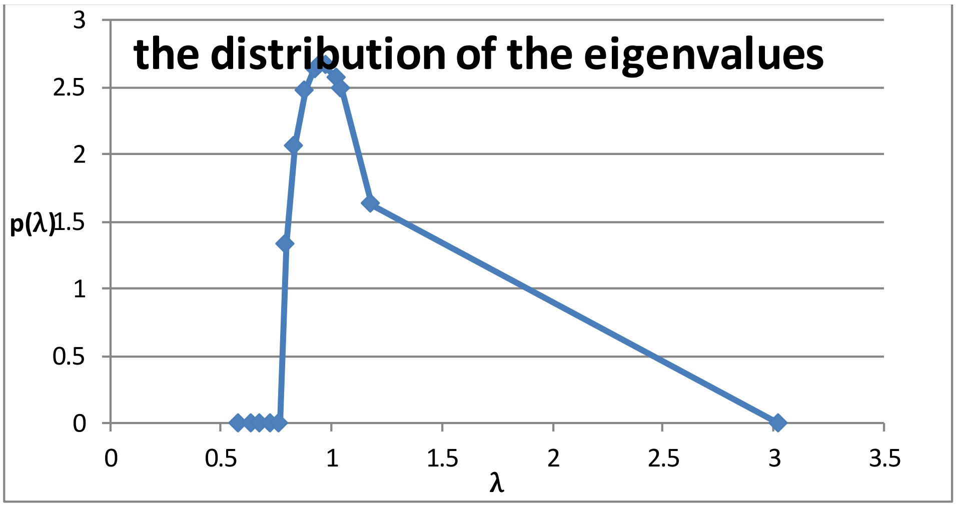


Figure 7.6: Theoretical (Marcenko-Pastur) empirical eigenvalues for banks in NSM.

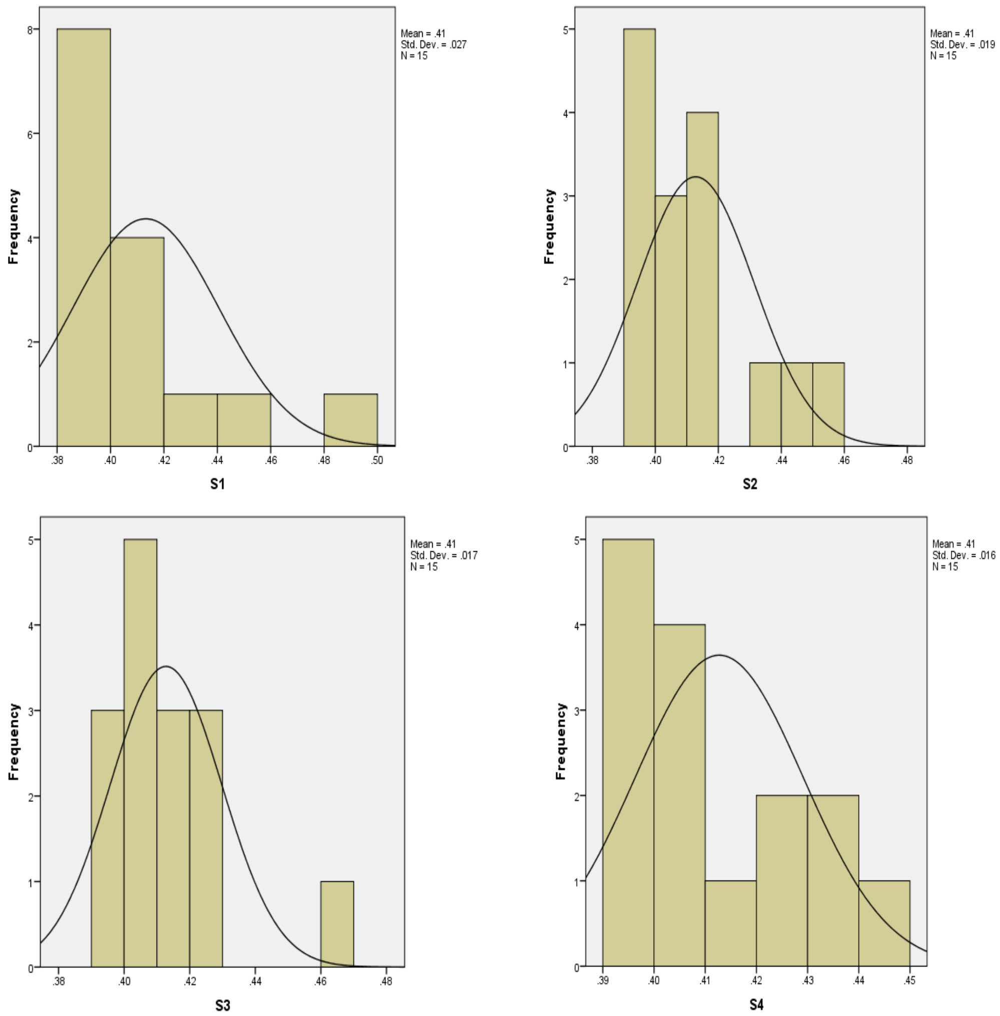


Figure 7.7: Distribution of eigenvector components of bank stocks in NSM

Figure (7.7) above presents the distribution of eigenvectors for the various eigenvalues in the empirical correlation matrix. The diagram labelled S1 represents an eigenvector component for deviating eigenvalue in the theoretical region where as the other 3 are the eigenvector components of the eigenvalue within the regions predicted from the random matrix theory.

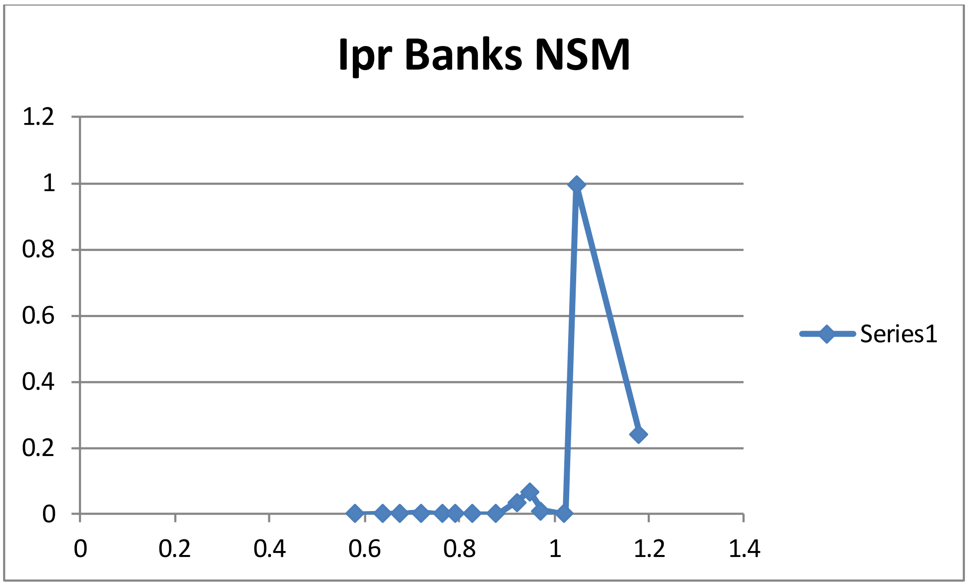


Figure 7.8: Inverse participation ratio and their ranks for NSM.

The inverse participation ratio (IPR) is the multiplicative inverse of the number of eigenvector components that contribute significantly to the eigenmode, Plerou, V. (2002). For the largest eigenvalue (deviating from the RMT bounds) almost all the stocks contribute to the corresponding eigenvector thereby justifying treating this eigenvector as the market factor. The eigenvector corresponding to other deviating eigenvalues also exhibit that their corresponding stocks contribute slightly to the overall market features in the NSM. The average IPR value is around $^3/_{15}\, for\ NSM$ larger than would be expected $^1/_N = 0.01 for\ NSM$, if all components contributed to each eigenvector, Guhr, T. (1998). The remaining eigenvectors appear to be random with some deviations from the predicted value of $^3/_N = 0.20$ possibly as a result of the existence of fat tails and high kurtosis of the return distributions.

7.8.5 Implications of the findings

The research has provided an insight into the dynamics of bank assets price correlation in the Nigerian Stock Market and consequently the information on the best risk management practices for investors in the Exchange. The empirical correlation matrix so obtained has shown that most of the bank stocks of NSM move in the same direction except the Union bank and Unity banks that have negative correlations with the other banks. For an investor in the NSM, it therefore, pays to have stakes in other non-bank stocks if he wants to diversify his portfolio in the market. It is, therefore, advisable to include derivative asset products due for introduction in the NSM to hedge against risk associated with the banking sector when the stock prices of bank assets go down.

7.9 Conclusion and hints on future work

It was observed that 6 out of 15 bank assets considered that have their corresponding eigenvalues lie outside this theoretical bound of eigenvalues, therefore, 60% of the information from the return distributions is purely random thereby leaving us with the alternative hypothesis of the RMT which states that the information on the market lies on the deviating eigenvalues. This means then that for NSM banks the true market characteristic lies with a significant number of the stocks resulting to 40% of the banks considered.

It can be observed from the correlation matrix obtained that each pairs have positive coefficients meaning that the respective stock move in the same direction as expected and that assets in the same industry should be more correlated together (Kawee and Nattachai Numpacharoen , 2013). However, as the correlation coefficients of the assets are not very high, spreading the investment portfolio within the banks is not a bad investment but one should note that diversification method within the banking sector only is not an optimal portfolio strategy. It is therefore better to invest in some derivative products like call and or put option to hedge against the risk associated with such investments.

CHAPTER EIGHT

USING RMT TO ESTIMATE REALISTIC CORRELATION MATRIX IN OPTION PRICES

8.0 Introduction

We propose here a method of finding realistic implied correlation matrix from a hypothetical portfolio of some assets of the Nigerian Stock Market using empirical correlation matrix. The empirical correlation matrix was obtained in the preceding chapter from a times series data on assets in the NSM for a period covering 2009 to 2013. Correlations amongst the volatility of different assets are very useful, not only for portfolio selection, but also in pricing of options and certain multivariate econometric models for price forecasting and volatility estimations Engle and Figlewski (2014). They assert that with regards to Black-Scholes (1973) option pricing model, the variance of portfolio, ρ of options exposed to vega risk only is given by

$$Var(\rho) = \sum_{i,j,k,l} \frac{w_i w_l \Lambda_{ij} \Lambda_{lk} C_{jk}}{v_j v_k \sigma_j \sigma_k} \tag{8.1}$$

where w_i are the weights in the portfolio, C_{ij} is the correlation coefficient between assets i and j and the vega matrix has $ij-th$ elements Λ_{ij} defined as

$$\Lambda_{ij} = \frac{\partial p_i}{\partial v_j} \tag{8.2}$$

with p_i as the price of option i, v_j is the implied volatility of asset underlying option j and σ_i is the standard deviation of the implied volatility v_i.

Kawee Numpacharoen (2012) asserts that not until recently when the financial markets world over were faced with financial crisis the comparative use of correlation testing and sensitivity analysis have always been underrated. He declares that fluctuations in correlation between different stocks in a financial market can definitely influence positions of investors concerning both market risk and credit risk.

It is noteworthy that most approaches of forecasting future correlation depend largely on the use of historical information only, but practitioners in the financial industry have to come realize that correlation actually varies through time as supported by researches carried out by Longin and Solnik (1995). To this end, it is recommendable to use the JP Morgan (1996) RiskMetrics method which is an exponentially weighted moving average correlation for forecasting correlation among stocks that takes into account the time-variability of correlation.

Furthermore, Skintzi and Refenes (2005) assert that there is a systematic tendency for implied correlation index returns to increase when the market index return go down or when there is an appreciable rise in stock market volatility, thus signifying a limited opportunity for portfolio diversification when it is needed most. They declare that one of the necessary properties required by investors to hold an efficient portfolio is the existing correlation between securities that are to be included in their portfolio and that these correlation estimates are desirable in most applications in finance including asset pricing models, capital allocation, risk management and option pricing and hedging. Thus, the study of stock price correlation in the Nigerian Stock Market is therefore desirable for proper modelling and pricing of proposed derivative products in the exchange.

It is known from our earlier study on implied volatility in chapter six of this thesis, that option prices reflect the market view and expectations which arguably contain useful information that are not included in the historical data. On the basis of this, therefore, implied correlation index otherwise called realistic implied correlation in this research will provide the market forecast of future average correlation between asset returns necessary for capital allocation and portfolio risk management in the Nigerian Stock Market. Skintzi and Refenes (2005) declare that very many option pricing formulas for instance foreign exchange options require correlation estimates, many others have used option prices to derive implied correlation measures for currency options including Lopez and Walter (2000) that derive option-implied correlation by using currency and cross-currency option data. They discover that implied correlations are essential in predicting future currency correlations. Similar to the process adopted in chapter six of this work, observed option prices are used to calculate the implied volatility by inverting the option pricing formula (Black-Scholes or other desired option pricing formula) from where we can, therefore, derive the market correlation forecast.

Kawee Numpachareon (2012) asserts that, not until recently, when the financial markets world over was faced with financial crises the comparative use of correlation testing and sensitivity analyses have always been neglected. He declares that fluctuations in correlation between different stocks in any given financial market can heavily influence positions of both market risk and credit risk.

A. Buss and G. Vilkov (2012) recall the standard Markowitz portfolio optimization result for which given a portfolio of N assets, the variance of portfolio σ_{port}^2 can be calculated using the formula

$$\sigma_{port}^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij} w_i w_j \sigma_i \sigma_j \qquad (8.3)$$

with σ_{port} = annual standard deviation or volatility of the portfolio, σ_i, σ_j = Annual standard deviation or volatility of asset $i\ and\ j$, w_i, w_j = Weights of asset i and j respectively, and C_{ij} = Correlation coefficient between asset $i\ and\ j$ with $C_{ij} = 1\ for\ i = j, 1 \leq i \leq N$.

Equation (8.3) can therefore be used in portfolio management and a portfolio that has minimum variance is said to be less risky. In our study, this can be illustrated by assigning some weight to a portfolio consisting of some stocks from NSM and then the value of C_{ij} the corresponding values in the empirical correlation matrix coefficient of the respective stocks to determine better portfolio choice(s).

As a result of symmetry of the correlation matrix, Pollet and Wilson (2010) propose an equivalent formula to that in equation (8.3) as given in equation (8.4) below:

$$\sigma^2 = \sum_{1=1}^{N} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{N-1}\sum_{j>1}^{N} C_{ij} w_i w_j \sigma_i \sigma_j \qquad (8.4)$$

When the portfolio variance at time t, given by $(\sigma_{port}^{Q})^2$ are as gotten from equation (8.4) above, Skintzi and Refenes (2005) derived the formula for computing the implied correlation index, CIX at a time t, using

$$CIX_t = \frac{(\sigma_{port})^2 - \sum_{i=1}^{N} w_i^2 \sigma_i^2}{2\sum_{i=1}^{N-1}\sum_{j>i} w_i w_j \sigma_i \sigma_j} \qquad (8.4a)$$

It can be observed from equation (8.4a) that the knowledge of the respective different pairwise correlation $C_{i,j}$ is no longer directly required, rather all we now need is portfolio and asset volatilities to obtain the required future correlation index which is useful in asset allocation and risk management when properly applied in diversification process of asset management of portfolios. Bourgoin (2001) declare that one of the notable properties of the correlation index is that, for sufficiently large portfolio, implied correlation index, CIX lies in the closed interval $0 \leq CIX \leq 1$. For any given weights and volatilities of N assets, the portfolio variance in equation (8.4a) is minimum when $CIX = 0$ and maximum when $CIX = 1$ thus giving a portfolio variance from (8.4) to be

$$\sigma_{port,min}^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 \qquad (8.4b)$$

and

$$\sigma_{port,max}^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{N-1}\sum_{j>i} w_i w_j \sigma_i \sigma_j \qquad (8.4c)$$

for minimum and maximum portfolio variances respectively.

Algebraic manipulation of equations (8.4b) and (8.4c) will transform the implied correlation index (8.4a) into the expression

$$CIX_t = \frac{\sigma_{port}^2 - \sigma_{port,min}^2}{\sigma_{port,max}^2 - \sigma_{port,min}^2} \qquad (8.4d)$$

These measurements of portfolio variances, therefore, provide a measure of the rate of portfolio diversifications. For minimum variance portfolio, the portfolio is fully diversified while in the case of maximum variance portfolio the portfolio lacks any diversification.

Black and Scholes (1973) propounded an option pricing formula which with the underlying assumptions can be used to calculate the equilibrium price of stock options. One of the assumptions of the model is the constant volatility but from evidences of implied volatility surfaces which have smiles and skews, as demonstrated in chapter 6 of this work, against a flat surface predicted by the model, the earlier constant variance assumption throughout the life span of an option, is therefore, not satisfied. Hence, implied volatility is seen not as a constant but rather a parameter that varies with respect to time to maturity and moneyness or strike price of the option Kim, M. et al. (2010).

In the Black-Scholes option pricing model, historical stock price data is used to estimate the volatility parameter which can be plugged into the model to derive the option values. Alternatively, in a bid to overcome the shortcoming witnessed from the constant volatility assumption of Black-Scholes, one may observe the market price of the option, and then invert the option pricing formula to determine the volatility implied by the option price. This market assessment of the underlying asset's volatility as reflected in the option price is called implied volatility of the option, Stewart Mayhew (1995). Given these developments therefore, the study of implied volatility and its relation to correlation matrix becomes indispensable in the exploration of methods of risk management and portfolio optimization, especially in an emerging market like the Nigerian Stock Market, where trade on derivative products are still at the formative stage. Therefore, the study of implied volatility and by extension Random Matrix Theory is very important to emerging markets including NSM for hedging currency risk, which is known to be one of the challenges to risk management faced by investors in emerging economies.

Krishnan and Nelken (2001) assert that in the recent past most large corporations are getting more interested in the use of basket options to hedge against the risk associated with their exposure to foreign currencies. The corresponding interaction between the respective currencies of interest are usually represented in a correlation matrix and through the associated correlation index; investors and entrepreneurs alike will be able to predict the degree of fluctuations in the currencies, and therefore, be able to guide against huge loses in their portfolio occasioned by the fluctuations in the exchange rate. They demonstrated that an American company that has chains of investments scattered over some Latin American countries, for instance, will be faced with the exchange rate risk in the local currencies of those countries with respect to the United States of America Dollars. Thus, if the American company expects to sell her products usually by the end of each year and in order to maintain its local production in those Latin American countries when the respective currencies of Latin American countries were to appreciate against the United States of American Dollars, the company is expected to use a basket of option on the respective country's currencies to mitigate risk associated with its investments in those countries.

It is indeed better for a company that is exposed to a variety of currency fluctuations to hedge directly its aggregate risk on their investments using basket of options than hedging individual exposures separately using call or put options. Krishnan et al. (2001) propose that the company in most cases can purchase an option on a basket of currencies at a cheaper rate that it can get through buying a combination of many separate options on the respective currencies. The price of a basket options is highly dependent on the correlation between the exchange rates, and the lesser the correlation coefficients between the currencies the lower the volatility of the basket, and consequently, the smaller the fair value of the basket of option. So, an increase in the correlation coefficient demands an increase in volatility which leads to the increase in the fair value of the basket of option and conversely, a decrease in the correlation will reduce the volatility which in turn will reduce the fair value of the basket of options.

Thus, a matrix of correlation constructed from various currencies of interest and indeed other assets could be studied through a systemic sensitivity analysis of the basket of changes in the correlation matrix for asset allocation and risk management. Therefore, the value of the constructed basket of options depends upon the correlation matrix we obtained from the historical prices of the assets or implied correlation index for the case of forecasting the asset prices by implied correlation. Krishnan and Nelken (2001) declare that more importantly the option value should depend upon future correlations which are the correlations that will actually be observed during the life of the option. This is analogous to implied volatility and historical volatility in the evaluation of underlying stock dynamics discussed earlier in this thesis. In like manner, implied correlation matrix in a basket of option is preferred to historical correlation matrix among constituent assets in the portfolio of investment and implied correlation is the correlations that will be observed during the life span of the option contract. In conclusion, Krishnan et al. (2001) declare that when the implied correlation matrix obtained from a basket of option on foreign currencies is much higher than the historical correlation matrix over any specified period of time then it is more logical to sell the basket and hedge the risk associated with such currencies using separate options on individual currencies.

8.1 Algorithm for Calculating Realistic Implied Correlation Matrix, R^Q

Kawee and Nattachai Numpacharoen [2013]defined a valid empirical correlation matrix from an nxn matrix as a matrix with the following properties: (a) All the diagonal entries must be one which is the case for the empirical correlation matrix obtained from the sample of stocks considered with the NSM in this Thesis (b) Non-diagonal entries of C_{ij} are real numbers in the closed interval $-1 \leq C_{ij} \leq 1$ (c) The empirical correlation matrix is symmetric (d) The empirical correlation matrix must be positive (semi) definite to accommodate matrix decomposition for some desired purposes like Monte-Carlo simulation Kawee Numpacharoen [2013]. They further stated that when the empirical correlation matrix are not identical as is the case with the matrix derived from the asset return distribution of stocks selected from NSM, the implied volatility of the portfolio σ^Q_{port} is given by

$$(\sigma^Q_{port})^2 = W * S^Q * C^Q * S^Q * W' \qquad (8.5)$$

Similarly, if σ^P_{port} is the implied volatility of the portfolio obtained from C^P then it can also be described as

$$(\sigma^P_{port})^2 = W * S^Q * C^P * S^Q * W' \qquad (8.6)$$

so that $$\sigma^P_{port} = \sqrt{W * S^Q * C^P * S^Q * W'}$$

where $W = [w_1 \quad \cdots \quad w_n]$ are the weights of the respective stocks in the portfolio;

$S^Q = \begin{bmatrix} \sigma^Q_1 & 0 \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma^Q_n \end{bmatrix}$ is a diagonal matrix got from the implied standard deviation of the respective assets being considered.

$C^Q = \begin{bmatrix} 1 & C^Q_{2,1} \cdots C^Q_{n-1,1} & C^Q_{n,1} \\ \vdots & \ddots & \vdots \\ C^Q_{n,1} & C^Q_{n,2} \cdots C^Q_{n-1,n} & 1 \end{bmatrix}$ is the desired realistic implied correlation matrix;

and C^P is a valid correlation matrix obtained from historical asset return correlations.

or and analogously from (8.3) we have

$$(\sigma^Q_{port})^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} C^Q_{i,j} w_i w_j \sigma^Q_i \sigma^Q_j \qquad (8.7)$$

In the same vain from equation (8.4) we can also re-write equation (8.5) as

$$\rightarrow (\sigma^Q_{port})^2 = \sum_{i=1}^{N} w_i^2 (\sigma^Q_i)^2 + 2 \sum_{i=1}^{N-1} \sum_{j>i}^{N} C^Q_{i,j} w_i w_j \sigma^Q_i \sigma^Q_j \qquad (8.8)$$

As w_i, σ_i^Q and σ_{port}^Q are all non-negative quantities, the terms $\sum_{i=1}^{n} w_i^2 (\sigma_i^Q)^2$ and $w_i w_j \sigma_i^Q \sigma_j^Q$ are also nonnegative hence any increase in $C_{i,j}^Q$ in equation (8.8) will induce an appropriate rise in σ_{port}^Q (the portfolio variance) and consequently the risk on the investment. Conversely a decrease in $C_{i,j}$ (the valid correlation matrix) will lead to a corresponding drop in the portfolio variance thereby reducing the risk associated with the respective asset portfolios. From the empirical correlation matrix C^P obtained from the 82 stocks examined from the NSM assets, which of course is not an equicorrelation matrix for the 20 stocks sampled, hence we can apply the method of Buss and Vilkov (2012).

Thus, since it is always easier to go short using derivative, we can therefore reduce the risk associated with some portfolios of investments for an increasing or decreasing correlation coefficient in an obtained valid implied correlation matrix by going short or long on the derivative products. In a like manner (as stated earlier), for a company using a basket of currency options to hedge its risk, if the implied correlation matrix obtained is substantially larger than the historical correlation matrix obtained from the constituent return on the respective currencies, then the manager is advised to sell the basket of options and go for separate call or put options to hedge his exposure to various currencies in his portfolio. However, if on the contrary the implied correlation matrix obtained is significantly less than the historical correlation matrix then the basket of option is a better risk management strategy for the company exposed to various currency risks its company is confronted with, Krishnan et al. (2001).

Kawee and Nattachai Numpachareon (2013) declare that for realistic correlation coefficient $C_{i,j}^Q$ can be written as $C_{i,j}^Q = C_{i,j}^P - \varphi(1 - C_{i,j}^P)$ where $\varphi \; \varepsilon \; (-1{,}0]$ with $C_{i,j}^P - C_{i,j}^Q$ defined as correlation risk premium of the assets under consideration. Consequently, for an nxn square matrix obtained from an empirical correlation matrix, Buss and Vilkov [2012] assert that to identify $Nx(N-1)/2$ correlations that satisfy equation (8.7), one can propose the following parametric form:

$$C^Q = C^P - \varphi * (I_{nxn} - C^P) \tag{8.9}$$

where, C^Q is the expected correlation under the objective measure and φ is the parameter to be identified.

By substituting equation (8.9) into equation (8.5) we shall have:

$$\varphi = -\frac{(\sigma_{port}^Q)^2 - W * S^Q * C^P * S^Q * W'}{W * S^Q * (I_{nxn} - C^P) * S^Q * W'} \tag{8.10}$$

As soon as we can compute the value of φ from (8.10) above, we can therefore, obtain the realistic empirical correlation matrix C^Q from equation (8.9) as it were.

and from equations (8.5) and (8.6), equation (8.10) above is equivalent to:

$$\varphi = -\frac{(\sigma_{port}^P)^2 - W * S^Q * C^Q * S^Q * W'}{W * S^Q * (I_{nxn} - C^P) * S^Q * W'} \tag{8.11}$$

Buss and Vilkov [2012] impose a restriction on the values φ to be in the region $-1 < \varphi \leq 0$for it to satisfy the technical conditions on the correlation matrix which includes that all the correlation $C_{i,j}^{Q}$, do not exceed one and that the correlation matrix is positive definite. Since $(I_{nxn} - C^{P}) \geq 0$ and to avoid the possibility of obtaining an invalid correlation matrix as a result of the value of φ that we got from equation (8.11), Kawee and Nattachai Numpacharoen [2013] propose a formula for valid correlation matrix that will take care of this shortcoming as stated below. Kawee Numpacharoen (2013) proved that given any two valid correlation matrices C and D of dimensions nxn and F a matrix of the same dimension given by

$$F = w * C + (1 - w) * D \tag{8.12}$$

then F must be a valid correlation matrix; where w (asset weight) is a real number in the closed interval $0 \leq w \leq 1$.

It therefore depends on the nature of the inequality existing between σ_{port}^{P} andσ_{port}^{Q} respectively that will inform our decision on the equivalent upper or lower bound equicorrelation matrix C to be used in obtaining a realistic implied correlation matrix. The corresponding equicorrelation matrices are represented by I_{nxn} for upper equicorrelation matrix and by a square matrix, L_{nxn} whose non-principal diagonal entries are $-\frac{1}{n-1}\ i.e\ for\ i \neq j\ and 1\ for\ i = j$ (i.e the principal diagonal entries) as the lower equicorrelation matrix.

Replacing F by C^{Q} and D by C^{P}in (8.12) we will obtain

$$C^{Q} = C^{P} + w * (C - C^{P}) \tag{8.13}$$

and from equations (8.5), (8.6) and (8.13) we shall have:

$$w = \frac{(\sigma_{port}^{Q})^{2} - (\sigma_{port}^{P})^{2}}{W * S^{Q} * (C - C^{P}) * S^{Q} * W'} \qquad (8\ .14)$$

The choice of a valid correlation matrix C_{nxn} to be substituted in equation (8.14) above in order to obtain the desired realistic correlation matrix R^{Q} depends on the results from the following steps:

Step 1: We calculate σ_{port}^{P} by using equation (8.6);
Step 2: Here we adopt Kawee Numpacharoen (2013) method for the adjustment of valid correlation matrix by Weighted Average Correlation Matrices (WACM) for the selection of lower bound matrix, L and upper bound matrix U, to obtain the realistic implied correlation matrix. The choice from either of L or U depends on the relationship between the two implied volatilities of the portfolio given by σ_{port}^{P} and σ_{port}^{Q}. If $\sigma_{port}^{P} > \sigma_{port}^{Q}$, we select the valid correlation matrix $C = L$ meaning that we adjust the valid correlaton matrix downwards to obtain the realistic correlation matrix, C^{Q}. However, if $\sigma_{port}^{P} \leq \sigma_{port}^{Q}$ we choose $C = U$ meaning that we are going to adjust the valid correlation matrix upwards to obtain the desires realistic correlation matrix, C^{Q}.
Step 3: We then compute w, from equation (8.14).

8.2 Empirical Result and Data Analysis

As stated in chapter four of the methodology for this research, we use the constructed valid empirical correlation matrix to estimate the realistic correlation matrix from a given sample of option prices. This approach is useful in assigning the respective weights to different assets in our portfolio as seen in table 8.1 below which will help in maximizing the returns and minimizing the risk on our portfolio of investments. To this effect and as an empirical demonstration, we therefore use the correlation matrix obtained from NSM stock returns on the various assets considered in the NSM from 2009 to 2013 for some selected assets. The assets are 7UP, ABCTransport, Access Bank, AgLevent, AIICO Insurance, Air service, Ashaka Cement, Julius Berger, Cadbury Nigeria Plc, CAP, CCNN, Cileasing, Conoil, Continsure, Cornerstone, Costain Construction, Courtvile, Custodian, Cutix Cables and Dangote Cement. We therefore, want to compute the realistic empirical correlation matrix for some assets already considered in the RMT before in chapter seven as below:

	A7UP	ABCTRANS	ACCESS	AGLEVENT	AIICO	AIRSERVIC	ASHAKACI	BERGER	CADBURY	CAP	CCNN	CILEASING	CONOIL	CONTINSU	CORNERST	COSTAIN	COURTVIL	CUSTODYI	CUTIX	DANGCEM
A7UP	1	-0.05084	-0.00262	0.00322	-0.00143	0.01566	-0.0029	0.041205	0.035481	0.014103	0.015787	0.010815	0.009635	0.028702	0.033092	-0.01957	-0.01251	-0.02231	0.008773	0.040708
ABCTRANS	-0.05084	1	0.056511	0.107324	0.026388	-0.00063	0.057519	-0.03212	0.016378	0.046174	0.054952	-0.00091	-0.02445	0.040527	0.027983	-0.01186	-0.03699	-0.0094	-0.05143	-0.01032
ACCESS	-0.00262	0.056511	1	0.041798	0.165593	0.005204	0.096497	0.054031	0.134552	0.03997	0.175307	0.055145	-0.04035	0.062478	-0.01304	0.062428	0.016785	0.01893	-0.04865	0.037644
AGLEVENT	0.00322	0.107324	0.041798	1	0.026699	0.001187	0.061421	0.009062	0.040987	0.019843	0.04584	0.009799	0.000529	0.005	0.031901	0.003182	0.015911	0.052039	0.00793	0.009541
AIICO	-0.00143	0.026388	0.165593	0.026699	1	-0.0073	0.079441	-0.06684	0.034305	0.035806	0.127068	0.084493	0.013289	0.052223	0.011167	0.079949	-0.03631	0.016342	0.039694	0.000341
AIRSERVIC	0.01566	-0.00063	0.005204	0.001187	-0.0073	1	0.013793	0.01008	0.019304	0.027947	0.014157	0.008397	-0.00943	-0.01882	0.037871	0.03219	0.024177	0.0165	-0.02333	-0.02294
ASHAKACI	-0.0029	0.057519	0.096497	0.061421	0.079441	0.013793	1	0.040604	0.131813	-0.01865	0.136209	0.023381	0.068711	0.041732	0.024245	0.033759	-0.05434	0.062872	-0.00385	0.051222
BERGER	0.041205	-0.03212	0.054031	0.009062	-0.06684	0.01008	0.040604	1	0.004316	-0.05637	-0.01496	-0.00019	0.003384	-0.02062	0.031925	0.001533	0.027304	0.002867	0.01409	0.045171
CADBURY	0.035481	0.016378	0.134552	0.040987	0.034305	0.019304	0.131813	0.004316	1	0.039896	0.06141	-0.02738	0.044002	-0.05896	-0.01341	0.078438	-0.00591	0.003203	0.006094	-0.00317
CAP	0.014103	0.046174	0.03997	0.019843	0.035806	0.027947	-0.01865	-0.05637	0.039896	1	0.032908	0.040034	-0.02318	-0.01104	0.011431	0.03451	0.021587	-0.00672	-0.02004	0.084764
CCNN	0.015787	0.054952	0.175307	0.04584	0.127068	0.014157	0.136209	-0.01496	0.06141	0.032908	1	0.049229	-0.06661	0.024031	-0.00524	0.040984	0.007728	0.065341	0.001566	0.030894
CILEASING	0.010815	-0.00091	0.055145	0.009799	0.084493	0.008397	0.023381	-0.00019	-0.02738	0.040034	0.049229	1	0.042932	0.032846	0.030753	0.00305	-0.01163	0.075287	-0.00421	0.037988
CONOIL	0.009635	-0.02445	-0.04035	0.000529	0.013289	-0.00943	0.068711	0.003384	0.044002	-0.02318	-0.06661	0.042932	1	0.017264	0.018679	-0.01089	-0.07688	0.059827	0.00496	0.041872
CONTINSU	0.028702	0.040527	0.062478	0.005	0.052223	-0.01882	0.041732	-0.02062	-0.05896	-0.01104	0.024031	0.032846	0.017264	1	0.07067	0.022625	0.008551	-0.08108	0.007992	0.008142
CORNERST	0.033092	0.027983	-0.01304	0.031901	0.011167	0.037871	0.024245	0.031925	-0.01341	0.011431	-0.00524	0.030753	0.018679	0.07067	1	0.016227	-0.01527	0.013627	-0.04751	0.011117
COSTAIN	-0.01957	-0.01186	0.062428	0.003182	0.079949	0.03219	0.033759	0.001533	0.078438	0.03451	0.040984	0.00305	-0.01089	0.022625	0.016227	1	-0.03288	0.030068	-0.02884	-0.0262
COURTVIL	-0.01251	-0.03699	0.016785	0.015911	-0.03631	0.024177	-0.05434	0.027304	-0.00591	0.021587	0.007728	-0.01163	-0.07688	0.008551	-0.01527	-0.03288	1	-0.00295	0.009347	0.006857
CUSTODYI	-0.02231	-0.0094	0.01893	0.052039	0.016342	0.0165	0.062872	0.002867	0.003203	-0.00672	0.065341	0.075287	0.059827	-0.08108	0.013627	0.030068	-0.00295	1	-0.00166	0.027586
CUTIX	0.008773	-0.05143	-0.04865	0.00793	0.039694	-0.02333	-0.00385	0.01409	0.006094	-0.02004	0.001566	-0.00421	0.00496	0.007992	-0.04751	-0.02884	0.009347	-0.00166	1	0.079732
DANGCEM	0.040708	-0.01032	0.037644	0.009541	0.000341	-0.02294	0.051222	0.045171	-0.00317	0.084764	0.030894	0.037988	0.041872	0.008142	0.011117	-0.0262	0.006857	0.027586	0.079732	1

Table 8.1: Empirical correlation matrix from NSM price return

8.3 Realistic Implied Correlation matrix computations:

Suppose we had the following weights and implied volatility (computed from option prices) for the under listed assets drawn from the Nigerian Stocks Market.

The hypothetical or assumed weights and implied volatilities are represented as weight $W = [.05, .08, .01, .04, .03, .06, .01, .03, .05, .07, .02, .04, .02, .07, .09, .04, .02, .07, .12, .08]$ and the corresponding implied volatility $S^Q = [.36, .26, .30, .10, .15, .20, .25, .40, .19, .24, .38, .27, .10, .22, .21, .40, .28, .30, .16, .29]'$ respectively.

Thus, with empirical correlation matrix given in table 1 drawn from stocks in the NSM we shall have

$C^P =$

1	-0.05084	-0.00262	0.00322	-0.00143	0.01566	-0.0029	0.041205	0.035481	0.014103	0.015787	0.010815	0.009635	0.028702	0.033092	-0.01957	-0.01251	-0.02231	0.008773	0.040708
-0.05084	1	0.056511	0.107324	0.026388	-0.00063	0.057519	-0.03212	0.016378	0.046174	0.054952	-0.00091	-0.02445	0.040527	0.027983	-0.01186	-0.03699	-0.0094	-0.05143	-0.01032
-0.00262	0.056511	1	0.041798	0.165593	0.005204	0.096497	0.054031	0.134552	0.03997	0.175307	0.055145	-0.04035	0.062478	-0.01304	0.062428	0.016785	0.01893	-0.04865	0.037644
0.00322	0.107324	0.041798	1	0.026699	0.001187	0.061421	0.009062	0.040987	0.019843	0.04584	0.009799	0.000529	0.005	0.031901	0.003182	0.015911	0.052039	0.00793	0.009541
-0.00143	0.026388	0.165593	0.026699	1	-0.0073	0.079441	-0.06684	0.034305	0.035806	0.127068	0.084493	0.013289	0.052223	0.011167	0.079949	-0.03631	0.016342	0.039694	0.000341
0.01566	-0.00063	0.005204	0.001187	-0.0073	1	0.013793	0.01008	0.019304	0.027947	0.014157	0.008397	-0.00943	-0.01882	0.037871	0.03219	0.024177	0.0165	-0.02333	-0.02294
-0.0029	0.057519	0.096497	0.061421	0.079441	0.013793	1	0.040604	0.131813	-0.01865	0.136209	0.023381	0.068711	0.041732	0.024245	0.033759	-0.05434	0.062872	-0.00385	0.051222
0.041205	-0.03212	0.054031	0.009062	-0.06684	0.01008	0.040604	1	0.004316	-0.05637	-0.01496	-0.00019	0.003384	-0.02062	0.031925	0.001533	0.027304	0.002867	0.01409	0.045171
0.035481	0.016378	0.134552	0.040987	0.034305	0.019304	0.131813	0.004316	1	0.039896	0.06141	-0.02738	0.044002	-0.05896	-0.01341	0.078438	-0.00591	0.003203	0.006094	-0.00317
0.014103	0.046174	0.03997	0.019843	0.035806	0.027947	-0.01865	-0.05637	0.039896	1	0.032908	0.040034	-0.02318	-0.01104	0.011431	0.03451	0.021587	-0.00672	-0.02004	0.084764
0.015787	0.054952	0.175307	0.04584	0.127068	0.014157	0.136209	-0.01496	0.06141	0.032908	1	0.049229	-0.06661	0.024031	-0.00524	0.040984	0.007728	0.065341	0.001566	0.030894
0.010815	-0.00091	0.055145	0.009799	0.084493	0.008397	0.023381	-0.00019	-0.02738	0.040034	0.049229	1	0.042932	0.032846	0.030753	0.00305	-0.01163	0.075287	-0.00421	0.037988
0.009635	-0.02445	-0.04035	0.000529	0.013289	-0.00943	0.068711	0.003384	0.044002	-0.02318	-0.06661	0.042932	1	0.017264	0.018679	-0.01089	-0.07688	0.059827	0.00496	0.041872
0.028702	0.040527	0.062478	0.005	0.052223	-0.01882	0.041732	-0.02062	-0.05896	-0.01104	0.024031	0.032846	0.017264	1	0.07067	0.022625	0.008551	-0.08108	0.007992	0.008142
0.033092	0.027983	-0.01304	0.031901	0.011167	0.037871	0.024245	0.031925	-0.01341	0.011431	-0.00524	0.030753	0.018679	0.07067	1	0.016227	-0.01527	0.013627	-0.04751	0.011117
-0.01957	-0.01186	0.062428	0.003182	0.079949	0.03219	0.033759	0.001533	0.078438	0.03451	0.040984	0.00305	-0.01089	0.022625	0.016227	1	-0.03288	0.030068	-0.02884	-0.0262
-0.01251	-0.03699	0.016785	0.015911	-0.03631	0.024177	-0.05434	0.027304	-0.00591	0.021587	0.007728	-0.01163	-0.07688	0.008551	-0.01527	-0.03288	1	-0.00295	0.009347	0.006857
-0.02231	-0.0094	0.01893	0.052039	0.016342	0.0165	0.062872	0.002867	0.003203	-0.00672	0.065341	0.075287	0.059827	-0.08108	0.013627	0.030068	-0.00295	1	-0.00166	0.027586
0.008773	-0.05143	-0.04865	0.00793	0.039694	-0.02333	-0.00385	0.01409	0.006094	-0.02004	0.001566	-0.00421	0.00496	0.007992	-0.04751	-0.02884	0.009347	-0.00166	1	0.079732
0.040708	-0.01032	0.037644	0.009541	0.000341	-0.02294	0.051222	0.045171	-0.00317	0.084764	0.030894	0.037988	0.041872	0.008142	0.011117	-0.0262	0.006857	0.027586	0.079732	1

Table 8.2: Empirical correlation matrix

The eigenvalues of the above twenty by twenty matrix designated by C^P = $[0.71, 0.79, 0.81, 0.82, 0.83, 0.86, 0.89, 0.92, 0.95, 0.95, 0.98, 0.99, 1.06, 1.094, 1.11, 1.12, 1.15, 1.16, 1.121, 1.60]'$ Thus the minimum eigenvalue of $C^P = 0.71$ which shows that C^P is a valid correlation matrix. Therefore, to estimate the realistic implied correlation matrix C^Q from the given twenty assets, we assume that the implied volatility of portfolio $\sigma_{port}^{Q} = 0.05$. Thus, by putting the implied volatilities of the respective assets in a matrix form we shall obtain, $S^Q =$

0.36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0.26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0.3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0.1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0.15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0.2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0.25	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0.4	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0.19	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0.24	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0.38	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0.27	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0.1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0.22	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.21	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.4	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.28	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.3	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.16	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.29

Table 8.3 Matrix of Implied Volatility

We now use equation (8.6) and the respective values of W, S^Q, C^P, as given above to calculate σ^P_{port}:

$$\sigma^P_{port} = SQRT(W * S^Q * C^P * S^Q * W') = 0.0361.$$

Since $0.05 > 0.0361 \rightarrow \sigma^Q_{port} > \sigma^P_{port}$, therefore we shall replace C in equation (8.14) by an equivalent identity 20x20 equicorrelation matrix to obtain the value of w:

$$w = \frac{(\sigma^Q_{port})^2 - (\sigma^P_{port})^2}{W * S^Q * (I_{20x20} - C^P) * S^Q * W\prime} = \frac{0.0016 - 0.0013}{W * S^Q * (I_{20x20} - C^P) * S^Q * W\prime} = \frac{0.0003}{-6.5737e-04} = -0.4564$$

Therefore, $C^Q = C^P + w * (I_{20x20} - C^P)$ is a twenty by twenty square matrix with I_{20x20} an equivalent identity matrix. The eigenvalues of

$C^Q =$
$[.57, .69, .72, .74, .76, .80, .84, .89, .93, .93, .96, .98, 1.09, 1.14, 1.16, 1.18, 1.22, 1.23, 1.30, 1.98]'$

from where we obtain the minimum eigenvalue to be 0.57 showing that C^Q is also positive semi-definite thus certifying the required condition for a realistic empirical correlation matrix.

8.4 Summary and Conclusion

As was stated earlier in the literature, these correlation matrices contain some relevant information for option pricing and hedging, (John Hull, 1997). The realistic implied correlation matrix C^Q has positive coefficients meaning that the respective stocks move in the same direction hence the diversification method in the portfolio is not an optimal portfolio strategy. It is, therefore, better to invest in some derivative products like call and put options to hedge against the risk on the portfolio for the hypothetical weight and implied volatility used in the estimated implied correlation matrix. The process will undoubtedly be useful in deploying derivative products for portfolio risk management in NSM when the trade on derivatives are fully operational in the Nigerian Capital Market.

We also observe that the concept of implied correlation could be used in options trading and hedging the risks associated with the portfolio of investment including the use of a basket of options in hedging foreign exchange risks. Thus, as currency option is one of the products earmarked for introduction into the Nigerian market, this research also provides some useful information on the use of basket of options and some knowledge of implied correlation index to manage foreign exchange risk in the NSM.

Chapter Nine

INTERPRETATION OF RESULTS AND DISCUSSION

9.0 Introduction

In this chapter, we interpret and discuss the research findings from the analysis outlined in the previous chapters 5-8. Chapter 5 looked at some stochastic calculus models including the seminal paper-Black-Scholes (1973) option pricing model, Ito calculus and the use of stochastic models in estimating the trajectory of stock market price dynamics. In particular, we used the stochastic calculus model from the Black-Scholes option pricing model to determine the trajectory of some assets in the Nigerian Stock Market including the bank stock - Access Bank, Nigeria, plc.

Ito calculus is known to be indispensable in the theory of derivative asset pricing. Hence, a greater part of chapter 5 was dedicated to exploitation of the algebra of Ito derivatives and integrals. Numerical approximations to stochastic differential equations using Euler-Maruyama method in stock price dynamics were computed for some chosen assets in the Nigerian Stock Market.

In chapter 6 the concept of Black-Scholes option pricing models and the application of some variants (extensions) of Black-Scholes model, including particularly the use of practitioners Black-Scholes model (Ad-Hoc Black-Scholes) were studied. Here we used Excel Visual Basics for Applications (excel VBA) programs to solve the Black-Scholes model for call options, and thereafter calculated the implied volatility model parameters. Some of the known feasible practitioners Black-Scholes were tested using some standard option prices obtained from yahoo finance, to determine their appropriateness or otherwise for pricing derivative call options, including the proposed derivative products in the Nigerian Stock Market.

By extension such models will also be suitable for pricing derivative put options using the put/call parity as shown in equation (2.4). The suitability or otherwise of several models within the categories of absolute smile and relative smile were explored using multiple regression models in combination with the excel VBA program for estimating implied volatility to determine the most suitable for derivative option pricing.

In chapter 7 emphases were on the underlying stock return for both the Nigerian Stock market (NSM) and the Johannesburg Stock Exchange (JSE). It is pertinent to note that since derivative contracts are written upon various underlying stocks and derive their value from the underlying stocks, efficient pricing and valuation of derivative products in the NSM have to reckon with SDEs that define their price dynamics. The dynamics of equal-time cross-correlation matrix of the multivariate times series is studied for the two exchanges of interest through an in-depth examination and analysis of the eigenvalue spectrum over some prescribed interval of time. The relevant information obtained from the eigenvalue spectrum of the cross-correlation matrix from the stock price return of the market being considered serves as compass with which we could view the market dynamics and compare the statistical properties for proper pricing and valuation of assets. We comment further below on the nature and heuristics of future work these RMT analyses entail practically developing suitable derivative products in the NSM, by vicariously working back from what is known in the benchmark JSE.

Finally, in chapter 8, our interest shifted to how we can use implied volatility to compute the realistic implied correlation matrix for a given set of option prices. In this context, we use the constructed valid empirical correlation matrix to estimate the realistic correlation matrix from a given sample of option prices. The rest of the chapter is organised as follows:

Section 9.1 is a quick recall of the research questions and some related study themes that will enhance easy follow through of the research. The next section discusses the result of theme 1 (stochastic calculus models). Section 9.3 is about implied volatility and the traditional Black-Scholes model and subsequently the practitioners/Ad-Hoc Black-Scholes. Immediately after this is the meaning and application of Random Matrix Theory in the study of the dynamics of stock market returns using data from Nigerian Stock Market and The Johannesburg Stock Exchange. Finally, we look at the nature of heuristics for future work with NSM in section 9.5.

9.1 Research Questions (RQs) and associated study themes

The researcher where appropriate reads the research questions in addressing the fundamental aims of the work. The research questions are as follows:

RQ1: What are the differentiating characteristics, performance trade-offs, assumptions, equations, and parameters, among stochastic calculus models used in derivative pricing, and how are the model parameters typically determined from market data?

RQ2: What is the links between the model features/derivative products and key investment objectives fulfilled by the products in financial markets, for instance risk hedging, arbitrage and speculation?

RQ3: Which stylized facts of stock markets are particularly associated with derivative pricing models, and how do they inform adaptations of these and related derivatives to the NSM?

RQ4: How do the research ideas including findings from the Random Matrix Theory apply to the NSM, for example how can the ideas be used to implement relevant experimental modelling for comparing the investment performance of selected derivative pricing models under different market scenarios in the NSM?

As clearly specified in chapter 1, there are two main aims of the research:

1) To explore the stylized facts and financial market characteristics of developing and emerging markets that will encourage derivatives trading in the Nigeria Stock market (NSM)

2) To compare these market features with those in (developed) markets with successful derivative trading, in order to develop the theoretical underpinning and some practical results in favour of trading in such derivatives in the NSM.

To achieve the aims of the research, the themes identified for the research include the following: History of derivatives trade in Nigeria; Approved Derivative products and their features; stochastic calculus models; Implied Volatility; Black-Scholes Model; Ad-Hoc Black-Scholes; Random Matrix Theory; Valid and Realistic Correlation Matrix for Option prices.

The subthemes are options, Foreign exchange options, Forwards (outright and non-deliverables), Foreign Exchange Swaps, Cross-currency interest rate swaps, Black-Scholes and its extensions, stylized facts of asset returns (volatility, implied volatility, moneyness, bubbles, market efficiency, predictability, valuation, anomalies), Wieners process, Ito calculus, numerical solution to stochastic differential/integral equations, Euler-Maruyama approximations, estimation of stock prices using Euler-Maruyama approximations.

Methods of estimating implied volatility, computing volatility, eigenvalue spectrum of the correlation matrix, distribution of eigenvector components, inverse participation ratio, Realistic correlation matrix computations.

9.2 The use of stochastic calculus models in finding the paths of assets using Monte-Carlo simulation

We studied the fundamental properties of stochastic calculus including Ito calculus properties which are the desired tools for evaluating stochastic calculus models for derivative assets price dynamics. The study also looked at the properties of Wieners process/Brownian motion as applied to stochastic calculus. As is required in the Euler-Maruyama approximations for the dynamics of asset price, the stochastic integrals and some of the relationship that exist between them and the Wieners process were considered in the work. The major concern to the researcher in stochastic calculus is some numerical solutions to stochastic differential equations and the best estimation to such equations even for non-feasible analytic solution.

To this end, the work looked at the Euler-Maruyama method for numerical approximations of stochastic integrals. The work has shown how to use Euler-Maruyama approximation in estimating stock return for an estimated mean μ and volatility δ (which is a measure of risk/variance) in a given asset. Thus, for an appropriate estimate of drift (mean) and volatility we can determine the evolution of a stock price dynamics given by

$$S(t) = S(0)exp[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma \int_0^t dw].$$

This was illustrated using some time series data of an Access bank stock price dynamics from the Nigerian Stock Market. The error associated with the Euler-Maruyama approximation was also estimated. Also demonstrated is the Langevin equation (Ornstein-Uhlenbeck process) which is applicable in modelling interest rate.

Remarks:

The Euler-Maruyama approximation could be used when there is no known analytical solution to a given stochastic process, and the process can therefore be compared with the analytical solution for situations where a solution to some stochastic process of interest to us have both numerical and analytical solutions.

For some of the differential equations we considered, it was observed that the convergence of numerical solution to the analytical solution depends largely on the choice of R. Greater accuracy are known to exist from the numerical simulation when smaller values of R are used and R as it were is defined in the relation $Dt = R * dt$ for Euler-Maruyama approximation.

It was demonstrated in the research work how to use the Euler-Maruyama to simulate future stock price for any given asset using the appropriate integral equation. In pursuance of this I took some time series data of asset price return in a Nigerian bank (Access bank) to illustrate this process.

9.3 Implied Volatility

In chapter 6 we looked at various computation techniques for obtaining implied volatility for a given set of option prices. When the desired implied volatility parameter is for a unique option price, the Excel Visual Basics for Applications (excel VBA) method of goal seek will suffice for the computation. However, in most practical computations involving implied volatility, we usually have series of call/put option prices in which case the excel VBA goal seek approach fails as it is very cumbersome to carry out the computations of the respective implied volatilities one at a time by goal seek approach. To save the computation time, other methods of estimating implied volatilities are therefore recommended which include newton Raphson and the Bisection methods.

In this research, we adopted the Bisection method in an excel VBA program environment. This approach enables us to obtain the desired Black-Scholes implied volatilities which can therefore be inserted into the Black-Scholes model for the computation of the desired call/put option prices.

The codes necessary for the Excel VBA program computations are stored in files called modules for use when desired by recalling the relevant modules by clicking on the appropriate file name(s). To address the constant volatility assumption of Black-Scholes (1973) model (this has been found to be generally untrue from this work), we examined various aspects of practitioners/Ad-Hoc Black-Scholes models under two main subdivisions: Absolute smile and Relative smiles to determine best model(s) for estimating implied volatilities. The relative smile models look at the effects of moneyness and time to maturity on implied volatility whereas the absolute smile models are concerned with the impact of the strike price and time to maturity on the implied volatility. The practitioners Black-Scholes model functions of moneyness, time to maturity and strike price considered in this work are as follows:

$$DVF_{R1}: \sigma_{iv} = a_0 + a_1(S/K) + a_2T + a_3(S/K)T$$

$$DVF_{R2}: \sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T$$

$$DVF_{R3}: \sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T + a_4(S/K)T$$

$$DVF_{R4}: \sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T + a_4T^2$$

$$DVF_{A1}: \sigma_{iv} = a_0 + a_1K + a_2T + a_3KT$$

$$DVF_{A2}: \sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T$$

$$DVF_{A3}: \sigma_{iv} = a_0 + a_1K + a_2T + a_3T^2 + a_4KT$$

$$DVF_{A4}: \sigma_{iv} = a_0 + a_1K + a_2T + a_3K^2 + a_4KT$$

$$DVF_{A5}: \sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T + a_4T^2 + a_5KT$$

The models that are suitable for pricing the derivative option based on the given option data used for this analysis are DVF_{A1}, DVF_{A4}, DVF_{R1}, DVF_{R2} and DVF_{R3} with the most appropriate of the models considered as DVF_{R3} based on the values of p, R^2 and adjusted R^2. It suffices to mention here that the estimation of implied volatility parameters by practitioners Black-Scholes model is a case of multiple regression analysis as there are more than one explanatory (predictor) variables. We show below the interpretations of the model parameter estimations and consequent meaning in relation to the predictor variables in comparison with the response variable - the implied volatility.

9.3.1 Interpretation of results

For the model $\sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T + a_4T^2 + a_5KT$ even though the R^2 value is high as high as 72% the parameter a_4 which estimates the variable T^2 does not improve the model parameter value estimation as the coefficient is 0.478 (value of the parameter) which is bigger than the admissible value of 0.05.

Next, we considered the implied volatility model given by $\sigma_{iv} = a_0 + a_1K + a_2T + a_3T^2 + a_4KT$. Here again the parameter estimation corresponding to the quadratic term in time to maturity exceeds the bound of 0.05 as its value is 0.56. Thus, the model does not best estimate implied volatility for the given set of option prices.

We considered another type of absolute smile model for implied volatility parameter estimation given by $\sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T$. The test statistics on the p-values for all the predictor variables which is a test of the null hypothesis which states that all the coefficient is all equal to zero and thus of no effect is as usual carried out. As is the practice, a low p-value (in particular, p-values less or equal to 0.05) shows that we reject the null hypothesis meaning that for the case(s) where all the predictor variables have admissible p-values and that the explanatory variables corresponding to the respective p-value(s) is(are) likely to be meaningful addition to the model.

Changes in the corresponding explanatory variables for all the p-values less than or equal to 0.05 are likely to affect the value of the response variable (implied volatility). Therefore, for the absolute smile model $\sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T$ the p-values are acceptable although the inclusion of T for estimation of the parameter a_3 is almost at the boundary of the acceptable value since as can be seen in the appendix the coefficient of the parameter (p-value) is 0.0489.

Furthermore, on absolute smile model types for implied volatility we considered $\sigma_{iv} = a_0 + a_1K + a_2T + a_3KT$, we see that the parametric estimation of the explanatory variables is within the region as $p < 0.05$ in all cases with a relatively large R-Squared value. Thus, the model is recommendable for the estimation of implied volatility for the call option data considered and consequently for some contemporary put option prices as stated in equation (2.4). We now consider another model which an increment on the parameters is above. The difference in the two parametrizations being an introduction of K^2 as an additional variable for to be estimated in the response variable (implied volatility).

For an absolute smile implied volatility model where the quadratic terms are functions of exercise price and product of exercise price and time to maturity given respectively by K^2 and KT represented by the equation: $\sigma_{iv} = a_0 + a_1K + a_2T + a_3K^2 + a_4KT$, we discover that the model best approximates implied volatility for the given set of option prices. The R-squared value is as high as 72% with all the p-values comfortably lying within the desired region of strictly less than 0.05.

Remarks:
The absolute smile models $\sigma_{iv} = a_0 + a_1K + a_2T + a_3K^2 + a_4KT$ and $\sigma_{iv} = a_0 + a_1K + a_2T + a_3KT$ are similar only that the former has more variables (predictors) than the latter. We then use the adjusted R-squared to determine which of the models that best estimate the parameters. Experience shows that an increase in the number of explanatory variables will naturally either increase the value of R^2 or keeps it value constant as it were. In this case, as the explanatory (predictors) increased from four parameters to five with an introduction of the predictor K^2 in the second implied volatility model, the R^2 increased from 58% to 72% and the adjusted R^2 also increased from 57% to 71%, which is a necessary condition for us to accept the multiple regression model that has the increased number of predictors. We also compare the percentage increase in the difference between the R^2 and adjusted R^2 in both cases. It is observable that the difference between both estimators is 1unit since there is also a corresponding increase on the value of the adjusted R^2 for the regression equation with more variable we then conclude that the additional variable improves the model parameter estimation.

For the other implied volatility model called relative smile models considered, they were found to be good estimators of implied volatility except $\sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T + a_4T^2$. The model $\sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T + a_4(S/K)T$ which has increased number of explanatory variable in comparison with the $\sigma_{iv} = a_0 + a_1(S/K) + a_2T + a_3(S/K)T$ turns out to be a better estimation of the response variable (implied volatility). The p-values in both models are very good and the R^2 in conjunction with the adjusted R^2 values represents an improvement in the model parameter estimation in the latter as there are increments in the values of both R^2 and adjusted R^2.

Lastly, from the relative smile model considered, $\sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T + a_4T^2$ which is an extension of $\sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T$, it can be seen from the summary statistics that the increment on the number of variables does not improve the parametrization as the p-values for the predictor variables T and T^2 in the new model lie outside the acceptable p-values hence we accept the null hypothesis which states that all the parameter coefficient are all zero. It is also observable that the adjusted R^2 also diminishes in value showing that the increment in the number of explanatory variables does not improve the implied volatility estimation.

Summary

Thus, for the data considered in these multiple regression analyses in implied volatility estimation, the admissible models in both relative and absolute smile models are as follows:

Absolute smile: $\sigma_{iv} = a_0 + a_1K + a_2T + a_3KT$

$$\sigma_{iv} = a_0 + a_1K + a_2T + a_3K^2 + a_4KT$$

Relative smile: $\sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T$

$$\sigma_{iv} = a_0 + a_1(S/K) + a_2T + a_3(S/K)T$$

$$\sigma_{iv} = a_0 + a_1(S/K) + a_2(S/K)^2 + a_3T + a_4(S/K)T$$

It is also observable that out of the nine models considered four from relative smile and the rest from absolute smile type of models we discovered that 75% and 40% respectively were found to good models for the parameter estimations thus showing that relative smile models are better form of practitioners/Ad-Hoc Black - Scholes model.

Recommendation: Relative smile models are preferable to absolute smile models in estimating implied volatility parameters.

9.4 Random matrix Theory

The Random Matrix Theory (RMT) provides investors with the best choice in their portfolio for derivative products and underlying assets through a spectral analysis of the dynamics of the correlation matrix obtained from the desired assets in a given market. For the assets whose returns go in the same direction, their coefficients are known to be large in the empirical correlation matrix. In order to avoid investing in stocks that have the propensity of rising/falling in price at the same time, investors are advised to, in adopting the stock diversification method of risk management choose from stocks that have least coefficient in the empirical correlation matrix or better still mixing those that have positive coefficients with other assets that have negative coefficients to guard against the risk of having all the stocks in the investor's portfolio falling in price at the same time.

Another approach is the use of derivative products where the investor mixes in his portfolio underlying stocks with high magnitude absolute value coefficients (very close to one) in the empirical correlation matrix with derivative products (for example put options) on those assets whose magnitude in the empirical correlation matrix are close to zero. The investor may also mix high positive with high negative values in the empirical correlation matrix to avoid colossal loss in their investment when prices of many assets are going down in value. This approach forms the basis of Markowitz (1952a) fundamental portfolio optimization theory aimed at providing a recipe for the selection of portfolio of assets such that the associated risk to investment is minimized for a desired expected return. We can therefore from the empirical correlation matrix so obtained propose the necessary stocks assets combination for investors wishing to diversify their portfolio for some assets, and in particular banks, within the Nigerian Stock Market.

9.4.1 Empirical correlation and general assets in NSM

Investors in underlying stocks such as flourmills are not encouraged to invest in Dangote cement, as they have high positive correlation with one another unless they want a combination of their investment in both the underlying stocks of flourmills with some derivative products possible put options in Dangote cement. If the investor has some premonition that both stocks will fall in price in the near future and he wishes to maximize his profit, he may have to add some put options to the underlying stocks he has purchased in both the Flourmills and Dangote cement stocks.

Alternatively, an investor who has some stake in Dangote cement or Flourmills should diversify his investment by complimenting his portfolio with some stocks such as Costain construction, 7Up bottling company and or Cadburys plc whose asset prices are seen to move in opposite direction with those of Dangote cement and the Flourmills. Furthermore, 7Up bottling company stock can go with any of the following stocks for any investors who may wish to diversify their portfolio in the NSM. They include: ABC Transport service, Access bank, AIICO Insurance, Ashaka cement, Costain construction, Dunlop, Guiness Nigeria PLC, JapauOil. Similarly, Conoil can also be taken along other stocks in the NSM which have negative correlation with it in the empirical correlation matrix and those stocks include assets like ABC Transport, Access bank, AirService, Costain Construction, Fidelity bank Guinness Breweries and May and Baker.

9.4.2 Empirical correlation and Bank stocks in NSM

For the Banking sector, investors who are interested in the bank stocks exclusively, have the only diversification method available to them to be the investment in any of the other banks shown in the empirical correlation matrix combined with Union bank or Unity bank stocks. Apart from Union bank stock and Unity bank all the other bank stocks cannot provide any diversification method for hedging the risk associated with the investments for investors whose portfolio are comprised of only bank stocks in the NSM as they are seen to have similar coefficients in the empirical correlation matrix coefficient in confirmation of the saying that assets in the same industry should be more correlated, (Kawee and Nattachai Numpacharoen, 2013). Thus, by this observation it means that the bank stocks in the empirical correlation matrix obtained from the NSM bank move in the same direction as expected.

9.4.3 Eigenvalue Analysis and average values of empirical correlation matrices

The eigenvalues obtained from a comparison of the corresponding Wishart matrices with that of the empirical correlation matrices in both exchanges have some information about the stock market dynamics as there are some significant deviations of the $\lambda's$ from the theoretical bounds. These deviations obtained from the eigenvalue analysis of correlation matrices signify that we can deduce some information from the covariance matrices as they are not dominated by noise. In the NSM stocks, the largest eigenvalue $\lambda_i = 4.87$ a value which is approximately three times larger than the predicted RMT value of 1.64. Laloux et al. (2000) assert that the smallest eigenvalues of the correlation matrix are the most sensitive to noise in the system and the eigenvector corresponding to the smallest eigenvalue are precisely those ones that represent the least risky portfolio for the assets considered. More so, about 5% of the eigenvalues exceed the upper theoretical bound of the eigenvalue representing mostly the oil sectors and bank stocks which are the key assets in the NSM and known to be the drivers of the entire economy. This means that 5% of the stocks carry information about the market (NSM) signifying that investors in the NSM can maximize their expected returns as they minimize the risk associated with their portfolios by properly scrutinizing the market features of the driving forces in the Nigerian economy which include the oil and banking industry in addition to other market strategies for risk management which include diversification of portfolios that they may wish to adopt.

For the JSE, the largest eigenvalue obtained from the empirical correlation matrix has a value of 11.86 which is five times larger than the RMT prediction of 2.37 units, thus showing that the stocks in the JSE and indeed in the two exchanges have some similar characteristics. 9% of the eigenvalues are higher than the predicted RMT values for the largest eigenvalue bound and it is the eigenvectors corresponding to these stocks that drive the market in the Johannesburg Stock Exchange. The corresponding stocks represented by those eigenvalues (large) are mostly from the mining sector and banking in the JSE stocks.

The average $\langle C_{ij} \rangle$ of the elements in the market empirical correlation matrices in both markets are 0.041 and 0.168 for NSM and JSE respectively showing that although both markets are emerging, assets in NSM are about four times more correlated than those of the JSE which implies that Johannesburg market is much more emerging that the Nigerian market, (Shen and Zheng, 2009). The implication of this result suggests that different macroeconomic forces are driving the two markets, Fenn, D.J. (2011), hence policy makers and investors alike in the Nigerian Stock Market should be wary of this fact especially as it concerns the interest of policy makers in NSM towards adaptation of the derivative products perceived to be working well in the JSE into the NSM.

Another important feature observed from the two markets which may serve as some source of joy or succour to investors in the NSM, trying to mimic successful products in the JSE into NSM is the fact that it was discovered that in the volatile periods, average value of $\langle C_{ij} \rangle$ are observed to be highest in the two markets which also agrees with the observations of Plerou, V. et al. (2001) for matured markets.

It was observed that a very high percentage 54%of the eigenvalues from the empirical correlation matrix of stock price return analysis lie below λ_{min} for JSE confirming the report earlier obtained by Wilcox and Gebbie (2007) which is attributable to the fact that many of the liquid stocks behave differently when compared with the rest of the assets in the market. However, for the NSM only 3.7% lie below the λ_{min} meaning that in contrast to the JSE, NSM liquid stocks behave in similar way when compared with rest of the assets in the Stock Market, (see fig. 7.0 and fig. 7.1). This, therefore, is another source of dissimilarity observed between NSM and JSE. For the socks larger than the theoretical predicted bounds of RMT, λ_{max}, we have a total of 4.9% in the NSM whereas that of the JSE there were a total of 8.6% of the eigenvalues lying above the predicted bound of the RMT which represents some measure of similarity in both exchanges. As it is this upper bound deviation of the eigenvalues that drive the market(s) it is worthy of note to the policy makers the comparable or same (fewness in number) of the assets driving the markets should be seen as a source of similarity in modelling the derivative products for NSM through successful products in the JSE.

9.4.4 Eigenvector Analysis

From the eigenvector distribution of the two exchanges, it is observed that for NSM many of the stocks do not move in the same direction of the dominant stocks - banks and oil industry unlike the JSE where almost all the stock move in the same direction with the dominant stocks (see fig.7.2 and fig.7.3) respectively. The overwhelming non-in formativeness of the remaining 92.7% and 91.43% for NSM and JSE respectively of the overall market from the eigenvalue range of values represented by the eigenvectors further suggests typical random behaviour of the two markets. However, the NSM assets price return dynamics is more random that that of the JSE based on the volume of assets that lie outside the predicted regions of RMT eigenvalue spectra. Moreover, further look at the distribution of first three eigenvectors in the NSM indicate the key features of mean, standard deviation and kurtosis of the markets. In comparison with the properties of normal distribution, stocks in the NSM are therefore seen to be skewed and leptokurtic in mean and standard deviation but fairly symmetric in kurtosis. The JSE stocks exhibits similar no symmetric behaviours although they are fairly symmetric in kurtosis. Thus, NSM appear to follow a beta-gamma family of distribution that are positively skewed as opposed to the JSE that are negatively skewed.

The overall analysis of the eigenvectors spectra in the two exchanges show that they have the same profile from the first few eigenvectors which suggest that there are persistence of market features and similar underlying driving economic forces. The RMT ability to reveal the fact that there exists market information outside the RMT range notwithstanding; the results suggest potential market inefficiency and ability to make money arbitrarily from both markets. A further comparative analysis of the two sets of eigenvectors distribution for NSM and JSE respectively suggest a flipping over or reverse dynamics in assets from JSE when compared to those in NSM for example U_2 and V_3 are minor reflections of each other. This feature is intuitively meaningful since the NSM is an oil-dependent and erratic in its price dynamics and market microstructure whereas for JSE which is mining dependent, and therefore, has a price regime that is relatively stable in nature. Thus, the declared interest of policy makers to model the non-existent derivatives pricing in Nigeria by adopting the existing products and price mechanisms in the JSE should be treated cautiously.

9.4.5 Inverse participation Ratio (IPR)

The average IPR value is around ${}^{3}/_{82} = 0.04$ for the Nigerian stock market which is larger than what is expected 0.01 whereas average IPR for JSE is ${}^{3}/_{35} = 0.09$ which is also greater than the expected IPR of 0.03 for all the components of the eigenvectors to contribute equally to the market mode, (Conlon et al., 2007). The distinction between the average IPR and expected IPR for both markets are as a result of the existence of fat tails and high kurtosis in the distributions probably due to noise in the system, see figures 7.4 and 7.5 for detailed illustration of the IPR.

9.4.6 Stock price return dynamics and analysis in Nigerian Banks using RMT

The cross-correlation matrix for JSE banks was found to have no information for the period under review. This is as a result of the fact that there are only five (5) bank stocks that have comprehensive market return for the period considered thereby making the system to be dominated by noise. The ratio, $r = \frac{N}{L} = \frac{5}{1148} = 0.004$, for bank stocks in the JSE is very infinitesimal; hence the empirical correlation matrix is dominated by noise. Thus, we will only evaluate the dynamics of bank stock returns for NSM whose ratio, r is large enough and can therefore not be dominated by noise. We considered fifteen (15) Nigerian bank stock price returns using the Random matrix theory. It was observed form the empirical correlation matrix so obtained that there are predominantly positive correlations (though not very high) and thus offers few opportunities for diversification among the bank assets for investors interested in spreading their portfolio among different bank assets in the NSM, (especially the Unity Bank and Union Bank).

Joost Driessen et al. (2009) assert that a market wide increase in correlations negatively has negative effects on investors' choices as it lowers the diversification benefits and from their findings, investing on solely underlying stocks with high correlations may be expensive. Thus, the surest alternative left to investors for cases of very high correlations among constituent underlying stocks, is therefore, taking some stakes in the derivative products, hence the need for Nigerian policy makers to expedite action on full implementation of the new derivative products earmarked for introduction into NSM, to avail its vast investors and other stakeholders the opportunities available to participants who trade on derivative products.

Moreso, since the bank stocks studied are not highly positively correlated coupled with the fact that unity bank and union bank are mostly negatively correlated with the rest of the other banks, it implies that diversification method of risk management could be adopted by the investing public in the NSM who wants to have most of their portfolio comprised of the bank stocks. It is also pertinent to mention here that with the upcoming derivative products into the Nigerian capital market, some call options in some assets and put options in others for the bank stocks in Nigeria could also be explored by investors to hedge risks associated with their portfolio of investments in the Nigerian bank stocks. Finally, we also note that when the asset diversification approach for risk management fails consequent upon an obtained high correlation between the respective stocks both for banks or any other assets, investors are then required to adopt investing on derivative products as a hedge tool or other forms of risk management on the underlying assets by using call/put options for assets whose price returns move in an opposite direction in the calculated empirical correlation matrix.

9.4.7 Realistic Implied Correlation matrix

We looked at the application of Random Matrix Theory in derivative asset pricing especially the option pricing. The research in this chapter further showed how to measure the risk in a given portfolio using Black-Scholes option pricing model when the assets in the portfolio of investments are exposed to Vega risk. Vega as stated earlier is the change in options price for a percentage change in volatility which like the Delta and Gamma is used in hedging risks in asset securities. For derivative products, we have implied correlation which is derived from option premium (the call and put option) and we recall that the premium paid in any given contract on an underlying asset is a measure of risk associated with holding some contract on that asset. Thus, by comparing the implied volatilities as reflected in implied correlation index, we are measuring indirectly the risks inherent in the underlying assets that bear the corresponding derivative contracts.

Kawee and Nattachai Numpacharoen (2013) observe that the implied correlation index is a measure of diversification level among index constituents that are being considered. Also, Skintzi and Refenes (2005) assert that for stocks whose implied correlation index decrease provide an opportunity for investors to diversity their portfolio on the constituent stocks but the method of risk management through diversification are discouraged when the implied correlation index increases or when there are appreciable rise in the stock market volatility. Bourgoin (2001) also provided an implied correlation index formula to measure portfolio variances and went further to illustrate how we can use same to gauge the rate of asset diversifications. He concludes that portfolio with minimum variance are known to provide full diversification opportunities while those with maximum variance lack any opportunity for asset diversification.

We also studied in chapter eight how Nigeria can solve its foreign exchange risk through the knowledge of implied correlation index for assets especially foreign currencies. Nigeria, like most emerging markets are continually faced with exchange rate risk occasioned by erratic fluctuations in its national currency - the naira as against currencies of developed economies like USA, UK, China and other European economies who are their trading partners. We therefore looked at measure to reduce these risks of exchange rate fluctuations through the use of implied correlation, which, no doubt, will help investors in the NSM manage effectively the risk associated with trading the naira with other currencies of the rest of the world. Walter and Lopez (2000) discover that option-implied correlations are essential for predicting future correlations using currency and cross-currency option data.

Furthermore, we also considered how to hedge the risk associated with foreign currencies, which is one of the derivative products Nigeria is introducing into her capital market from the perspective of Krishnan and Nelken (2001). They provide an algorithm termed currency triangle that could be used to predict the degree of fluctuations in some foreign currencies of interest to guide against huge losses on investors that are continually faced with foreign exchange fluctuations in a mixture of foreign countries where they have much stakes. Their approach provides a good direction on how to use basket of options to hedge conglomeration of foreign currency risks as against using separate options to hedge various risks associated with the currencies the manager/investor is faced with. The choice of whether to use the basket of options for the conglomerate of foreign currencies or individual options on the respective foreign currencies exposures depends on the future correlations of the respective currencies, and these future correlations themselves depend on the correlations that will actually be observed during the life span of the option. This technique will however be very necessary to investors and other stakeholders in the Nigerian capital market, especially when the derivative products trade becomes fully operational in the NSM.

We used in this research work the method of BUSS and Vilkov (2012) to find the realistic implied correlation matrix through an adjustment of correlation matrix obtained from the empirical correlation that existed with some selected stocks in the NSM. This adjustment is necessary so as to eliminate the risk of obtaining an invalid realistic correlation matrix based on the existing implied correlation of portfolio of the underlying assets. Kawee Nampacharoen (2013) declare that the choice and nature of the adjustment of the correlation matrix (for upwards or downward adjustment) depends largely on the existing implied volatility of the portfolio as stated in chapter 8.

9.5 The nature and heuristics of future work which these RMT analyses entail

We now present a vista of future work which will use the above results to aid the development of useful derivatives in the NSM. For this, we recall the following comments earlier made in this chapter:

'We comment further below on the nature and heuristics of future work which these RMT analyses entail for future work on practically developing suitable derivative products in the NSM, by vicariously working back from what is known in the benchmark JSE'.

In a related paper submitted recently to the Central Bank of Nigeria Journal of Applied Statistics (CBN JAS), we note as follows.

9.5.1 Some notes on heuristic modelling of JSE-NSM asset and derivative price dynamics

The management of the Nigerian Stock Exchange (NSE) indicated in a meeting with the researchers in 2014 that: the NSM was interested in using derivative products to deepen the markets; enable such products to play traditional roles in risk hedging, speculation and arbitrage; and successfully benchmark its performance on existing JSE derivatives, given the relatively more advanced status of the latter.

Hence, the heuristics aims to combine JSE derivatives data with broader NSM stylized facts and characterisations, especially based on Random Matrix Theory (RMT), to simulate plausible derivative models and prices that will fit the Nigerian stylized facts and RMT results better. For example, to cover the essential scope in this initial modelling of derivatives in the NSM, we will look at key sectors and products that will be more useful for achieving the stated derivative modelling objectives – risk hedging, speculation and arbitrage – especially those which the NSE management mentioned that major NSM investors are interested in. As mentioned in the introduction to this Thesis, these products include currency options, cross-currency swaps, deliverable and non-deliverable forwards. Also, important market sectors in these considerations are banking and financial services (as in this paper), energy and (agricultural) commodity derivatives such as futures, because of the strategic relevance of energy and agriculture sectors in the Nigerian economy. For instance, banking and financial services are fundamental sources of development finance for investors (households, firms and government). Oil and gas provide the energy inputs into manufacturing and production of goods and services, and revenues for Nigeria, and agricultural products support other industries.

The ingredients for the derivative pricing heuristics are the correlation structures from RMT analyses and Black-Scholes derivative pricing models, observed stylized facts of underlying asset prices, and implied volatility dynamics in the JSE. These facts will be categorised as Generalised Stylised Facts (GSFs) and Implied Volatility Stylised Facts (IVSFs). Sequential modelling in form of models M1, M2, M3, for example, will exploit the JSE data, based on comparative analysis of the performance of selected derivative models against the standard BS model, for suitable derivative products mentioned above. The reason for this approach is to understand which BS models or extensions of the BS model are typically used in the JSE for specific asset prices, and whether the derivative prices from competing models are more accurate than the ones used.

This knowledge will be very useful to NSE management, as they optimise the decision choices facing them in introducing derivative products and models in the NSM. It will also be useful to the JSE management, if they become aware that existing models used in pricing JSE derivatives are not as good as alternative models revealed by this research. This, therefore, will be a crucial contribution of the heuristics to knowledge.

We, however, recall that the underlying asset prices are available in Nigeria, but not derivative prices. Given that derivative models are common theoretical knowledge across the two markets, we represent the Nigerian information as NBS for BS model, NGSFs for General Stylised Facts (GSFs), NUAPs for underlying asset prices. We use the known NGSFs in Nigeria to estimate the unknown Nigerian implied volatility stylised facts (NIVSFs). Similar notations are adopted for JSE by replacing by S. The key research question now is: How do we overcome the lack of research data on the IVSFs which underpin derivative pricing in the NSM?

In brief, the following steps are involved: a) compare the stylised facts (GSF information) on underlying prices for JSE and NSM, to gauge how close the two data sets are in behaviour (using, say, the first four moments and distributions of the data sets); b) explore the correlation or heuristic links between the full data on SGSFs and SIVSFs in South Africa, and across Nigeria and South Africa; c) infer therefore the likely range of values for the unknown Nigerian IVSFs; d) run RMT analyses on asset prices from key market sectors in NSM and JSM (for example selected banks, oil and gas, commodities), to characterise mainstream tendencies in the markets, and further refine the initial correlational and heuristic links in b) above; e) using knowledge of the RMT comparisons, simulate plausible data that fits the Nigerian modelling scenarios and repeat the sequence of modelling M1, M2, M3, …, on the data (Voss, 2014). This will produce indicative results which will inform possible decisions on the models and projected prices that could obtain in the NSM, under different modelling. A schematic illustration of these heuristics is presented below.

9.5.2 Further description of the heuristics

This section explains the strategy for modelling the as yet non-existing data on derivative pricing in Nigeria. Heuristics generally refers to the use of creative common-sense reasoning to perform tasks that ordinarily would have been (near) impossible to do. This impossibility trait explains why there is no known result on derivative trading and pricing in Nigeria because qualified financial engineers argue that there is no historical data to work with. We note again that successful application of this strategy in future work will constitute a novel methodological, theoretical and practical contribution of the research to knowledge.

It should be noted that management of the Nigerian Stock Exchange (NSE) indicated in a meeting with the researchers in 2014 that the NSM was interested in using derivative products to deepen the markets, and at the same time enable such products to play traditional roles in risk hedging, speculation and arbitrage. The NSE management also noted that the NSM is benchmarking its performance on the Johannesburg Stock Exchange (JSE), given the relatively more advanced status of the JSE.

The researcher knows that the JSE has been trading on different types of derivatives. Hence, the purpose of this statement of strategy is to explain how existing knowledge of derivatives in the JSE will be combined with analysis of broader stock market features (stylized facts) and characterisations, especially based on Random Matrix Theory (RMT), to simulate plausible derivative models and prices that will fit the Nigerian data (stylized facts and RMT results) better. The following schema in Figure 9.1 explains visually the steps involved in this strategy.

Products sectors/models	Information in Johannesburg (South Africa) Stock Market (JSM)	Information in Nigerian Stock Market (NSM)
	BS Stock Derivatives **GSFs** **IVSFs**	BS GS GD **GSFs** **IVSFs?**
Models	**Steps in the modelling** 1. Test Black-Scholes alternative derivative pricing models on some South African data 2. Use known model assumptions and data behaviour (stylized facts) to obtain M_1, M_2, M_3 models 3. Run suitable RMT analysis 4. If possible ascertain underpinning data distributions 5. Determine optimal models from M_1, M_2, M_3	**Steps in the modelling** 1. repeat Random matrix theory analysis on similar NSM data as in JSM 2. Fit suitable distribution to NSM data (Generalized distribution, (GD and Generalized stylized facts, GS) 3.Compare 1 and 2 with JSM results and simulate likely Implied volatility surfaces (IVSFs) 4. Use insights from 1 - 3 above to simulate corresponding NSM data and M_1, M_2, M_3 models from Nigeria.

Figure 9.1: A visual schematic for comparative modelling of derivative in JSM and NSM

9.5.3 Discussion of the key steps stated in Figure 9.1

The figure is presented as a quadrant with one half representing the Nigerian side of the intended analysis, and the other side the South Africa side. Information from the South African side will underpin the specific (simulated data modelling in Nigeria).

Column 1 of the figure summarises the nature of products of interest. For example, to cover the essential scope in this initial modelling of derivatives in the NSM the researcher could look at key sectors and products that will be more useful for achieving the stated derivative modelling objectives – risk hedging, speculation and arbitrage – especially those that the NSE management mentioned that major NSM investors are interested in. These products include currency options, cross-currency swaps, deliverable and non-deliverable forwards. Also, important in these considerations are energy and (agricultural) commodity derivatives such as futures, because of the strategic relevance of energy and agriculture sectors in the Nigerian economy. For instance, oil and gas in addition to providing the energy inputs into manufacturing and production of goods and services, are key revenue earners for Nigeria and agricultural products support other industries. Another key sector of potential interest in this work, especially in connection with RMT characterisation is the banking sector, again because of the overarching importance of banks source of development finance for investors (households, firms and government). The researcher will explain in simple terms the role of RMT in the research shortly.

Column 2 depicts the nature of empirical modelling to be performed on existing South African data. The upper-left quadrant of the column uses the symbolisms B, S, and D to portray indicative analyses using the Black-Scholes derivative pricing model (B) on observed stylized facts (S) of data on underlying asset prices and implied volatility dynamics (D) in the JSE. These data are represented as Generalised Stylised Facts (GSFs) and Implied Volatility Stylised Facts (IVSFs). The lower-left quadrant depicts the nature of sequential modelling M1, M2, M3 … which will exploit the JSE data, based on comparative analysis of the performance of selected derivative models (see Chapter 2 on literature review of the various models) against the standard BS model, for suitable derivative products mentioned above, as appropriate.

The reason for this approach is to understand which models, may be the BS model are typically used in the JSE for specific asset prices, and whether the derivative pricing from competing models is more accurate than the ones used. This knowledge will be very useful to the NSE management as they optimise the decision choices facing them in introducing derivative products and models in the NSM. It will also be useful to the JSE management if they become aware that existing models used in pricing JSE derivatives are not as good as alternative models revealed by the research. This, therefore, will be a crucial contribution of this research to knowledge. Again, this sequence of models is denoted by M1, M2, M3, and so on here.

Column 3 depicts the nature of empirical modelling to be performed on the as yet unavailable existing Nigerian data. We, however, recall that the underlying asset prices are available in Nigeria, but not derivative prices. Given that derivative models are common theoretical knowledge across the two markets, we represent the Nigerian information as B for BS model, GS for General Stylised Facts (GSFs), GD for underlying asset prices, in the upper-right quadrant of the column. We use GSFs below this information structure to show that the General Stylised Facts of the underlying prices are known, and the IVSFs? (with a question mark) to show that the Implied Volatility Stylised Facts are not available. The research question now is:

How do we overcome the lack of research data on the IVSFs which underpin derivative pricing in the NSM? This is discussed further below in using the symbolism in the lower-right quadrant of Column 3.

The steps summarised in this lower-right quadrant are as follows: a) compare the stylised facts (GS information) on both underlying prices for JSM and NSM – the reason for this is to gauge how close the two data sets are in behaviour (using, say, the first four moments and distributions of the data sets); b) we explore the correlation between the full data on GSFs and IVSFs in South Africa; c) we then infer the likely range of values for the unknown Nigerian IVSFs; d) we run RMT analyses on asset prices from key market sectors in NSM and JSM (for example selected banks, oil and gas, commodities), in order to characterise mainstream tendencies in the markets; e) using knowledge of the RMT comparisons, we simulate plausible data that fits the Nigerian modelling scenarios and repeat the sequence of modelling M1, M2, M3, …, on the data. This will produce indicative results which will inform possible decisions on the models and projected prices that could obtain in the NSM, under different modelling scenarios and assumptions.

The above-mentioned steps in modelling derivatives in Nigeria through revealed affinities between the NSM and JSM trading data will be achieved within a broad-based characterisation work using suitable systems of sector- and asset-based stylised facts and RMT results can be described as a heuristic approach. RMT is a theoretical suite of techniques which originated from modern physics (astrophysics and theoretical particle physics) and is widely applied in statistical physics and econophysics. It relates to using correlation measures among clusters of measurements, based on eigenvalue and eigenvector analyses, among other techniques in multivariate statistics, underpinned by assumptions about the likely types of probability distributions which generate the data clusters, to explore the relationships among the data clusters.

In this section of the thesis we simply note that these RMT techniques are used as baseline tools for initially studying the closeness or otherwise among the selected data clusters from sectors and sets of asset prices in the JSM and NSM. The results will then be combined with further knowledge of a) the statistical distributions which govern the respective data cluster, b) the extent to which the data behaviours support the assumptions of different derivative pricing models, with the BS model as a reference point, hence the plausible validity of the models in deciphering derivative prices, in order to simulate supposed NSM data that fit the distributions and models, and thereby produce plausible derivative prices for the NSM.

It is expected that the research will serve as a point of departure in further modelling of derivative prices in NSM post-introduction of such products in the market. Importantly, the results will provide theoretical knowledge of the limitations of different derivative pricing models reviewed in the literature presented in Chapter 3 of this thesis, which will be useful for further theoretical research on the models and their applications in Nigeria and similar emerging markets, with emphasis on Sub-Sahara African markets such as Algeria, Ghana, Egypt, and Kenya. We presented the crucial touch-points of RMT and the JSM-NSM characterisation results in the thesis. We followed this up with the remaining steps in modelling selected JSM-NSM data and derivative prices for risk hedging, speculation and arbitrage investment goals, in the subsequent chapters to the RMT chapter.

Chapter 10

CONTRIBUTIONS TO KNOWLEDGE, RECOMMENDATIONS AND CONCLUSION

10.0 Introduction

In chapter 9, we interpreted and discussed the research findings from the data carried out in the previous chapters 5-8. The discussions were centred on methods of forecasting the asset price dynamics for stocks in the Nigerian Stock Market, (NSM), error estimates in the various methods, finding solutions to stochastic calculus models and their comparison with analytic solutions where feasible.

We also discussed various implied volatility models and parameter estimation with appropriate choices of models, based on the calculations carried out with the data considered, with a good estimation of the errors involved in the respective models.

10.1 Summary of Findings

This research identified some more appropriate implied volatility models from the list of conventional possible relative and absolute smile volatility models that could be adopted in the pricing of European call and put options using Black-Scholes (1973) model, for the yet-to-be-introduced derivative products in the NSM. The findings reveal that generally relative smile models are preferable to the absolute smile counterparts in addressing the shortcomings of constant volatility assumption of the original Black-Scholes model in pricing derivative options.

The research identified the importance and use of Random Matrix Theory and indeed stock price return correlations in portfolio and risk management for investors in the NSM. Prominent among these is the fact that one of the properties needed by investors to hold an efficient portfolio is the existing correlation between securities that are to be included in the portfolio. These correlation estimates are desirable in most applications in finance, for example asset pricing models (including derivative assets), capital allocation, risk management and option pricing, Skentzi and Refenes (2005). The work also recognizes that implied correlation index obtained through the measurement of portfolio variances are useful in determining the rate of portfolio diversifications.

10.2 Contributions to Knowledge (CsTK)

We now explore the findings from the research and their respective contributions to knowledge. We carefully examined the research questions of this study and a tabular presentation of findings with their corresponding contribution to knowledge is presented below. Table 10.1 highlights the contributions to knowledge linked to findings and the appropriate research questions.

Table 10.1: Summary of Findings and CsTK

Research Question	Findings	Contributions to Knowledge (CsTK)
Q1: What are the differentiating characteristics, performance trade-offs, assumptions, equations, and parameters, among stochastic calculus models used in derivative pricing, and how are the model parameters typically determined from market data?	We examined relevant stochastic calculus models especially their uses in derivative assets and option pricing. Determination of the parameters of the models	The research demonstrated suitable derivative pricing methods for the NSM including interest rate models which NSM investors need for effective portfolio risk management. Proposes ways of determining the parameters of the stochastic calculus models necessary for derivative assets pricing were elucidated.
	Estimation of solutions to stochastic calculus by Euler-Maruyama.	It was shown that depending on the step size $\Delta t = R * dt$, the Euler-Maruyama approximation to solutions of stochastic differential equation coincides with the analytical solution (where the analytical solution exists) for appropriate choice of R.
	The research shows the use of Euler-Maruyama in finding an approximation to solutions of stochastic calculus models Parameter estimation for determining the variables in Ad-Hoc/Practitioners Black-Scholes models.	Hence, in absence of tractable analytic solutions, the method of Euler-Maruyama approximation is shown to be applicable in the estimation of the models in this research work Relative volatility smiles are preferable to absolute smile models in the estimation of implied volatility for use in the pricing of call and put option by Black-Scholes model
Q2: What are the links between the model features/derivative products and key investment objectives fulfilled by the products in financial markets, for instance risk hedging, arbitrage and speculation?	Proposed derivatives trade in NSM will provide risk-hedging opportunities to investors and entrepreneurs with the NSM	This work has clarified the anticipated benefits of derivatives trade in the NSM, including proposed pioneer products, especially for managing credit, interest rate and operational risks in the Nigerian market.
	Hedging foreign currency risk	This research offers some solution to important problem of foreign exchange risks, using implied correlation index, basket of options or put options.
	Arbitrage	The NSM favours arbitragers due to lack of proper dissemination of market

		information, especially for agricultural products, thereby encouraging riskless profit to few market participants who buy those products. The insights were applied in Nigeria to risk hedging and speculation in wheat farming, breweries industry, and crude oil prices, for example. There are also some known cases of insider knowledge in trade within NSM which makes it possible for some investors to earn excess profit by using the privileged information they possess to outsmart other investors in the market.
Q3: Which stylized facts of stock markets are particularly associated with derivative pricing models, and how do they inform adaptations of these and related derivatives to the NSM?	Derivative products are used to speculate possible future changes in the market prices of commodities.	The research agrees with the findings of Skentzi and Refenes (2005) that prominent among the properties desired by investors in the NSM to hold efficient portfolio is the existing correlation between securities that are to be included in the investor's portfolio and that these correlation estimates are required in derivative asset pricing models, risk management and option pricing, Bourgoin (2001). We have shown as stated by Bourgoin (2001) that from implied correlation index, portfolio with minimum variances offer full diversification opportunities, unlike the ones with higher variances that do not encourage diversification among constituent stocks
	Historical volatility	Historical volatility estimates from various asset returns would be used in estimating the call and put option prices for traditional Black-Scholes (1973) model when the actual trades on derivatives fully commences in NSM, hence the need for a demonstration of the methods in this research.
	Implied volatility	The original Black-Scholes model and other extensions of Black-Scholes including Ad-Hoc or Practitioners Black-Scholes models for derivative asset

		pricing and evaluation were studied in this research. As shown in this research on the implied volatility surface, volatility is not constant throughout the option lifespan as postulated in original Black-Scholes (1973) option pricing model hence the research recommends the use of implied volatility in estimating the value of call and put option for the derivative products due to be introduced into the NSM.
	Predictability	Nigerian Stock Market as an emerging economy is faced with constant fluctuations in interest and exchange rates, hence mitigating the unfavourable changes in these variables are of utmost importance to the policy makers in the NSM. The possible derivatives products and their pricing models that address these problems were studied in this research. The asset price dynamics and future prediction of asset prices in the short run can be successfully carried out using the Monte-Carlo simulations as shown in chapter 5 of this research. The method shown therein can be used to forecast the possible prices of assets that will serve as the underlying stocks for derivative contracts using Euler-Maruyama approximations. Successful prediction of such values for the underlying stocks would reduce the risk on portfolio held by investors and other stakeholders in the NSM.
	Anomalies	Anomalies in the NSM would also affect the pricing of derivative products being introduced in the exchange, for instance the recently witnessed economic recession that have befallen Nigeria since 2015. The recession has forced the price of goods and services to skyrocket as a result of increased scarcity of foreign exchange and the inability of entrepreneurs to access foreign exchange because of a depleted

		foreign reserve and an astronomical rise in interest rate regime from single digit to as high as 21%. The economy is believed to have technical exited from recession, but prices of goods and services are still very high.
	Asset return correlation	It was also observed from the computation on empirical correlation matrix that there are evidences of momentum effect as we discovered that high past returns lead to high future returns in asset prices as was observed by Jegadesh and Titman (1993). It was discovered from a comparative analysis from the empirical correlation matrices of Johannesburg Stock exchange with that of the Nigerian Stock Market that the average $\langle C_{ij} \rangle$ of stocks in NSM is 0.041 while that of the JSE is 0.168. This means that even though the two markets are both emerging, the NSM is about four times more correlated than that of the JSE implying that the market dynamics in the two markets are significantly different, so this research is important on how to use existing derivative trading information in the JSE to develop suitable prices for the NSM, Shen and Zheng (2009), D.J. Fenn et al. (2011). In NSM, 7.3% of the eigenvalues lie outside the theoretical value of the RMT correlation matrix (therefore contain information about the market), whereas for the JSE assets, the percentage of the eigenvalues that carry information about the entire market is 8.57%. RMT was applied to reveal more detailed knowledge of the price dynamics in the two markets, which are fundamental to future work in developing suitable derivative products in the NSM. A heuristic approach for doing this was elucidated for the first time in this line of work.

		The research shows how we can use empirical correlation matrix to measure the risk associated with assets in portfolio. From the research evidence demonstrated in chapter eight of this thesis, when the correlation among assets increase it will induce an appropriate rise in the portfolio variance and thus the risk on investment. Conversely, a decrease in valid empirical correlation matrix will lead to a decrease in portfolio variance thereby reducing the risk associated with the respective portfolios. Implied correlation matrix as shown in this research is applicable in hedging the risks associated with foreign exchange and the policy makers in NSM have included some currency related derivative contracts among the pioneer products to be introduced into the NSM.
Q4: How do the research ideas including findings from the Random Matrix Theory apply to the NSM, for example how can the ideas be used to implement relevant experimental modelling for comparing the investment performance of selected derivative pricing models under different market scenarios in the NSM?		Amongst the six key market features used in empirical finance (Bubbles, Anomalies Efficiency, Predictability and valuation Volatility-both historical and implied), this work mainly explored implied volatility within the models for derivative pricing. The derivative pricing models include ideas on valuations which extended to valuation of investment firms that substantially trade in derivatives. Future work on developing suitable derivative products for the NSM using the RMT and heuristics in the thesis will exploit the remaining market features or stylized facts in more details.

The contributions to knowledge in this research work foreground suitable models for the pricing and evaluation of proposed derivative products in the NSM. The underpinning heuristics for the immediate follow-on work was elucidated at the end of chapter 9, based on RMT correlational analyses of the NSM and JSE market data. The analyses explored the similarities and differences in stock price return for the two most dominant emerging markets in the Sub-Saharan Africa especially as it concerns the modelling of trade on derivatives in the NSM after the corresponding successful products in the Johannesburg Stock Exchange.

The following sub-sections again link the contributions to knowledge to the research questions, to assist the researcher in evaluating the success of the study in exploring the original research questions. In the same vein, this approach also generates insights for future work.

10.3.1 What are the differentiating characteristics, performance trade-offs, assumptions, equations, and parameters, among stochastic calculus models used in derivative pricing, and how are the model parameters typically determined from market data?

T.C. Urama et al. (2016) show how to estimate the volatility parameter using historical prices from the desired asset returns. To this end, in this research work, in addition to estimating historical volatility, we looked at the Adhoc Black-Scholes or Practitioners Black-Scholes, which makes use of implied volatility in estimating the volatility parameters that is needed in the evaluation of European call and put options for Black-Scholes option pricing models.

10.3.2 What are the links between the model features/derivative products and key investment objectives fulfilled by the products in financial markets, for instance risk hedging, arbitrage and speculation?

The use of currency derivatives, for instance currency swaps, foreign exchange options and similar derivative contracts, depends on the exchange rate exposures being encountered by firms, government and individuals in that economy. Nigerian economy being import driven (as a consuming nation for most of her daily needs in exchange for exportation of crude oil) is therefore heavily exposed to the exchange/interest rate fluctuations, and therefore needs foreign currency derivatives in the day-to-day running of the economy. More so, the major exports from Nigeria, crude oil and with few export earnings from agricultural products are highly connected with the fluctuations in the exchange rates, especially the United States of American dollars; hence, the need for trade on derivatives to mitigate the risks associated with those changes in the value of naira with respect to other major currencies of the world.

Geczy et al. (1997) find that firms with large sizes and exposures to exchange rate emanating from trade/sales in the Nigerian capital market are more likely to use currency derivatives. Multinational companies in Nigeria that have other outlets in other Sub-Saharan Africa, like Shell Petroleum BP, Mobil Telecommunication Network of Nigeria (MTN) will therefore need a basket of options to protect their investments in Nigeria, and other African countries where they have some stakes against fluctuations and some adverse changes in the value of currencies in the respective countries where they have their lines of investments. David Haushalter (2000) studied the hedging policies of oil and gas producers and discovers that the extent of hedging is proportional to financing cost. He declares that the basis risk is important determinant for oil and gas producers' risk management policies and that companies that are primarily gas producers hedge production more extensively than their oil-based counterparts.

Allayannis and Ofek (2001) opine that in addition to foreign currency derivatives, firms can also use foreign debt to protect themselves from exposure to exchange rate risks. They assert that a firm which has its revenue denominated in foreign currencies (cash inflows) can issue foreign debt, since this process will create a stream of cash outflows in a foreign currency. Nigeria's main source of revenue from the NNPC is denominated in the United States of America dollars and the issuance of foreign debt on same will help to mitigate the risk on foreign exchange, as a result of depreciating value of the naira in relation to the value of the American dollars, which is known to have been adversely affecting the Nigerian economy.

We note, however, that conversely the imports into a country which represents cash outflows in a foreign currency cannot be hedged through foreign debt and therefore other approaches to fluctuations in the exchange rate, for example foreign currency swaps and currency options are preferable for hedging the risk associated with capital outflows in a country. Thus, firms mostly use currency derivatives to hedge their exchange rate related risk exposures but not to speculate in the foreign exchange markets.

Michael Chui (2012) asserts that in derivatives market we have two active participants - hedgers or speculators. While hedgers in the derivative market seek to protect themselves against adverse changes in the values of their assets and liabilities, speculators aim at profiting from anticipated changes in market prices or rates in the associated derivative contacts that they have been engaged in.

Hence, the goal of hedgers and speculators in the derivative market are two sides of the same coin. Speculators in the market are exposed to higher risks than the hedgers. Hence, NSM policy makers should aim at proper regulation of the activities of market participants, especially the speculators to avoid excessive risk taking in the system which might lead to colossal losses and subsequent market crash. This is very important since the capital required to enter into a derivative contract on the part of the speculators is very infinitesimal compared to the value of the contract, hence the speculative market participant may therefore be tempted to take excessive risk more than the threshold their revenue stream could cope with.

It is also important to mention however, that the risks in any contract(s) are never eradicated but rather most of the risk management strategies involve transferring the risks from the risk averse investors to those willing and able to manage risk at some fixed cost on the part of the risk averse investor. There is, therefore, the need for speculators in the Nigerian market to be mindful of the possible losses associated with the various risks that they have in their portfolio.

10.3.3 Which stylized facts of stock markets are particularly associated with derivative pricing models, and how do they inform adaptations of these and related derivatives to the NSM?

Prominent among the stylized facts and market features of the NSM that influence derivative trade is the volatility parameter. Option pricing consists mainly of estimation of some or all ot its parameters and the desired accurate option prices depend largely on the method of estimation of those parameters among which is the volatility. Andersen and Bollerslev (1998) declare that volatility parameter(s) in finance and by extension for derivative products is indispensable in asset pricing and evaluation. From their observation, the variation in economy-wide risk factors is useful in pricing of financial securities, including derivative products, and that return volatility is the key input to option pricing and portfolio allocation needs. In this regard, accurate measures and efficient forecasts of volatility are very important towards an effective implementation of trade on the proposed derivative products in the NSM including better choices in the pricing models as indicated in this research for effective risk management in the NSM.

Andersen et al. (2006) declare that the trade-off between risk and expected return, where risk is related to price volatility constitute one of the major concepts in modern finance. In the light of this, therefore, measuring risk and ability to accurately forecast volatility is arguably one of the most important pursuits in empirical asset pricing and risk management. There has been an ongoing argument as to how to estimate the volatility parameter that would be used in the evaluation of derivative products (including options). Some researchers subscribe to the use of historical estimate of volatility parameter while others subscribe to the forward-looking method of volatility estimation (implied volatility). In this research work, we looked at both estimators of volatility parameter but with the findings from the research; implied volatility is preferable to historical volatility especially as it concerns the Black-Scholes option pricing formula in evaluating European call and put option being the type of option recommended for use in NSM by the financial regulators.

Andersen et al. (1999) also indicate that the precise estimation of diffusion volatility does not require a long calendar span of data; rather an acceptable estimate of volatility can be obtained from a short span of data, as long as the returns are sampled sufficiently frequently. In this work, for the estimation of implied volatility we took few samples accordingly, whereas for asset return correlation we took a very large data as it requires sufficiently large spectrum of data to reduce noise in the results to be obtained. Andersen et al. (1999) reiterate that good forecast of volatility and correlations are very useful in portfolio allocation and asset risk management.

The predictability and forecast of future correlation among constituent stocks in a portfolio depend largely on the historical correlations. For the associated derivative assets, implied correlation index which makes use of some measurement of portfolio variances as shown in the research can be used in determining the rate of portfolio diversification, (Bourgoin, 2001). This research has shown how to diversify portfolio from an estimation of portfolio variances by choosing those assets that have minimum portfolio variances in the estimated implied correlation index. This process is applicable in managing the risk in a given basket of options and could therefore be applied for mitigating risks connected with trade on foreign currency options proposed for introduction into the Nigerian Stock Market.

This approach will give investors especially multinational companies investing in Nigeria and other Sub-Saharan African economies the opportunity of hedging individual risks associated with exchange rate fluctuations by method of lumping up the collective risks on those currencies with a basket of option or in the alternative adopting the method of separate derivative options on the respective individual currencies when and where it is more appropriate.

10.3.4 How do the research ideas including findings from the Random Matrix Theory apply to the NSM, for example how can the ideas be used to implement relevant experimental modelling for comparing the investment performance of selected derivative pricing models under different market scenarios in the NSM?

This research has shown that the dynamics of asset return correlation as shown in Random Matrix Theory (RMT) is very useful in portfolio optimization (including asset diversification) and as well in derivative asset risk management process. The result from RMT has shown that the stock price returns in the NSM when compared with normal distribution are skewed and leptokurtic in mean and standard deviation but fairly symmetric in kurtosis. This underlying stock return characteristic is very important in modelling derivative products due for introduction in the NSM as efficient modelling and pricing of proposed derivatives depend largely on the observed market features and characteristics of the underlying stocks in the NSM upon which the derivatives contracts are built. The comparative study of stocks price dynamics in the NSM with that of the JSE provides the necessary information on major similarities and differences between the two exchanges which is desirable for empirical modelling of derivative products in the NSM especially as it is the policy decision of the NSM to model trade on derivative in NSM from the perceived successful derivative products being traded in Johannesburg Stock Exchange as stated by the financial regulators during our scientific visit to Nigeria.

The implied correlation index studied in chapter eight is applicable in hedging risks relating to foreign exchange which is one of the derivative products earmarked for introduction in the NSM. Large corporations are always interested in hedging their currency exposures by using a basket of options instead of taking separate put options to hedge the risks that they are exposed to in the respective countries where they have their investments, (M. Bensman, 1997). Furthermore, Krishnan and Nelken (2001) declare that companies that are exposed to a variety of currency fluctuations find it more profitable to hedge directly their aggregate risk through the use of basket of options by deploying the knowledge of estimated valid correlation matrix.

It is known that most of the countries in the Sub-Saharan Africa including Nigeria, Ghana and South Africa export raw materials to more advanced economies in Europe, America and Asia and in return import finished products form those developed economies. To this end many of the manufacturing companies from advanced economies that supply the finished products to Nigeria and other contemporary African nations have different production lines of investments in Nigeria, Ghana and South Africa, for instance, and therefore needs to mitigate the risks emanating from the fluctuations in the values of Naira, Cedi and Rand when compared to the United States of American Dollars. Thus, these multinational companies will need to have a firm grip of the past and predicted future correlation of the these currencies that they are exposed to in the various investments portfolios they have in these African nations through the analysis of future correlation among the respective currencies to be able to use effectively basket of option to hedge those risks that they are exposed to as a result their stakes in Nigeria, Ghana and South African markets.

Limitations of the Research

The research was conducted with limited funding for a designated period from the Nigerian government. Hence, the focus was to develop the theoretical foundations for successful introduction of derivatives trading in Nigeria. Further work should, therefore, build on current findings as summarised below.

Implications for Future Research and Conclusion

We have elucidated the nature of future work in the NSM-JSE RMT heuristics in chapter 9. We recommend a full implementation of the research intentions in the heuristics, and a wider use of the heuristics and other techniques researched under the broad theme of Systematic Stock Market Characterisation and Development (SSMCD), by all PhD students in the SIMFIM Research Group, Sheffield Hallam University, UK.

References

Acharya, V. & Johnson, T. (2007). Insider trading in credit derivatives, Journal of Financial Economics 84, 110 - 141.

Adelegan, O. Janet (2009). The derivatives market in South Africa: Lessons for Sub-Saharan Africa countries, IMF working paper series WP/09/06, September 2009.

Adi Ben-Israel (1966). A Newton-Raphson method for solution to systems of equations. Journal of Mathematical Analysis and Applications 15, 243-252.

Adrian B. & Vilkov G. (2012). Measuring equity risk with option-implied correlations. The Review of Financial Studies, 25(10), 3113-3140.

African development Bank Group (2010). African Fixed Income and Derivative Guidebook May 2010.

Aichison, J. & Brown, J.A.C. (1963). The Lognormal Distribution, Cambridge University Press.

Allayannis G. & Eli Ofek (2001). Exchange rate exposure, hedging and the use of foreign currency derivatives. Journal of International Money and Finance 20, 273-296

Ameer, R. (2009). Value-reliance of foreign exchange and interest rate de discourse: The case of Malaysia firms. Journal of Risk Finance, 10(1), 78-90.

Amin, K.I & Ng, V.K (1993). Option valuation with systematic stochastic volatility. The Journal of Finance, 48(3), Papers and Proceedings of the 53rd annual general meeting of the American Finance Association: Anaheim, California, January 5-7, 1993, 881-910.

Andersen, T.G. & Bollerslev, T. (1998). Answering the skeptics: Yes, standard volatility models do provide accurate forecasts, International Economic Review 39(4), Symposium on forecasting and Empirical Methods in Macroeconomics and Finance, 885-905; published by Wiley for the Economics Department of the University of Pennsylvania and Institute of Social and economic Research, Osaka University.

Anderson, T.G., Bollerslev, T., Diebold, F. X., & Labys, P., (1999). Understanding, Optimizing, Using and Forecasting Realized Volatility and Correlation, Working paper, NorthWestern University, USA.

Anderson, T.G., Bollerslev, T., Christoffersen, P.F., & Diebold, F. X. (2006). Volatility and correlation Forecasting, Handbook of Economic Forecasting, volume 1 Edited by Graham Elliot, Clive W.J. Granger and Allan Timmermann, Elsevier.

Arnold T., Erwin, G., Nail, L., & Nixon, T. (2006). Do options markets substitute for stock market(s)? Evidence from trading on anticipated tender offer announcements. International Review of Financial Analysis, 15, 247 - 255.

Aysun, U. & Guldi, M. (2011). Derivatives market activity in emerging markets and exchange rate exposure. Journal of Emerging Markets Finance and Trade, 47(6), 46 - 67.

Baba, N., Parker, F. & Nagano, T. (2008). The spill over of money market turbulence to FX swap and cross-currency swap markets. BIS Quarterly Review, March 2008.

Bai, Z.D. (1999). Methodologies in spectral analysis of large dimensional random matrices, A Review, Statistica Sinica 9, 611- 677.

Bakaert, G. & Wu, G. (2000). Asymmetric volatility and risk in equity markets. The Review of Financial Studies, 13(1), 1-42.

Ballotta Laura & Kyprianou Andreas, E. (2001). A note on the α-quantile option. Journal of Applied Mathematical Finance 8, 137-144.

Bakshi G., Chades Cao, & Zhiwu Chen (1997). Empirical performance of alternative option pricing models. The Journal of Finance 52(5), 2003 - 2049.

Balsam, S.; Kim, S. (2001). Effects of Interest rate swaps. Journal of Economics and Business 53, 547- 562.

Banz, R. (1981). The relationship between return and market value of common stock. Journal of Financial Economics 9, 3 - 18.

Basu, P. & Samanta, P. (2001). Volatility and stock prices: Implications from production model of asset pricing models, Economic Letters Journal, 70, 229 - 235.

Basu, S. (1977). Investment performance of common stock in relation to their price-earnings ratios: A test of the efficient market hypothesis. Journal of Finance, 32, 663 - 682.

Basu, S. (1983). The relationship between earnings' yield, market value and returns for NYSE common stocks: Further evidence, Journal of Financial Economics, 12, 129 - 156.

Baurdoux, E.J. & Kyprianou, A.E. (2004). Further calculations for Israeli options. Journal of Stochastics and Stochastics Reports, 76(6), 549-569.

Baxter, M. (1995). International Business Cycles in: G. Grossman and K. Rogoff, eds., Handbook of International Economics, vol.3 (North Holland, Amsterdam).

Beckers, S. (1980). The constant elasticity of variance model and its implication for option pricing. The Journal of Finance, 35, (3), 661 - 673.

Beckers, Stan (1981). Standard deviations implied in option prices as predictors of future stock price variability. Journal of Banking and Finance, 363 - 381.

Bekaert, G.E., & Harvey, C.R. (1997). Emerging Equity market volatility. Journal of Financial Economics, 43, 29-77.

Bekaert, G.E.,Erb, C.B., Harvey, C.R., & Vyskanta, T.E. (1998). Distribution characteristics of Emerging Markets returns and asset allocation. Journal of Portfolio Management, 24, 102-116.

Beladi, H. and Choi, E.K. (2008). Book series: Frontiers of Economics and Globalization, Chapter 9: A Source of Long Memory in Volatility by Hyung, Namwon, Poon, Ser-Huang, Granger Clive, W.J. 329-380, Emerald Insight publishers.

Bensman, M. (1997). Hedging multiple currency exposures: The easy way derivatives strategy.

Bharadia, M., Christofides, N., & Salkin, G. (1996) A quadratic method for the calculation of Implied Volatility using the Garman-Kohlhangen model. Financial Analysts Journal 52, 61-64.

Black, Fisher(1975) - Fact and Fantasy in the use of options; Financial Analysts Journal, July-August, 36-72.

Black, Fischer (1976). The pricing of commodity contracts. Journal of Financial Economics 3, 167-179.

Black, F. & Scholes, M. (1973). The Pricing of Options and Corporate Liabilities, Journal of Political Economy, 3, 637-654.

Blair, B.J., Poon, S.H., & Taylor, S.J.(2001). Forecasting S&P 100 volatility: The incremental information content of implied volatilities and high frequency returns. Journal of Econometrics 105, 5-26.

Blanchard, O.J. & Watson, M. (1982). Bubbles, rational expectations and financial markets, in: P. Watchel, ed; Crisis in the economic and financial structure (Lexington Books, Lexington, MA).

Bolerslev, T., Ray, Y.C. & Kroner, K.F. (1992). Arch modelling in finance: A review of the theory and empirical evidence. Journal of Econometrics 51, 5-59.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroscedasticity (GARCH), Journal of Econometrica, 31, 307 - 327.

Bollerslev, T. & Zhou, H. (2002). Estimating stochastic volatility diffusion using conditional moments of integrated volatility. Journal of Econometrics 109, 35-65.

Bouchaud, J-P. & Potters, M. (2000). Theory of Financial Risks: From Statistical Physics to Risk Management, Cambridge University Press, First published 2000, Reprinted 2001.

Bouchaud, J-P., Gefen, Y., Potters, M. & Wyart, M. (2004) Fluctuations and response in financial markets: The subtle nature of 'random' price changes. Quantitative Finance 4 (2), 176-190.

Bourgoin, F. (2001). Stress-testing correlations: An application to portfolio risk management, New York: Wiley.

Boyle, P.P. (1977). Options: A Monte Carlo Approach. Journal of Financial Economics 4, 323-338.

Brenner, M. & Subrahmanyan, G.M. (1988). A simple formula to compute the Implied Standard Deviation. Financial Analysts Journal, 80-83.

Brewer, K.D., Feng, Yi & Kwan, C.C.Y. (2012). Geometric Brownian motion, Option Pricing and Simulation: Some spreadsheet Based Exercises in Financial Modelling, e-Journal Spreadsheets in Education (eJSiE), 5(3), Article 4.

Brock, W., Lakonishok, J. & LeBaron, B. (1992). Some technical rules and the stochastic properties of stock returns. The Journal of Finance, 47(5), 1731-1764.

Campbell, J.Y., Hilscher, J. & Szilagyi, J. (2008). In search of distress risk. Journal of Finance 63, 2899 - 2939.

Campbell, J.Y., Grossman, S. J., & Wang, J. (1993). Trading volume and serial correlation in stock returns. The Quarterly Journal of Economics, 905 - 939.

Campbell, J.Y., Lo, A.W. & Mackinlay, A.C., (1997).The Econometrics of Financial Markets, Princeton University Press, Princeton, New Jersey.

Castilla, A.F., Kyprianou, A.E. & Scheichl, R. (2016). An Euler-Poisson scheme for Levy driven stochastic differential equations. Journal of Applied Probability 53, 262-278.

Chambers, D.R. & Nawalkha, S.K. (2001) An Improved approach to Computing Implied Volatility. The Financial Review 38, 89-100.

Chan, K., McQueen, G. & Steven, T. (1998). Are there rational speculative bubbles in Asian markets? Pacific Basin Finance Journal, 6, 125 - 151.

Chandradew S. & Banerjee, K. (2015). A study of correlations in the stock market. Physica A, 432, 321-330.

Charterjee A. & Chakrabarti B.K. (2006). Econophysics of Stock and other markets, Springer.

Cheang, G. H. L. & Chiarella C. (2011). A modern view on Merton's jump-diffusion model, Quantitative Finance Research Centre, Research Paper 287, January 2011; http://ssrn.com/abstract=2167155.

Cheridito, P. (2000). Regularized fractional Brownian motion and option pricing. preprint ETH Zurich; http://www.math.ethz.ch/dito/,2000.

Cheridito, P., Kawaguchi, H., & Maejima, M. (2003). Fractional Ornstein-Uhlenbeck processes, Electronic Journal of Probability 8 (3), 1 - 14.

Chiras, D. & Manaster, S. (1978). The information content of option prices and a test of market efficiency. Journal of Financial Economics, 6, 213 - 234.

Christie, A.A. (1982). The stochastic behaviour of common stock variances: Value, leverage and interest rate effects. Journal of Financial Economics 10, 407-432.

Chui, Michael (2012). Derivatives markets, products and participants: an overview, IFC Bulletin No 35, Data requirements for monitoring derivative transactions. Proceedings of the workshop organised by the people's Bank of China and the Irving Fisher Committee, Zhengzhou, 27-29 September 2010, produced by Bank for International Settlements.

Conlon, T., Ruskin, H.J. & Crane, M. (2007). Random Matrix Theory and fund of funds portfolio optimization. Physica A, 382, 565-576.

Conlon, T., Ruskin, H.J. & Crane, M. (2009) Cross-correlation dynamics in financial times series. Physica A, 388, 705-714.

Constantinides, G. M. (1978). Market risk adjustment in project valuation. The Journal of Finance, 33(2), 603-616.

Cont, R. (2001). Empirical properties of asset returns: Stylized facts and statistical issues. Quantitative Finance Journal 1, 223-236.

Cont, R., & Fonseca, J. (2002). Dynamics of implied volatility surfaces. Quantitative Finance Journal 2, 45-60.

Corrado C., J. & Miller T., W. Jr (1996). A note on a simple, accurate formula to compute implied standard deviations. Journal of Banking and Finance 20, 595 - 603.

Corrado Charles & Su Tie (1996). Skewness and Kurtosis in S&P 500 Index returns implied by option prices. The Journal of financial Research, 19 (2), 175 - 192.

Cox, J.C. (1975). Notes on option pricing I: Constant elasticity of variance diffusion, working paper, Stanford University, Graduate School of Business, Stanford, CA.

Cox, J.C. (1996). The constant elasticity of variance option pricing model. The Journal of Portfolio Management, 22, 16 -17.

Cox, J.C., Ingersoll, J.E. Jr & Ross, S.A. (1985). A Theory of the Term Structure of Interest Rates. Econometrica, 53(2), 385-408.

Cox, J.C. & Ross, S. A. (1975). The pricing of options for jump processes, Rodney L. White Center Working Paper number -75, University of Pennsylvania, Philadelphia, PA

Cox, J.C. & Ross, S. A. (1976). The valuation of options for alternative stochastic processes. Journal of Financial Economics, 3, 145 - 160.

Cox, J.C., Ross, S.A. & Rubinstein, M. (1979). Option pricing: A simplified approach. Journal of Financial Economics, 7, 229 - 263.

Cukur, S., Eryigit, M. & Eryigit, R. (2007). Cross-correlation in an emerging market financial data. Physica A, 376, 555-564.

David Haushalter (2000). Financing policy, basis risk, and corporate hedging: Evidence from oil and gas producers. The Journal of Finance 55(1), 107-152.

Davydov, D. & Linetsky, V. (2001). The valuation and hedging of barrier and look-back option under the constant elasticity of variance process. Management Science Journal 47,949 - 965.

De Santis, G. & Imrohoroglu, S. (1997). Stock market and volatility in emerging markets. Journal International Money and Financ,16(4), 561 - 579.

Dempster, M.A.H. & Hutton, J.P. (1996). Numerical valuation of cross-currency swaps and swaptions. http://ssrn.com/abstract:40872.

Desmond, J. Higham (2001). An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations, Society for Industrial & Applied Mathematics, SIAM Review Journal 43(3) 525 - 546.

Desmond, J. Higham (2010), Euler-Maruyama Approximation to Stochastic Differential Equation, Working Paper; Department of Mathematics and Statistics, University of Strathclyde (Science), Basel February 2010.

DeTemple, J. & Selden, L. (1988) Option listing and stock returns, First Boston working paper series, Columbia University.

Diba, B.T. & Grossman, H.I. (1987). On the inception of rational bubbles. Quarterly Journal of Economics, 102, 697 - 700.

Ding, Z. & Granger, C.W.J. (1996). Modelling volatility persistence of speculative returns: A new Approach. Journal of Econometrics, 73, 185-215.

Ding, Z. Granger, C.W.J. & Engle, R.F. (1993). A long memory property of stock market returns and a new model. Journal of Empirical Finance 1, 83-106.

Dodd, R. & Griffith-Jones, S. (2007). Report on derivatives markets: stabilizing or speculative impact on Chile and a comparison with Brazil, A project document of United Nations in Economic Commission for Latin America and The Caribbean (ECLAC).

Drozdz, S., Grummer, F. & Speth, J. (2001). Towards identifying the world stock market cross-correlations: DAX versus Dow Jones. Physica A: Statistical Mechanics and its Applications, 226-234.

Duan, J.C., Gauthier, G. & Simonato, J.G. (1999). An analytical approximation for GARCH option pricing model. Journal of Computational Finance 2, 75-116.

Duan, J.C., Gauthier, G., Simonato, J.G. & Sasseville, C. (2006). Approximating the GJR-GARCH and EGARCH option pricing models analytically. Journal of Computational Finance, 9(3), 1-29.

Duan, J.C., Gauthier, G. & Simonato, J.G. (1999). Fast valuation of derivatives contracts by simulation, working paper.

Duitermaat, J.J., Kyprianou, A.E. & van-Schaik, K. (2005). Finite expiry Russian options. Journal of Stochastic Process and their Applications 115, 609-638.

Dumas, B., Flemming, J. & Whaley, R.E. (1998). Implied volatility Functions: Empirical Tests. The Journal of Finance, 53(6), 2059-2106.

Dunbar Steven, R. (2014). Stochastic Processes and Advanced Mathematical Finance: Stochastic Differential Equations and the Euler-Maruyama Method, A Working Paper, Department of Mathematics, 203 Avery Hall, University of Nebraska-Lincoln, http://www.math.unl.edu/ `sdunbarl

Edelman A. (1988). Eigenvalues and condition numbers of random matrices. SIAM Journal on Matrix Analysis and Applications, 9(4), 543-560.

El- Alaoui, Marwane (2015). Random matrix theory and portfolio optimization in Moroccan Stock Exchange. Physica A , 92-99.

Elliot, W.B., Huffman, S. P. & Makar, D.S. (2003). Foreign-dominated debt and foreign currency derivatives: Complements or substitutes in hedging foreign currency risk? Journal of Multinational Financial Management 13, 123-139.

Emenike, K.O. (2010). Modelling stock returns volatility in Nigeria using GARCH models, MPRA paper No. 22723; https://mpra.nb,uni-muenchen.de/22723/

Emira, K., Jukan, M.K. & Civic, B. (2012). The use of derivatives in financial risk management by companies in Bosnia and Herzegovina, Economic Review. A Journal of Economics and Business, 10 (2), 36 - 48.

Emmanuel, D. & Macbeth, J.C. (1982). Further results on the constant elasticity of variance call option pricing model. Journal of Financial and Quantitative Analysis, 17 (4), 53 - 54.

Engle, R.F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of the UK inflation. Econometrica, 50 (3), 987 - 1008.

Engle, R.F. and Figlewski, S. (2014). Modelling the dynamics of correlation among implied volatilities. Rev. Finance, 1-28.

Engle, R.F. & Patton, A.J. (2001). What good is a volatility model? Quantitative Finance Journal 1, 237-245. http://www.tandfonline.com/action/journal/

Ezepue, P.O. & Omar, T. O. (2012). Weak-form Market Efficiency of the Nigerian Stock Market in the context of Financial Reforms and global Financial Crisis. Journal of Business 13(3), 209-220.

Ezepue, P.O. & Solarin, A.R.T (2009). The Meta-heuristics of global financial crisis in the eyes of the credit squeeze: Any lessons for modelling emerging financial market(s)? In Ale et al (Eds.) proceeding of the 2008 International Conference on Mathematical Modelling of Global Challenging problems in the 21st Century, Abuja - Nigeria: National Mathematical Centre, 278 - 288.

F.J. Dyson (1971) Rev. Mex. Fis 20, 231.

Farmer, J.D. & Lo, A.W. (1999). Frontiers of Finance: Evolution and efficient markets, Proceedings of National Academies of Science and Engineering, Irvine, CA, USA.

Fekete, D., Fontbona, J., & Kyprianou, A.E. (2017). Skeletal stochastic differential equations for continuous-state branching process, arXiv: 1702.03533v2 [Math.PR], 31 March 2017

Feng Ma, Y. W. & Dengshi H. (2013). Multifractal detrended cross-correlations analysis between the Chinese Stock Market and surrounding stock markets. Physica A, 392, 1659-1670.

Fenn, D.J., Porter, M., Williams, S., McDonald, M., Johnson, N., & Jones, N. (2011). Temporal evolution of financial market correlations. Physical Review E 84, 026109.

Figlewski, S. (1994). Forecasting volatility using historical data; New York University, Stern School of Business, Finance Department working paper series, FD-94-32.

Fisher, S. (1975). The demand for Index bonds, Journal of Political Economy, 83 (3), 509 - 534.

Froot, K. A., David, S.F. & Stein, J.C. (1993). Risk Management: Coordinating Corporate Investment and Financing Policies. Journal of Finance 48, 1629-1658.

Galluccio, S., Bouchaud, J. & Potters, M. (1998) Rational decisions, random matrices and spin glasses. Physica A 259, 449-456.

Gamde (2014). Nigerian Stock Exchange public interaction: A Paper he presented during the scientific visit of Research team - Statistics, Information Modelling and Financial Mathematics Research Group, Materials and Engineering Research Institute (MERI), Sheffield Hallam University, UK to the NSE, May, 2014

Garman, M. B. & Kohlhagen, S.W. (1983). Foreign currency option values. Journal of International Money and Finance, 2 (3), 231 - 237.

Geczy, C., Minton, B.A. & Schrand, C. (1997). Why firms use currency derivatives. The Journal of Finance, 52(4), 512-540

Gee-rae K. & Markowitz H.M. (1989). Investment rules, margin and market volatility. The Journal of Portfolio Management, 16(1), 45-52.

Gentle, J. (1998). Singular value factorization, Berlin; Springer, 1998.

Gentle, J. (1998). Numerical Linear Algebra for applications in statistics (Statistics & Computing), Springer.

German, M. B. & Kohlhagen, S.W. (1983). Foreign currency option values. Journal of International Money and Finance, 2 (3), 231 - 237.

Giot, P. (2003). The information content of implied volatility in agricultural commodity markets. Journal of Future Markets 23, 441- 454.

Gonzalez, J.M. & Alatorre, A. (2015). Random matrix approach to correlation matrix of financial data (Mexican Stock Market case). Modern Economy 6, 1033-1042. Scientific Research publishing. http://www.scirp.org/journal/mc 2015.9099,

Gopikrishnan, P., Rosenow, B., Plerou, V. & Stanley, H.E. (2001). Quantifying and interpreting collective behaviour in financial markets. Physical Review E, 64, 035106.

Grabbe, J.O. (1983). The pricing of call and put options on Foreign Exchange. Journal of International Money and Finance 2, 239-253.

Guhr, T., Muller, A. & Weidenmuller, H. (1998). Random matrix theories in Quantum Physics: Common concepts. Physics Reports 299, 189-425.

Hentschel L. & Kothari, S.P. (2001). Are corporations reducing or taking risks with derivatives. Journal of Financial Quantitative Analysis, 36 (1), 89 - 115.

Heston, S.L. (1993). A closed form solution for options with stochastic volatility with application to bond and currency options. Review of Financial Studies Journal 6, 327 - 343.

Heston, S.L. & Nandi, S. (2000). A closed-form GARCH option pricing model. Review of Financial Studies 13, 585 - 626.

Higgins, P. & Humpage, F.O. (2005). Nondeliverable forwards: Can we tell where the Renminbi is headed? Working paper, Federal Reserve Bank of Cleveland, USA, permit No 385.

Hooyman, C.J. (1994). The use of Foreign Exchange Swaps by Central Banks. Journal of International Monetary Fund, Staff Papers, March 1, 1994, 41(1), periodicals archives online 149, IMF papers.

Hoque, A., Chan, F. & Manzur, M. (2008). Efficiency of the foreign currency options market. Global Finance Journal 19, 157 - 170.

Hsieh, K.C. & Ritchken, P. (2005). An empirical comparison of GARCH option pricing models, published online: Rev. derivative Res (2005), 8, 129 - 150, DOI 10.1007/s11147 - 006 - 900- 3.

Hull, J. (1997). Option, Futures and Other derivatives Securities, Third Edition. New Jersey: Prentice-Hall.

Hull, John, (2002). Options, Futures and Other Derivatives, 5th Edition, Prentice Hall, 151-252.

Hull, J. & White, A. (1987). The pricing of options on assets with stochastic volatilities. Journal of Finance 43, 281 - 300.

Hulle V.C. (1998) . On the Nature of European Holdings Groups. International Review of Law and Economics 18, 255-277.

Hyung N., Ser-Huang P., Granger, & Clive, C. W. J. (2008). Chapter 9 : A source of long memory in volatility, in David E. Rapach, Mark E. Wohar (ed) Forecasting in the presence of structural breaks and model uncertainty (Frontiers of Economics and Globalization, volume 3, Emerald Group Publishing Limited), 329 - 380.

Ishii, J. Otker-Robe, I. & Cui, L. (2001). Measures to limit the offshore use of currencies: Pros and cons, IMF Working paper, WPO 1/43, April 2001.

Islam, S. & Watanapalachaikul, S. (2005). Empirical finance modelling and analysis of emerging financial and stock markets, 1st edition, Physica - Verlag- Heidelberg. New York.

Ivailo, I., Daimov, P.N., Kolm, L., Maclin, D. & Shiber, Y.C. (2012). Hidden noise structure and random matrix models of stock correlations. Quantitative Finance, 567-572.

Jackson, E. (2003). A user's Guide to Principal Components, A Wiley Interscience publication, John Wiley and Sons Inc. New York,

Jae Dong Noh (2000). A model for correlations in stock markets. Physical Review E, 61(5), 1-2.

Jaimungal, S. & Wang, T. (2006). Catastrophe options with stochastic interest rates and compound Poisson losses. Elsevier Journal of Insurance: Mathematics and Economics 38, 469-483.

Jarrow, R. & Rudd, A. (1982). Approximate option valuation for arbitrary stochastic processes. Journal of Financial Economics, 10, 347-369.

Jarrow, R. & Turnbull, S., (1999). Derivatives securities Cincinnati South-Western College Publishing.

Jegadesh, N. & Titman, S. (1993). Returns to buying winners and selling losers: Implications for market efficiency. Journal of Finance 48, 65 - 91.

Jensen, M.C. (1968). The performance of mutual Funds in the period 1945-64. Journal of Finance, 23(2), 389-446.

Jiang, G. & McCauley, R. (2004). Asian local currency bond markets. BIS Quarterly Review, 67-79.

Jiang, G.J. & Van der Sluis, P.J. (2000). Index options pricing models with stochastic volatility and stochastic interest rate, Centre of Economic Research, Tilburg University working paper number 2000-36.

Jianwu X., Hong, Z. & Qin, C. (2007). The constant elasticity of variance model and the Legendre Transform-dual solution for annuity contracts. Insurance, Mathematics and Economics Journal 40, 302 - 310.

Jorion, P. & Sweeney, R.J. (1996). Mean reversion in real exchange rates: Evidence and implications for forecasting. Journal of International Money and Finance, 15(4), 535-550.

Joshua, M. P. & Mungo W. (2010). Average correlation and stock market returns. Journal of Financial Economics 96, 364-380.

JP Morgan (1996). RiskMetrics technical document (4^{th} edition) New York. JP Morgan/Reuters.

Keim, D.B. (1983). Size related anomalies and stock return seasonality: Further empirical evidence. Journal of Financial Economics 12, 13 - 32.

Keith, Sill (1997). The economic benefits and risk of derivatives securities. Business Review, Federal Reserve Bank of Philadelphia, 15 - 26.

Kendell, M. & Stuart A. (1977). The Advanced Theory of Statistics 1. Distribution Theory, London Griffin Publishers, 4^{th} edition.

Keynes, J.M. (1923). A tract on Monetary Reform. London: Macmillan.

Kim, S. (2009). The performance of Traders' rule in options market. The Journal of Futures Market, 29(11), 999-1020.

Kloeden, P.E. & Platen, E. (1989). A Survey of Numerical Methods for Stochastic Differential Equations, Stochastic Hydrology & Hydraulics 3, 15 - 178.

Kloeden, P.E. & Platen, E. (1992). Numerical Solutions of Stochastic Differential Equations, Springer, Berlin.

Kloeden, P.E., Platen, E. & Scurz, H. (1993). Numerical Solution of Stochastic Differential Equations Through Computer Experiments, Springer, Berlin.

Kondor I. & Kertesz, J. (1999). EconoPhysics: An Emerging Science, Kluwer, Dordrecht.

Koutmos, G. & Knif, J. (2002). Estimating systemic risk using time varying distributions. European Financial Management, 8(1), 59-73.

Krishnan, H. & Nelken, I. (2001). Estimating implied correlations for currency basket options using the maximum entropy method. Derivative Use, Trading & Regulation, 7(3)1357-0927.

Kritzman, M. (1991). What practitioners need to know about estimating volatility part 1. Financial Analysts Journal, 22-25.

Kumar R.and Sinha, S. (2007). Collective behaviour of stock price movements in an emerging market. Physical Review E 76, 046116.

Kuria A. M. & Riro G. K. (2013). Stock market anomalies: A study of seasonal effects on average returns of Nairobi Securities Exchange. Research Journal of Finance and Accounting 4 (7), 207 - 216.

Laloux, L., Cizeau, P., Bouchaud, J. Potters, M. (1999). Noise dressing of financial correlation matrices. Physical Review Letters, 83, 1467-1470.

Laloux, L., Cizeau, P., Potters, M., & Bouchaud, J. (2000). Random matrix theory and financial correlations. International Journal of Theoretical and Applied Finance, 3, 1-7.

Larose, T. D.; Lorose, C.D. (2015). Data Mining and predictive Analytics, 2nd edition, Wiley publishers.

Latane, H. A. & Rendleman R.J. Jr (1976). Standard deviations of stock price ratios implied in option prices. The Journal of Finance 31 (2), 369 - 381.

Leon, A. and Sentana E. (1997). Pricing options on assets with predictable white noise returns, Seminar paper in CEMFI and LSE Conference, Casado del Alisal 5, 28014, Madrid, Spain.

Lipscomb, L. (2005). An overview of Non-deliverable foreign exchange forward markets. Federal Reserve Bank of New York.

Lo, A. & Wang, J. (1995). Implementing option pricing models when asset returns are predictable. The Journal of Finance 50 (1), 87 - 129.

Longin, F. & Solnik, B. (1995). Is correlation in International equity markets constant? Journal of International Money and Finance, 14(1), 3- 26.

Lopez, J.A. & Walter, C.A. (2000). Is implied correlation worth calculating? Evidence from foreign exchange options and historical data. Journal of derivatives, 7, 65-82.

Kim, M., Lee, S., Hwang, D., Kim, S. & Ko I.(2010). Dynamics of implied volatility surfaces from random matrix theory. Physica A, 389, 2762-2769.

Kyprianou Andreas, E. (2004). Some calculations for Israeli options. Finance and Stochastics Journal 8, 73-86.

Ma, C.K and Rao, R.P. (1988), Information Asymmetry and Options Trading. The Financial Review: The official publication of the Eastern Financial Association, Knoxville, 23(1), 39-51.

Ma, G., Ho, C. and McCauley, R.N. (2004), The markets for non-deliverable forwards in Asian currencies, Bank for International Settlements, BIS Quarterly Review.

MacBeth, D.J. and Merville, J.L. (1979). An empirical examination of the Black-Scholes option pricing model. The Journal of Finance 34 (5), 1173 - 1186.

Malliaris, A.G. and Brock, W. (1982). Stochastic Methods in Economics and Finance: Advanced textbooks in Economics, North Holland Publishers, Elsevier, and Amsterdam

Malliaris, A.G. and Malliaris, M.E. (1991). Inflation rates and Inflation: a continuous time stochastic approach. Economics Letters 37, 351-356.

Malliaris, A.G. and Urrutia, J.L. (1991). An empirical investigation among real, monetary and financial variables. Economics Letters 37, 151-158.

Malliaris, A.G. and Urrutia, J.L. (1991). Test of random walk of hedge ratios and measures of hedging effectiveness for stock indexes and foreign currencies. The Journal of Futures Markets, 11(1), 55-68.

Malliaris, A.G. and Urrutia, J.L. (1991). The impact of the lengths of estimation periods and hedging horizons on the effectiveness of a hedge: Evidence from Foreign Currency Futures. The Journal of Futures Markets, 11(3), 271-289.

Manaster, S. and Koehler, G. (1982). The calculation of Implied Variances from the Black-Scholes Model: A note. The Journal of Finance, 37(1), 227-230.

Manaster, S. and Rendleman, R. (1982). Option prices as predictors of equilibrium stock prices. Journal of Finance 37, 1043-1057.

Mandelbort, B.B. (1963). The variation of certain speculative prices. Journal of Business 36, 392-417.

Mantegna, R.N. (1999). Hierarchical structure in financial markets. The European Physical Journal B, 11, 193-197.

Mantegna, R.N. and Stanley, H. E. (2000). An Introduction to Econophysics, Correlations and Computing in Finance, Cambridge University Press, First Edition, 2000.

Marcenko, V.A. and Pastur, L.A (1967) Distribution of eigenvalues for some sets of random matrices, Math.USSR sbornik 1(4), 457-483.

Mark Richardson (2009), Stochastic Differential Equations, Case Study, A Working Paper for Mathematical Modelling and Scientific Computing Group, mathematical Institute, University of Oxford, 1 - 25.

Markowitz, H. (1952A) Portfolio Selection. Journal of Finance, 2, 77-91.

Markowitz H. (1952B) Portfolio Selection: Efficient Diversification of Investment, John Wiley and sons, New York, 1952.

Maruyama, G. (1955). Continuous Markov Processes and Stochastic Equations. Rend. Circolo Math. Palermo 4, 48 - 90.

Mayhew Stewart (1995). Implied volatility: Literature Review. Financial Analysis Journal, 8 - 10.

Mbeng Mezui, C.A., Rutten, L., Sekioua, S., Zhang, J., N'Diaye, M.M., Kabanyane, N., Arvanitis, Y., Duru, U., & Nekati, B. (2013). Guidebook on African Commodity and Derivative Exchanges. African Development Bank.

McShane, E. J. (1974). Stochastic Calculus and Stochastic Models, Academic Press, New York.

Merton R. C. (1973). Theory of rational option pricing. Bell Journal of Economics & Management Science, 4, 141-183.

Merton Robert, C. (1976). Option pricing when underlying stock returns are discontinuous. Journal of Financial economics 3, 125 - 144.

Mihaljek, Dubravko and Frank Peter (2010). Derivatives in Emerging Markets. BIS Quarterly Review, The Bank for International Settlements, 43 - 58.

Milstein, G. N. (1974). Approximate Integration of Stochastic Differential Equations. Theory of Probability Applications 19, 557- 562

Morrison, D.F. (1990). Multivariate Statistical Methods, Third Edition, McGraw-Hill, Inc. New York.

Moschini, G. & Myers, R.J (2002). Testing for constant hedge ratios in commodity markets: A multivariate approach. Journal of Empirical Finance 9, 589-603.

Mullins, M. & Murphy, F. (2009). Financial Globalization, State autonomy and modern Financial Instruments: The case of Brazil. Globalisation Journal, 6 (4), 433-449.

Nance, D.R., Smith, C.W. Jr & Smithson, C.W. (1993). On the determinants of corporate hedging. Journal of Finance, 48(1), 267-284.

Nnanwa, C.P., Urama, T.C., Ezepue, P.O. (2017) Random matrix theory analysis of cross-correlation in the Nigerian Stock Exchange, Proceedings of International Scientific Forum 2017 accepted to appear.

Nnanwa, C.P., Urama, T.C., Ezepue, P.O. (2016) Portfolio optimization of financial services stocks in the Nigerian Stock Exchange. American Review of Mathematics and Statistics, 4(2) 1-9.

Neftci, Salih N. (1996). An Introduction to the Mathematics of Financial Derivatives, Academy Press, New York, First Edition.

Neftci Salih, N. (2000). An Introduction to the Mathematics of Financial Derivatives, Academy Press, New York; Second Edition.

Nelson, D. (1991). ARCH models as diffusion approximations. Journal of Econometrics 45, 7 - 38.

Newton, N. J. (1991). Asymptotically Efficient Runge-Kutta Methods for a Class of Ito and Stratonovich Equations. SIAM Journal of Applied Mathematics, 52, 542 - 567

Nguyen, H. and Faff, R. (2003). Can the use of foreign currency derivatives explain variations in foreign exchange exposure? Evidence from Australian companies. Journal of Multinational Financial Management 13, 193-215.

Nobi, A., Maeng, S., Gyun, G. & Lee J. (2013). Random matrix theory and cross-correlations in global financial indices and local stock market indices. Journal of Korean Physical Society, 62(4), 569-574.

Numpacharoen Kawee (2013). Weighted average correlation matrices method for correlation stress testing and sensitivity analysis. The Journal of Derivatives, 67-74.

Numpacharoen, K. & Numpacharoen, N. (2013) Estimating realistic implied correlation matrix from option prices. Journal of Mathematical Finance 3, 401-406; http://www.scirp.org/journal/jmf

Omar Mahmoud, A.T. (2012). Stochastic models in emerging financial market analysis: A case study of the Nigerian Stock Market, PhD Dissertation Thesis, Sheffield Hallam University, U.K.

Omotosho, D. (2014). Paper he presented during the scientific visit of Research team - Statistics, Information Modelling and Financial Mathematics Research Group, Materials and Engineering Research Institute (MERI), Sheffield Hallam University, UK to the NSM, May, 2014

Onadele, P. (2014). Paper presented during the scientific visit of Research team - Statistics, Information Modelling and Financial Mathematics Research Group, Materials and Engineering Research Institute (MERI), Sheffield Hallam University, UK to the NSM, May, 2014

Osuoha J. I. (2010). Commodity Trading and Futures, Emmaesth Printing & Publishing, Lagos Nigeria.

Ovidiu, C. (2012). An Introduction to Stochastic Calculus with Applications, Eastern Michigan University, Ypsilanti, MI 48197, USA, lecture series.

Owolabi, O.F. (2011). Central Bank of Nigeria- Guidelines for FX derivatives and Modalities for CBN Foreign Exchange Forwards, Financial Markets Department, March 2011.

Ozaki, T. (1985). Nonlinear Time-Series Models and Dynamical Systems (eds E.J Hannan, P. R. Krishnaiah and M. M. Rao). Handbook of Statistics 5, North Holland, Amsterdam

Ozaki, T. (1992). A Bridge Between Nonlinear Time Series Models and Nonlinear Stochastic Dynamical Systems: A Local Linearization Approach. Statistica Sinica 2, 113 - 135.

Pafka, S. & Kondor, I. (2003) Noisy covariance matrices and portfolio optimization 2. Physica A, 319, 487-494.

Pafka, S. and Kondor, I. (2004). Estimated correlation matrices and portfolio optimization. Physica A, 343, 623-634.

Pagan, A.R., Hall, A.D., and Martin, V.(1995) Modelling the term structure; School of Business Discussion Papers - paper 60. http://epublications.bond.edu.au/discussion_papers/60.

Pagan Adrian (1996). The econometrics of financial markets. Journal of Empirical Finance 3, 12 - 102.

Palan, Stefan (2009). Bubbles and Crashes in Experimental Assets Markets, Springer, London.

Palmer, R.G., Arthur, W.B., Holland, J.H. & LeBaron, B. (1994). Artificial economic life: A simple model of a stock market. Physica D, 75, 264-274.

Panzar, L. and Cipu, E.C. (2004). Using of stochastic Ito and Stratonovich integrals in derived security pricing, Proceedings of The 3-rd International Colloquium "Mathematics in Engineering and numerical Physics" , October 7-9, 2004, Bucharest, Romania, 222-229.

Pesaran, H. M. (2010). Predictability of asset returns and efficient market hypothesis, CESIFO working paper number 3116, category 12: Empirical and theoretical methods, www.SSRN.com.

Peterside, T. (2014). Paper presented during the scientific visit of Research team - Statistics, Information Modelling and Financial Mathematics Research Group, Materials and Engineering Research Institute (MERI), Sheffield Hallam University, UK to the NSM, May 2014

Plerou, V., Gopikrishnan, P., Rosenow, B., Amaral, L.A.N. & Stanley, H.E. (1999). Universal and nonuniversal properties of cross-correlations in financial time series. Physical Review Letters, 83, 1471-1474.

Plerou, V., Gopikrishnan, P., Rosenow, B., Amaral, L.A.N. & Stanley, H.E. (2000). A random matrix theory approach to financial cross-correlations. Physica A, 287, 374-382.

Plerou, V., Gopikrishnan, P., Rosenow, B., Amaral, L.A.N. & Stanley, H.E. (2001) Collective behaviour of stock price movements: A random matrix theory approach. Physica A, 299, 175-180.

Plerou, V., Gopikrishnan, P., Rosenow, B., Amaral, L.A.N., Guhr, T. & Stanley, H.E. (2002). Random matrix approach to cross-correlations in financial data. Physical Review E 65, 066126.

Podobnik, B., Wang, D., Horvatec, D., Grosse, I. & Stanley, H.E. (2010). Time-lag cross-correlations in collective phenomena. Euro - Physics Letters, EPL 90, 68001, 1 - 6.

Poon, S. and Granger, C.W.J. (2003). Forecasting volatility in financial markets: A review. Journal of Economic Literature, 41, 478-539.

Potters Marc, Bouchaud Jean-phillipe & Laloux laurent (2005). Financial applications of random matrix theory: Old laces and new pieces. ACTA Physica Polonica B, 36(9), 2767-2783.

Rashid, S., Winter-Nelson, A. & Garcia, P. (2010). Purposes and potential for commodity exchanges in African economies. International Food Policy Research Institute (01035)

Reinganum, M.R. (1981). Misspecification of capital asset pricing: Empirical anomalies based on earnings yield and market values. Journal of Financial Economics 9, 19 - 46.

Reinganum, M.R. (1983). The anomalous stock market behaviour of small firms in January: Empirical tests for tax-loss selling effects. Journal of Financial Economics, 12, 89 - 104.

Riadh Baazaoui and Ouerdiane, H. (2014). A Comparative Study of Numerical Simulation of Stochastic Differential Equations; Application to Stochastic Duffing-Oscilllator. International Journal of Mathematical Analysis, 8(5), 229 - 245

Rosenow, B., Plerou, V. & Gopikrishnan, P. (2000). Application of random matrix theory to study cross-correlations of stock prices. International Journal of Theoretical and Applied Finance, 3(3), 399-403.

Rosenow, B., Gopkrishnan, P., Plerou, V. & Stanley H.E. (2002). Random magnets and correlations of stock price fluctuations. Physica A, 314, 762-767.

Rosenow, B., Gopkrishnan, P., Plerou, V., & Stanley, H.E. (2003). Dynamics of cross-correlations in the stock market. Physica A, 324, 241-246.

Saito, Y. and Mitsui, T. (1992). Discrete Approximation for Stochastic Differential Equations. Transactions of the Japan Society for Industrial and Applied Mathematics, 2, 1 - 16

Samuelson, P. (1965). Proof that properly anticipated prices fluctuate randomly. Industrial Management Review 6, 41 - 49.

Sandoval L. Jr and Italo De P. Franca (2012). Correlation of financial markets in times of crisis. Physica A, 391, 187-208.

Sang-Hyeon, P. & Jeong-Hoon, K. (2011). Asymptotic option pricing under the constant elasticity of variance diffusion. Journal of Mathematical Analysis and Applications 375, 490 - 501.

Sangita, M. & Behera, H. (2006). Non - Deliverable foreign exchange forward market: An Overview. Reserve Bank of India occasional papers 27(3), 25 - 56.

Sargent, T. J. (1973). Interest rates and prices in the long run: A study of the Gibson paradox. Journal of Money, Credit and Banking; Blackwell Publishing 5 (1), 385 - 449.

Sauer, T. (2013). Computational Solution of Stochastic Differential Equations, Wiley Advanced Review Periodicals: Computational Statistics, 5(5), 362 - 371.

Schroder Mark (1989). Computing the constant elasticity of variance options pricing formula. The Journal of Finance 44 (1), 211 - 19.

Schwert, W. G. (2002). Anomalies and market efficiency. National Bureau of Economic Research, 1050 Massachusetts Avenue Cambridge, NBER working paper series 9277.

Scott, L. (1997). Pricing Stock Options in a jump diffusion model with stochastic volatility and interest rates: Application of Fourier inversion methods. Mathematical Finance Journal, 7, 413-26.

Scruggs, J.T. and Glabadanidis, P. (2003). Risk premia and the dynamic covariance between stock and bond returns. Journal of Financial and Quantitative Analysis 38, 295-316.

Selloua Chakri (2016). The development of commodity derivatives markets in Africa; United Nations Conference on Trade and development, (UNCTAD 14), Nairobi Kenya, 17-22 July, 2016.

Sengupta A.M. & Mitra P.P. (1999). Distributions of singular values for some random matrices. Physical Review E, (60)3, 1-4.

Sensoy, A., Yuksel, S. & Erturk, M. (2013). Analysis of cross-correlations between financial markets after the 2008 crisis. Physica A, 392, 5027-5045.

Sentana, E. and Wadhwani, S. (1992). Feedback traders and stock return autocorrelations: Evidence from a century of daily data. The Economic Journal 102, 415-425.

Sharifi, S., Crane, M., Shamaie, A. & Ruskin, H. (2004). Random matrix theory for portfolio optimisation: a stability approach. Physica A, 335, 629-643.

Shen J. & Zheng, B. (2009). Cross-correlation in financial dynamics. Letters Journal Exploring the Frontiers of Physics, EPL 86, 48005

Shiu, Y., Moles, P. & Shin, Y. (2010). What motivates banks to use derivatives: Evidence from Taiwan. Journal of Derivatives 17 (4), 67 - 78.

Skintzi, V.D. and Refenes, A.N. (2005). Implied correlation index: A new measure of diversification. The Journal of Futures Markets 25(2), 171-197.

Sloan, R.G. (1996). Do stock prices fully reflect information in accruals and cash flows about future earnings? Journal of Accounting Review 71, 289 - 315.

Smith, C.W and Stulz, R.M. (1985). The determinants of firms' hedging policies. Journal of Financial and Quantitative Analysis 20, 391-405.

Stambaugh, R.F., Jianfeng Yu, & Yuan Y. (2012). The short of it: Investor sentiment and anomalies. Journal of Financial Economics, 104, 288 - 302.

Stoll, H.R. (1969). The Relationship between Put and Call Option Prices. Journal of Finance, 26(5), 802-824.

Stout Lynn, A. (1999). Why the law hates speculators: Regulation and private ordering in the market for OTC derivatives. Duke Law Journal 48 (4), 701 - 786.

Stuart, R. M. and Ord, K. (1987). Kendall's Advanced Theory of Statistics, volume 1, 5th edition, Oxford University Press, NewYork.

Stulz, R.M. (1984). Optimal hedging policies. Journal of Financial and Quantitative Analysis 19, 127-40.

Stulz, R.M. (2004). Should we fear derivatives? Journal of Economic Perspectives 18 (3), 173 - 192.

Sun, T., Sundaresan, S. & Wang, C. (1993). Interest rate swaps: An empirical investigation. Journal of Financial Economics 34, 77-99.

Sundaram, R.K (2013). Derivatives in Financial market development. International Growth Centre, Working Paper, February 2013.

Sundaram, R.K (2013). CDS Auctions and Recovery Rates: An Appraisal. Journal of Derivatives, 97 - 102.

Taylor Stephen, J. (2005). Modelling Financial Times Series (Second Edition), World Scientific Publishing.

Taylor, S.J. (2011). Asset Price Dynamics, Volatility and Prediction, Princeton University Press, New Jersey.

Thakoor, N., Tangman, D. & Bhuruth, M. (2013). A new fourth-order numerical scheme for option pricing under the constant elasticity of variance model. Applied Mathematics Letters Journal 26, 160 - 64.

Thompson, S.B. (2011). Simple formulas for standard errors that cluster by both firm and time. Journal of Financial Economics 99, 1-10.

Timothy Sauer (2008). Finance Conference Proceedings, International Conference of Management, developed and hosted by The College of Information Sciences and Technology, The Pennsylvania State University, USA.

Urama, T.C. and Ezepue, P.O. (2016). Proceedings of the International Symposium on Mathematical and Statistical Finance, 1-3 September 2015, Mathematics and Statistics Complex, University of Ibadan, Oyo State, Nigeria, Publications of the SIMFIM 3E-ICMCS Research Consortium, ISBN: 978-37246-5-7.

Urama, T.C., Ezepue, P.O. & Nnanwa, P.C. (2017a). Analysis of cross-correlation in Emerging Markets Using Random Matrix Theory. Journal of Mathematical Finance, 2017, 7, 291- 301. http://www.scirp.org/journal/jmf

Urama, T.C., Ezepue, P.O. & Nnanwa, P.C. (2017b). Analysis of cross-correlation in Emerging Markets Using Random Matrix Theory, Proceedings of 6th annual International Conference on Computational Mathematics, Computational Geometry & Statistics, (CMCGS), 2017 and 5th annual International Conference on Operations Research and Statistics (ORS) 2017, Singapore; Global Science and Technology Forum.

Urama, T.C., Ezepue, P.O. & Nnanwa, P.C. (2017c). Application of Random Matrix Theory in Estimating Realistic Implied Correlation Matrix from Option Prices, Proceedings of 6th annual International Conference on Computational Mathematics, Computational Geometry & Statistics, (CMCGS), and 5th annual International Conference on Operations Research and Statistics (ORS) 2017, Singapore; Global Science and Technology Forum.

Utsugi, A., Ino, K. & Oshikawa M.(2004). Random matrix theory analysis of cross-correlations in financial markets. Physical Review E, 70, 026110.

Varsha Kulkarni and Nivedita Deo (2007) Correlation and volatility in an Indian Stock Market: A random matrix approach. European Physical Journal B, 60, 101-109.

Vasicek, O. (1977). An equilibrium characterization of the term structure. Journal of Financial Economics 5, 177-188.

Vecer Jan and Xu Mingxin (2004). Pricing Asian options in a semi martingale model. Quantitative Finance 4; 170-175, Institute of Physics Publishing.

Verchenko, Olesia (2011). Testing option pricing models: Complete and Incomplete markets, Kyiv School of Economics and Kyiv Economics Institute; Discussion paper series; www.kse.org.ua

Voit, J. (2001). The Statistical Mechanics of Financial Markets, Springer, Berlin.

Wall, L.D. and Pringle, J.J. (1989). Alternative Explanations of Interest Rate Swaps: A theoretical and empirical analysis, Journal of Financial Management, 59-73.

Wang, G., Xie, C., Chen, S.,Yang, J. & Yang, M. (2013). Random matrix theory analysis of cross-correlations in the US stock market: Evidence from Pearson's correlation coefficient and detrended cross-correlation coefficient. Physica A, 392, 3715-3730.

Watanabe, K., Akuma, H. and Mifune, J. (2002) The effectiveness of capital controls and monitoring: The case of non-internationalisation of emerging market currencies, Executives' Meeting of East Asia and Pacific Central Banks, EMEAP, discussion paper, January 2002.

Whitehead, E (2013). Africa's agricultural commodity exchanges take root. 15 August 2013. This is Africa of the Financial Times. Online. [Available]

http://www.thisisafricaonline.com/Business/Africa-s-agriculture-commodity-exchange-take-root?ct=true

Whitehill, Sam (2009). An Introduction to pricing correlation products using a pairwise correlation matrix. The Journal of Credit Risk, 5(1), 97-110.

Wilcox, D.and Gebbie, T. (2004) On the analysis of cross-correlations in South African market data. Physica A, 344, 294-298.

Wilcox Diane & Gebbie, Tim (2007). An Analysis of cross-correlations in an emerging market. Physica A, 375, 584-598.

Wong, H.Y. and Zhao, J. (2008). An artificial boundary method for American option pricing under the constant elasticity of variance model, SIAM Journal of Numerical Analysis 46, 2183 - 2209.

Appendices

Day of the week	Date	Price of Stock, Si	Price Rel. Si/Si-1	Daily Ret. ln(Si/Si-1)	Daily Ret. Squared
Monday	09/05/2016	4.28			
Tuesday	10/05/2016	4.3	1.0046729	0.004662013	2.17344E-05
Wed.	11/05/2016	4.31	1.00232558	0.002322881	5.39578E-06
Thursday	12/05/2016	4.46	1.03480278	0.034210862	0.001170383
Friday	13/05/2016	4.83	1.08295964	0.079697702	0.006351724
Monday	16/05/2016	5.21	1.07867495	0.075733388	0.005735546
Tuesday	17/05/2016	5.05	0.96928983	-0.03119161	0.000972917
Wed.	18/05/2016	5.1	1.00990099	0.009852296	9.70677E-05
Thursday	19/05/2016	5.35	1.04901961	0.047856021	0.002290199
Friday	20/05/2016	5.35	1	0	0
Monday	23/05/2016	5.1	0.95327103	-0.04785602	0.002290199
Tuesday	24/05/2016	5.2	1.01960784	0.019418086	0.000377062
Wed.	25/05/2016	5.61	1.07884615	0.075892094	0.00575961
Thursday	26/05/2016	5.98	1.06595365	0.063869848	0.004079358
Friday	27/05/2016	5.96	0.99665552	-0.00335009	1.12231E-05
Monday	30/05/2016	5.96	1	0	0
Tuesday	31/05/2016	5.67	0.95134228	-0.04988136	0.00248815
Wed.	01/06/2016	5.28	0.93121693	-0.07126302	0.005078418
Thursday	02/06/2016	5.29	1.00189394	0.001892148	3.58022E-06
Friday	03/06/2016	5.56	1.0510397	0.049779862	0.002478035
Monday	06/06/2016	5.3	0.95323741	-0.04789129	0.002293575
Tuesday	07/06/2016	5.3	1	0	0
Wed.	08/06/2016	5.35	1.00943396	0.00938974	8.81672E-05
Thursday	09/06/2016	5.5	1.02803738	0.027651531	0.000764607
Friday	10/06/2016	5.52	1.00363636	0.003629768	1.31752E-05
Monday	13/06/2016	5.75	1.04166667	0.040821995	0.001666435
Tuesday	14/06/2016	5.79	1.00695652	0.006932437	4.80587E-05
Wed.	15/06/2016	6	1.03626943	0.035627178	0.001269296
Thursday	16/06/2016	5.99	0.99833333	-0.00166806	2.78241E-06
Friday	17/06/2016	5.92	0.98831386	-0.01175496	0.000138179
Monday	20/06/2016	5.8	0.97972973	-0.02047853	0.00041937
Tuesday	21/06/2016	6.1	1.05172414	0.050430854	0.002543271
Wed.	22/06/2016	6.2	1.01639344	0.016260521	0.000264405
Thursday	23/06/2016	6.16	0.99354839	-0.00647251	4.18934E-05
Friday	24/06/2016	5.93	0.96266234	-0.03805256	0.001447998
Monday	27/06/2016	5.9	0.99494098	-0.00507186	2.57238E-05
Tuesday	28/06/2016	5.8	0.98305085	-0.01709443	0.00029222
Wed.	29/06/2016	5.71	0.98448276	-0.01563889	0.000244575
Thursday	30/06/2016	5.71	1	0	0

Friday	01/07/2016	5.75	1.00700525	0.006980831	4.8732E-05
Sum				0.295246845	0.050823065

Table 5.1: Some sample of stock price in the NSM.

Table 6.0: Some sample of call and put options from yahoo finance for model testing										
June 16 calls										
Strike	**Contract Name**	**Last Price**	**Bid**	**Ask**	**Change**	**% Change**	**Volume**	**Open Interest**	**Implied Volatility**	
40	AAPL170616C00040000	113.26	115.7	116.65	0	0.00%	73	0	232.72%	
47.5	AAPL170616C00047500	105.65	108.2	109.15	0	0.00%	25	1	204.59%	
50	AAPL170616C00050000	103.22	105.7	106.65	0	0.00%	17	1	196.19%	
55	AAPL170616C00055000	98.15	100.75	101.65	0	0.00%	15	5	182.32%	
60	AAPL170616C00060000	93.6	95.7	96.65	0	0.00%	770	214	166.80%	
65	AAPL170616C00065000	90.66	90.7	91.6	2.51	2.85%	3	13	152.44%	
70	AAPL170616C00070000	85.65	84.45	86.85	2.04	2.44%	8	57	167.68%	
75	AAPL170616C00075000	80.37	80.1	82.05	2.87	3.70%	12	59	125.78%	

80	AAPL170616C00080000	75.7	74.75	76.6	2.9	3.98%	41	262	135.79%	
82.5	AAPL170616C00082500	49.55	0	0	0	0.00%	0	0	0.00%	
85	AAPL170616C00085000	68.4	70.8	71.7	0	0.00%	5	238	113.67%	
87.5	AAPL170616C00087500	57.2	55.9	56.5	0	0.00%	1	72	0.00%	
90	AAPL170616C00090000	66.13	65.75	66.7	2.63	4.14%	1	99	103.37%	
92.5	AAPL170616C00092500	63.55	63.3	64.05	2.5	4.10%	6	60	97.07%	
95	AAPL170616C00095000	61.1	60.8	61.25	2	3.38%	7	233	86.82%	
97.5	AAPL170616C00097500	55.35	58.25	59.2	0	0.00%	2,220	46	90.14%	
100	AAPL170616C00100000	56.12	55.8	56.75	2.79	5.23%	12	1,314	87.50%	
105	AAPL170616C00105000	48.75	50.8	51.75	0	0.00%	53	645	79.25%	
110	AAPL170616C00110000	45.57	45.6	46.3	-0.59	-1.28%	5	1,806	61.13%	
115	AAPL170616C00115000	40.5	40.5	41	-0.99	-2.39%	4	1,541	56.15%	
120	AAPL170616C00120000	36.1	35.55	36.35	-0.1	-0.28%	27	2,225	57.76%	
125	AAPL170616C00125000	30.65	30.65	31.1	-0.71	-2.26%	81	523	45.34%	40-155 (ITM)
130	AAPL170616C00130000	25.8	25.65	26.05	-0.4	-1.53%	869	2,308	37.55%	

135	AAPL170616C00135000	21.28	20.65	21.1	0.02	0.09%	21	2,147	32.03%	
140	AAPL170616C00140000	16	15.85	16.15	-0.5	-3.03%	229	5,492	26.22%	
145	AAPL170616C00145000	11.25	11.1	11.4	-0.52	-4.42%	623	56,752	22.19%	
150	AAPL170616C00150000	7	6.95	7.1	-0.43	-5.79%	4,726	51,064	19.63%	
155	AAPL170616C00155000	3.65	3.65	3.7	-0.31	-7.83%	9,978	80,529	18.19%	
160	AAPL170616C00160000	1.64	1.62	1.65	-0.14	-7.87%	9,899	26,443	18.09%	
165	AAPL170616C00165000	0.69	0.67	0.7	-0.04	-5.48%	2,959	14,686	18.93%	
170	AAPL170616C00170000	0.31	0.3	0.32	0	0.00%	1,777	8,961	20.41%	
175	AAPL170616C00175000	0.17	0.15	0.17	0.01	6.25%	1,086	6,154	22.36%	
180	AAPL170616C00180000	0.09	0.08	0.09	0.01	12.50%	6,005	9,826	23.98%	
185	AAPL170616C00185000	0.04	0.04	0.06	0	0.00%	6,446	6,203	26.17%	
190	AAPL170616C00190000	0.02	0.02	0.03	0.01	100.00%	1	120	27.15%	
195	AAPL170616C00195000	0.02	0	0.02	0.01	100.00%	25	78	28.91%	
200	AAPL170616C00200000	0.02	0	0.02	0.01	100.00%	10	45	31.64%	
205	AAPL170616C00205000	0.02	0	0.04	0	0.00%	50	50	37.31%	

225	AAPL170616C00225000	0.02	0	0.02	0	0.00%	20	0	44.53%	
250	AAPL170616C00250000	0.02	0	0.02	0	0.00%	3	3	52.34%	
255	AAPL170616C00255000	0.01	0	0.02	0	0.00%	3	3	54.69%	
CallsforJuly 21, 2017										
Strike	**Contract Name**	**Last Price**	**Bid**	**Ask**	**Change**	**% Change**	**Volume**	**Open Interest**	**Implied Volatility**	
25	AAPL170721C00025000	86.8	83.3	84.8	0	0.00%	13	0	0.00%	
40	AAPL170721C00040000	71.74	0	0	0	0.00%	40	0	0.00%	
45	AAPL170721C00045000	108.09	110.75	111.7	0	0.00%	159	0	149.90%	
50	AAPL170721C00050000	103.12	105.75	106.7	0	0.00%	90	0	137.89%	
60	AAPL170721C00060000	93.25	95.85	96.75	0	0.00%	64	21	119.92%	
70	AAPL170721C00070000	83.05	85.85	86.75	0	0.00%	25	3	102.15%	
75	AAPL170721C00075000	78.11	80.85	81.8	0	0.00%	5	0	94.92%	
80	AAPL170721C00080000	73.2	75.9	76.8	0	0.00%	240	102	88.09%	
85	AAPL170721C00085000	68.08	70.85	71.8	0	0.00%	4	2	80.42%	
90	AAPL170721C00090000	63.2	65.9	66.85	0	0.00%	240	9	74.85%	
95	AAPL170721C00095000	59.25	60.9	61.8	0	0.00%	22	429	67.97%	

100	AAPL170721C00100000	53.5	55.95	56.85	0	0.00%	96	1,071	62.84%	
105	AAPL170721C00105000	51.31	50.95	51.9	3.22	6.70%	5	332	57.40%	
110	AAPL170721C00110000	44	46	46.9	0	0.00%	85	808	52.10%	25-155 (ITM
115	AAPL170721C00115000	40.7	40.65	41.25	-0.8	-1.93%	22	81	43.58%	
120	AAPL170721C00120000	36.4	36.05	36.8	2.4	7.06%	119	767	45.26%	
125	AAPL170721C00125000	30.93	30.8	31.3	-0.12	-0.39%	28	208	34.18%	
130	AAPL170721C00130000	26.15	25.9	26.4	-0.2	-0.76%	100	7,982	30.37%	
135	AAPL170721C00135000	21.28	21.1	21.55	-0.02	-0.09%	59	12,840	26.73%	
140	AAPL170721C00140000	16.66	16.4	16.8	-0.28	-1.65%	70	14,185	23.40%	
145	AAPL170721C00145000	12.25	12.15	12.35	-0.32	-2.55%	858	14,936	21.00%	
150	AAPL170721C00150000	8.33	8.35	8.5	-0.42	-4.80%	1,225	32,836	19.74%	
155	AAPL170721C00155000	5.25	5.2	5.35	-0.2	-3.67%	3,527	30,611	18.80%	
160	AAPL170721C00160000	3.05	3	3.1	-0.13	-4.09%	4,393	14,713	18.37%	
165	AAPL170721C00165000	1.65	1.63	1.67	-0.04	-2.37%	3,254	23,404	18.26%	
170	AAPL170721C00170000	0.86	0.85	0.89	-0.03	-3.37%	1,019	3,260	18.63%	

175	AAPL170721C00175000	0.46	0.46	0.48	0	0.00%	760	1,644	19.24%	
180	AAPL170721C00180000	0.27	0.26	0.28	0.02	8.00%	661	15,465	20.19%	
185	AAPL170721C00185000	0.15	0.12	0.15	0.08	114.29%	3,086	7,741	20.75%	
190	AAPL170721C00190000	0.09	0.08	0.09	0.01	12.50%	11,400	9,421	21.63%	
195	AAPL170721C00195000	0.05	0.04	0.05	0.01	25.00%	6,120	6,428	22.17%	
200	AAPL170721C00200000	0.03	0.02	0.03	0.01	50.00%	140	1,002	22.95%	
205	AAPL170721C00205000	0.02	0	0.05	0	0.00%	3	3	26.37%	
210	AAPL170721C00210000	0.01	0	0.03	0	0.00%	1	0	26.76%	
Calls for August 18, 2017										
Strike	**Contract Name**	**Last Price**	**Bid**	**Ask**	**Change**	**% Change**	**Volume**	**Open Interest**	**Implied Volatility**	
2.5	AAPL170818C00002500	153.3	153.15	154.1	0	0.00%	1	0	457.42%	
50	AAPL170818C00050000	103.65	105.8	106.75	0	0.00%	89	89	117.58%	
75	AAPL170818C00075000	77.96	80.9	81.8	0	0.00%	208	1	80.27%	
80	AAPL170818C00080000	73.02	75.95	76.85	0	0.00%	70	59	75.00%	
85	AAPL170818C00085000	59.7	58.4	59	0	0.00%	1	1	0.00%	
90	AAPL170818C00090000	62.97	65.95	66.85	0	0.00%	8	4	63.33%	

100	AAPL170818C00100000	55.72	55	57.05	2.31	4.33%	10	86	61.06%	
105	AAPL170818C00105000	47.95	51	51.95	0	0.00%	30	20	54.50%	
110	AAPL170818C00110000	36.8	42.75	43.35	0	0.00%	1	66	0.00%	
115	AAPL170818C00115000	41.51	41.15	42.05	3.42	8.98%	3	268	45.12%	
120	AAPL170818C00120000	36.19	35.6	36.55	-0.32	-0.88%	5	314	35.65%	2.50-155 (ITM)
125	AAPL170818C00125000	31.07	31.1	31.5	-0.64	-2.02%	32	11,097	30.71%	
130	AAPL170818C00130000	26.85	26.3	26.8	-0.03	-0.11%	3	1,850	28.74%	
135	AAPL170818C00135000	21.6	21.7	22.15	-0.82	-3.66%	78	2,005	26.38%	
140	AAPL170818C00140000	17.15	17.35	17.6	-0.35	-2.00%	31	7,201	23.88%	
145	AAPL170818C00145000	13.4	13.35	13.5	-0.28	-2.05%	178	9,937	22.44%	
150	AAPL170818C00150000	9.8	9.8	9.95	-0.33	-3.26%	648	21,262	21.60%	
155	AAPL170818C00155000	6.85	6.85	6.95	-0.21	-2.97%	1,031	17,222	20.86%	
160	AAPL170818C00160000	4.55	4.55	4.65	-0.14	-2.99%	1,769	14,178	20.48%	
165	AAPL170818C00165000	2.88	2.86	2.92	-0.08	-2.70%	280	9,296	20.07%	
170	AAPL170818C00170000	1.77	1.75	1.8	-0.02	-1.12%	569	3,523	20.05%	

175	AAPL170818C00175000	1.08	1.06	1.09	0.01	0.93%	196	1,901	20.19%	
180	AAPL170818C00180000	0.64	0.64	0.67	0.03	4.92%	826	2,044	20.57%	
185	AAPL170818C00185000	0.4	0.39	0.42	0.04	11.11%	993	1,668	21.07%	
190	AAPL170818C00190000	0.27	0.25	0.27	0.02	8.00%	1,117	758	21.66%	
195	AAPL170818C00195000	0.18	0.16	0.18	0.04	28.57%	21	1,260	22.34%	
200	AAPL170818C00200000	0.11	0.1	0.12	0.04	57.14%	136	1,923	22.95%	
205	AAPL170818C00205000	0.08	0.06	0.08	0.03	60.00%	2,010	612	23.54%	
280	AAPL170818C00280000	0.01	0	0.02	0	0.00%	1	1	39.06%	
Calls for October 20, 2017										
Strike	**Contract Name**	**Last Price**	**Bid**	**Ask**	**Change**	**% Change**	**Volume**	**Open Interest**	**Implied Volatility**	
35	AAPL171020C00035000	118.05	120.75	121.7	0	0.00%	2	0	116.55%	
50	AAPL171020C00050000	81.76	81.85	82.65	0	0.00%	20	0	0.00%	
55	AAPL171020C00055000	88.68	88.4	89.1	0	0.00%	6	2	0.00%	
60	AAPL171020C00060000	95.8	95.85	96.75	0	0.00%	1	0	78.03%	
65	AAPL171020C00065000	66.75	66.9	67.65	0	0.00%	7	0	0.00%	
70	AAPL171020C00070000	83.1	85.85	86.8	0	0.00%	33	1	66.99%	

75	AAPL171020C00075000	77.98	80.9	81.8	0	0.00%	24	0	62.23%	
80	AAPL171020C00080000	76.57	75.95	76.85	3.62	4.96%	2	217	58.15%	
85	AAPL171020C00085000	67.85	70.95	71.9	0	0.00%	8	0	53.86%	
90	AAPL171020C00090000	63.05	66	66.9	0	0.00%	225	11	54.97%	
95	AAPL171020C00095000	58	61	61.9	0	0.00%	42	14	50.39%	35-155 (ITM)
100	AAPL171020C00100000	56.06	55	57.15	-0.44	-0.78%	2	278	48.17%	
105	AAPL171020C00105000	51.43	51.15	52	2.63	5.39%	3	118	42.65%	
110	AAPL171020C00110000	46.1	45.5	46.5	-0.5	-1.07%	3	251	34.39%	
115	AAPL171020C00115000	38.54	41.35	42.2	0	0.00%	668	3,064	35.96%	
120	AAPL171020C00120000	37	36.2	36.8	0.5	1.37%	5	4,817	29.47%	
125	AAPL171020C00125000	31.65	31.6	32.05	-0.35	-1.09%	5	4,301	27.34%	
130	AAPL171020C00130000	26.83	27	27.4	-0.32	-1.18%	24	5,199	25.40%	
135	AAPL171020C00135000	22.6	22.65	23	-0.58	-2.50%	1,967	7,510	24.06%	
140	AAPL171020C00140000	18.64	18.65	18.85	-0.01	-0.05%	2,301	21,989	22.91%	
145	AAPL171020C00145000	15	14.95	15.15	-0.3	-1.96%	64	21,944	22.27%	

150	AAPL171020C00150000	11.58	11.65	11.8	-0.02	-0.17%	319	16,994	21.58%	
155	AAPL171020C00155000	8.89	8.8	8.95	-0.14	-1.55%	4,553	18,929	21.09%	
160	AAPL171020C00160000	6.5	6.45	6.6	-0.15	-2.26%	5,028	14,930	20.72%	
165	AAPL171020C00165000	4.59	4.6	4.7	-0.09	-1.92%	3,233	18,991	20.35%	
170	AAPL171020C00170000	3.19	3.15	3.25	-0.06	-1.85%	3,133	4,582	20.07%	
175	AAPL171020C00175000	2.13	2.15	2.19	-0.04	-1.84%	250	2,296	19.87%	
180	AAPL171020C00180000	1.45	1.44	1.48	0.02	1.40%	243	1,641	19.90%	
185	AAPL171020C00185000	0.95	0.96	0.99	-0.01	-1.04%	290	1,658	19.98%	
190	AAPL171020C00190000	0.65	0.65	0.67	-0.15	-18.75%	52	378	20.18%	
195	AAPL171020C00195000	0.49	0.44	0.46	0	0.00%	30	523	20.47%	
200	AAPL171020C00200000	0.3	0.3	0.32	0	0.00%	105	1,660	20.80%	
205	AAPL171020C00205000	0.2	0.2	0.23	0	0.00%	16	655	21.24%	
210	AAPL171020C00210000	0.15	0.14	0.16	0	0.00%	65	861	21.53%	
215	AAPL171020C00215000	0.1	0.07	0.1	0.04	66.67%	105	148	21.49%	
220	AAPL171020C00220000	0.07	0.05	0.07	0.03	75.00%	25	264	21.78%	

225	AAPL171020C00225000	0.05	0.04	0.06	0.01	25.00%	50	20	22.56%	
230	AAPL171020C00230000	0.04	0.02	0.04	0.01	33.33%	10	5	22.66%	
Calls for November 17, 2017										
Strike	**Contract Name**	**Last Price**	**Bid**	**Ask**	**Change**	**% Change**	**Volume**	**Open Interest**	**Implied Volatility**	
47.5	AAPL171117C00047500	105.66	108.3	109.25	0	0.00%	704	1	87.55%	
50	AAPL171117C00050000	103.35	105.85	106.75	0	0.00%	2	1	84.62%	
55	AAPL171117C00055000	76.72	76.9	77.7	0	0.00%	4	2	0.00%	
60	AAPL171117C00060000	93.2	95.8	96.75	0	0.00%	180	2	71.39%	
65	AAPL171117C00065000	40.85	45.1	45.75	0	0.00%	2	14	0.00%	
70	AAPL171117C00070000	82.3	85.85	86.8	0	0.00%	180	25	61.72%	
75	AAPL171117C00075000	56.84	57	57.75	0	0.00%	45	0	0.00%	
80	AAPL171117C00080000	73.3	75.95	76.85	0	0.00%	2	29	53.56%	
85	AAPL171117C00085000	67.3	70.95	71.9	0	0.00%	450	115	55.10%	
90	AAPL171117C00090000	63.2	66	66.9	0	0.00%	724	97	50.66%	
92.5	AAPL171117C00092500	60.65	63.5	64.45	0	0.00%	400	36	48.98%	
95	AAPL171117C00095000	60.77	60.75	62.1	2.12	3.61%	20	211	48.15%	

97.5	AAPL171117C00097500	55.65	58.55	59.45	0	0.00%	990	34	44.82%	
100	AAPL171117C00100000	56.25	54.9	57.3	-0.35	-0.62%	10	603	45.50%	
105	AAPL171117C00105000	48.35	51.15	52.05	0	0.00%	1,005	217	39.67%	47.5-155 (ITM)
110	AAPL171117C00110000	46.01	46	46.55	-0.69	-1.48%	16	3,892	32.12%	
115	AAPL171117C00115000	41.4	40.7	41.7	-0.49	-1.17%	6	6,413	29.87%	
120	AAPL171117C00120000	36.45	36.45	37	-0.25	-0.68%	4	8,452	28.37%	
125	AAPL171117C00125000	31.75	31.9	32.35	-0.61	-1.89%	4	8,651	26.73%	
130	AAPL171117C00130000	27.48	27.55	27.9	-0.42	-1.51%	26	17,452	25.53%	
135	AAPL171117C00135000	23.2	23.35	23.7	-0.32	-1.36%	24	5,716	24.62%	
140	AAPL171117C00140000	19.31	19.45	19.65	-0.49	-2.47%	121	14,722	23.49%	
145	AAPL171117C00145000	16	15.9	16.1	-0.14	-0.87%	45	18,015	23.00%	
150	AAPL171117C00150000	12.75	12.7	12.9	-0.19	-1.47%	1,488	16,668	22.50%	
155	AAPL171117C00155000	10	9.95	10.1	-0.16	-1.57%	97	7,163	22.04%	
160	AAPL171117C00160000	7.6	7.6	7.75	-0.16	-2.06%	186	5,811	21.70%	
165	AAPL171117C00165000	5.65	5.65	5.8	-0.1	-1.74%	96	3,254	21.38%	

170	AAPL171117C00170000	4.15	4.15	4.25	-0.1	-2.35%	128	3,000	21.12%	
175	AAPL171117C00175000	3.01	2.98	3.05	-0.03	-0.99%	44	696	20.91%	
180	AAPL171117C00180000	2.09	2.11	2.16	-0.06	-2.79%	135	494	20.80%	
185	AAPL171117C00185000	1.48	1.49	1.53	-0.02	-1.33%	50	648	20.81%	
190	AAPL171117C00190000	1.06	1.05	1.09	0.05	4.95%	19	1,581	20.93%	
195	AAPL171117C00195000	0.73	0.75	0.77	-0.16	-17.98%	20	72	21.05%	
200	AAPL171117C00200000	0.53	0.53	0.55	-0.11	-17.19%	126	274	21.24%	
205	AAPL171117C00205000	0.38	0.38	0.4	-0.02	-5.00%	30	41	21.51%	
210	AAPL171117C00210000	0.29	0.24	0.26	0	0.00%	65	0	21.39%	
225	AAPL171117C00225000	0.11	0.08	0.1	0	0.00%	20	0	22.17%	
January 19 2018										
2.5	AAPL180119C00002500	153.13	151.6	154.55	0	0.00%	6	0	270.31%	
5	AAPL180119C00005000	147.98	150.65	151.6	0	0.00%	8	0	258.59%	
10	AAPL180119C00010000	142.97	145.65	146.55	0	0.00%	8	0	193.36%	
40	AAPL180119C00040000	101.2	98.45	103	0	0.00%	1	1	0.00%	
42.5	AAPL180119C00042500	110.76	113.3	114.25	0	0.00%	4	28	91.75%	

47.5	AAPL180119C00047500	105.7	108.3	109.25	0	0.00%	277	24	84.28%	
50	AAPL180119C00050000	105.74	104.45	107.1	1.94	1.87%	50	3,442	71.78%	
55	AAPL180119C00055000	100.98	99.25	102.05	3.08	3.15%	15	1,257	63.28%	
60	AAPL180119C00060000	95.75	95.85	96.8	2.85	3.07%	30	2,373	69.48%	
65	AAPL180119C00065000	91.3	90.85	91.8	3.4	3.87%	5	548	64.18%	
70	AAPL180119C00070000	86.05	85.85	86.8	2.15	2.56%	74	1,833	59.30%	
75	AAPL180119C00075000	81.05	80.95	81.85	2.15	2.72%	41	1,595	55.52%	
80	AAPL180119C00080000	75.62	75.7	76.25	-0.38	-0.50%	4	3,146	49.74%	
82.5	AAPL180119C00082500	73.4	72.15	74.75	2.9	4.11%	8	45	55.92%	2.5-155 (ITM)
85	AAPL180119C00085000	70.65	70.4	70.9	-1.16	-1.62%	2	3,547	42.07%	
87.5	AAPL180119C00087500	66.3	68.5	69.4	0	0.00%	4	492	49.15%	
90	AAPL180119C00090000	65.69	65.75	66.35	-0.66	-0.99%	13	7,412	42.92%	
92.5	AAPL180119C00092500	64.06	63.5	64.45	3.56	5.88%	1	893	45.51%	
95	AAPL180119C00095000	61.1	60.4	61	0.14	0.23%	1	7,048	36.27%	
97.5	AAPL180119C00097500	58.6	57.3	59.4	2.86	5.13%	4	1,328	41.38%	

100	AAPL180119C00100000	55.2	55.55	55.85	-0.6	-1.08%	77	41,021	31.60%	
105	AAPL180119C00105000	51.27	50.7	51.15	0.35	0.69%	92	12,159	31.14%	
110	AAPL180119C00110000	46.1	45.85	46.35	-0.3	-0.65%	103	28,151	29.41%	
115	AAPL180119C00115000	41.6	41.5	41.95	-0.37	-0.88%	95	17,595	29.54%	
120	AAPL180119C00120000	36.82	36.8	37	-0.18	-0.49%	54	49,325	26.54%	
125	AAPL180119C00125000	32.1	32.2	32.5	-0.55	-1.68%	38	39,131	25.34%	
130	AAPL180119C00130000	28.16	28.15	28.25	-0.24	-0.85%	611	48,035	24.53%	
135	AAPL180119C00135000	24.2	24.15	24.3	-0.15	-0.62%	35	36,412	24.01%	
140	AAPL180119C00140000	20.5	20.45	20.65	-0.2	-0.97%	643	70,654	23.61%	
145	AAPL180119C00145000	17.15	17.1	17.2	-0.2	-1.15%	74	22,303	23.00%	
150	AAPL180119C00150000	14.1	14.05	14.15	-0.25	-1.74%	415	52,379	22.58%	
155	AAPL180119C00155000	11.41	11.4	11.45	-0.13	-1.13%	432	18,901	22.20%	
160	AAPL180119C00160000	8.9	9	9.15	-0.25	-2.73%	1,033	23,265	21.93%	
165	AAPL180119C00165000	7.06	7.05	7.15	-0.14	-1.94%	653	15,724	21.59%	
170	AAPL180119C00170000	5.42	5.4	5.5	-0.06	-1.09%	273	21,504	21.30%	

175	AAPL180119C00175000	4.05	4.1	4.2	-0.1	-2.41%	61	10,040	21.14%	
180	AAPL180119C00180000	3.1	3.05	3.15	-0.04	-1.27%	273	13,704	20.97%	
185	AAPL180119C00185000	2.31	2.29	2.33	0.02	0.87%	207	5,705	20.81%	
190	AAPL180119C00190000	1.73	1.7	1.73	0.03	1.76%	10	2,536	20.79%	
195	AAPL180119C00195000	1.26	1.26	1.29	0.01	0.80%	45	2,256	20.84%	
200	AAPL180119C00200000	0.95	0.95	0.97	0.01	1.06%	62	4,297	20.97%	
205	AAPL180119C00205000	0.74	0.7	0.72	0.05	7.25%	27	1,078	21.05%	
210	AAPL180119C00210000	0.57	0.52	0.54	-0.05	-8.06%	70	1,326	21.19%	
215	AAPL180119C00215000	0.4	0.4	0.42	0	0.00%	22	1,625	21.46%	
220	AAPL180119C00220000	0.33	0.31	0.33	-0.02	-5.71%	72	865	21.75%	
225	AAPL180119C00225000	0.25	0.24	0.26	-0.02	-7.41%	40	263	22.05%	
230	AAPL180119C00230000	0.19	0.19	0.21	0.02	11.76%	155	575	22.39%	
235	AAPL180119C00235000	0.17	0.16	0.17	0.03	21.43%	10	784	22.71%	
240	AAPL180119C00240000	0.12	0.09	0.1	0.07	140.00%	10	303	22.12%	
245	AAPL180119C00245000	0.1	0.06	0.08	0.02	25.00%	5	621	22.41%	

250	AAPL180119C00250000	0.05	0.05	0.07	0.02	66.67%	2	150	22.90%	
255	AAPL180119C00255000	0.04	0	0.05	0	0.00%	10	0	22.85%	
260	AAPL180119C00260000	0.06	0.05	0.06	0.03	100.00%	21	870	24.12%	
270	AAPL180119C00270000	0.01	0	0.02	0	0.00%	55	0	23.05%	
280	AAPL180119C00280000	0.02	0	0.06	0	0.00%	113	412	27.05%	

Tables showing some computations of call options used in the model testing are as shown below:

June 16 calls					
Strike	Bid	Ask			
100	55.8	56.75	15/05/2017	16/06/2017	25
105	50.8	51.75	15/05/2017	16/06/2017	25
110	45.6	46.3	15/05/2017	16/06/2017	25
115	40.5	41	15/05/2017	16/06/2017	25
120	35.55	36.35	15/05/2017	16/06/2017	25
125	30.65	31.1	15/05/2017	16/06/2017	25
130	25.65	26.05	15/05/2017	16/06/2017	25
135	20.65	21.1	15/05/2017	16/06/2017	25
140	15.85	16.15	15/05/2017	16/06/2017	25
145	11.1	11.4	15/05/2017	16/06/2017	25
150	6.95	7.1	15/05/2017	16/06/2017	25
155	3.65	3.7	15/05/2017	16/06/2017	25
160	1.62	1.65	15/05/2017	16/06/2017	25
165	0.67	0.7	15/05/2017	16/06/2017	25
170	0.3	0.32	15/05/2017	16/06/2017	25
175	0.15	0.17	15/05/2017	16/06/2017	25
180	0.08	0.09	15/05/2017	16/06/2017	25
185	0.04	0.06	15/05/2017	16/06/2017	25
190	0.02	0.03	15/05/2017	16/06/2017	25
100	55.95	56.85	15/05/2017	21/07/2017	50
105	50.95	51.9	15/05/2017	21/07/2017	50
110	46	46.9	15/05/2017	21/07/2017	50
115	40.65	41.25	15/05/2017	21/07/2017	50
120	36.05	36.8	15/05/2017	21/07/2017	50

125	30.8	31.3	15/05/2017	21/07/2017	50
130	25.9	26.4	15/05/2017	21/07/2017	50
135	21.1	21.55	15/05/2017	21/07/2017	50
140	16.4	16.8	15/05/2017	21/07/2017	50
145	12.15	12.35	15/05/2017	21/07/2017	50
150	8.35	8.5	15/05/2017	21/07/2017	50
155	5.2	5.35	15/05/2017	21/07/2017	50
160	3	3.1	15/05/2017	21/07/2017	50
165	1.63	1.67	15/05/2017	21/07/2017	50
170	0.85	0.89	15/05/2017	21/07/2017	50
175	0.46	0.48	15/05/2017	21/07/2017	50
180	0.26	0.28	15/05/2017	21/07/2017	50
185	0.12	0.15	15/05/2017	21/07/2017	50
190	0.08	0.09	15/05/2017	21/07/2017	50
100	55	57.05	15/05/2017	18/08/2017	70
105	51	51.95	15/05/2017	18/08/2017	70
110	42.75	43.35	15/05/2017	18/08/2017	70
115	41.15	42.05	15/05/2017	18/08/2017	70
120	35.6	36.55	15/05/2017	18/08/2017	70
125	31.1	31.5	15/05/2017	18/08/2017	70
130	26.3	26.8	15/05/2017	18/08/2017	70
135	21.7	22.15	15/05/2017	18/08/2017	70
140	17.35	17.6	15/05/2017	18/08/2017	70
145	13.35	13.5	15/05/2017	18/08/2017	70
150	9.8	9.95	15/05/2017	18/08/2017	70
155	6.85	6.95	15/05/2017	18/08/2017	70

160	4.55	4.65	15/05/2017	18/08/2017	70
165	2.86	2.92	15/05/2017	18/08/2017	70
170	1.75	1.8	15/05/2017	18/08/2017	70
175	1.06	1.09	15/05/2017	18/08/2017	70
180	0.64	0.67	15/05/2017	18/08/2017	70
185	0.39	0.42	15/05/2017	18/08/2017	70
190	0.25	0.27	15/05/2017	18/08/2017	70
100	55	57.15	15/05/2017	20/10/2017	115
105	51.15	52	15/05/2017	20/10/2017	115
110	45.5	46.5	15/05/2017	20/10/2017	115
115	41.35	42.2	15/05/2017	20/10/2017	115
120	36.2	36.8	15/05/2017	20/10/2017	115
125	31.6	32.05	15/05/2017	20/10/2017	115
130	27	27.4	15/05/2017	20/10/2017	115
135	22.65	23	15/05/2017	20/10/2017	115
140	18.65	18.85	15/05/2017	20/10/2017	115
145	14.95	15.15	15/05/2017	20/10/2017	115
150	11.65	11.8	15/05/2017	20/10/2017	115
155	8.8	8.95	15/05/2017	20/10/2017	115
160	6.45	6.6	15/05/2017	20/10/2017	115
165	4.6	4.7	15/05/2017	20/10/2017	115
170	3.15	3.25	15/05/2017	20/10/2017	115
175	2.15	2.19	15/05/2017	20/10/2017	115
180	1.44	1.48	15/05/2017	20/10/2017	115
185	0.96	0.99	15/05/2017	20/10/2017	115
190	0.65	0.67	15/05/2017	20/10/2017	115

100	54.9	57.3	15/05/2017	17/11/2017	135
105	51.15	52.05	15/05/2017	17/11/2017	135
110	46	46.55	15/05/2017	17/11/2017	135
115	40.7	41.7	15/05/2017	17/11/2017	135
120	36.45	37	15/05/2017	17/11/2017	135
125	31.9	32.35	15/05/2017	17/11/2017	135
130	27.55	27.9	15/05/2017	17/11/2017	135
135	23.35	23.7	15/05/2017	17/11/2017	135
140	19.45	19.65	15/05/2017	17/11/2017	135
145	15.9	16.1	15/05/2017	17/11/2017	135
150	12.7	12.9	15/05/2017	17/11/2017	135
155	9.95	10.1	15/05/2017	17/11/2017	135
160	7.6	7.75	15/05/2017	17/11/2017	135
165	5.65	5.8	15/05/2017	17/11/2017	135
170	4.15	4.25	15/05/2017	17/11/2017	135
175	2.98	3.05	15/05/2017	17/11/2017	135
180	2.11	2.16	15/05/2017	17/11/2017	135
185	1.49	1.53	15/05/2017	17/11/2017	135
190	1.05	1.09	15/05/2017	17/11/2017	135
100	55.55	55.85	15/05/2017	19/01/2018	180
105	50.7	51.15	15/05/2017	19/01/2018	180
110	45.85	46.35	15/05/2017	19/01/2018	180
115	41.5	41.95	15/05/2017	19/01/2018	180
120	36.8	37	15/05/2017	19/01/2018	180
125	32.2	32.5	15/05/2017	19/01/2018	180
130	28.15	28.25	15/05/2017	19/01/2018	180

135	24.15	24.3	15/05/2017	19/01/2018	180
140	20.45	20.65	15/05/2017	19/01/2018	180
145	17.1	17.2	15/05/2017	19/01/2018	180
150	14.05	14.15	15/05/2017	19/01/2018	180
155	11.4	11.45	15/05/2017	19/01/2018	180
160	9	9.15	15/05/2017	19/01/2018	180
165	7.05	7.15	15/05/2017	19/01/2018	180
170	5.4	5.5	15/05/2017	19/01/2018	180
175	4.1	4.2	15/05/2017	19/01/2018	180
180	3.05	3.15	15/05/2017	19/01/2018	180
185	2.29	2.33	15/05/2017	19/01/2018	180
190	1.7	1.73	15/05/2017	19/01/2018	180

Table 6.1: Computation of call options

Table 6.2: Computation for the Implied volatility surface

	BSCImVol(S, K, r, q, T, callmktprice)				
Stock price	Strike	Bid	Ask	Time in days	IV
155.7	100	55.95	56.85	50	0.774191
155.7	105	50.95	51.9	50	0.706708
155.7	110	46	46.9	50	0.641367
155.7	115	40.65	41.25	50	0.491893
155.7	120	36.05	36.8	50	0.506019
155.7	125	30.8	31.3	50	0.392778
155.7	130	25.9	26.4	50	0.348723
155.7	135	21.1	21.55	50	0.309455
155.7	140	16.4	16.8	50	0.272412
155.7	145	12.15	12.35	50	0.249722
155.7	150	8.35	8.5	50	0.234246
155.7	155	5.2	5.35	50	0.221639
155.7	160	3	3.1	50	0.216386
155.7	165	1.63	1.67	50	0.215601
155.7	170	0.85	0.89	50	0.219031
155.7	175	0.46	0.48	50	0.226274
155.7	180	0.26	0.28	50	0.236576
155.7	185	0.12	0.15	50	0.240526
155.7	190	0.08	0.09	50	0.252824
155.7	100	55	57.05	70	0.609

155.7	105	51	51.95	70	0.617533
155.7	110	42.75	43.35	70	4.66E-09
155.7	115	41.15	42.05	70	0.513328
155.7	120	35.6	36.55	70	0.397738
155.7	125	31.1	31.5	70	0.371379
155.7	130	26.3	26.8	70	0.341073
155.7	135	21.7	22.15	70	0.31383
155.7	140	17.35	17.6	70	0.288633
155.7	145	13.35	13.5	70	0.27175
155.7	150	9.8	9.95	70	0.259803
155.7	155	6.85	6.95	70	0.250469
155.7	160	4.55	4.65	70	0.244817
155.7	165	2.86	2.92	70	0.239838
155.7	170	1.75	1.8	70	0.239119
155.7	175	1.06	1.09	70	0.240804
155.7	180	0.64	0.67	70	0.244649
155.7	185	0.39	0.42	70	0.249899
155.7	190	0.25	0.27	70	0.256856
155.7	100	55	57.15	115	0.511666
155.7	105	51.15	52	115	0.513542
155.7	110	45.5	46.5	115	0.410411
155.7	115	41.35	42.2	115	0.432147
155.7	120	36.2	36.8	115	0.365417
155.7	125	31.6	32.05	115	0.341927
155.7	130	27	27.4	115	0.316885
155.7	135	22.65	23	115	0.29878

155.7	140	18.65	18.85	115	0.285154
155.7	145	14.95	15.15	115	0.275033
155.7	150	11.65	11.8	115	0.265647
155.7	155	8.8	8.95	115	0.258328
155.7	160	6.45	6.6	115	0.252759
155.7	165	4.6	4.7	115	0.248229
155.7	170	3.15	3.25	115	0.244139
155.7	175	2.15	2.19	115	0.242347
155.7	180	1.44	1.48	115	0.242289
155.7	185	0.96	0.99	115	0.243176
155.7	190	0.65	0.67	115	0.245681
155.7	100	54.9	57.3	135	0.486075
155.7	105	51.15	52.05	135	0.484665
155.7	110	46	46.55	135	0.413072
155.7	115	40.7	41.7	135	0.364338
155.7	120	36.45	37	135	0.358648
155.7	125	31.9	32.35	135	0.337788
155.7	130	27.55	27.9	135	0.321797
155.7	135	23.35	23.7	135	0.307602
155.7	140	19.45	19.65	135	0.294102
155.7	145	15.9	16.1	135	0.285768
155.7	150	12.7	12.9	135	0.277995
155.7	155	9.95	10.1	135	0.271833
155.7	160	7.6	7.75	135	0.266689
155.7	165	5.65	5.8	135	0.261922
155.7	170	4.15	4.25	135	0.258884

155.7	175	2.98	3.05	135	0.256348
155.7	180	2.11	2.16	135	0.254877
155.7	185	1.49	1.53	135	0.254948
155.7	190	1.05	1.09	135	0.25612
155.7	100	55.55	55.85	180	0.405807
155.7	105	50.7	51.15	180	0.387179
155.7	110	45.85	46.35	180	0.362114
155.7	115	41.5	41.95	180	0.362237
155.7	120	36.8	37	180	0.331937
155.7	125	32.2	32.5	180	0.313033
155.7	130	28.15	28.25	180	0.30544
155.7	135	24.15	24.3	180	0.296787
155.7	140	20.45	20.65	180	0.290072
155.7	145	17.1	17.2	180	0.283333
155.7	150	14.05	14.15	180	0.277756
155.7	155	11.4	11.45	180	0.273324
155.7	160	9	9.15	180	0.26863
155.7	165	7.05	7.15	180	0.264927
155.7	170	5.4	5.5	180	0.261395
155.7	175	4.1	4.2	180	0.259391
155.7	180	3.05	3.15	180	0.257234
155.7	185	2.29	2.33	180	0.256431
155.7	190	1.7	1.73	180	0.256389

Table 6.2: Summary statistics for $\sigma_{iv} = a_0 + a_1 K + a_2 K^2 + a_3 T + a_4 T^2 + a_5 KT$.

SUMMARY OUTPUT								
		$\sigma_{iv} = a_0 + a_1 K + a_2 K^2 + a_3 T + a_4 T^2 + a_5 KT$						
Regression Statistics								
Multiple R	0.8484986							
R Square	0.7199499							
Adjusted R Square	0.7042167							
Standard Error	0.0634237							
Observations	95							
ANOVA								
	df	*SS*	*MS*	*F*	*Significance F*			
Regression	5	0.920363	0.184073	45.76005	3.63E-23			
Residual	89	0.358008	0.004023					
Total	94	1.278371						
	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	2.4869899	0.220595	11.27401	7.82E-19	2.048672	2.925308	2.048672	2.92530751
K	-0.024085	0.00289	-8.3329	8.98E-13	-0.02983	-0.01834	-0.02983	-0.0183419
K^2	6.371E-05	9.74E-06	6.540223	3.77E-09	4.44E-05	8.31E-05	4.44E-05	8.3063E-05
T	-1.582438	0.407399	-3.88424	0.000197	-2.39193	-0.77294	-2.39193	-0.7729439
T^2	0.3419867	0.481145	0.710777	0.479081	-0.61404	1.298012	-0.61404	1.29801188

KT	0.0086661	[illegible]	[illegible]	[illegible]	[illegible]	0.012382	0.00495	0.01238248

$$\sigma_{iv} = a_0 + a_1K + a_2K^2 + a_3T + a_4T^2 + a_5KT$$

Stock price	Strike	Bid	Ask	Time in days	IV		K	K^2	T	T^2	KT
155.7	100	55.95	56.85	50	0.774191		100	10000	0.136986	0.018765	13.69863
155.7	105	50.95	51.9	50	0.706708		105	11025	0.136986	0.018765	14.38356
155.7	110	46	46.9	50	0.641367		110	12100	0.136986	0.018765	15.06849
155.7	115	40.65	41.25	50	0.491893		115	13225	0.136986	0.018765	15.75342
155.7	120	36.05	36.8	50	0.506019		120	14400	0.136986	0.018765	16.43836
155.7	125	30.8	31.3	50	0.392778		125	15625	0.136986	0.018765	17.12329
155.7	130	25.9	26.4	50	0.348723		130	16900	0.136986	0.018765	17.80822
155.7	135	21.1	21.55	50	0.309455		135	18225	0.136986	0.018765	18.49315
155.7	140	16.4	16.8	50	0.272412		140	19600	0.136986	0.018765	19.17808
155.7	145	12.15	12.35	50	0.249722		145	21025	0.136986	0.018765	19.86301
155.7	150	8.35	8.5	50	0.234246		150	22500	0.136986	0.018765	20.54795
155.7	155	5.2	5.35	50	0.221639		155	24025	0.136986	0.018765	21.23288
155.7	160	3	3.1	50	0.216386		160	25600	0.136986	0.018765	21.91781
155.7	165	1.63	1.67	50	0.215601		165	27225	0.136986	0.018765	22.60274
155.7	170	0.85	0.89	50	0.219031		170	28900	0.136986	0.018765	23.28767
155.7	175	0.46	0.48	50	0.226274		175	30625	0.136986	0.018765	23.9726
155.7	180	0.26	0.28	50	0.236576		180	32400	0.136986	0.018765	24.65753
155.7	185	0.12	0.15	50	0.240526		185	34225	0.136986	0.018765	25.34247
155.7	190	0.08	0.09	50	0.252824		190	36100	0.136986	0.018765	26.0274
155.7	100	55	57.05	70	0.609		100	10000	0.191781	0.03678	19.17808

155.7	105	51	51.95	70	0.617533		105	11025	0.191781	0.03678	20.13699
155.7	110	42.75	43.35	70	4.66E-09		110	12100	0.191781	0.03678	21.09589
155.7	115	41.15	42.05	70	0.513328		115	13225	0.191781	0.03678	22.05479
155.7	120	35.6	36.55	70	0.397738		120	14400	0.191781	0.03678	23.0137
155.7	125	31.1	31.5	70	0.371379		125	15625	0.191781	0.03678	23.9726
155.7	130	26.3	26.8	70	0.341073		130	16900	0.191781	0.03678	24.93151
155.7	135	21.7	22.15	70	0.31383		135	18225	0.191781	0.03678	25.89041
155.7	140	17.35	17.6	70	0.288633		140	19600	0.191781	0.03678	26.84932
155.7	145	13.35	13.5	70	0.27175		145	21025	0.191781	0.03678	27.80822
155.7	150	9.8	9.95	70	0.259803		150	22500	0.191781	0.03678	28.76712
155.7	155	6.85	6.95	70	0.250469		155	24025	0.191781	0.03678	29.72603
155.7	160	4.55	4.65	70	0.244817		160	25600	0.191781	0.03678	30.68493
155.7	165	2.86	2.92	70	0.239838		165	27225	0.191781	0.03678	31.64384
155.7	170	1.75	1.8	70	0.239119		170	28900	0.191781	0.03678	32.60274
155.7	175	1.06	1.09	70	0.240804		175	30625	0.191781	0.03678	33.56164
155.7	180	0.64	0.67	70	0.244649		180	32400	0.191781	0.03678	34.52055
155.7	185	0.39	0.42	70	0.249899		185	34225	0.191781	0.03678	35.47945
155.7	190	0.25	0.27	70	0.256856		190	36100	0.191781	0.03678	36.43836
155.7	100	55	57.15	115	0.511666		100	10000	0.315068	0.099268	31.50685
155.7	105	51.15	52	115	0.513542		105	11025	0.315068	0.099268	33.08219
155.7	110	45.5	46.5	115	0.410411		110	12100	0.315068	0.099268	34.65753
155.7	115	41.35	42.2	115	0.432147		115	13225	0.315068	0.099268	36.23288
155.7	120	36.2	36.8	115	0.365417		120	14400	0.315068	0.099268	37.80822
155.7	125	31.6	32.05	115	0.341927		125	15625	0.315068	0.099268	39.38356
155.7	130	27	27.4	115	0.316885		130	16900	0.315068	0.099268	40.9589
155.7	135	22.65	23	115	0.29878		135	18225	0.315068	0.099268	42.53425

155.7	140	18.65	18.85	115	0.285154		140	19600	0.315068	0.099268	44.10959
155.7	145	14.95	15.15	115	0.275033		145	21025	0.315068	0.099268	45.68493
155.7	150	11.65	11.8	115	0.265647		150	22500	0.315068	0.099268	47.26027
155.7	155	8.8	8.95	115	0.258328		155	24025	0.315068	0.099268	48.83562
155.7	160	6.45	6.6	115	0.252759		160	25600	0.315068	0.099268	50.41096
155.7	165	4.6	4.7	115	0.248229		165	27225	0.315068	0.099268	51.9863
155.7	170	3.15	3.25	115	0.244139		170	28900	0.315068	0.099268	53.56164
155.7	175	2.15	2.19	115	0.242347		175	30625	0.315068	0.099268	55.13699
155.7	180	1.44	1.48	115	0.242289		180	32400	0.315068	0.099268	56.71233
155.7	185	0.96	0.99	115	0.243176		185	34225	0.315068	0.099268	58.28767
155.7	190	0.65	0.67	115	0.245681		190	36100	0.315068	0.099268	59.86301
155.7	100	54.9	57.3	135	0.486075		100	10000	0.369863	0.136799	36.9863
155.7	105	51.15	52.05	135	0.484665		105	11025	0.369863	0.136799	38.83562
155.7	110	46	46.55	135	0.413072		110	12100	0.369863	0.136799	40.68493
155.7	115	40.7	41.7	135	0.364338		115	13225	0.369863	0.136799	42.53425
155.7	120	36.45	37	135	0.358648		120	14400	0.369863	0.136799	44.38356
155.7	125	31.9	32.35	135	0.337788		125	15625	0.369863	0.136799	46.23288
155.7	130	27.55	27.9	135	0.321797		130	16900	0.369863	0.136799	48.08219
155.7	135	23.35	23.7	135	0.307602		135	18225	0.369863	0.136799	49.93151
155.7	140	19.45	19.65	135	0.294102		140	19600	0.369863	0.136799	51.78082
155.7	145	15.9	16.1	135	0.285768		145	21025	0.369863	0.136799	53.63014
155.7	150	12.7	12.9	135	0.277995		150	22500	0.369863	0.136799	55.47945
155.7	155	9.95	10.1	135	0.271833		155	24025	0.369863	0.136799	57.32877
155.7	160	7.6	7.75	135	0.266689		160	25600	0.369863	0.136799	59.17808
155.7	165	5.65	5.8	135	0.261922		165	27225	0.369863	0.136799	61.0274
155.7	170	4.15	4.25	135	0.258884		170	28900	0.369863	0.136799	62.87671

155.7	175	2.98	3.05	135	0.256348		175	30625	0.369863	0.136799	64.72603
155.7	180	2.11	2.16	135	0.254877		180	32400	0.369863	0.136799	66.57534
155.7	185	1.49	1.53	135	0.254948		185	34225	0.369863	0.136799	68.42466
155.7	190	1.05	1.09	135	0.25612		190	36100	0.369863	0.136799	70.27397
155.7	100	55.55	55.85	180	0.405807		100	10000	0.493151	0.243198	49.31507
155.7	105	50.7	51.15	180	0.387179		105	11025	0.493151	0.243198	51.78082
155.7	110	45.85	46.35	180	0.362114		110	12100	0.493151	0.243198	54.24658
155.7	115	41.5	41.95	180	0.362237		115	13225	0.493151	0.243198	56.71233
155.7	120	36.8	37	180	0.331937		120	14400	0.493151	0.243198	59.17808
155.7	125	32.2	32.5	180	0.313033		125	15625	0.493151	0.243198	61.64384
155.7	130	28.15	28.25	180	0.30544		130	16900	0.493151	0.243198	64.10959
155.7	135	24.15	24.3	180	0.296787		135	18225	0.493151	0.243198	66.57534
155.7	140	20.45	20.65	180	0.290072		140	19600	0.493151	0.243198	69.0411
155.7	145	17.1	17.2	180	0.283333		145	21025	0.493151	0.243198	71.50685
155.7	150	14.05	14.15	180	0.277756		150	22500	0.493151	0.243198	73.9726
155.7	155	11.4	11.45	180	0.273324		155	24025	0.493151	0.243198	76.43836
155.7	160	9	9.15	180	0.26863		160	25600	0.493151	0.243198	78.90411
155.7	165	7.05	7.15	180	0.264927		165	27225	0.493151	0.243198	81.36986
155.7	170	5.4	5.5	180	0.261395		170	28900	0.493151	0.243198	83.83562
155.7	175	4.1	4.2	180	0.259391		175	30625	0.493151	0.243198	86.30137
155.7	180	3.05	3.15	180	0.257234		180	32400	0.493151	0.243198	88.76712
155.7	185	2.29	2.33	180	0.256431		185	34225	0.493151	0.243198	91.23288
155.7	190	1.7	1.73	180	0.256389		190	36100	0.493151	0.243198	93.69863

				BSC(S, K, r, q, sigma, T)							
										IV Fit	Call Price
										0.623932987	55.76035664
										0.574744502	50.79899621
										0.52874141	45.85036114
										0.48592371	40.92102085
SUMMARY OUTPUT										0.446291403	36.02126566
										0.409844488	31.1675711
Regression Statistics										0.376582966	26.38663877
Multiple R	0.8484986									0.346506836	21.72167313
R Square	0.7199499									0.319616099	17.2410731
Adjusted R Square	0.7042167									0.295910755	13.04766328
Standard Error	0.0634237									0.275390803	9.281737367
Observations	95									0.258056244	6.104776576
										0.243907078	3.651859187
ANOVA										0.232943304	1.964173903
	df	*SS*	*MS*	*F*	*Significance F*					0.225164923	0.950242158
Regression	5	0.920363	0.184073	45.76005	3.63E-23					0.220571934	0.421399534
Residual	89	0.358008	0.004023							0.219164338	0.178350542
Total	94	1.278371								0.220942135	0.076329687
										0.225905324	0.035240614
	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*		0.590870071	55.92969866
Intercept	2.4869899	0.220595	11.27401	7.82E-19	2.048672	2.925308	2.048672	2.92530751		0.544055847	50.98189855
K	-0.024085	0.00289	-8.3329	8.98E-13	-0.02983	-0.01834	-0.02983	-0.0183419		0.500427016	46.05177091
K^2	6.371E-05	9.74E-06	6.540223	3.77E-09	4.44E-05	8.31E-05	4.44E-05	8.3063E-05		0.459983577	41.14776522

T	-1.582438	0.407399	-3.88424	0.000197	-2.39193	-0.77294	-2.39193	-0.7729439		0.422725531	36.28253552
T^2	0.3419867	0.481145	0.710777	0.479081	-0.61404	1.298012	-0.61404	1.29801188		0.388652877	31.47542096
KT	0.0086661	0.00187	4.633283	1.22E-05	0.00495	0.012382	0.00495	0.01238248		0.357765616	26.75623248
										0.330063748	22.17062435
										0.305547272	17.78663836
										0.284216189	13.70010503
										0.266070498	10.03317116
										0.2511102	6.9172726
										0.239335295	4.455710337
										0.230745782	2.67765811
										0.225341662	1.514791904
										0.223122935	0.824861379
										0.2240896	0.447659193
										0.228241657	0.252537268
										0.235579108	0.154289914
										0.523986944	56.1819195
										0.482514807	51.24874028
										0.444228063	46.34262812
										0.409126712	41.47526468
										0.377210753	36.66342025
										0.348480187	31.93153703
										0.322935014	27.31521724
										0.300575233	22.86533124
										0.281400845	18.65151427
										0.265411849	14.76222823
										0.252608246	11.29710281
										0.242990036	8.348294341
										0.236557218	5.973764067
										0.233309793	4.174683046

										0.23324776	2.892014186
										0.23637112	2.026225025
										0.242679873	1.467787695
										0.252174018	1.121487486
										0.264853556	0.916812095
										0.497598191	56.21515753
										0.458500316	51.28784489
										0.422587833	46.39186985
										0.389860743	41.54032254
										0.360319045	36.75185746
										0.33396274	32.05334773
										0.310791828	27.48325765
										0.290806308	23.09519926
										0.274006181	18.9601278
										0.260391447	15.16427341
										0.249962105	11.79930334
										0.242718155	8.943491371
										0.238659599	6.638903378
										0.237786434	4.875988798
										0.240098663	3.595745728
										0.245596284	2.708828201
										0.254279298	2.120236948
										0.266147704	1.74768648
										0.281201503	1.529352378
										0.445731931	56.15390641
										0.411976143	51.24781012
										0.381405748	46.38402811
										0.354020745	41.5794131
										0.329821135	36.85775895

										0.308806918	32.2525368
										0.290978093	27.8097297
										0.276334661	23.58962018
										0.264876621	19.66549037
										0.256603974	16.11684189
										0.25151672	13.01620137
										0.249614858	10.41241404
										0.250898389	8.317435722
										0.255367312	6.703724628
										0.263021628	5.513753064
										0.273861337	4.676264133
										0.287886438	4.121575954
										0.305096932	3.791202631
										0.325492818	3.641385115

Table 6.3: Summary Statistics for $\sigma_{iv} = a_0 + a_1 K + a_2 K^2 + a_3 T + a_4 T^2 + a_5 KT$

Table 6.4: summary statistics for (see directly IV model below)

$$\sigma_{iv} = a_0 + a_1K + a_2T + a_3T^2 + a_4KT$$

Stock price	Strike	Bid	Ask	Time in days	IV
155.7	100	55.95	56.85	50	0.774191
155.7	105	50.95	51.9	50	0.706708
155.7	110	46	46.9	50	0.641367
155.7	115	40.65	41.25	50	0.491893
155.7	120	36.05	36.8	50	0.506019
155.7	125	30.8	31.3	50	0.392778
155.7	130	25.9	26.4	50	0.348723
155.7	135	21.1	21.55	50	0.309455
155.7	140	16.4	16.8	50	0.272412
155.7	145	12.15	12.35	50	0.249722
155.7	150	8.35	8.5	50	0.234246
155.7	155	5.2	5.35	50	0.221639
155.7	160	3	3.1	50	0.216386
155.7	165	1.63	1.67	50	0.215601
155.7	170	0.85	0.89	50	0.219031
155.7	175	0.46	0.48	50	0.226274
155.7	180	0.26	0.28	50	0.236576
155.7	185	0.12	0.15	50	0.240526
155.7	190	0.08	0.09	50	0.252824
155.7	100	55	57.05	70	0.609
155.7	105	51	51.95	70	0.617533
155.7	110	42.75	43.35	70	4.66E-09
155.7	115	41.15	42.05	70	0.513328
155.7	120	35.6	36.55	70	0.397738
155.7	125	31.1	31.5	70	0.371379
155.7	130	26.3	26.8	70	0.341073
155.7	135	21.7	22.15	70	0.31383
155.7	140	17.35	17.6	70	0.288633
155.7	145	13.35	13.5	70	0.27175
155.7	150	9.8	9.95	70	0.259803
155.7	155	6.85	6.95	70	0.250469
155.7	160	4.55	4.65	70	0.244817
155.7	165	2.86	2.92	70	0.239838
155.7	170	1.75	1.8	70	0.239119
155.7	175	1.06	1.09	70	0.240804
155.7	180	0.64	0.67	70	0.244649

155.7	185	0.39	0.42	70	0.249899
155.7	190	0.25	0.27	70	0.256856
155.7	100	55	57.15	115	0.511666
155.7	105	51.15	52	115	0.513542
155.7	110	45.5	46.5	115	0.410411
155.7	115	41.35	42.2	115	0.432147
155.7	120	36.2	36.8	115	0.365417
155.7	125	31.6	32.05	115	0.341927
155.7	130	27	27.4	115	0.316885
155.7	135	22.65	23	115	0.29878
155.7	140	18.65	18.85	115	0.285154
155.7	145	14.95	15.15	115	0.275033
155.7	150	11.65	11.8	115	0.265647
155.7	155	8.8	8.95	115	0.258328
155.7	160	6.45	6.6	115	0.252759
155.7	165	4.6	4.7	115	0.248229
155.7	170	3.15	3.25	115	0.244139
155.7	175	2.15	2.19	115	0.242347
155.7	180	1.44	1.48	115	0.242289
155.7	185	0.96	0.99	115	0.243176
155.7	190	0.65	0.67	115	0.245681
155.7	100	54.9	57.3	135	0.486075
155.7	105	51.15	52.05	135	0.484665
155.7	110	46	46.55	135	0.413072
155.7	115	40.7	41.7	135	0.364338
155.7	120	36.45	37	135	0.358648
155.7	125	31.9	32.35	135	0.337788
155.7	130	27.55	27.9	135	0.321797
155.7	135	23.35	23.7	135	0.307602
155.7	140	19.45	19.65	135	0.294102
155.7	145	15.9	16.1	135	0.285768
155.7	150	12.7	12.9	135	0.277995
155.7	155	9.95	10.1	135	0.271833
155.7	160	7.6	7.75	135	0.266689
155.7	165	5.65	5.8	135	0.261922
155.7	170	4.15	4.25	135	0.258884
155.7	175	2.98	3.05	135	0.256348
155.7	180	2.11	2.16	135	0.254877
155.7	185	1.49	1.53	135	0.254948
155.7	190	1.05	1.09	135	0.25612
155.7	100	55.55	55.85	180	0.405807
155.7	105	50.7	51.15	180	0.387179
155.7	110	45.85	46.35	180	0.362114

155.7	115	41.5	41.95	180	0.362237
155.7	120	36.8	37	180	0.331937
155.7	125	32.2	32.5	180	0.313033
155.7	130	28.15	28.25	180	0.30544
155.7	135	24.15	24.3	180	0.296787
155.7	140	20.45	20.65	180	0.290072
155.7	145	17.1	17.2	180	0.283333
155.7	150	14.05	14.15	180	0.277756
155.7	155	11.4	11.45	180	0.273324
155.7	160	9	9.15	180	0.26863
155.7	165	7.05	7.15	180	0.264927
155.7	170	5.4	5.5	180	0.261395
155.7	175	4.1	4.2	180	0.259391
155.7	180	3.05	3.15	180	0.257234
155.7	185	2.29	2.33	180	0.256431
155.7	190	1.7	1.73	180	0.256389

We now determine the values of the parameters from the multiple regression as shown below:

K	T	T^2	KT
100	0.136986	0.018765	13.69863
105	0.136986	0.018765	14.38356
110	0.136986	0.018765	15.06849
115	0.136986	0.018765	15.75342
120	0.136986	0.018765	16.43836
125	0.136986	0.018765	17.12329
130	0.136986	0.018765	17.80822
135	0.136986	0.018765	18.49315
140	0.136986	0.018765	19.17808
145	0.136986	0.018765	19.86301
150	0.136986	0.018765	20.54795
155	0.136986	0.018765	21.23288
160	0.136986	0.018765	21.91781
165	0.136986	0.018765	22.60274
170	0.136986	0.018765	23.28767
175	0.136986	0.018765	23.9726
180	0.136986	0.018765	24.65753
185	0.136986	0.018765	25.34247
190	0.136986	0.018765	26.0274
100	0.191781	0.03678	19.17808
105	0.191781	0.03678	20.13699
110	0.191781	0.03678	21.09589
115	0.191781	0.03678	22.05479
120	0.191781	0.03678	23.0137
125	0.191781	0.03678	23.9726
130	0.191781	0.03678	24.93151

SUMMARY OUTPUT

Regression Statistics	
Multiple F	0.765084
R Square	0.585354
Adjusted I	0.566926
Standard E	0.076744
Observati	95

ANOVA

	df	*SS*	*MS*	*F*	*gnificance F*
Regressio	4	0.7483	0.187075	31.76319	1.71E-16
Residual	90	0.530071	0.00589		
Total	94	1.278371			

	Coefficients	*andard Err*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*ower 95.0%*	*Upper 95.0%*
Intercept	1.195313	0.118909	10.05234	2.22E-16	0.95908	1.431547	0.95908	1.431546557
K	-0.00561	0.00074	-7.5787	2.98E-11	-0.00708	-0.00414	-0.00708	-0.004139149
T	-1.58244	0.492963	-3.21005	0.00184	-2.5618	-0.60308	-2.5618	-0.603079631
T^2	0.341987	0.582198	0.587406	0.558402	-0.81465	1.498624	-0.81465	1.498624287
KT	0.008666	0.002263	3.829079	0.000237	0.00417	0.013162	0.00417	0.013162337

125	0.191781	0.03678	23.9726	155	0.369863	0.136799	57.32877
130	0.191781	0.03678	24.93151	160	0.369863	0.136799	59.17808
135	0.191781	0.03678	25.89041	165	0.369863	0.136799	61.0274
140	0.191781	0.03678	26.84932	170	0.369863	0.136799	62.87671
145	0.191781	0.03678	27.80822	175	0.369863	0.136799	64.72603
150	0.191781	0.03678	28.76712	180	0.369863	0.136799	66.57534
155	0.191781	0.03678	29.72603	185	0.369863	0.136799	68.42466
160	0.191781	0.03678	30.68493	190	0.369863	0.136799	70.27397
165	0.191781	0.03678	31.64384	100	0.493151	0.243198	49.31507
170	0.191781	0.03678	32.60274	105	0.493151	0.243198	51.78082
175	0.191781	0.03678	33.56164	110	0.493151	0.243198	54.24658
180	0.191781	0.03678	34.52055	115	0.493151	0.243198	56.71233
185	0.191781	0.03678	35.47945	120	0.493151	0.243198	59.17808
190	0.191781	0.03678	36.43836	125	0.493151	0.243198	61.64384
100	0.315068	0.099268	31.50685	130	0.493151	0.243198	64.10959
105	0.315068	0.099268	33.08219	135	0.493151	0.243198	66.57534
110	0.315068	0.099268	34.65753	140	0.493151	0.243198	69.0411
115	0.315068	0.099268	36.23288	145	0.493151	0.243198	71.50685
120	0.315068	0.099268	37.80822	150	0.493151	0.243198	73.9726
125	0.315068	0.099268	39.38356	155	0.493151	0.243198	76.43836
130	0.315068	0.099268	40.9589	160	0.493151	0.243198	78.90411
135	0.315068	0.099268	42.53425	165	0.493151	0.243198	81.36986
140	0.315068	0.099268	44.10959	170	0.493151	0.243198	83.83562
145	0.315068	0.099268	45.68493	175	0.493151	0.243198	86.30137
150	0.315068	0.099268	47.26027	180	0.493151	0.243198	88.76712
155	0.315068	0.099268	48.83562	185	0.493151	0.243198	91.23288
160	0.315068	0.099268	50.41096	190	0.493151	0.243198	93.69863
165	0.315068	0.099268	51.9863				
170	0.315068	0.099268	53.56164				
175	0.315068	0.099268	55.13699				
180	0.315068	0.099268	56.71233				
185	0.315068	0.099268	58.28767				
190	0.315068	0.099268	59.86301				
100	0.369863	0.136799	36.9863				
105	0.369863	0.136799	38.83562				
110	0.369863	0.136799	40.68493				
115	0.369863	0.136799	42.53425				
120	0.369863	0.136799	44.38356				
125	0.369863	0.136799	46.23288				
130	0.369863	0.136799	48.08219				
135	0.369863	0.136799	49.93151				
+140	0.369863	0.136799	51.78082				
145	0.369863	0.136799	53.63014				

150	0.369863	0.136799	55.47945

Table 6.5: Regression Statistics/ANOVA table for $\sigma_{iv} = a_0 + a_1K + a_2T + a_3K^2 + a_4KT$

Stock price	Strike	Bid	Ask	Time in days
155.7	100	55.95	56.85	50
155.7	105	50.95	51.9	50
155.7	110	46	46.9	50
155.7	115	40.65	41.25	50
155.7	120	36.05	36.8	50
155.7	125	30.8	31.3	50
155.7	130	25.9	26.4	50
155.7	135	21.1	21.55	50
155.7	140	16.4	16.8	50
155.7	145	12.15	12.35	50
155.7	150	8.35	8.5	50
155.7	155	5.2	5.35	50
155.7	160	3	3.1	50
155.7	165	1.63	1.67	50
155.7	170	0.85	0.89	50
155.7	175	0.46	0.48	50
155.7	180	0.26	0.28	50
155.7	185	0.12	0.15	50
155.7	190	0.08	0.09	50
155.7	100	55	57.05	70
155.7	105	51	51.95	70
155.7	110	42.75	43.35	70
155.7	115	41.15	42.05	70
155.7	120	35.6	36.55	70
155.7	125	31.1	31.5	70
155.7	130	26.3	26.8	70
155.7	135	21.7	22.15	70
155.7	140	17.35	17.6	70
155.7	145	13.35	13.5	70
155.7	150	9.8	9.95	70
155.7	155	6.85	6.95	70
155.7	160	4.55	4.65	70
155.7	165	2.86	2.92	70
155.7	170	1.75	1.8	70
155.7	175	1.06	1.09	70

155.7	180	0.64	0.67	70
155.7	185	0.39	0.42	70
155.7	190	0.25	0.27	70
155.7	100	55	57.15	115
155.7	105	51.15	52	115
155.7	110	45.5	46.5	115
155.7	115	41.35	42.2	115
155.7	120	36.2	36.8	115
155.7	125	31.6	32.05	115
155.7	130	27	27.4	115
155.7	135	22.65	23	115
155.7	140	18.65	18.85	115
155.7	145	14.95	15.15	115
155.7	150	11.65	11.8	115
155.7	155	8.8	8.95	115
155.7	160	6.45	6.6	115
155.7	165	4.6	4.7	115
155.7	170	3.15	3.25	115
155.7	175	2.15	2.19	115
155.7	180	1.44	1.48	115
155.7	185	0.96	0.99	115
155.7	190	0.65	0.67	115
155.7	100	54.9	57.3	135
155.7	105	51.15	52.05	135
155.7	110	46	46.55	135
155.7	115	40.7	41.7	135
155.7	120	36.45	37	135
155.7	125	31.9	32.35	135
155.7	130	27.55	27.9	135
155.7	135	23.35	23.7	135
155.7	140	19.45	19.65	135
155.7	145	15.9	16.1	135
155.7	150	12.7	12.9	135
155.7	155	9.95	10.1	135
155.7	160	7.6	7.75	135
155.7	165	5.65	5.8	135
155.7	170	4.15	4.25	135
155.7	175	2.98	3.05	135
155.7	180	2.11	2.16	135
155.7	185	1.49	1.53	135
155.7	190	1.05	1.09	135
155.7	100	55.55	55.85	180
155.7	105	50.7	51.15	180

155.7	110	45.85	46.35	180
155.7	115	41.5	41.95	180
155.7	120	36.8	37	180
155.7	125	32.2	32.5	180
155.7	130	28.15	28.25	180
155.7	135	24.15	24.3	180
155.7	140	20.45	20.65	180
155.7	145	17.1	17.2	180
155.7	150	14.05	14.15	180
155.7	155	11.4	11.45	180
155.7	160	9	9.15	180
155.7	165	7.05	7.15	180
155.7	170	5.4	5.5	180
155.7	175	4.1	4.2	180
155.7	180	3.05	3.15	180
155.7	185	2.29	2.33	180
155.7	190	1.7	1.73	180

For the implied volatility parameter estimations we shall have:

$$\sigma_{iv} = a_0 + a_1 K + a_2 T + a_3 K^2 + a_4 KT$$

IV		T	K	K^2	KT
0.774191		0.136986	100	10000	13.69863
0.706708		0.136986	105	11025	14.38356
0.641367		0.136986	110	12100	15.06849
0.491893		0.136986	115	13225	15.75342
0.506019		0.136986	120	14400	16.43836
0.392778		0.136986	125	15625	17.12329
0.348723		0.136986	130	16900	17.80822
0.309455		0.136986	135	18225	18.49315
0.272412		0.136986	140	19600	19.17808
0.249722		0.136986	145	21025	19.86301
0.234246		0.136986	150	22500	20.54795
0.221639		0.136986	155	24025	21.23288
0.216386		0.136986	160	25600	21.91781
0.215601		0.136986	165	27225	22.60274
0.219031		0.136986	170	28900	23.28767
0.226274		0.136986	175	30625	23.9726
0.236576		0.136986	180	32400	24.65753
0.240526		0.136986	185	34225	25.34247
0.252824		0.136986	190	36100	26.0274

0.609		0.191781	100	10000	19.17808
0.617533		0.191781	105	11025	20.13699
4.66E-09		0.191781	110	12100	21.09589
0.513328		0.191781	115	13225	22.05479
0.397738		0.191781	120	14400	23.0137
0.371379		0.191781	125	15625	23.9726
0.341073		0.191781	130	16900	24.93151
0.31383		0.191781	135	18225	25.89041
0.288633		0.191781	140	19600	26.84932
0.27175		0.191781	145	21025	27.80822
0.259803		0.191781	150	22500	28.76712
0.250469		0.191781	155	24025	29.72603
0.244817		0.191781	160	25600	30.68493
0.239838		0.191781	165	27225	31.64384
0.239119		0.191781	170	28900	32.60274
0.240804		0.191781	175	30625	33.56164
0.244649		0.191781	180	32400	34.52055
0.249899		0.191781	185	34225	35.47945
0.256856		0.191781	190	36100	36.43836
0.511666		0.315068	100	10000	31.50685
0.513542		0.315068	105	11025	33.08219
0.410411		0.315068	110	12100	34.65753
0.432147		0.315068	115	13225	36.23288
0.365417		0.315068	120	14400	37.80822
0.341927		0.315068	125	15625	39.38356
0.316885		0.315068	130	16900	40.9589
0.29878		0.315068	135	18225	42.53425
0.285154		0.315068	140	19600	44.10959
0.275033		0.315068	145	21025	45.68493
0.265647		0.315068	150	22500	47.26027
0.258328		0.315068	155	24025	48.83562
0.252759		0.315068	160	25600	50.41096
0.248229		0.315068	165	27225	51.9863
0.244139		0.315068	170	28900	53.56164
0.242347		0.315068	175	30625	55.13699
0.242289		0.315068	180	32400	56.71233
0.243176		0.315068	185	34225	58.28767
0.245681		0.315068	190	36100	59.86301
0.486075		0.369863	100	10000	36.9863
0.484665		0.369863	105	11025	38.83562
0.413072		0.369863	110	12100	40.68493
0.364338		0.369863	115	13225	42.53425
0.358648		0.369863	120	14400	44.38356

0.337788		0.369863	125	15625	46.23288
0.321797		0.369863	130	16900	48.08219
0.307602		0.369863	135	18225	49.93151
0.294102		0.369863	140	19600	51.78082
0.285768		0.369863	145	21025	53.63014
0.277995		0.369863	150	22500	55.47945
0.271833		0.369863	155	24025	57.32877
0.266689		0.369863	160	25600	59.17808
0.261922		0.369863	165	27225	61.0274
0.258884		0.369863	170	28900	62.87671
0.256348		0.369863	175	30625	64.72603
0.254877		0.369863	180	32400	66.57534
0.254948		0.369863	185	34225	68.42466
0.25612		0.369863	190	36100	70.27397
0.405807		0.493151	100	10000	49.31507
0.387179		0.493151	105	11025	51.78082
0.362114		0.493151	110	12100	54.24658
0.362237		0.493151	115	13225	56.71233
0.331937		0.493151	120	14400	59.17808
0.313033		0.493151	125	15625	61.64384
0.30544		0.493151	130	16900	64.10959
0.296787		0.493151	135	18225	66.57534
0.290072		0.493151	140	19600	69.0411
0.283333		0.493151	145	21025	71.50685
0.277756		0.493151	150	22500	73.9726
0.273324		0.493151	155	24025	76.43836
0.26863		0.493151	160	25600	78.90411
0.264927		0.493151	165	27225	81.36986
0.261395		0.493151	170	28900	83.83562
0.259391		0.493151	175	30625	86.30137
0.257234		0.493151	180	32400	88.76712
0.256431		0.493151	185	34225	91.23288
0.256389		0.493151	190	36100	93.69863

Table 6.6: Summary Statistics for IV model shown below

$$\sigma_{iv} = a_0 + a_1 S/K + a_2(S/K)^2 + a_3 T$$

Stock price	Strike	Bid	Ask	Time in days
155.7	100	55.95	56.85	50
155.7	105	50.95	51.9	50
155.7	110	46	46.9	50
155.7	115	40.65	41.25	50
155.7	120	36.05	36.8	50
155.7	125	30.8	31.3	50
155.7	130	25.9	26.4	50
155.7	135	21.1	21.55	50
155.7	140	16.4	16.8	50
155.7	145	12.15	12.35	50
155.7	150	8.35	8.5	50
155.7	155	5.2	5.35	50
155.7	160	3	3.1	50
155.7	165	1.63	1.67	50
155.7	170	0.85	0.89	50
155.7	175	0.46	0.48	50
155.7	180	0.26	0.28	50
155.7	185	0.12	0.15	50
155.7	190	0.08	0.09	50
155.7	100	55	57.05	70
155.7	105	51	51.95	70
155.7	110	42.75	43.35	70
155.7	115	41.15	42.05	70
155.7	120	35.6	36.55	70
155.7	125	31.1	31.5	70
155.7	130	26.3	26.8	70
155.7	135	21.7	22.15	70
155.7	140	17.35	17.6	70
155.7	145	13.35	13.5	70

155.7	150	9.8	9.95	70
155.7	155	6.85	6.95	70
155.7	160	4.55	4.65	70
155.7	165	2.86	2.92	70
155.7	170	1.75	1.8	70
155.7	175	1.06	1.09	70
155.7	180	0.64	0.67	70
155.7	185	0.39	0.42	70
155.7	190	0.25	0.27	70
155.7	100	55	57.15	115
155.7	105	51.15	52	115
155.7	110	45.5	46.5	115
155.7	115	41.35	42.2	115
155.7	120	36.2	36.8	115
155.7	125	31.6	32.05	115
155.7	130	27	27.4	115
155.7	135	22.65	23	115
155.7	140	18.65	18.85	115
155.7	145	14.95	15.15	115
155.7	150	11.65	11.8	115
155.7	155	8.8	8.95	115
155.7	160	6.45	6.6	115
155.7	165	4.6	4.7	115
155.7	170	3.15	3.25	115
155.7	175	2.15	2.19	115
155.7	180	1.44	1.48	115
155.7	185	0.96	0.99	115
155.7	190	0.65	0.67	115
155.7	100	54.9	57.3	135
155.7	105	51.15	52.05	135
155.7	110	46	46.55	135
155.7	115	40.7	41.7	135
155.7	120	36.45	37	135
155.7	125	31.9	32.35	135
155.7	130	27.55	27.9	135
155.7	135	23.35	23.7	135
155.7	140	19.45	19.65	135
155.7	145	15.9	16.1	135
155.7	150	12.7	12.9	135
155.7	155	9.95	10.1	135
155.7	160	7.6	7.75	135
155.7	165	5.65	5.8	135
155.7	170	4.15	4.25	135

155.7	175	2.98	3.05	135
155.7	180	2.11	2.16	135
155.7	185	1.49	1.53	135
155.7	190	1.05	1.09	135
155.7	100	55.55	55.85	180
155.7	105	50.7	51.15	180
155.7	110	45.85	46.35	180
155.7	115	41.5	41.95	180
155.7	120	36.8	37	180
155.7	125	32.2	32.5	180
155.7	130	28.15	28.25	180
155.7	135	24.15	24.3	180
155.7	140	20.45	20.65	180
155.7	145	17.1	17.2	180
155.7	150	14.05	14.15	180
155.7	155	11.4	11.45	180
155.7	160	9	9.15	180
155.7	165	7.05	7.15	180
155.7	170	5.4	5.5	180
155.7	175	4.1	4.2	180
155.7	180	3.05	3.15	180
155.7	185	2.29	2.33	180
155.7	190	1.7	1.73	180

IV		S/K	S/K^2	T
0.774191		1.557	2.424249	0.136986
0.706708		1.482857	2.198865	0.136986
0.641367		1.415455	2.003512	0.136986
0.491893		1.353913	1.833081	0.136986
0.506019		1.2975	1.683506	0.136986
0.392778		1.2456	1.551519	0.136986
0.348723		1.197692	1.434467	0.136986
0.309455		1.153333	1.330178	0.136986
0.272412		1.112143	1.236862	0.136986
0.249722		1.073793	1.153032	0.136986
0.234246		1.038	1.077444	0.136986
0.221639		1.004516	1.009053	0.136986
0.216386		0.973125	0.946972	0.136986
0.215601		0.943636	0.89045	0.136986
0.219031		0.915882	0.83884	0.136986
0.226274		0.889714	0.791592	0.136986
0.236576		0.865	0.748225	0.136986

0.240526		0.841622	0.708327	0.136986
0.252824		0.819474	0.671537	0.136986
0.609		1.557	2.424249	0.191781
0.617533		1.482857	2.198865	0.191781
4.66E-09		1.415455	2.003512	0.191781
0.513328		1.353913	1.833081	0.191781
0.397738		1.2975	1.683506	0.191781
0.371379		1.2456	1.551519	0.191781
0.341073		1.197692	1.434467	0.191781
0.31383		1.153333	1.330178	0.191781
0.288633		1.112143	1.236862	0.191781
0.27175		1.073793	1.153032	0.191781
0.259803		1.038	1.077444	0.191781
0.250469		1.004516	1.009053	0.191781
0.244817		0.973125	0.946972	0.191781
0.239838		0.943636	0.89045	0.191781
0.239119		0.915882	0.83884	0.191781
0.240804		0.889714	0.791592	0.191781
0.244649		0.865	0.748225	0.191781
0.249899		0.841622	0.708327	0.191781
0.256856		0.819474	0.671537	0.191781
0.511666		1.557	2.424249	0.315068
0.513542		1.482857	2.198865	0.315068
0.410411		1.415455	2.003512	0.315068
0.432147		1.353913	1.833081	0.315068
0.365417		1.2975	1.683506	0.315068
0.341927		1.2456	1.551519	0.315068
0.316885		1.197692	1.434467	0.315068
0.29878		1.153333	1.330178	0.315068
0.285154		1.112143	1.236862	0.315068
0.275033		1.073793	1.153032	0.315068
0.265647		1.038	1.077444	0.315068
0.258328		1.004516	1.009053	0.315068
0.252759		0.973125	0.946972	0.315068
0.248229		0.943636	0.89045	0.315068
0.244139		0.915882	0.83884	0.315068
0.242347		0.889714	0.791592	0.315068
0.242289		0.865	0.748225	0.315068
0.243176		0.841622	0.708327	0.315068
0.245681		0.819474	0.671537	0.315068
0.486075		1.557	2.424249	0.369863
0.484665		1.482857	2.198865	0.369863
0.413072		1.415455	2.003512	0.369863

0.364338		1.353913	1.833081	0.369863
0.358648		1.2975	1.683506	0.369863
0.337788		1.2456	1.551519	0.369863
0.321797		1.197692	1.434467	0.369863
0.307602		1.153333	1.330178	0.369863
0.294102		1.112143	1.236862	0.369863
0.285768		1.073793	1.153032	0.369863
0.277995		1.038	1.077444	0.369863
0.271833		1.004516	1.009053	0.369863
0.266689		0.973125	0.946972	0.369863
0.261922		0.943636	0.89045	0.369863
0.258884		0.915882	0.83884	0.369863
0.256348		0.889714	0.791592	0.369863
0.254877		0.865	0.748225	0.369863
0.254948		0.841622	0.708327	0.369863
0.25612		0.819474	0.671537	0.369863
0.405807		1.557	2.424249	0.493151
0.387179		1.482857	2.198865	0.493151
0.362114		1.415455	2.003512	0.493151
0.362237		1.353913	1.833081	0.493151
0.331937		1.2975	1.683506	0.493151
0.313033		1.2456	1.551519	0.493151
0.30544		1.197692	1.434467	0.493151
0.296787		1.153333	1.330178	0.493151
0.290072		1.112143	1.236862	0.493151
0.283333		1.073793	1.153032	0.493151
0.277756		1.038	1.077444	0.493151
0.273324		1.004516	1.009053	0.493151
0.26863		0.973125	0.946972	0.493151
0.264927		0.943636	0.89045	0.493151
0.261395		0.915882	0.83884	0.493151
0.259391		0.889714	0.791592	0.493151
0.257234		0.865	0.748225	0.493151
0.256431		0.841622	0.708327	0.493151
0.256389		0.819474	0.671537	0.493151

We now present a summarized **ANOVA** table and **SUMMARY STATISTICS** for the absolute and relative smile models considered in this thesis to three decimals places.

(1) For the absolute smile model DVF_{AS}: $\boldsymbol{\sigma_{iv} = a_0 + a_1 K + a_2 K^2 + a_3 T + a_4 T^2 + a_5 KT}$ we have the following Summary Statistics/ANOVA table:

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.848
R Square	0.720
Adjusted R Square	0.704
Standard Error	0.063
Observations	95

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	5	0.920	0.184	45.760	0.000
Residual	89	0.358	0.004		
Total	94	1.278			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	2.487	0.221	11.274	0.000	2.049	2.925	2.049	2.925
K	-0.024	0.003	-8.333	0.000	-0.030	-0.018	-0.030	-0.018
K^2	0.000	0.000	6.540	0.000	0.000	0.000	0.000	0.000
T	-1.582	0.407	-3.884	0.000	-2.392	-0.773	-2.392	-0.773
T^2	0.342	0.481	0.711	0.479	-0.614	1.298	-0.614	1.298
KT	0.009	0.002	4.633	0.000	0.005	0.012	0.005	0.012

Table 6.7: Summary Output table for $\boldsymbol{\sigma_{iv} = a_0 + a_1 K + a_2 K^2 + a_3 T + a_4 T^2 + a_5 KT}$.

(2) For the absolute smile model given by $\boldsymbol{\sigma_{iv} = a_0 + a_1K + a_2T + a_3T^2 + a_4KT}$ we have the following output:

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.765
R Square	0.585
Adjusted R Square	0.567
Standard Error	0.077
Observations	95.000

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	4.000	0.748	0.187	31.763	0.000
Residual	90.000	0.530	0.006		
Total	94.000	1.278			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	1.195	0.119	10.052	0.000	0.959	1.432	0.959	1.432
K	-0.006	0.001	-7.579	0.000	-0.007	-0.004	-0.007	-0.004
T	-1.582	0.493	-3.210	0.002	-2.562	-0.603	-2.562	-0.603
T^2	0.342	0.582	0.587	0.558	-0.815	1.499	-0.815	1.499
KT	0.009	0.002	3.829	0.000	0.004	0.013	0.004	0.013

Table 6.8: The Summary statistics of the model $\boldsymbol{\sigma_{iv} = a_0 + a_1K + a_2T + a_3T^2 + a_4KT.}$

(3) Next we look at another absolute smile model given by $\boldsymbol{\sigma_{iv} = a_0 + a_1K + a_2T + a_3K^2 + a_4KT}$ that is the same with the earlier absolute smile model treated in (1) above except that the quadratic term here is K^2 instead of T^2 as before. We then compute the Summary Statistics/ANOVA table to determine the better model for implied volatility estimations.

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.848
R Square	0.718
Adjusted R Squa	0.706
Standard Error	0.063
Observations	95

ANOVA

	df	SS	MS	F	Significance F
Regression	4	0.918	0.230	57.389	0.000
Residual	90	0.360	0.004		
Total	94	1.278			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95.0%	Upper 95.0%
Intercept	2.459	0.217	11.357	0.000	2.029	2.890	2.029	2.890
T	-1.369	0.275	-4.975	0.000	-1.916	-0.823	-1.916	-0.823
K	-0.024	0.003	-8.356	0.000	-0.030	-0.018	-0.030	-0.018
K^2	0.000	0.000	6.558	0.000	0.000	0.000	0.000	0.000
KT	0.009	0.002	4.646	0.000	0.005	0.012	0.005	0.012

Table 6.9: The Summary statistics for the model $\boldsymbol{\sigma_{iv} = a_0 + a_1 K + a_2 T + a_3 K^2 + a_4 KT}$

(4) Still on absolute smile models we consider $\boldsymbol{\sigma_{iv} = a_0 + a_1 K + a_2 K^2 + a_3 T}$ which has the following table for the summary output:

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.807
R Square	0.651
Adjusted R Square	0.639
Standard Error	0.070
Observations	95

ANOVA

	df	SS	MS	F	Significance F
Regression	3	0.832	0.277	56.534	0.000
Residual	91	0.446	0.005		
Total	94	1.278			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95.0%	Upper 95.0%
Intercept	2.081	0.222	9.366	0.000	1.639	2.522	1.639	2.522
K	-0.021	0.003	-6.859	0.000	-0.028	-0.015	-0.028	-0.015
K^2	0.000	0.000	5.923	0.000	0.000	0.000	0.000	0.000
T	-0.113	0.057	-1.995	0.049	-0.225	-0.001	-0.225	-0.001

Table 6.10: The Summary Statistics for the model given by $\boldsymbol{\sigma_{iv} = a_0 + a_1 K + a_2 K^2 + a_3 T}$.

(5) Finally on the absolute smile models we have, $\boldsymbol{\sigma_{iv} = a_0 + a_1 K + a_2 T + a_3 KT}$ with the only quadratic term being the product of K and T. The Summary Statistics as shown below:

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.764
R Square	0.584
Adjusted R Square	0.570
Standard Error	0.076
Observations	95

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	3	0.746	0.249	42.542	0.000
Residual	91	0.532	0.006		
Total	94	1.278			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	1.168	0.109	10.730	0.000	0.952	1.384	0.952	1.384
K	-0.006	0.001	-7.606	0.000	-0.007	-0.004	-0.007	-0.004
T	-1.369	0.333	-4.115	0.000	-2.030	-0.708	-2.030	-0.708
KT	0.009	0.002	3.843	0.000	0.004	0.013	0.004	0.013

Table 6.11: The Summary output for the absolute smile model $\boldsymbol{\sigma_{iv} = a_0 + a_1 K + a_2 T + a_3 KT.}$

We now consider the relative smile model for estimating the implied volatility models. In relative smile models, the estimation of the models parameters are in terms for underlying stock prices, time to maturity and strike price of the given option.

(1) For relative smile model given by $\sigma_{iv} = a_o + a_1 {}^S/_K + a_2 ({}^S/_K)^2 + a_3 T$ the Summary Statistics is given by:

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.810
R Square	0.657
Adjusted R Square	0.645
Standard Error	0.069
Observations	95

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	3	0.840	0.280	58.037	0.000
Residual	91	0.439	0.005		
Total	94	1.278			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	0.750	0.214	3.511	0.001	0.326	1.174	0.326	1.174
S/K	-1.101	0.375	-2.934	0.004	-1.846	-0.356	-1.846	-0.356
S/K^2	0.644	0.160	4.020	0.000	0.326	0.962	0.326	0.962
T	-0.113	0.056	-2.013	0.047	-0.224	-0.001	-0.224	-0.001

Table 6.12: Summary Statistics for relative smile model: $\sigma_{iv} = a_o + a_1 {}^S/_K + a_2 ({}^S/_K)^2 + a_3 T$.

(2) We now consider another relative smile implied volatility model $\boldsymbol{\sigma_{iv} = a_0 + a_1 {}^{S}/_{K} + a_2 T + a_3 ({}^{S}/_{K}) T}$. The Summary Output for this model is given by:

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.824
R Square	0.679
Adjusted R Square	0.668
Standard Error	0.067
Observations	95

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	3	0.868	0.289	64.069	0.000
Residual	91	0.411	0.005		
Total	94	1.278			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	-0.494	0.092	-5.397	0.000	-0.676	-0.312	-0.676	-0.312
S/K	0.761	0.081	9.445	0.000	0.601	0.921	0.601	0.921
T	1.218	0.280	4.350	0.000	0.662	1.774	0.662	1.774
S/K*T	-1.194	0.246	-4.845	0.000	-1.684	-0.704	-1.684	-0.704

Table 6.13: Summary Statistics for the model: $\boldsymbol{\sigma_{iv} = a_0 + a_1 {}^{S}/_{K} + a_2 T + a_3 ({}^{S}/_{K}) T}$.

(3) For the relative smile model represented by $\boldsymbol{\sigma_{iv} = a_0 + a_1 S/_K + a_2 (S/_K)^2 + a_3 T + a_4 (S/_K) T}$ we have the following Summary Statistics:

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.860
R Square	0.740
Adjusted R Square	0.728
Standard Error	0.061
Observations	95

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	4	0.946	0.236	63.914	0.000
Residual	90	0.333	0.004		
Total	94	1.278			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	0.349	0.202	1.731	0.087	-0.051	0.749	-0.051	0.749
S/K	-0.741	0.335	-2.210	0.030	-1.407	-0.075	-1.407	-0.075
S/K^2	0.644	0.140	4.590	0.000	0.365	0.922	0.365	0.922
T	1.218	0.253	4.806	0.000	0.715	1.722	0.715	1.722
S/K*T	-1.194	0.223	-5.352	0.000	-1.637	-0.751	-1.637	-0.751

Table 6.14: The Summary Statistics $\boldsymbol{\sigma_{iv} = a_0 + a_1 S/_K + a_2 (S/_K)^2 + a_3 T + a_4 (S/_K) T}$.

(4) Finally, on relative smile model we have: $\boldsymbol{\sigma_{iv}} = \boldsymbol{a_0} + \boldsymbol{a_1}\, {}^{\boldsymbol{S}}\!/_{\boldsymbol{K}} + \boldsymbol{a_2}({}^{\boldsymbol{S}}\!/_{\boldsymbol{K}})^2 + \boldsymbol{a_3 T} + \boldsymbol{a_4 T^2}$ whose Summary Statistics is given by:

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.811
R Square	0.658
Adjusted R Square	0.643
Standard Error	0.070
Observations	95

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	4	0.842	0.210	43.355	0.000
Residual	90	0.437	0.005		
Total	94	1.278			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
Intercept	0.778	0.219	3.558	0.001	0.344	1.212	0.344	1.212
S/K	-1.101	0.376	-2.925	0.004	-1.848	-0.353	-1.848	-0.353
S/K^2	0.644	0.161	4.007	0.000	0.325	0.963	0.325	0.963
T	-0.326	0.334	-0.976	0.332	-0.989	0.338	-0.989	0.338
T^2	0.342	0.528	0.647	0.519	-0.708	1.392	-0.708	1.392

Table 6.15: Summary Statistics for the model $\boldsymbol{\sigma_{iv}} = \boldsymbol{a_0} + \boldsymbol{a_1}\, {}^{\boldsymbol{S}}\!/_{\boldsymbol{K}} + \boldsymbol{a_2}({}^{\boldsymbol{S}}\!/_{\boldsymbol{K}})^2 + \boldsymbol{a_3 T} + \boldsymbol{a_4 T^2}$.

DERIVATION OF MARCENKO-PASTUR LAW (DISTRIBUTION)

The Marcenko-Pastur (M-P) law investigates the level density for various ensembles of positive matrices of a Wishart-like structure which is denoted by $W = XX^T$, where X stands for a random matrix. In particular, for some stocks in the Nigerian Stock Market (NSM), we have $R = \frac{1}{L}X^TX$ with L as the period of time considered in the time series and we make use of the Cauchy transform to derive the M-P distribution.

To derive the level density associated with a given ensembles of random matrices, and in a more general sense some free convolutions of the M-P law, we will use the Voicucescu S-transform and the Cauchy functions.

Suppose that $X = (X_1, X_2, \dots, X_n) \in \mathbb{R}^{pxl}$ where X_i, are independent and identically distributed with mean zero and variance one. Furthermore, let's define

$$\mathbb{R}_n = \frac{1}{L}XX^T \in \mathbb{R}^{pxp} \qquad (1)$$

and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ denote the eigenvalues of the matrix $\mathbb{R}_n$. In particular, from the data used in my research for the stock from Nigerian Stock Market, $L = 1018, P = 82$. Suppose we define the random spectral measure by

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} f(\lambda_i) \qquad (2)$$

where $\lambda_{i's}$ are the eigenvalues of the random matrix, we can then state the, M-P distribution as follows:

Marcenko-Pastur Law (Distribution): If $\mathbb{R}_n$, μ_n are defined as in (1) and (2) above, and suppose further that $^p/_L$ approaches $Q \in (0,1)$ where p, L are sufficiently large, then we have

$\mu_n(.,w) \Rightarrow \mu$ almost surely (a.s) with μ known to have a deterministic measure whose density is given by

$$\wp(\lambda) = \frac{d\mu}{dx} = \frac{Q}{2\pi x}\sqrt{(b-x)(x-a)}\,|_{a\leq x\leq b} \qquad (3)$$

Here, a and b are functions of Q given by $a(Q) = (1-\sqrt{Q})^2, b(Q) = (1+\sqrt{Q})^2$ with a and b representing λ_{min} and λ_{max} respectively in the thesis.

Remark: We observe that when the rectangular parameter $Q = 1, with\, a = 0, b = 4$ we shall have

$$\wp(\lambda) = \frac{d\mu}{dx} = \frac{1}{2\pi x}\sqrt{(4-x)x}\,|_{0\leq x\leq 4} \equiv \frac{1}{2\pi}\sqrt{\frac{4-x}{x}} \qquad (4)$$

which yields the image of a semicircle distribution under the mapping $x \to x^2$.

The variable x represents a suitably rescaled eigenvalue λ of $\mathbb{R}_n$. For normalized random Wishart matrix, with respect to the trace condition $Tr\ \mathbb{R}_n = 1$, the rescaled variable is $x = \lambda N$, where N is the size of matrix, X.

We now use the S-transform that corresponds to an unknown probability measure defined on a complex variable ω, on the x-axis for the analysis of M-P distribution defined as

$$S_{M-P}(\omega) = \frac{1}{1+\omega} \tag{5}$$

To infer this measure and the spectral density $\wp(\lambda)$, Mlotkowski et al. (2015) write the S-transform as

$$S(\omega) = \frac{1+\omega}{\omega}\,\chi(\omega) \tag{6}$$

where, $$\frac{1}{\chi(\omega)}\,G\left\{\frac{1}{\chi(\omega)}\right\} - 1 = \omega \tag{7}$$

Suppose we set the characteristics function χ(ω) as

$$\frac{1}{\chi(\omega)} = z \; with \; z \in \mathbb{C}^{+} \equiv \{z \in \mathbb{C} : Im\, z > 0\} \tag{8}$$

This will enable us to obtain the implicit solution to the Green's function G(z) which can also be referred to as the Cauchy function written as:

$$G(z) = \frac{1}{n}\,tr(A - zI)^{-1} \tag{9}$$

where, A connotes a random matrix from the ensemble investigated (which in this work represents the 82 stocks considered drawn from market prices in the Nigerian Stock Market).

Putting equation (8) into (7) we shall have:

$$zG(z) - 1 = \omega \Rightarrow zG(z) = 1 + \omega \; or \; G(z) = \frac{1+\omega}{\omega}$$

Thus from (9) above $G(z) = \frac{1}{n}\,tr\,(A - zI)^{-1} = \frac{1+\omega}{\omega}$ (10)

Furthermore, from (6) and (8) we shall have: $s(\omega) = \frac{1+\omega}{z\omega}$ which implies that

$$zwS\big(\omega(z)\big) = 1 + \omega(z) \tag{11}$$

We now demonstrate how to obtain the general form of the M-P distribution which describes the asymptotic level density $\wp(\lambda)$ of random states of $\wp = \frac{XX^T}{TrXX^T}$, where X is the rectangular complex Ginibre matrix of size NxM, with the chosen rectangular parameter $Q = M/N \leq 1$.

Consider another S-transform similar to that of equation (5) defined as:

$$S_c(\omega) = \frac{1}{1+c\omega} \tag{12}$$

which reduces to equation (5) for $c = 1$ and putting equation (12) into (11) we shall obtain:

$$z\omega(z)\left(\frac{1}{1+c\omega}\right) = 1 + \omega(z) \text{ or } z\omega = (1+\omega)(1+c\omega)$$

$$\Rightarrow c\omega^2 + \omega(c + 1 - z) + 1 = 0 \quad (13)$$

By solving the quadratic equation in terms of ω using the general formula we shall obtain:

$$\omega = \frac{-(c+1-z) \pm \sqrt{(c+1-z)^2 - 4c}}{2c}$$

$$= \frac{-(c+1-z) \pm \sqrt{(c^2 - 2c(1+z) + (1-z)^2}}{2c}$$

$$= \frac{-(c+1-z) \pm \sqrt{(c-1-\sqrt{z})^2 - (c-1+\sqrt{z})^2}}{2c}$$

Thus, the imaginary part of ω is zero when c lies outside the interval$[(1 - \sqrt{z})^2, (1 + \sqrt{z})^2]$. Finally, to obtain the spectral density as shown in equation (3), we apply the Stieltjes inversion formula and since the negative imaginary part of the Green's function yields the spectral function,

$$\wp(\lambda) = -\frac{1}{\pi}\lim_{\varepsilon \to 0} \mathcal{I}mG(z)|_{z=\lambda+i\varepsilon} \quad (14)$$

we shall have:

$$\wp(\lambda) = \frac{d\mu}{dx} = \frac{Q}{2\pi x}\sqrt{(b-x)(x-a)}\,|_{a \le x \le b}, \text{as required;}$$

where $a(Q) = (1 - \sqrt{z})^2 \equiv (1 - \sqrt{Q})^2$ and

$b(Q) = (1 + \sqrt{z})^2 \equiv (1 + \sqrt{Q})^2$ and x being a dummy variable is represented by c.

The M-P equation has undergone several reformulations since its first appearance in the original paper of Marcenko-Pastur (1967). Some of this reformulation process was the instantiation of the equation by Silverstein and Bai under four different assumptions for the derivation of the theorem. To this end therefore, one derives the Marcenko-Pastur law as above using Marcenko-Pastur Theorem or the Silverstein and Bai Theorem (1995) as stated below:

Consider an NxN matrix, B_N . Assume that

(a) X_n is annxN matrix such that the matrix elementsX_{ij}^n are independent identically distributed (i.i.d) complex variables with mean zero and variance 1,$i.e\ X_{ij}^n \in ₵, E(X_{ij}^n) = 0\ \&\ E(||X_{ij}^n||^2) = 1$

(b)$n = n(N) with\ {}^{n}/_{N} \to c > 0\ as\ N \to \infty$. In particular, for the Nigerian stocks considered, $n = 82, N = 1018\ \ni\ {}^{n}/_{N} = 0.08 > 0$, the same applies to JSE stocks.

(c) $T_n = diag(\tau_1^n, \tau_2^n, \dots, \tau_n^n)$ where $\tau_i^n \in \mathbb{R}$ and the eigenvalue distribution function (e.d.f) of $\{\tau_1^n, \tau_2^n, \dots, \tau_n^n\}$ converges almost surely in distribution to a probability distribution function (p.d.f) $H(\tau) as\ N \to \infty >$

(d) $B_N = A_N + \frac{1}{N} X_n^* T_n X_n$, where $A_N$is a Hermitian NxN matrix for which F^{A_N} converges vaguely to Æ almost surely, Æ being a possibly defective (i.e with discontinuities) nonrandom distribution function

(e) $X_n, T_n \; and \; A_n$ are independent.

Then, almost surely, F^{B_N}converges vaguely, almost surely, as $N \to \infty$ to a nonrandom distribution function (d.f) F^B whose Stieltjes transform $m(z), z \in ₵$ satisfies the canonical equation

$$m(z) = m_A\left(z - c\int \frac{\tau dH(\tau)}{1+\tau m(z)}\right) \tag{15}$$

We begin by defining the Stieltjes transform in an eigenvalue distribution which has proven to be an efficient tool for determining a limiting density. For every non-real$z$, the Stieltjes (or Cauchy) transform of the probability measure $F^A(x) = F[A(x)](z)$ is given by

$$m_A(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} dF^A(x) \tag{16}$$

with $z = x + iy \in ₵, y \neq 0.$

Suppose $A_N = 0$, from (d) above, $B_N = \frac{1}{N} X_n^* T_n X_n$. The Stieltjes transform of A_N, from definition (16) above will then be

$$m_A(z) = \frac{1}{0-z} = -\frac{1}{z}$$

and using Marcenko-Pastur theorem as expressed in equation (15) above, the Stieltjes transform $m(z)$ of B_N is given by

$$m(z) = -\frac{1}{z - c\int \frac{\tau dH(\tau)}{1+\tau m(z)}} \tag{17}$$

we can therefore find that the inverse of $m(z)$ will be given by

$$z = -\frac{1}{m} + c\int \frac{\tau}{1+\tau m} dH(\tau) \tag{18}$$

Equation (18) can be seen as an expression of relationship between the Stieltjes transform variable m and the probability space z which can alternatively be referred to as a canonical equation or functional inverse of $m(z)$.

Thus, to determine the density of B_N as defined in (d) above using inversion formula (14) we need to solve (18) for $m(z)$. Hence, to be able to simplify the relationship between m and z we need to obtain $dH(\tau)$ from equation (18). Theoretically, $dH(\tau)$ could be regarded as any density which satisfies the conditions of Marcenko-Pastur theorem. In Particular, for some specific distribution of $dH(\tau)$ we can obtain the density analytically.

For $T_n = 1$, in (c) above of the theorem, which coincidentally is the same as was observed from the empirical matrix (as the diagonal elements of T_n are non-random) with distribution function). We note here that for general forms of the probability distribution $H(\tau)$ it is not possible to find an analytic solution for m in (18) above, however, for the well-known white Parcenko -Pastur or canonical form of the distribution, equation (18) can be solved using the relation $dH(\tau) = \delta(\tau - 1)$ to obtain $z = -\frac{1}{m} + \frac{c}{1+m}$. Thus, with $T_n = 1$ we obtain from equation (18)

$$z = -\frac{1}{m} + \frac{c}{1+m} \Rightarrow z(m)(1+m) = -(1+m) + cm$$

$$\text{or } m^2 z + m(1 - c + z) + 1 = 0 \qquad (20)$$

which is analogous to the expression represented by equation (13) as solving as before we can therefore obtain the solutions of m in terms of z.

Thus, to obtain the density which is usually referred to as the Marcenko-Pastur distribution we solve the quadratic equation in (20) above and make use of equation (14) to get:

$$\wp(\lambda) = \frac{dF^B(x)}{dx} = \frac{d\mu}{dx} = \frac{Q}{2\pi x}\sqrt{(b-x)(x-a)}\,|_{a \le x \le b}, \text{ as obtained before.}$$

For the Nigerian stocks the probability density function for the eigenvalues is given by:

$$\wp(\lambda) = 1.975 * SQRT((1.65 - x) * (x - 0.56))/x$$

0.13	#NUM!
0.2092	#NUM!
0.2236	#NUM!
0.2376	#NUM!
0.2487	#NUM!
0.2521	#NUM!
0.277	#NUM!
0.2943	#NUM!
0.3118	#NUM!
0.326	#NUM!
0.3294	#NUM!
0.3625	#NUM!
0.3723	#NUM!
0.3726	#NUM!
0.3959	#NUM!
0.4049	#NUM!
0.4171	#NUM!
0.4328	#NUM!
0.4572	#NUM!

0.4586	#NUM!
0.4644	#NUM!
0.4918	#NUM!
0.4999	#NUM!
0.5293	#NUM!
0.5387	#NUM!
0.5414	#NUM!
0.5484	#NUM!
0.5522	#NUM!
0.5757	0.445537
0.6022	0.689639
0.6047	0.705995
0.6194	0.788922
0.6227	0.804953
0.6336	0.852556
0.6508	0.914089
0.69	1.011173
0.7069	1.039918
0.7108	1.045681
0.7257	1.065069
0.7324	1.072541
0.7396	1.07979
0.7671	1.100933
0.7854	1.110097
0.7979	1.114451
0.8194	1.118799
0.8269	1.119476
0.8452	1.119506
0.8681	1.116654
0.876	1.115006
0.9157	1.102282
0.9296	1.096286
0.9493	1.086605
0.9653	1.077813
0.9942	1.060044
1.0094	1.049817
1.0303	1.03486
1.0407	1.027056
1.0578	1.013737
1.0799	0.995678
1.1065	0.972771
1.1148	0.965376
1.158	0.925106
1.1866	0.896883

1.235	0.846401
1.2396	0.841426
1.2922	0.782298
1.3072	0.764652
1.3326	0.733919
1.3475	0.715363
1.3896	0.66059
1.4435	0.584405
1.4679	0.547072
1.4923	0.507463
1.5071	0.482102
1.605	0.266843
1.68	#NUM!
1.7654	#NUM!
1.8508	#NUM!
3.3282	#NUM!
3.9053	#NUM!
4.4287	#NUM!

For the South African stocks the probability density function for the eigenvalues is given by:

$$\wp(\lambda) = 5.2155/x) * SQRT((1.381 - x) * (x - 0.683))$$

0.000356	
0.005846	#NUM!
0.006855	#NUM!
0.007528	#NUM!
0.008439	#NUM!
0.008614	#NUM!
0.009819	#NUM!
0.010088	#NUM!
0.010953	#NUM!
0.011272	#NUM!
0.011712	#NUM!
0.012484	#NUM!
0.014506	#NUM!
0.016954	#NUM!
0.017789	#NUM!
0.020088	#NUM!
0.022645	#NUM!
0.026253	#NUM!
0.038805	#NUM!
0.598543	#NUM!

0.918852	1.8739617
0.970954	1.8457596
0.987239	1.8285104
0.991409	1.8235223
0.99647	1.8171669
1.001014	1.8111846
1.009146	1.7998413
1.016027	1.7896199
1.032413	1.7630621
1.06708	1.6971464
1.401997	#NUM!
2.001833	#NUM!
2.975673	#NUM!
5.907029	#NUM!
11.86331	#NUM!

The stocks considered from the Nigerian Stock Market are as follows:

- 7Up Bottling Company plc
- ABC Transport Service
- Access Bank plc
- Aglevent
- AIICO Insurance
- Air Services Company
- Ashaka Cement
- Berger Paints
- Cadbury Nigeria plc
- CAP
- CCNN
- Cileasing
- Conoil
- Continsure
- Cornerstone
- Costain Construction
- Courtville
- Custodyins
- Cutix Cables
- Dangote Cement
- Dangotye Sugar
- Diamond Bank
- Dunlop Tyres
- Eternal Oil
- Equitorial Trust Bank
- Evans Medical
- First Bank of Nigeria (FBN) plc
- First City Monument Bank (FCMB)
- Fidelity Bank
- Fidson
- Flour Mills
- FO
- Glaxosmith
- Guaranty Trust Bank (GTB)
- Guinness Breweries plc
- Honey Flour
- Ikeja Hotels
- International Breweries
- Intenegins

- Japaul Oil
- Julius Berger Construction
- Learn Africa
- Livestock
- Mansurd
- May & Baker
- Mbenefit
- Mobil Oil
- MRS Oil
- NAHCO
- NASCON
- Nigerian Breweries (NB)
- Neimeth
- NEM
- Nestle Foods
- Nigerins
- Oando Oil
- Okomuo Oil
- PRESCO
- Prestige
- PZ
- Redstar
- Royalex
- RTBrisco
- SkyeBank
- STANBIC IBTC Bank
- Sterling Bank
- Tiger Bra
- Total Oil
- Transcorp hotels
- UAC Nigeria
- UAC Properties
- United Bank for Africa (UBA)
- Union Bank of Nigeria (UBN)
- Uniliver
- Unity Bank
- UPL
- UTC
- Vitafoam
- WAPCO
- WAPIC

- Wema Bank
- Zenith Bank.

For Johannesburg Stock exchange we have

- ABSA
- African Bank
- AngloAmerican
- AngloAshanti
- Aspen Pharmacare
- BHP
- Bidvest
- Compagnie
- Exxaro Resources
- FirstRand Limited
- Gold Fields
- Growth Point Properties
- Harmony
- Investec Limited
- Investec Plc
- Kumbaron
- Lonmin Plc
- Massmart Holdings
- Mondi Limited
- Mondi Plc
- MTN Group
- Naspers
- Nedbank Group
- Old Mutual
- Remgro
- RMB Holdings
- SAB Miller
- Sanlam
- Sasol
- Shoprite
- Standard Bank Steinhoff
- Steinhoff
- Tiger Brands
- Truworths
- Vodacom.