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Spectral Analysis of Non-selfadjoint Differential Operators

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by

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FOR MY PARENTS

and

MY WIFE AND SON

who were very patient

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Abstract.

This thesis is concerned with the extension of classical Titchmarsh-Weyl theory to non-selfadjoint Sturm-Liouville operators on the half-line. We introduce the thesis with some mathematical background which is needed for the development of the main results. This includes a brief summary of relevant aspects of Lebesgue measure and integration, analytic function theory, unbounded operator theory, and selfadjoint extensions of symmetric operators, and is given in the first two chapters. An introduction to Weyl theory and related topics for the self adjoint case can be found in Chapter 3. The main work on non-selfadjoint second order differential operators associated with the equation $-y'' + qy = \lambda y$ begins in Chapter 4. We first describe Sims' extension of Weyl's limit point, limit circle theory to the non-selfadjoint case, and some later generalisations by McLeod. Some standard results of Titchmarsh are then extended from the selfadjoint to the non-selfadjoint case and an important result on the stability of the essential spectrum of a non-selfadjoint differential equation of form $-(py')'+qy = \lambda y$ is also obtained. In Chapter 5, we extend some results of Chaudhuri-Everitt to non-selfadjoint operators in which the potential satisfies the condition $\lim_{x\to\infty} \Im q(x) = L$. Some worked examples at the end of Chapter 5 show that in certain cases where there is a complex boundary condition and real coefficient, or a complex coefficient with real boundary condition, some complex eigenvalues can be explicitly calculated. Finally in Chapter 6 we describe a physical problem which gives rise to a non-selfadjoint eigenvalue problem on the half-line.

Keywords: Differential operator, Non-selfadjoint, Spectrum, Eigenvalue problem, Weyl m-function.

Introduction

The theory of singular differential operators, which is a very familiar field of work amongst physicists and mathematicians, was founded in 1909 by Hermann Weyl, who published a famous paper entitled "Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen wilkurlicher Funktionen" [50]. After the fundamental work of Weyl in 1910, there was no further work in this area until 1926, when Erwin Schrödinger published two papers "Quantisierung als Eigenwertproblem" [41], which are known as the mathematical foundation of Quantum Mechanics. He obtained a stationary equation for the electron which is known as the Schrödinger wave equation

$$-\Delta\psi + V\psi = \lambda\psi$$

where ψ is the wave function, V is a potential function and λ is the energy level of the particle.

Further development of the theory of singular differential operators was due to researchers who were interested in working on the problem of determining the energy spectrum of concrete systems. In quantum mechanics in particular investigation of the spectrum of the one dimensional Schrödinger operator made further progress during 1946-1952, due to A. Wintner, P. Hartman and K. Putnam. The theory was continued by R. Phillips who considered boundedness of the imaginary part of the potential q and determined the behaviour of the spectrum and resolvent when a real potential q is associated with a small complex perturbation [36]. J. Schwartz considered the same problem as Phillips but with an extra condition on the potential q, giving singularities for q at both end points of the interval (a, b) [42].

There is an extensive literature concerning one dimensional Schrödinger operators $\frac{-d^2}{dx^2} + v$ which are self adjoint differential operators of Sturm-Liouville type. E. C. Titchmarsh developed the theory of self adjoint singular differential operators from the point of view of complex analysis using the calculus of residues [46].

In August, 1948 K. Kodaira obtained the same results independently of Titchmarsh's work [24], and it was M. Stone who rebuilt the theory from the view point of linear operators and obtained some new results [45].

This thesis is not concerned with selfadjoint problems but rather with non-selfadjoint problems, for which there is not a lot of literature available. The earliest work in this connection goes back to 1952, when I. M. Gelfand considered the differential operator defined in the Hilbert space $L^2(\mathbb{R}^3)$ by the differential expression

$$-\Delta u + pu + qu,$$

where p is a real function such that $-\Delta u + pu$ is a selfadjoint operator, and q is a continuous complex function, vanishing outside of a bounded domain [13]. Also M. A. Naimark in a comprehensive article obtained results on the spectrum of a non-selfadjoint second order differential operator on the positive half-line for the first time [33]. He considered the case where the coefficient function q is integrable and the boundary condition are complex. R. R. D. Kemp [22] obtained similar results on the interval $(-\infty, \infty)$.

Investigation of the spectrum of a non-selfadjoint problem when the coefficient q is continuous and certain other conditions on q and the boundary condition α are satisfied, was also carried out in 1957 by A. Sims [44]. He obtained a very important result about the limit-point limit-circle geometry that in the selfadjoint case had been discovered by Weyl. In non-selfadjoint problems which are defined on $[0, \infty)$ by

$$\begin{cases} -\phi'' + q\phi = \lambda\phi \\ \phi(0,\lambda)\cos\alpha + \phi'(0,\lambda)\sin\alpha = 0 \end{cases}$$

it is possible to have more than one solution of the corresponding differential equation which is square integrable when q is in the limit point case at infinity. Later in 1961 J. B. McLeod [30], [31] contributed to this area by extending Sims' result; he succeeded in removing some conditions and obtained two types of eigenfunctions in the case of regular Sturm-Liouville problems and in the singular case on the interval $[0, \infty)$. An interesting result is that the eigenfunctions can be complete in the Hilbert space sense if some functions which he called adjoint functions are added; moreover for large $|\lambda|$ the zeros of certain functions corresponding to the L^2 solutions are simple. He also showed that it is possible for there to be no L^2 solution of the equation for any $\lambda \in \mathbb{C}$ in the complex case if $\Im q(x)$ is unbounded from both above and below.

Chapter One contains the mathematical background relating to Lebesgue measure and integration, Hilbert spaces and operator theory. Chapter Two contains some basic materials related to real and complex analysis and gives some standard theorems of the theory of differential equations. In Chapter Three there is some history about boundary value problems, both regular and singular, and we include some material relating to selfadjoint Sturm-Liouville problems and a brief introduction to J-symmetry and J-selfadjoint operators. Chapter Four and Five are concerned with the main results, namely the extension of some material given in Titchmarsh's book [46] and the extension of some results of Chaudhuri-Everitt on the m-function and its relationship to the resolvent set [4]. Chapter Six is mainly concerned with the conclusion. It also states some results about the singular continuous spectrum of selfadjoint differential operators and asks whether in the non-selfadjoint case there is a singular continuous part of the spectrum and if so, how we can obtain and characterize it.

Contents

Chapter 1

Mathematical background

1.1 Introduction

In this chapter we collect some definitions and results about Lebesgue measure and Lebesgue-Stieltjes measure and integration with respect to those measures. Of course some results about Hilbert spaces and linear operators on a Hilbert space will also be considered.

1.2 Measure and Integral

Definition 1.2.1 A *measure* is an extended real-valued set function μ having the following properties:

i) The domain $\mathcal{A}$ of μ is a σ-algebra.

ii) $\mu(E) \geq 0$ for each $E \in \mathcal{A}$

iii) μ is σ-additive i.e. $\mu(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ where the E_n are mutually disjoint sets of $\mathcal{A}$.

iv) $\mu(\emptyset) = 0$.

By property iii) we see that for each sequence $\{E_n\}$ of subsets of $\mathcal{A}$ we have

$$\mu(\cup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

If $\mathcal{A}$ is a σ-algebra of subsets of X and μ is a countably additive measure on $\mathcal{A}$, then the triple $(X, \mathcal{A}, \mu)$ is called a *measure space.* Let X be a set and μ an outer measure on the class $P(X)$ of all subsets of X. A subset A of X is said to be *μ-measurable* if

$$\mu(B) = \mu(B \cap A) + \mu(B \cap A')$$

for all $B \subseteq X$, where A' denotes the complement of A in X.

Let $\mathcal{A}$ be the class of all open intervals of real axis $\mathbb{R}$ which forms a sequential covering of $\mathbb{R}$. Let μ^* be given by

i) $\mu^*(\emptyset) = 0$

ii) $\mu^*((a, b)) = b - a$ if $a \neq b$

The complete measure μ determined by the measure μ^* is called the *Lebesgue measure* [12].

Let f be a real-valued, monotone-increasing function defined on the real axis $\mathbb{R}$. We assume that f is right-continuous i.e.

$$\lim_{x \to a^+} f(x) = f(a)$$

Let $\mathcal{A}$ consist of all the open intervals (a, b) of $\mathbb{R}$ covering $\mathbb{R}$ sequentially as before and define μ^* by

i) $\mu^*(\emptyset) = 0$

ii) $\mu^*((a, b)) = f(b) - f(a)$

The measure μ corresponding to this μ^* is called the *Lebesgue-Stieltjes measure* induced by f. So $\mu((a, b]) = f(b) - f(a)$.

Theorem 1.2.1 *There exists a set on the real line which is not Lebesgue measurable.*

Proof: see [12] Thm.1.6.1.

Let f be defined on a measurable set E_0. Denote by $E(f > A)$ the set of points of E_0 at which $f(x) > A$, with a similar meaning for $E(f \geq A)$, etc. The function f is said to be measurable if, for all values of the constant A, the sets of one of the four families

$$E(f > A), \quad E(f < A), \quad E(f \geq A), \quad E(f \leq A)$$

are measurable. Any one of these four conditions implies the other three.
Let $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ be a simple function defined on a measure space $(X, \mathcal{A}, \mu)$ where χ_{E_i} is the characteristic function of the measurable set E_i. We say that f is integrable on X if and only if $\mu(E_i) < \infty$ for each $i = 1, 2, \ldots, n$. An extended real-valued, measurable function f, on a measure space $(X, \mathcal{A}, \mu)$ is said to be integrable if there exists a sequence $\{f_n\}$ of integrable simple functions having the following properties:
i) $\{f_n\}$ is a Cauchy sequence in the mean
ii) $\lim_{n\to\infty} f_n(x) = f(x)$ a.e.
The integral of f is denoted by $\int_X f(x) d\mu(x) = \lim_{n\to\infty} \int_X f_n d\mu$. If E is a measurable set then the integral of a measurable function f over E is defined by

$$\int_E f(x) d\mu(x) = \int_X \chi_E f(x) d\mu(x)$$

Three of the most important theorems in the theory of integration are

Theorem 1.2.2 (Lebesgue's Bounded Convergence Theorem) *Let $\{f_n\}$ be a sequence of integrable functions that converges almost everywhere(in brief a.e.) to a measurable function f. If $|f_n(x)| \leq g(x)$ a.e. for all n, where g is an integrable function, then f is integrable and*

$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0$$

Proof: [12] Thm. 2.9.1.

Theorem 1.2.3 (Lebesgue's Monotone Convergence Theorem) *Let $\{f_n\}$ be a sequence of measurable functions on X and suppose that* *i)* $0 \le f_1(x) \le f_2(x) \le \ldots \le \infty$ *for every* $x \in X$

ii) $f_n(x) \to f(x)$ *as* $n \to \infty$*, for every* $x \in X$.

then f *is measurable, and*

$$\int_X f_n d\mu \to \int_X f d\mu \quad \text{as} \quad n \to \infty$$

Proof: [38] Thm. 1.26.

Theorem 1.2.4 (Fatou's Lemma) *If* $f_n : X \to [0, \infty]$ *is measurable, for each positive integer* n*, then*

$$\int_X (\liminf_{n\to\infty} f_n) d\mu \le \liminf_{n\to\infty} \int_X f_n d\mu \tag{1.2.1}$$

Strict inequality can occur in (1.2.1).

Proof: [38] Thm. 1.28.

1.3 Hilbert Space and Linear Operators

A separable Hilbert space $\mathcal{H}$ is a linear vector space over the field $\mathbb{C}$ of complex numbers having the following properties:

i) There exists a strictly positive scalar product in $\mathcal{H}$, i.e. a mapping $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$, denoted by $< \cdot, \cdot >$, which is linear in the first and conjugate linear in the second argument. Thus

$$< \alpha f + g, h > = \alpha < f, h > + < g, h > \quad \text{for all} \quad \alpha \in \mathbb{C}, \ \text{and} \ f, g, h \in \mathcal{H}$$

$$< f, g > = \overline{< g, f >}$$

$$< f, f >^{\frac{1}{2}} = \|f\| > 0 \quad \text{unless} \quad f \equiv 0 \tag{1.3.2}$$

ii)$\mathcal{H}$ is complete in the norm (1.3.2), i.e. if $\{f_n\}$ is a Cauchy sequence, so that $\|f_n - f_m\| \to 0$ as $n, m \to \infty$, then there exists $f \in \mathcal{H}$ such that $\|f_n - f\| \to 0$ as $n \to 0$.

iii) There exists a denumerable sequence $\{f_n\}$ which is dense in $\mathcal{H}$.

Definition 1.3.1 A subset $\mathcal{D}$ of $\mathcal{H}$ is *dense* in $\mathcal{H}$ if, given any $f \in \mathcal{H}$ and any $\epsilon > 0$, there exists $g \in \mathcal{D}$ such that $\|g - f\| < \epsilon$.

The two following inequalities are basic to the contents of this chapter, and hold for each f and g in the Hilbert space $\mathcal{H}$.

i) *Cauchy-Schwarz inequality.*

$$| < f \cdot g > | \leq \|f\|\|g\|. \tag{1.3.3}$$

ii) *Triangle inequality.*

$$\|f + g\| \leq \|f\| + \|g\|. \tag{1.3.4}$$

Definition 1.3.2 i) f is orthogonal to g, written $f \perp g$, if $< f, g >= 0$, where $f, g \in \mathcal{H}$.

ii) The orthogonal complement $\mathcal{D}^{\perp}$ of $\mathcal{D}$ is defined as

$$\mathcal{D}^{\perp} = \{f \in \mathcal{H} : f \perp g \;\; \text{for all} \;\; g \in \mathcal{D}\}$$

Theorem 1.3.1 *Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Then*

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$$

Proof: see [39], Thm. 12.4.

By Theorem 1.3.1 each $f \in \mathcal{H}$ may be written as

$$f = f_1 + f_2 \tag{1.3.5}$$

with $f_1 \in \mathcal{M}$ and $f_2 \in \mathcal{M}^{\perp}$. Equation (1.3.5) enables us to define a mapping

$$P : \mathcal{H} \rightarrow \mathcal{M}$$

by $Pf = f_1 \in \mathcal{M}$, for each $f \in \mathcal{H}$. We call P the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$.

Let $\{e_n\}$ be an orthonormal sequence in $\mathcal{H}$. Then for every $f \in \mathcal{H}$

$$\sum_{n=1}^{\infty} | < f, e_n > |^2 \leq \|f\|^2$$

This is known as Bessel's inequality.

Definition 1.3.3 A *complete* set in $\mathcal{H}$ is a subset $\mathcal{M} \subset \mathcal{H}$ whose span is dense in $\mathcal{H}$. A total orthonormal set is sometimes called an orthonormal basis.

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then for each $f \in \mathcal{H}$ we have

$$f = \sum_{n=1}^{\infty} < f, e_n > e_n$$

and Parseval's equation

$$\sum_{n=1}^{\infty} | < f, e_n > |^2 = ||f||^2$$

The coefficient $< f, e_n >$ is called a *Fourier coefficient.* In a separable Hilbert space every orthonormal basis is countable and if $\mathcal{H}$ contains a total orthonormal sequence, then it is in other words, a separable Hilbert space.

Theorem 1.3.2 *The Hilbert space $\mathcal{H}$ is separable if and only if $\mathcal{H}$ has a complete orthonormal sequence.*

Proof: see [1].

Definition 1.3.4 A *linear operator* T in $\mathcal{H}$ is a linear mapping defined on a linear manifold $D(T) \subseteq \mathcal{H}$ with values in $\mathcal{H}$. Thus we have

$$T : D(T) \to \mathcal{H} \quad \text{with} \quad T(\alpha f + g) = \alpha T(f) + T(g)$$

where $f, g \in D(T)$ and $\alpha \in \mathbb{C}$

Let T and S be two linear operators in $\mathcal{H}$. Then $T = S$ iff $D(T) = D(S) = D$ and $Tf = Sf$ for all $f \in D$. We mean by an extension S of T denoted by $T \subseteq S$ an operator S with domain $D(S)$ such that $D(T) \subset D(S)$ and $Tf = Sf$ for all $f \in D(T)$. A linear operator T is said to be bounded if there exists a positive constant $M < \infty$ such that

$$||Tf|| \leq M||f|| \quad \text{for all} \quad f \in D(T)$$

The norm $||T||$ of T is then defined as the minimum of all possible values of M and we have

$$||T|| = \inf_{0 \neq f \in \mathcal{H}} \frac{||Tf||}{||f||}.$$

The space of all bounded operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. A linear operator $T \in \mathcal{B}(\mathcal{H})$ is called a finite rank operator if the range $R(T) = \{Tf : f \in D(T)\}$ of T is finite dimensional. The representation of a finite rank operator on $\mathcal{H}$ is as follows

$$Tf = \sum_{i=1}^{N} < g_i, f > h_i,$$

where $\{g_i, h_i\}$ are $2N$ vectors in $\mathcal{H}$ and $N < \infty$.

Definition 1.3.5 An operator $T \in \mathcal{B}(\mathcal{H})$ is *compact* if, for every bounded sequence $\{f_n\}$ in $\mathcal{H}$, the sequence $\{Tf_n\}$ has a convergent subsequence.

We obtain an extremely important realization of a separable Hilbert space by taking, as the vectors, functions of points in a real Euclidean space. The Hilbert space obtained is denoted by $L^2(a,b)$ and it is the set of all complex-valued measurable functions on (a,b) such that the square of their absolute value is Lebesgue integrable. In this space two functions are regarded as identical if they differ only on a set of Lebesgue measure zero. The inner product operation in the linear space $L^2(a,b)$ satisfies

$$< f, g >= \int_a^b f(t)\overline{g(t)}dt.$$

for all $f, g \in L^2(a,b)$, and is a finite complex number since

$$| < f, g > | \leq \int_a^b |f(t)\bar{g}(t)|dt \leq (\int_a^b |f(t)|^2)^{\frac{1}{2}}(\int_a^b |g(t)|^2)^{\frac{1}{2}} < \infty$$

We point out that the space $L^2(a,b)$ may be regarded as a subspace of $L^2(c,d)$ if $c \leq a < b \leq d$, and in particular, as a subspace of $L^2(-\infty, +\infty)$. We also mention that convergence in the topology of L^2 is based on the metric induced by the norm and is called convergence in the mean. This is denoted by l.i.m. and is such that if $\{f_n\}$ is a convergent sequence in L^2 that converges to $f \in \mathcal{H}$, then

$$\text{l.i.m.}_{n\to\infty} f_n(t) = f(t) \text{ iff } \lim_{n\to\infty} \int_a^b |f_n(t) - f(t)|^2 dt = 0$$

Let σ be a non-decreasing function of bounded variation on the real axis, and assume that σ is right-continuous, i.e.

$$\sigma(t+0) = \lim_{\epsilon\to 0^+} \sigma(t+\epsilon) = \sigma(t)$$

With the aid of the function σ we can construct a measure called the Lebesgue-Stieltjes measure. This is analogous to Lebesgue measure, but a bit different from it, since the Lebesgue measure of the interval (a,b) is its length $b-a$ and the Lebegue-Stieltjes measure of (a,b) corresponding to σ is $\sigma(b-0)-\sigma(a)$ which might be zero; also the Lebegue-Stieltjes measure of a singleton may be not equal to zero. We consider the linear space of all σ-measurable functions f for which the Lebesgue-Stieltjes integral

$$\int_{-\infty}^{+\infty} |f(t)|^2 d\sigma(t)$$

exists, and with inner product

$$< f,g >= \int_{-\infty}^{+\infty} f(t)\overline{g(t)}d\sigma(t)$$

This is a separable Hilbert space which is denoted by $\mathcal{L}^2_\sigma$ [1].

Definition 1.3.6 Let T be a densely defined operator on $\mathcal{H}$. Corresponding to T there exists an *adjoint operator* T^* with domain $D(T^*)$ consisting of all elements $f \in \mathcal{H}$ such that $f \in D(T^*)$ if and only if there exists an element $f^* \in \mathcal{H}$ satisfying

$$< Tg, f >=< g, f^* > \quad \text{for all} \quad g \in D(T)$$

in which case $T^*f = f^*$.

Let T be a densely defined operator on $\mathcal{H}$. Then T is called selfadjoint iff $T^* = T$ and T is said to be closed iff graph $G(T) = \{(f,Tf) : f \in D(T)\}$ of T is a closed subspace of $\mathcal{H} \times \mathcal{H}$. We say that T is closable if T has a closed extension. It is equivalent to the condition that the graph $G(T)$ of T is a submanifold of a closed linear manifold which is at the same time a graph, and it follows that T is closable iff the closure $\overline{G(T)}$ of $G(T)$ is a graph. We say that T is symmetric if $D(T)$ is dense in $\mathcal{H}$ and for all $f,g \in D(T)$ $\quad < Tf,g >=< f,Tg >$. Since $T \subseteq T^*$ and T^* is closed, T has a closed extension T^* (i.e. T is closable) and it may be shown that $(T^*)^* = \overline{T}$. Essentially selfadjoint operators are those that have selfadjoint closure. Obviously selfadjoint operators are closed and if T is finite rank operator, which is a bounded, then T is symmetric iff T is selfadjoint. However, in general symmetric bounded and unbounded operators may be non-selfadjoint, as we can see in the following example.

Example 1.3.1 Let $\mathcal{H} = L^2[0,1]$ and let $D(T)$ consist of all continuously differentiable functions f in $\mathcal{H}$ which satisfy the boundary conditions

$$f(1) = f(0) = 0 \tag{1.3.6}$$

and define T by $Tf = \frac{1}{i}\frac{df}{dx}$ for all $f \in D(T)$. Then

$$\begin{aligned} < Tf, g > &= \int_0^1 Tf(t)\overline{g(t)}dt = \int_0^1 \frac{1}{i} f'(t)\overline{g(t)}dt \\ &= \frac{1}{i}[f(t)\overline{g(t)}]_0^1 + \int_0^1 -\frac{1}{i}\overline{g'(t)}f(t)dt = \int_0^1 f(t)\overline{Tg(t)}dt \\ &= < f, Tg > \end{aligned}$$

We see that T is unbounded symmetric operator on $L^2[0,1]$ but $T \neq T^*$ since $D(T) \neq D(T^*)$ and there exists $g \in D(T^*)$ which does not satisfy the boundary condition 1.3.6.

Definition 1.3.7 Let T be an operator defined on $\mathcal{H}$. A complex number λ is called an *eigenvalue* of T if there is a non-zero vector $f \in D(T)$ such that

$$Tf = \lambda f$$

f is called an *eigenfunction* of T belonging to the eigenvalue λ. The set

$$\{f : Tf = \lambda f\}$$

which is denoted by $\mathcal{N}_\lambda$ is a linear manifold of $\mathcal{H}$ and is called the *eigenspace* of T for the eigenvalue λ and the dimension of $\mathcal{N}_\lambda$ is called the geometric *multiplicity* of λ. The algebraic multiplicity of λ will be defined in Chapter Two §2.2.3. The *eigenvalue problem* consists primarily in finding all eigenvalues and eigenfunctions of a given operator T, and the *boundary value problem* is an eigenvalue problem satisfying certain boundary conditions.

Definition 1.3.8 Let T be defined on $\mathcal{H}$. Then

i) The *resolvent set* $\rho(T)$ of T is defined to be the set of all complex numbers λ such that $(T-\lambda I)^{-1}$ exists as an everywhere defined bounded operator. It may be shown that $\rho(T)$ is an open set. For each $\lambda \in \rho(T)$ we denote by $R_\lambda(T)$ as the resolvent operator $(T-\lambda I)^{-1}$.

ii) The *spectrum* $\sigma(T)$ of T is defined to be the complement in $\mathbf{C}$ of the resolvent set $\rho(T)$.

iii) The *point spectrum* $\sigma_p(T)$ of T is defined to be the set of those complex numbers λ in $\sigma(T)$ for which $\overline{(T-\lambda I)D(T)} \neq \mathcal{H}$.

iv) The *continuous spectrum* σ_c of T is defined to be the set of all complex numbers $\lambda \in \sigma(T)$ for which $(T-\lambda I)^{-1}$ exists and is an unbounded operator defined on a dense subset of $\mathcal{H}$.

v) The *point continuous spectrum* σ_{pc} of T is defined to be the set of all complex numbers $\lambda \in \sigma(T)$ for which $(T-\lambda I)^{-1}$ exists and the manifold $(T-\lambda I)D(T)$ is not dense in $\mathcal{H}$ and also it is not closed.

Definition 1.3.9 Let T be an operator on $\mathcal{H}$. The dimension of the range $R(T)$ of T is called the *rank* of T and we denote it by rank$T$. The codimension of $R(T)$ is called the *deficiency* of T and is denoted by def$T$. Thus

$$\text{rank}T + \text{def}T = \dim\mathcal{H}$$

The inverse image $T^{-1}(0)$ of T is called the *kernel* or *null space* of T and is denoted by $N(T)$. The dimension of $N(T)$ is called the *nullity* of T which we shall denote by nul$T$. We have

$$\text{rank}T + \text{nul}T = \dim\mathcal{H}$$

It follows that if dimension $\dim\mathcal{H}$ of $\mathcal{H}$ is finite, then def$T$ = nul$T$; but if $\dim\mathcal{H} = \infty$, this may no longer be true.

1.4 Essential Spectrum

For selfadjoint operators in a Hilbert space there are several definitions of the essential spectrum σ_e.

Definition 1.4.1 The essential spectrum σ_e is the set of limit-points of the spectrum (where eigenvalues are counted according to their multiplicity and hence infinite dimensional eigenvalues are included.); i.e. all points of the spectrum except isolated eigenvalues of finite multiplicity.

Definition 1.4.2 σ_e is the largest subset of the spectrum remaining invariant under arbitrary compact perturbations, denoted by

$$\sigma_e(T) = \bigcap_{S \text{ compact}} \sigma(T+S).$$

In the general case for arbitrary operators such as non-selfadjoint operators the following definitions are sometimes convenient [11].

Definition 1.4.3 Let T be a closed, densely defined linear operator in $\mathcal{H}$ and define the following subsets of $\mathbb{C}$

$$\Phi^+(T) = \{\lambda : R(T - \lambda I) \text{ is closed and } \mathrm{nul}(T - \lambda I) < \infty\}$$

$$\Phi^-(T) = \{\lambda : R(T - \lambda I) \text{ is closed and } \mathrm{def}(T - \lambda I) < \infty\}$$

where $R(T - \lambda I), \mathrm{nul}(T - \lambda I)$ and $\mathrm{def}(T - \lambda I)$ denote the range, nullity and defficiency index of $T - \lambda I$ respectively.

$$\Phi_1(T) = \Phi^+(T) \cup \Phi^-(T)$$

$$\Phi_3(T) = \Phi^+(T) \cap \Phi^-(T)$$

$$\Phi_4(T) = \{\lambda \in \Phi_3(T) : \ \mathrm{nul}(T - \lambda I) = \ \mathrm{def}(T - \lambda I)\}$$

$$\Phi_5(T) = \{\lambda \in \Phi_4(T) : \text{ there is a deleted neighbourhood of } \lambda$$
$$\text{lying in the resolvent set } \rho(T) \text{ of } T\}$$

Let

$$\sigma_{ek}(T) = \mathbb{C} \setminus \Phi_k(T), \quad k = 1, 3, 4, 5$$

$$\sigma_{e2}^{\pm}(T) = \mathbb{C} \setminus \Phi^{\pm}(T).$$

Each of these sets $\sigma_{ek}(T)$, $\sigma_{e2}^{\pm}(T)$ is used for the essential spectrum in the literature. ([40] Chap 11).

The sets $\sigma_{ek}(T), \sigma_{e2}^{\pm}(T)$ are all closed subsets in $\mathbb{C}$ as their complements are open and clearly

$$\sigma_{e1}(T) \subseteq \sigma_{e2}^{\pm}(T) \subseteq \sigma_{e3}(T) \subseteq_{e4} (T) \subseteq \sigma_{e5}(T).$$

Since R(T) is closed if and only if R(T^*) is closed and nulT^*= def$T$, defT^*=nul$T$, it follows that $\lambda \in \sigma_{e2}^{+}(T)$ if and only if $\bar{\lambda} \in \sigma_{e2}^{-}(T^*)$. Also if $\lambda \in \sigma_{e4}(T) \setminus \sigma_{e2}^{+}(T) = \Phi^{+}(T) \setminus \Phi_4(T)$ then either λ is an eigenvalue of T (of finite geometric mutiplicity) or $\lambda \in \sigma_r(T) = \{\lambda : \ \text{nul}(T - \lambda I) = 0, \ \text{def}(T - \lambda I) \neq 0\}$, the so called residual spectrum of T.

Theorem 1.4.1 *a) If T is selfadjoint the sets $\sigma_{ek}(T), k = 1, 3, 4, 5,$ and $\sigma_{e2}^{\pm}(T)$ are identical and $\lambda \in \sigma(T) \setminus \sigma_{ek}(T)$ if and only if λ is an isolated eigenvalue of finite multiplicity.*
b) If T is J-selfadjoint, the sets $\sigma_{ek}(T), k = 1, 3, 4$ and $\sigma_{e2}^{\pm}(T)$ are identical [11].

Definition 1.4.4 Let T be a closed operator defined on the Hilbert space $\mathcal{H}$. A complex number λ belongs to the resolvent set $\rho(T)$ of T if and only if nul$(T - \lambda I)$ = def$(T - \lambda I)$. A complex number λ belongs to the *essential* spectrum of T if and only if either R$(T - \lambda I)$ is not closed, or R$(T - \lambda I)$ is closed but def$(T - \lambda I)$ = nul$(T - \lambda I) = \infty$.

Theorem 1.4.2 *A finite-dimensional perturbation leaves the continuous spectrum invariant* [53] 2.8.

Chapter 2

Analytic functions

2.1 Introduction

In this chapter we study some mathematical tools that will be required in the subsequent spectral analysis of differential operators, both in the case of selfadjoint boundary-value problems and for non-selfadjoint problems, which arise either from a complex potential, or a complex boundary condition, or both. We need in this chapter to summarise properties of analytic functions defined on a region and to describe some boundary properties of these functions. It is well-known that certain analytic functions which are defined on some specific regions play a very important role in characterising the different parts of the spectrum in the selfadjoint case. We recall that a region is a point set contained in the complex plane $\mathbb{C}$ which is connected and contains only interior points.

2.2 Complex Analysis

As a natural extension of real numbers one can step off the real axis to find more points to associate with the square root of negative real numbers. We need to give legitimacy to the operation $\sqrt{-1}$, since in the real numbers $\sqrt{-1}$ is meaningless, so some sense has to be given to the above operation. By extending the set of real numbers so as to form the smallest field containing $\mathbf{R} \cup \{i\}$, we arrive at the set of complex numbers $\mathbf{C}$. Each element of $\mathbf{C}$ may be represented as a unique point in the so-called complex plane, using rectangular Cartesian co-ordinates. We omit further details of the definition of the complex numbers and their properties, because they are extensively explained in every complex variable text book. Just as real functions may be defined on $\mathbf{R}$ or some subset of $\mathbf{R}$, so we may define complex-valued functions on $\mathbf{C}$ or some subset G of $\mathbf{C}$. We say that $f : G \to \mathbf{C}$ is differentiable at $z_0 \in G$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The limit $f'(z_0)$ is called derivative of $f(z)$ at z_0.

Definition 2.2.1 A function f which is defined and differentiable throughout a region R is called an analytic (regular) function on R. The region R is called a region of regularity of the function f.

For example elementary functions such as z^n, e^z, $\sin z$, $\cos z, \cdots$ are all regular; however the function $f(z) = \Re z$ is easily seen to be a function which is continuous in the entire plane but not a regular function in any region. We can denote a complex function $f(z)$ of the complex variable z by $w = f(z)$ or, more preciswely,

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real functions of the real variables x and y. Then we have the following theorem:

Theorem 2.2.1 (Cauchy-Riemann) *Let*

$$f(z) = u(x, y) + iv(x, y)$$

be a function of a complex variable defined on a domain G. Then a necessary and sufficient condition for $f(z)$ to be differentiable(as a function of a complex *variable) at the point* $z_0 = x_0 + iy_0 \in G$ *is that the functions* $u(x,y)$ *and* $v(x,y)$ *be differentiable(as functions of the two real variables* x *and* y*) at the point* (x_0, y_0) *and satisfy the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.2.1}$$

at (x_0, y_0)*. If these conditions are satisfied,* $f'(z_0)$ *can be represented in any of the forms*

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

where the partial derivatives are calculated at (x_0, y_0).

The equations (2.2.1), which are usually called the *Cauchy-Riemann equations*, are of basic importance in the theory of analytic functions and its applications to problems of physics and engineering. In fact, the equations (2.2.1) had already been studied in the eighteenth century by D'Alembert (1717-1783) and Euler (1707-1783), in researches devoted to the application of functions of a complex variable to hydrodynamics (D'Alembert and Euler), and to cartography and integral calculus (Euler).

2.2.1 Singularities

Let f be a single-valued function. Then any point of the complex plane at which f is analytic shall be called an *ordinary* point and any point at which f is not analytic is called a singular point or a singularity. A singularity is called *isolated* when no other singularities exist in its immediate neighborhood. The nature of the singularity of $f(z)$ at infinity is assumed to be the same as the nature of the singularity of $f(\frac{1}{z})$ at the origin. There are singularities which are removable. For example, the function $f(z) = \frac{\sin z}{z}$ has a singularity at the origin because it is not defined there. But, taking into account that $\lim_{z \to 0} f(z) = 1$ the singularity is removed by defining $f(0) = 1$. A pole is a point at which the function $f(z)$ is not analytic because its modulus becomes infinity large, but at which $\frac{1}{f(z)}$ is regular.

In other words if a is a pole of order n of $f(z)$, it is possible by Laurent's theorem to define a function $\phi(z)$ analytic at a, such that in a neighborhood of a (except the point a itself)

$$f(z) = \phi(z) + A_{-1}(z-a)^{-1} + A_{-2}(z-a)^{-2} + \ldots + A_{-n}(z-a)^{-n}$$

Evidently $f(z)$ is not analytic at a, but

$$\begin{aligned}\frac{1}{f(z)} &= \frac{1}{\phi(z) + A_{-1}(z-a)^{-1} + A_{-2}(z-a)^{-2} + \ldots + A_{-n}(z-a)^{-n}} \\ &= \frac{(z-a)^n}{(z-a)^n\phi(z) + A_{-1}(z-a)^{n-1} + A_{-2}(z-a)^{n-2} + \ldots + A_{-n}}\end{aligned}$$

and $\frac{1}{f(z)}$ vanishes at a. We say that $\frac{1}{f(z)}$ has a zero of order n at a and so is regular at a. If $n = 1$, the pole and the zero are called *simple.* All singularities of $f(z)$ which are not poles are called *essential singularities.* In a neighborhood of an isolated essential singularity $a, f(z)$ can be expanded into a Laurent series

$$f(z) = \phi(z) + \sum_{n=1}^{\infty} A_{-n}(z-a)^{-n}$$

where $\phi(z)$ is analytic at a. If the function $f(z)$ is analytic in a finite simply connected closed domain C, the integral $\frac{1}{2\pi i}\int_C f(z)dz$ evaluated along its boundary, vanishes. If the function is not analytic in C, such an integral may not be zero; if there is only one singularity in C, at the point a, the value of the above integral is called the *residue* of $f(z)$ at a.

2.2.2 Conformal mappings

Let $f(z)$ be an analytic function of a complex variable defined on a domain G. Suppose $f(z)$ has a non-zero derivative $f'(z)$ at a point $z_0 \in G$, and let l be a curve which passes through z_0 and has a tangent τ at z_0. Then $w = f(z)$ maps l into a curve L in the w-plane which passes through the point $w_0 = f(z_0)$ and has a tangent T at w_0. Moreover, the inclination of T exceeds the inclination of τ by the angle $Arg f'(z_0)$; it follows that the angle between any two curves passing through z_0 is preserved under f.

Definition 2.2.2 A mapping by a continuous function which preserves angles between curves passing through a given point z_0 is said to be a *conformal mapping*

For example the fractional linear transformation, or Möbius transformation,

$$L(z) = \frac{az+b}{cz+d} \tag{2.2.2}$$

where a, b, c and d are arbitrary complex numbers (except that c and d are not both zero, and $ad - bc \neq 0$) is a conformal mapping. Thus the function $w = L(z)$ maps the finite point $-\frac{d}{c}$ into the point at infinity, and maps the point at infinity into the finite point $\frac{a}{c}$. Moreover circles on the z-plane passing through the point $z = -\frac{b}{a}$ correspond to circles on the w-plane passing through the origin, circles on the z-plane passing through the point $z = -\frac{d}{c}$ correspond to straight lines on the w-plane, and, in general, circles and straight lines on the z-plane correspond to circles and straight lines on the w-plane.

2.2.3 Meromorphic functions

We are concerned with functions whose domain and range both lie in the extended complex plane $\mathbb{C} \cup \{\infty\}$. A function f is said to be *meromorphic* at a provided that the domain of f is a neighbourhood of a, that there exists $r > 0$ such that

$$\{z : |z - a| < r\}$$

is contained in the domain of f and f is analytic in

$$\{z : 0 < |z - a| < r\}$$

and finally that f is continuous at a. A function f is meromorphic in a region (or open set) Ω provided that f is meromorphic at each point of Ω. In other words, a meromorphic function is one whose only singularities are poles [19].

Let $f(z) = u(z) + iv(z)$ be an analytic function in the upper half-plane with positive imaginary part; thus if $z = x + iy$ then $v(z) \geq 0$ if $y > 0$. We can see that the class P of all such functions is a convex cone; i.e. if α and β are positive real numbers and f and g are two functions in the set P, then the function $\alpha f(z) + \beta g(z)$ is also in P. The class P is also closed under composition and $\frac{-1}{f(z)}$ also is in the class P if $f \in P$. Let (a, b) be an open interval of the real axis; by $P(a, b)$ we mean the subclass of P consisting of those functions in P which admit an analytic continuation across the interval into the lower half-plane and where the continuation is by reflection. Thus the functions in this class are real on the interval (a, b) and are continuable throughout the lower half-plane. It is clear that $P(a, b)$ is also a convex cone. If $f(z)$ is in $P(a, b)$ then $v(x) = 0$ if $a < x < b$, hence by the Cauchy-Rieman equations we see that $f(x) = u(x)$ is an increasing function on the interval. Consider that square root function $\sqrt{z}$ which is positive on the right half-axis; then $f(z) = \sqrt{z}$ is a function in the class P since the argument of $\sqrt{z}$ is a half of the argument of z and so the number $\sqrt{z}$ falls in the upper right quadrant of the complex plane if $\Im z > 0$. . Other examples of this kind of function are $f(z) = z^{\gamma}$, $0 < \gamma < 1$ and $f(z) = \tan z$. Let T be a selfadjoint transformation on a Hilbert space $\mathcal{H}$ and let $z \in \mathbb{C} \backslash \mathbb{R}$. Then the resolvent operator $R_z = (T - zI)^{-1}$ is a well defined and bounded operator, and it is well known that, for any $u \in \mathcal{H}$ the function $f(z) =< R_z u, u >$ is analytic in the upper half-plane ([49] Th. 5.16), and belongs to P since, if $R_z u = v$ then $(T - zI)v = u$ and therefore

$$f(z) =< R_z u, u >=< v, u >=< v, (T - zI)v >=< v, Tv > -\bar{z} < v, v >$$

The imaginary part of $f(z)$ is $y\|v\|$ since (v, Tv) is real and so $\Im f(z)$ has the same sign as $\Im z = y$.

The functions in the class P admit a canonical integral representation.

Theorem 2.2.2 *A function $f(z)$ in the class P has a unique canonical representation of the form*

$$f(z) = \alpha + \beta z + \int_{-\infty}^{\infty} [\frac{1}{t - z} - \frac{t}{t^2 + 1}] d\sigma(t),$$

where α and $\beta \geq 0$ are two real constants, and $\sigma(t)$ is a non-decreasing function on the real axis for which $\int_{-\infty}^{\infty} \frac{d\sigma(t)}{1+t^2} < \infty$.

Theorem 2.2.3 (Cauchy) *If a function $f(z)$ is analytic and one valued inside and on a simple closed contour C, then $\int_C f(z)dz = 0$.* [47]

Theorem 2.2.4 (Morera) *If $f(z)$ is a continuous function of z in a region D, and if the integral $\int f(z)dz$ taken round any closed contour in D is zero, then $f(z)$ is analytic inside D.* [47]

Theorem 2.2.5 (Liouville) *A function which is analytic for all finite values of z, and is bounded, is a constant* [47]. We know [47] §2.71 that in the neighbourhood of an isolated singularity $z = a$, a one valued analytic function $f(z)$ may be expanded in the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n-1}^{\infty} b_n(z-a)^{-n}.$$

The coefficient b_1 is of particular importance, and is called the residue of $f(z)$ at the point $z = a$. So by the Laurent's series expansion of $f(z)$

$$b_1 = \frac{1}{2\pi i}\int_C f(z)dz,$$

where C is any circle with centre $z = a$, which does not include all other singularities of the function. If $z = a$ is a simple pole of f then $b_1 = \lim_{z\to a}(z-a)f(z)$.

Theorem 2.2.6 *Let $f(z)$ be one-valued and analytic inside and on a simple closed contour C, except at a finite number of singularities $z_1, z_2, \ldots, z_n$. Let the residue of $f(z)$ at these points be $r_1, r_2, \ldots, r_n$. Then*

$$\int_c f(z)dz = 2\pi i(r_1 + r_2 + \ldots + r_n).$$

Definition 2.2.3 A root λ_0 of an algebraic equation

$$F(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$$

is called nondegenerate or simple if $F'(\lambda_0) \neq 0$. In other words λ_0 is simple if the decomposition

$$F(\lambda) = \Pi_{i=1}^{n}(\lambda - \lambda_i)$$

has $\lambda_i = \lambda_0$ for exactly one value of i.

A root λ_0 is said to have multiplicity m if

$$F'(\lambda_0) = \ldots = F^{(m-1)}(\lambda_0) = 0,\ F^{(m)}(\lambda_0) \neq 0.$$

Or equivalently if exactly m of the λ_i equal λ_0.

Chapter 3

Weyl Theory

3.1 Introduction

In the theory of differential operators we distinguish two cases, regular and singular, which lead to the study of boundary value problems. Most physical phenomena appear as singular boundary value problems; this involves finding solutions of an appropriate differential equation which satisfy certain boundary conditions in an interval (a, b) of real numbers, where either a or b or both may be singular points. Herman Weyl was (in 1909) the first to consider singular boundary value problems; in particular he considered a singular second order Sturm-Liouville type problem on the interval $(a, b) \subseteq \mathbf{R}$. In this chapter we try to give a fairly complete account of the theory of Weyl and will describe some later improvements to this theory. The most important result that makes the Weyl theory so considerable and remarkable is that we can extend the Fourier expansion of functions to the much more general (Sturm-Liouville) eigenfunction expansion.

3.2 Regular Boundary Value Problems

We consider one of the most important operators which is frequently encountered in applications, namely the operator T generated by the Sturm-Liouville type differential expression

$$\tau \equiv -\frac{d^2}{dx^2} + q(x),$$

where $q(x)$ is a continuous real-valued function which is regular and defined on the interval $[a, b]$. Consider the operator T_0 defined on the Hilbert space $\mathcal{H} = L^2[a, b]$, by $T_0 f = \tau f$ for each f in the domain of definition D_0 of T_0, where

$$D_0 = \{f \in L^2[a, b] : f \text{ and } f' \text{ are absolutely continuous, } \tau f \in L^2[a, b]\}.$$

The operator T_0 has a maximal domain on which τf is meaningful for each $f \in D_0$, and so is called the *maximal operator.* An operator T, a restriction of T_0, is said to be a selfadjoint operator if and only if

$$< Tf, g > - < f, Tg >= 0$$

for all f, g in the domain $D(T)$ of T. This means, in this case, that T is selfadjoint if and only if

$$\int_a^b (Tf)\bar{g} - \int_a^b f\overline{Tg} = W[f, g](b) - W[f, g](a) = 0 \tag{3.2.1}$$

for all $f, g \in D(T)$, where $Tf = \tau f$. So we can see, for example, that if $f(a) = f(b) = 0$ for all $f \in D(T)$, then T is a selfadjoint operator. In general, the regular boundary-value problem which we are dealing with, is of the form

$$\tau f = \lambda f \tag{3.2.2}$$

$$f(a) \cos\alpha + f'(a) \sin\alpha = 0 \tag{3.2.3}$$

$$f(b) \cos\beta + f'(b) \sin\beta = 0, \tag{3.2.4}$$

where λ is a complex parameter, $\alpha,\ \beta \in [0, \pi)$ and the function $f \in L^2[a, b]$ is a solution of the problem (3.2.2-4), so that f is an eigenfunction corresponding to the eigenvalue λ of the operator T. By (3.2.1) we can see that the above problem is a selfadjoint one, and we define the domain of definition $D_{\alpha,\beta}$ of T by

$$D_{\alpha,\beta} = \{f \in D_0 : f(a)\cos\alpha + f'(a)\sin\alpha = 0,\ f(b)\cos\beta + f'(b)\sin\beta = 0\}.$$

Lemma 3.2.1 *For each $f \in \mathcal{H}$ the equation*

$$\tau y = f$$

has a solution $y(x)$ which satisfies the boundary condition

$$y(a) = y'(a) = y(b) = y'(b) = 0$$

if and only if the function f is orthogonal to all solutions of the homogeneous equation $\tau y = 0$.

Proof: see [34]

If $\mathcal{S}$ denotes the set of all solutions of $\tau y = 0$ and $\mathcal{R}$ denotes the range of the operator T, then by Lemma 3.2.1 an element f of $\mathcal{H}$ lies in $\mathcal{R}$ if and only if it is orthogonal to $\mathcal{S}$ and so $\mathcal{H} = \mathcal{S} + \mathcal{R}$. Lemma 3.2.1 can be used to prove that D is dense in $\mathcal{H}$, and that T is closed [34] §17.3 Lemma 1.

Lemma 3.2.2 *i)Eigenfunctions corresponding to different eigenvalues are orthogonal. ii)Eigenvalues of the operator T are real.*

Proof: Suppose that $f(x,\lambda_1)$ and $g(x,\lambda_2)$ are eigenfunctions corresponding to the different eigenvalues λ_1 and λ_2 respectively. Then

$$\begin{aligned}\int_a^b f(x,\lambda_1)\tau g(x,\lambda_2)dx &- \int_a^b g(x,\lambda_2)\tau f(x,\lambda_1)dx \\ &= \int_a^b f(x,\lambda_1)(-g''(x,\lambda_2) + q(x)g(x,\lambda_2))dx \\ &- \int_a^b g(x,\lambda_2)(-f''(x,\lambda_1) + q(x)f(x,\lambda_1))dx \\ &= [-f(x,\lambda_1)g'(x,\lambda_2) + f'(x,\lambda_1)g(x,\lambda_2)]_a^b\end{aligned}$$

or equivalently

$$\begin{aligned}\int_a^b f(x,\lambda_1)\tau g(x,\lambda_2)dx \quad - \quad & \int_a^b g(x,\lambda_2)\tau f(x,\lambda_1)dx \\ = \quad & \int_a^b \lambda_2 f(x,\lambda_1)g(x,\lambda_2)dx - \int_a^b \lambda_1 f(x,\lambda_1)g(x,\lambda_2)dx \\ = \quad & (\lambda_2-\lambda_1)\int_a^b f(x,\lambda_1)g(x,\lambda_2)dx = W[f,g]_a^b \quad (3.2.5)\end{aligned}$$

where $W[f,g]_a^b = 0$. Therefore

$$\int_a^b f(x,\lambda_1)g(x,\lambda_2)dx = 0$$

since $\lambda_1 \neq \lambda_2$.

If an eigenvalue λ is a complex number i.e. $\Im\lambda \neq 0$ then $\bar{\lambda}$ is also an eigenvalue and from 3.2.5 with $\lambda_1 = \lambda$, $\lambda_2 = \bar{\lambda}$, and $g = \bar{f}$, where $f(x,\bar{\lambda}) = \bar{f}(x,\lambda)$, we obtain

$$(\lambda-\bar{\lambda})\int_a^b |f(x,\lambda)|^2 dx = 2i\Im\lambda \int_a^b |f(x,\lambda)|^2 dx = 0$$

which implies $f \equiv 0$. But this is a contradiction, since f is a non-trivial eigenfunction corresponding to the eigenvalue λ. Hence we conclude that $\Im\lambda = 0$, so that λ is a real number.

Suppose that $\phi(x,\lambda)$ and $\chi(x,\lambda)$ are two solutions of (3.2.2) satisfying

$$\phi(a,\lambda) = \sin\alpha, \;\; \phi'(a,\lambda) = -\cos\alpha \qquad (3.2.6)$$

$$\chi(b,\lambda) = \sin\beta, \;\; \chi'(b,\lambda) = -\cos\beta \qquad (3.2.7)$$

where α, β are two fixed real numbers in the interval $[0,\pi)$. We note that the solution $\phi(x,\lambda)$ satisfies (3.2.3). It is easy to see that the Wronskian of ϕ and χ at the point x in the interval $[a,b]$ satisfies

$$W[\phi(x,\lambda)\chi(x,\lambda)] = \phi(x,\lambda)\chi'(x,\lambda) - \phi'(x,\lambda)\chi(x,\lambda)$$

$$\frac{d}{dx}W[\phi(x,\lambda)\chi(x,\lambda)] = \phi(x,\lambda)\chi''(x,\lambda) - \phi''(x,\lambda)\chi(x,\lambda) = 0$$

so that $W[\phi(x,\lambda)\chi(x,\lambda)] =$ constant, with respect to the independent variable x. Hence we can write

$$W[\phi(x,\lambda)\chi(x,\lambda)] = \omega(\lambda) \qquad (3.2.8)$$

where $\omega(\lambda)$ has real zeros, since the zeros of $\omega(\lambda)$ are the eigenvalues of T and by Lemma 3.2.2 the eigenvalues are real. Noting that (3.2.5) is true for any solutions f_1 and f_2 of (3.2.2), set $\lambda_1 = \lambda,\ \lambda_2 = \lambda',\ f_1(x,\lambda) = \phi(x,\lambda)$ and $f_2(x,\lambda) = \phi(x,\lambda')$. Then we have

$$(\lambda - \lambda') \int_a^b \phi(x,\lambda)\phi(x,\lambda')dx = \phi(b,\lambda)\phi'(b,\lambda') - \phi'(b,\lambda)\phi(b,\lambda')$$

Also by (3.2.7) and (3.2.8)

$$\omega(\lambda)\phi(b,\lambda') - \omega(\lambda')\phi(b,\lambda) = \sin\beta[\phi(b,\lambda)\phi'(b,\lambda') - \phi'(b,\lambda)\phi(b,\lambda')]$$

Hence if $\sin\beta \neq 0$ then we have

$$(\lambda - \lambda')\sin\beta \int_a^b \phi(x,\lambda)\phi'(x,\lambda')dx = \omega(\lambda)\phi(b,\lambda') - \omega(\lambda')\phi(b,\lambda) \tag{3.2.9}$$

We are going to show by contradiction that the zeros of the function $\omega(\lambda)$ are all simple. First suppose $\sin\beta \neq 0$ and that λ_0 is a multiple zero of degree two. Then we can write $\omega(\lambda) = (\lambda - \lambda_0)^2\Omega(\lambda)$ where $\Omega(\lambda_0) \neq 0$ so that $\omega(\lambda_0 \pm i\nu) = -\nu^2\Omega(\lambda_0 \pm i\nu) = O(\nu^2)$ as $\nu \to 0$. Hence if we set $\lambda = \lambda_0 + i\nu$ and $\lambda' = \lambda_0 - i\nu$ in (3.2.9) then the right-hand side of (3.2.9) is $O(\nu^2)$, but the left-hand side is $2i\nu\sin\beta\int_a^b |\phi(x,\lambda)|^2dx$, which is $O(\nu)$. This is a contradiction. So $\omega(\lambda)$ has no zeros of order 2; similarly, $\omega(\lambda)$ has no zeros of order $n > 2$. Hence all zeros of $\omega(\lambda)$ are simple.

If $\sin\beta = 0$ then $\cos\beta \neq 0$ and we have

$$(\lambda - \lambda')\cos\beta \int_a^b \phi(x,\lambda)\phi'(x,\lambda')dx = \omega(\lambda')\phi'(b,\lambda) - \omega(\lambda)\phi'(b,\lambda')$$

Theorem 3.2.1 *Let q be a continuous function in the interval $[a,b]$ and let α be an arbitrary real number in the interval $[0,\pi)$. Then the equation (3.2.2) has a solution $\phi(x,\lambda)$ satisfying*

$$\phi(a,\lambda) = \sin\alpha, \quad \phi'(a,\lambda) = -\cos\alpha$$

and for each x in $[a,b]$ the function $\phi(x,\lambda)$ is an entire function of λ.

Proof: see [46] Theorem 1.5.

As a simple selfadjoint boundary-value problem on $[0, \pi]$ one can consider the problem

$$f'' + \lambda^2 f = 0, \; f(0) = f(\pi) = 0$$

in which $q \equiv 0$. We see that a solution is of the form $f(x) = c \sin \lambda x$ where c is a constant. Eigenvalues and the corresponding non-trivial eigenfunctions are as follows

$$\lambda_n = n, \quad f_n(x) = \sin nx$$

for all $n \in \mathbb{N}$. Investigation of the eigenfunction properties of this example shows that the eigenfunctions

$$\sin x, \; \sin 2x, \; \sin 3x, \; \ldots \; \sin nx$$

have two important properties

i) $\sin x$ has two zeros, $\sin 2x$ three zeros, $\sin 3x$ four zeros, ..., and $\sin nx$ has n zeros in the interval $[0, \pi]$.

ii) Between every two consecutive zeros of the eigenfunction $\sin nx$ the eigenfunction $\sin(n+1)x$ has a zero.

This example illustrates two important theorems due to Sturm and Liouville.

Theorem 3.2.2 (Comparison theorem) *Let two homogeneous equation*

$$u'' + q(x)u = 0 \tag{3.2.10}$$

$$v'' + r(x)v = 0 \tag{3.2.11}$$

be given, in the interval $[a, b]$. *If* $q(x) < r(x)$ *for all* $x \in [a, b]$, *there exists at least one zero of each solution of* (3.2.11) *between any two zeros of any solution of* (3.2.10)

Proof: see [28]

Theorem 3.2.3 *There is an indefinitely increasing sequence of eigenvalues* $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ *of the boundary-value problem* (3.2.2-4), *and the eigenfunction corresponding to the eigenvalue* λ_n *has exactly* n *zeros in the interval* (a, b).

Proof: also see [28]

Recalling (3.2.6) and (3.2.7), we note that the solutions $\phi(x,\lambda)$ and $\chi(x,\lambda)$ are linearly dependent, so that $W[\phi,\chi]=0$, if and only if $\phi(x,\lambda)$ is an eigenfunction of the problem (3.2.2-4). Therefore the eigenvalues of (3.2.2-4) coincide with the zeros of the Wronskian. We now give the integral representation of the resolvent. Let $W[\phi,\chi]=\omega(\lambda)\neq 0$ for some complex parameter λ; then for each continuous function g, the non-homogeneous equation

$$\tau f-\lambda f=g \tag{3.2.12}$$

has a solution $f(x,\lambda)=\int_a^b G(x,t;\lambda)g(t)dt$ where

$$G(x,t;\lambda)=\begin{cases}\frac{\phi(x,\lambda)\chi(t,\lambda)}{\omega(\lambda)}, & \text{if } a\le x\le t\le b\\ \frac{\phi(t,\lambda)\chi(x,\lambda)}{\omega(\lambda)}, & \text{if } a\le t\le x\le b\end{cases}$$

and is called Green's function. In other words we have

$$f(x,\lambda)=\frac{1}{\omega(\lambda)}\left\{\phi(x,\lambda)\int_a^x\chi(t,\lambda)g(t)dt+\chi(x,\lambda)\int_x^b\phi(t,\lambda)g(t)dt\right\}$$

The function f is called a resolvent and is a solution of (3.2.12) which satisfies (3.2.3-4).

Theorem 3.2.4 *Let g be a function with continuous second derivative and satisfying the boundary conditions* (3.2.3-4). *Then g can be expanded into an absolutely and uniformly convergent Fourier series of eigenfunctions of the boundary-value problem* (3.2.2-4).

Proof: see [28]

3.3 Singular Boundary Value Problems

In this section the study of boundary-value problem is extended to the singular case on the interval $[0,\infty)$, where the left endpoint is regular. An ordinary boundary condition can be assigned at the origin but the point at infinity poses a serious problem. The difficulty was resolved by Hermann Weyl in his dissertation of 1909. Weyl also wrote three memoirs on singular ordinary linear differential equations with completion dates in 1908, 1909, and 1910; these papers are reproduced, respectively, as paper numbers 6, 8 and 7 in Volume I of his collected mathematical works [50].

A translation into English of the relevant sections of the memoir of Weyl has been made by D. Race, in his M. Sc. thesis, at the University of Dundee, Scotland, U. K, in 1976. In the previous section we obtained the expansion theorem for the regular Sturm-Liouville problem. It is also possible to obtain the expansion theorem for a singular problem, if we consider it as the limit of a sequence of regular problems. We will start by considering the case in which the function q is real and continuous on the interval $[0,\infty)$; the analysis is similar if the interval is $[a,b)$ where the point b is a singular and a is a regular point.

Consider the operator T, defined on the Hilbert space $\mathcal{H} = L^2[0,\infty)$ and generated by the differential expression τ, whose domain of definition $\mathcal{D}$ is defined as follows

$$\mathcal{D} = \{f \in \mathcal{H} : f, f' \in AC_{\rm loc}[0,\infty),\ \tau f \in \mathcal{H}\},$$

where $AC_{\rm loc}[0,\infty)$ is the class of all locally absolutely continuous functions on $[0,\infty)$ i.e. $f \in AC_{\rm loc}[0,\infty)$ if and only if f is absolutely continuous on the finite interval $[0,X]$ for each $X > 0$, and $\tau f = -\frac{d^2 f}{dx^2} + q(x)$ in which q is regular at 0 and singular at infinity. We define T by $Tf = \tau f$ for all $f \in \mathcal{D}$. The domain $\mathcal{D}$ of T is the largest linear manifold that is dense in the Hilbert space $\mathcal{H}$ and is such that T can be well defined on it; we therefore denote this operator by $T_{\max}$. Among physicists the operator $T_{\max}$ is called the *Hamiltonian.*

We can also associate with T a minimal operator $T_{\min}$ by defining the operator T' with domain

$$\mathcal{D}' = \{f \in \mathcal{D} : f \text{ has compact support in } [0,\infty)\}$$

such that $T'f = \tau f$ for all $f \in \mathcal{D}'$, and $T_{\min} = \bar{T}'$ where $\bar{T}'$ is the closure of T'. It is well known that T' and hence $T_{\min}$ are densely defined and $(T_{\min})^* = T_{\max}$

[34].

Let $f(x) = f(x,\lambda), g(x) = g(x,\lambda')$ satisfy the differential equation

$$\tau y = -y'' + q(x)y = \lambda y \tag{3.3.13}$$

and the corresponding equation with λ' instead of λ, respectively. Then for $b > 0$

$$\int_0^b \{f(x)(-g''(x) + q(x)g(x)) - g(x)(-f''(x) + q(x)f(x))\}dx =$$

$$(\lambda'-\lambda)\int_0^b f(x)g(x)dx = \int_0^b \{g(x)f''(x) - f(x)g''(x)\}dx = W[g,f](b) - W[g,f](0)$$

Setting $\lambda' = \bar{\lambda}$ and $g = \bar{f}$ we obtain

$$-2i\Im\lambda \int_0^b |f(x)|^2 dx = W[\bar{f}, f](b) - W[\bar{f}, f](0) \tag{3.3.14}$$

Here we can identify conditions under which the operator $T_{\max}$ is selfadjoint. First of all $T_{\max}$ has to be symmetric i.e.

$$< T_{\max} f, g > - < f, T_{\max} g >= W[f,\bar{g}](\infty) - W[f,\bar{g}](0) = 0$$

for all $f, g \in D(T_{\max})$, where $W[f,\bar{g}](\infty) = \lim_{x\to\infty} W[f,\bar{g}](x)$ and this limit exists, since the integrals $\int_0^\infty (T_{\max} f)\bar{g}$ and $\int_0^\infty f\overline{(T_{\max} g)}$ are finite.

Now let $\phi(x,\lambda)$ and $\theta(x,\lambda)$ be solutions of (3.3.13) satisfying the boundary conditions

$$\begin{aligned} \phi(0,\lambda) &= \cos\alpha, \quad \phi'(0,\lambda) = -\sin\alpha \\ \theta(0,\lambda) &= \sin\alpha, \quad \theta'(0,\lambda) = \cos\alpha \end{aligned} \tag{3.3.15}$$

where λ is a complex parameter and α is a real number. We note that $\phi(x,\lambda)$ satisfies

$$\phi(0,\lambda)\sin\alpha + \phi'(0,\lambda)\cos\alpha = 0$$

and

$$W[\phi,\theta](x) = W[\phi,\theta](0) = \sin^2\alpha + \cos^2\alpha = 1$$

Consider the solution $\psi_b(x,\lambda)$ of (3.3.13) of the form $\theta(x,\lambda) + m_b(\lambda)\phi(x,\lambda)$. on $[0,b]$ which satisfies a real boundary condition β at the point $x = b$, so that

$$\psi_b(x,\lambda)\cos\beta + \psi_b'(x,\lambda)\sin\beta = 0.$$

This gives

$$m_b(\lambda) = -\frac{\theta(b,\lambda)\cos\beta + \theta'(b,\lambda)\sin\beta}{\phi(b,\lambda)\cos\beta + \phi'(b,\lambda)\sin\beta}$$

If we set $\cot\beta = z$, with $\beta \neq 0$ then

$$m_b(\lambda, z) = -\frac{\theta(b,\lambda)z + \theta'(b,\lambda)}{\phi(b,\lambda)z + \phi'(b,\lambda)} \tag{3.3.16}$$

This is a Möbius transformation on the complex plane $\mathbb{C}$, which is well defined, for fixed $\lambda \in \mathbb{C}_+$, fixed $b > 0$ and variable $z \in \mathbb{C}$ as we noe show. Since $\phi(x,\lambda)$ is a solution of (3.3.13), we can write

$$\phi''(x,\lambda)\bar{\phi}(x,\lambda) = (q(x) - \lambda))|\phi(x,\lambda)|^2$$

$$\phi''(x,\lambda)\bar{\phi}(x,\lambda) = \frac{d}{dx}[\phi'(x,\lambda)\bar{\phi}(x,\lambda)] - |\phi'(x,\lambda)|^2$$

Taking the integral from 0 to b, then

$$[\phi'(x,\lambda)\bar{\phi}(x,\lambda)]_0^b - \int_0^b |\phi'(x,\lambda)|^2 dx = \int_0^b (q(x) - \lambda)|\phi(x,\lambda)|^2 dx$$

so that

$$\Im(\phi'(b,\lambda)\bar{\phi}(b,\lambda) + \sin\alpha\cos\alpha - \int_0^b |\phi'(x,\lambda)|^2 dx) = \Im(\int_0^b (q(x) - \lambda)|\phi(x,\lambda)|^2 dx)$$

from which

$$\Im(\phi'(b,\lambda)\bar{\phi}(b,\lambda)) = \Im(-\lambda \int_0^b |\phi(x,\lambda)|^2)$$

Hence

$$\Im(\phi'(b,\lambda)\bar{\phi}(b,\lambda)) = -(\Im\lambda)\int_0^b |\phi(x,\lambda)|^2 dx < 0, \quad \text{if} \quad \Im\lambda > 0. \tag{3.3.17}$$

But we have the identity

$$\Im\frac{\phi'(b,\lambda)}{\phi(b,\lambda)} = \frac{\Im(\phi'(b,\lambda)\bar{\phi}(b,\lambda))}{|\phi(b,\lambda)|^2}$$

Hence if $\Im\lambda \neq 0$ then

$$\Im\frac{\phi'(b,\lambda)}{\phi(b,\lambda)} \neq 0$$

i.e. the denominator of (3.3.16) is not zero, and so (3.3.16) is well defined. On the other hand

$$\theta(b,\lambda)\phi'(b,\lambda) - \theta'(b,\lambda)\phi(b,\lambda) = W[\theta,\phi](b) = -1 \neq 0,$$

i.e. (3.3.16) is not constant. Also (3.3.16) does not degenerate into a linear transformation since for each $b > 0$ the function $\phi(b, \lambda)$ is a non-zero entire function except for a discrete set.

The real line corresponds to a circle C_b and the upper half-plane goes to the exterior of the circle C_b while the lower half-plane corresponds to the interior of the circle C_b, since from (3.3.16) we have

$$m_b(\lambda, \frac{-\phi'(b,\lambda)}{\phi(b,\lambda)}) = \infty$$

and this shows that if $\Im\lambda > 0$ by (3.3.17) $\Im\frac{-\phi'(b,\lambda)}{\phi(b,\lambda)} > 0$ i.e. the upper half-plane goes to the exterior of the circle C_b. The result for the lower half plane is obtained similarly.

Recalling properties of conformal mappings in complex analysis we obtain for each b as $z = \cot\beta$ varies, m_b describes a circle C_b in the complex plane $\mathbb{C}$. Also we obtain by (3.3.14) with $f = \phi$, that

$$\Im(-\frac{\phi'(b,\lambda)}{\phi(b,\lambda)}) = \frac{1}{2i}(-\frac{\phi'(b,\lambda)}{\phi(b,\lambda)} + \frac{\bar{\phi}'(b,\lambda)}{\bar{\phi}(b,\lambda)}) = -\frac{1}{2}i\frac{W[\phi,\bar{\phi}](b)}{|\phi(b,\lambda)|^2} = \nu\frac{\int_0^b |\phi(x,\lambda)|^2 dx}{|\phi(b,\lambda)|^2}$$

which has the same sign as $\nu = \Im\lambda$, since $W[\phi,\bar{\phi}](0) = 0$. Hence if $\nu > 0$, the exterior of C_b corresponds to the upper half of the complex plane $\Im z > 0$.

We are going to characterize the behaviour of the circles C_b when b approaches large values. We see that the real line from the complex plane $\mathbb{C}$ is transferred, by the conformal mapping (3.3.16), to a circle C_b and so the origin $z = 0$ maps to a point $m_b(\lambda) = -\frac{\theta'(b,\lambda)}{\phi'(b,\lambda)}$ on the circle C_b Also since $m_b = \infty$ corresponds to $z = -\frac{\phi'(b,\lambda)}{\phi(b,\lambda)}$, the centre of C_b, which is the conjugate of the point at infinity with respect to the circle C_b, corresponds to the conjugate $\bar{z} = -\frac{\bar{\phi}'(b,\lambda)}{\bar{\phi}(b,\lambda)}$ of $z = -\frac{\phi'(b,\lambda)}{\phi(b,\lambda)}$ with respect to the real line; that is $m_b(\lambda, -\frac{\bar{\phi}'(b,\lambda)}{\bar{\phi}(b,\lambda)}) = -\frac{W[\theta,\bar{\phi}](b)}{W[\phi,\bar{\phi}](b)}$. Hence the radius r_b of C_b satisfies

$$r_b = |-\frac{\theta'(b,\lambda)}{\phi'(b,\lambda)} + \frac{W[\theta,\bar{\phi}](b)}{W[\phi,\bar{\phi}](b)}| = |\frac{W[\theta,\phi](b)}{W[\phi,\bar{\phi}](b)}| = \frac{1}{2\nu\int_0^b |\phi(x,\lambda)|^2 dx} \qquad (3.3.18)$$

Now $m_b(\lambda)$ is inside of C_b if and only if $\Im z = \frac{1}{2i}(z - \bar{z}) < 0$ or $i(z - \bar{z}) > 0$. If we express z in terms of m_b we have

$$z = -\frac{m_b(\lambda)\phi'(b,\lambda) + \theta'(b,\lambda)}{m_b(\lambda)\phi(b,\lambda) + \theta(b,\lambda)} \tag{3.3.19}$$

and so

$$i(z-\bar{z}) = i\left\{-\frac{m_b(\lambda)\phi'(b,\lambda) + \theta'(b,\lambda)}{m_b(\lambda)\phi(b,\lambda) + \theta(b,\lambda)} + \frac{\bar{m}_b(\lambda)\bar{\phi}'(b,\lambda) + \bar{\theta}'(b,\lambda)}{\bar{m}_b(\lambda)\bar{\phi}(b,\lambda) + \bar{\theta}(b,\lambda)}\right\} > 0,$$

i.e.

$$i(|m_b|^2 W[\phi,\bar{\phi}](b) + m_b W[\theta,\bar{\phi}](b) + \bar{m}_b W[\theta,\bar{\phi}](b) + W[\theta,\bar{\theta}](b)) > 0,$$

or

$$iW[\psi_b(x,\lambda), \bar{\psi}_b(x,\lambda)](b) > 0, \tag{3.3.20}$$

Setting $f = \psi_b$ and $g = \bar{\psi}_b$ in (3.3.14) and (3.3.20) we obtain

$$2\nu \int_0^b |\theta(x,\lambda) + m_b(\lambda)\phi(x,\lambda)|^2 dx < iW[\theta + m_b\phi, \bar{\theta} + \bar{m}_b\bar{\phi}](0).$$

Since

$$W[\theta + m_b\phi, \bar{\theta} + \bar{m}_b\bar{\phi}](0) = W[\theta,\bar{\theta}](0) + |m_b|^2 W[\phi,\bar{\phi}](0) +$$

$$\bar{m}_b W[\theta,\bar{\phi}](0) + m_b W[\phi,\bar{\theta}](0) = \bar{m}_b W[\theta,\bar{\phi}](0) + m_b W[\phi,\bar{\theta}](0)$$

where we have used

$$W[\phi,\bar{\theta}](0) = 1,\ W[\theta,\bar{\phi}] = -1,\ W[\phi,\bar{\phi}](0) = W[\theta,\bar{\theta}](0) = 0$$

then

$$W[\theta + m_b\phi, \bar{\theta} + \bar{m}_b\bar{\phi}](0) = m_b - \bar{m}_b = 2i\Im m_b$$

Hence m_b is inside C_b if $\nu > 0$ and

$$\int_0^b |\theta + m_b\phi|^2 dx < -\frac{\Im m_b}{\nu}$$

The same result is obtained if $\nu < 0$. In each case the sign of $\Im m_b$ is opposite to that of ν. It follows that, if m_b is inside or on the circle C_b, and if $b > a > 0$ for some real a then

$$\int_0^a |\theta + m_b\phi|^2 dx < \int_0^b |\theta + m_b\phi|^2 dx \leq -\frac{\Im m_b}{\nu}$$

and therefore m_b is inside C_a. This means that C_a contains C_b in its interior if $0 < a < b$. Thus for a given λ with $\Im\lambda > 0$, as $b \to \infty$ the circles C_b converges either to a circle C_∞ or to a point m. In the case where the limit is a circle C_∞, then its radius $r_\infty = \lim_{b\to\infty} r_b$ is positive, and from (3.3.18) this implies that $\phi(x,\lambda) \in L^2[0,\infty)$. If $\tilde{m}$ is any point on C_∞, then $\tilde{m}$ is inside any circle C_b for $b > 0$. Hence

$$\int_0^b |\theta + \tilde{m}\phi|^2 dx < -\frac{\Im\tilde{m}}{\Im\lambda}$$

The same argument holds if $\tilde{m}$ reduces to the point m, i.e.

$$\int_0^b |\theta + m\phi|^2 dx \le -\frac{\Im m}{\Im\lambda}$$

for all values of $b > 0$. Hence

$$\int_0^\infty |\theta + m\phi|^2 dx \le -\frac{\Im m}{\Im\lambda}$$

and consequently for every value of λ with $\Im\lambda \neq 0$ (3.3.13) has a solution $\psi(x,\lambda)$ which belongs to $L^2[0,\infty)$.

Theorem 3.3.1 *For all non-real λ, there exists a square-integrable solution*

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda)$$

of the equation (3.3.13) *on the interval* $[0,\infty)$

Proof: see [50].

The solution $\psi(x,\lambda)$ is called a Weyl function [29].

Corollary 3.3.1 *For each non-real λ, we have*

$$\lim_{b\to\infty} \psi_b(x,\lambda) = \psi(x,\lambda)$$

$$\lim_{b\to\infty} \|\psi_b\|_b^2 = \lim_{b\to\infty} \int_0^b |\psi_b(x,\lambda)|^2 dx = \int_0^\infty |\psi(x,\lambda)|^2 dx = \|\psi\|^2$$

Proof: Let $\Im\lambda \neq 0$. Then by Theorem 3.3.1 we obtain $\psi(x,\lambda) \in L^2[0,\infty)$, and obviously we have

$$\psi_b(x,\lambda) = \psi(x,\lambda) + [m_b(\lambda) - m(\lambda)]\phi(x,\lambda)$$

Now in the case $C_b \to C_\infty$ we get $m_b \to m$ as $b \to \infty$, so $\psi_b(x,\lambda) \to \psi(x,\lambda)$ and since $\psi(x,\lambda) \in L^2[0,\infty)$ then

$$\lim_{b\to\infty} \int_0^b |\psi_b(x,\lambda)|^2 dx = \int_0^\infty |\psi(x,\lambda)|^2 dx$$

In the case $C_b \to m$ as $b \to \infty$ we get by (3.3.18) that

$$|m_b(\lambda) - m(\lambda)| \leq 2r_b = \frac{1}{\nu \int_0^b |\phi(x,\lambda)|^2 dx} \to 0$$

and hence $m_b \to m$ and $\psi_b(x,\lambda) \to \psi(x,\lambda)$. Moreover

$$\int_0^b |(m_b(\lambda) - m(\lambda))\phi(x,\lambda)|^2 dx = |m_b(\lambda) - m(\lambda)|^2 \int_0^b |\phi(x,\lambda)|^2 dx \leq$$

$$4r_b^2 \int_0^b |\phi(x,\lambda)|^2 dx \leq (\nu^2 \int_0^b |\phi(x,\lambda)|^2 dx)^{-1}$$

and therefore, since $\phi(x,\lambda) \notin L^2[0,\infty)$, we conclude that

$$\lim_{b\to\infty} \int_0^b |\psi_b(x,\lambda)|^2 dx = \int_0^\infty |\psi(x,\lambda)|^2 dx$$

3.4 The Limit-circle and Limit-point cases

In this section we still consider equation (3.3.13) on the interval $[0,\infty)$, where the function $q(x)$ is continuous in each finite subinterval of $[0,\infty)$. We recall that for each $b > 0$ the conformal mapping (3.3.16) transforms the real line to a circle C_b with radius

$$r_b = \frac{1}{2\nu \int_0^b |\phi(x,\lambda)|^2 dx}$$

When $b \to \infty$ then $r_b \to r$, where either $r > 0$ or $r = 0$. Therefore the circles C_b converge either to a circle C or to a point m. If $C_b \to C$ we say that τ is in the *limit-circle* case at infinity, and by (3.3.18) we have $\phi(x,\lambda) \in L^2[0,\infty)$ so that by Theorem 3.3.1 all solutions of the problem (3.3.13), (3.3.15) are square integrable. If $r = 0$ we say that τ is *limit-point* at infinity. In this case $\phi(x,\lambda)$ no longer belongs to the Hilbert space $L^2[0,\infty)$ and so not every solution of the problem (3.3.13), (3.3.15) is square integrable. Thus in the Limit-point case there is a solution which is not square integrable.

Example 3.4.1 Let $q \equiv 0$ in the problem (3.3.13), (3.3.15). Then we have

$$y'' + \lambda y = 0 \tag{3.4.21}$$

where λ is a real parameter and $y_1(x, \lambda) = e^{ix\sqrt{\lambda}}$, $y_2(x, \lambda) = e^{-ix\sqrt{\lambda}}$ are linearly independent solutions of (3.4.21). We see that y_2 is not a square integrable solution for $\Im\sqrt{\lambda} > 0$.

There is a question here that says: Does the limit-circle and limit-point classification depend on a particular value of λ_0, or does it only depend on the operator T ? The answer is provided in the following theorem.

Theorem 3.4.1 *If for some complex parameter λ_0, all solutions of (3.3.13) are in $L^2[0, \infty)$, then, for any complex parameter λ, all solutions of (3.3.13) also belong to $L^2[0, \infty)$. In other words, if the operator T is in the limit-circle case for some λ_0, then T is in the limit-circle case for all λ*

Proof: see [50].

Corollary 3.4.1 *The limit-circle case holds if and only if the equation*

$$-y'' + q(x)y = \lambda y$$

has two linearly independent solutions in $L^2[0, \infty)$.

Corollary 3.4.2 *The limit-point case holds either for all non-real values of λ or for no such values.*

There are various criteria for the operator T to be in the limit-point case, some of which are given in the following theorems.

Theorem 3.4.2 *If $q(x) \geq -kx^2$, where k is a positive constant, then the operator $T \equiv -\frac{d^2}{dx^2} + q(x)$ is in the limit-point case*

Proof: see [28]. Note that in Theorem 3.4.2, θ and ϕ satisfy $\phi(0, \lambda) = \sin\alpha$, $\phi'(0, \lambda) = -\cos\alpha$ and $\theta(0, \lambda) = \cos\alpha$, $\theta'(0, \lambda) = \sin\alpha$, which are not the same conditions as in (3.3.15.)

Theorem 3.4.3 *Let M be a positive differentiable function, and k_1 and k_2 positive constants such that for large values of x*

$$q(x) \geq -k_1 M(x),$$

where

$$\int_x^\infty \frac{1}{\sqrt{M(x)}} dx = \infty, \quad |\frac{M'(x)}{\sqrt{M(x)^3}}| \leq k_2$$

Then T is in the limit-point case at infinity.

Proof: see [5].

Note that in this theorem θ, ϕ satisfy

$$\begin{cases} \phi(0,\lambda) = \cos\alpha, \ \phi'(0,\lambda) = \sin\alpha \\ \theta(0,\lambda) = \sin\alpha, \ \theta'(0,\lambda) = -\cos\alpha. \end{cases}$$

Now consider the problem

$$\begin{cases} \tau f = -f'' + q(x)f = \lambda f \\ \sin\alpha f(0) + \cos\alpha f'(0) = 0 \\ \cos\beta f(b) + \sin\beta f'(b) = 0 \end{cases} \tag{3.4.22}$$

where $0 \leq \alpha, \ \beta < \pi$. By (3.3.14) the above problem is a selfadjoint boundary-value problem in the interval $[0, b]$, and hence there exists a sequence $\{\lambda_{n,b}\}, \ n = 1, 2, \ldots$ of real eigenvalues and a corresponding complete set of normalised eigenfunctions $\{f_{n,b}\}$ such that $\tau f_{n,b}(x) = \lambda_{n,b} f_{n,b}(x), \ n = 1, 2, \ldots$. Let $\theta, \ \phi$ be linearly independent solutions satisfying (3.3.15); then we can see that ϕ satisfies the first boundary condition in (3.4.22) so

$$f_{n,b}(x) = r_{n,b}\phi(x, \lambda_{n,b})$$

where $r_{n,b}$ is a constant independent of x. Now let the function ρ_b be a non-decreasing right continuous step function of λ which satisfies

$$\rho_b(0) = 0, \ \lim_{\epsilon \to 0^+} \rho_b(\sigma + \epsilon) = \rho(\sigma)$$

and is defined as follows:

$$\rho_b(\sigma) = \begin{cases} \sum_{0 < \lambda_{n,b} \leq \sigma} \frac{1}{\int_0^b f_{n,b}^2(x,\sigma)dx} & \text{if } \sigma > 0 \\ \sum_{\sigma < \lambda_{n,b} \leq 0} \frac{-1}{\int_0^b f_{n,b}^2(x,\sigma)dx} & \text{if } \sigma \leq 0 \end{cases}$$

Theorem 3.4.4 *Let τ be in the limit-point case at infinity. Then*

i) There exists a monotone nondecreasing function ρ on whole real axis such that

$$\rho(\nu) - \rho(\mu) = \lim_{b\to\infty}(\rho_b(\nu) - \rho_b(\mu))$$

at points of continuity ν, μ of ρ.

ii) Let $\mathcal{L}^2(\rho)$ denote the Hilbert space which consists of all real functions g, measurable with respect to the Lebesgue-Stieltjes measure generated by ρ and such that

$$\int_{-\infty}^{\infty} g(\sigma)d\rho(\sigma) < \infty$$

Then if $f \in L^2[0,\infty)$ there exists a function $g \in \mathcal{L}^2(\rho)$ such that

$$\lim_{a\to\infty}\int_{-\infty}^{\infty}|g(\sigma) - \int_0^a f(t)\phi(t,\sigma)dt|^2 d\rho(\sigma) = 0$$

and

$$\int_0^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|g(\sigma)|^2d\rho(\sigma)$$

iii) The integral

$$\int_{-\infty}^{\infty} g(\sigma)\phi(x,\sigma)d\rho(\sigma)$$

converges in $L^2[0,\infty)$ to f, that is,

$$\lim_{\omega\to\infty}\int_0^{\infty}|f(x) - \int_{-\omega}^{\omega} g(\sigma)\psi(x,\sigma)d\rho(\sigma)|^2dx = 0$$

iv) If m is the limit point, considered as a function of λ, then

$$\rho(\nu) - \rho(\mu) = \lim_{y\downarrow 0^+}\frac{1}{\pi}\int_{\mu}^{\nu}\Im m(x+iy)dx$$

at points of continuity ν, μ of ρ and inversely

$$m(\lambda) - m(\lambda_0) = \int_{-\infty}^{\infty}\left(\frac{1}{\sigma-\lambda} - \frac{1}{\sigma-\lambda_0}\right)d\rho(\sigma) + c(\lambda - \lambda_0)$$

where c is a nonnegative constant, $\Im\lambda_0 > 0, \Im\lambda > 0$ and $\Im m(\lambda) > 0$.

Proof: see [5].

The function ρ is called the *spectral function* for the problem

$$\tau f = \lambda f, \quad \sin\alpha f(0) + \cos\alpha f'(0) = 0$$

The value of g at σ is given by the formula in the second part of the above theorem and may be written

$$g(\sigma) = \text{l.i.m.}_{a\to\infty} \int_0^a f(x)\phi(x,\sigma)dx$$

where l.i.m. denotes the limit in the mean. This leads to the *Parseval equality*

$$\int_0^\infty |f(x)|^2 dx = \int_{-\infty}^\infty |g(\sigma)|^2 d\rho(\sigma) \tag{3.4.23}$$

and the inverse relationship

$$f(x) = \text{l.i.m.}_{\omega\to\infty} \int_{-\omega}^{\omega} g(\lambda)\phi(x,\lambda)d\rho(\lambda)$$

which is called the expansion formula.

Theorem 3.4.5 (Expansion theorem) *Let $f \in L^2[0,\infty)$ be a continuous function in each finite subinterval of $[0,\infty)$. Then there exists a function $g \in \mathcal{L}^2(\rho)$ such that*

$$f(x) = \text{l.i.m.}_{\omega\to\infty} \int_{-\omega}^{\omega} g(\sigma)\phi(x,\sigma)d\rho(\sigma)$$

where

$$g(\lambda) = \text{l.i.m.}_{a\to\infty} \int_0^a f(x)\phi(x,\sigma)dx.$$

where convergence is in the sense of Theorem 3.4.4.

Proof: see [5].

We also have the inverse statement of the expansion theorem, which is called the *Inversion Theorem.*

Theorem 3.4.6 (Inversion theorem) *If $g \in \mathcal{L}^2(\rho)$, there exists an $f \in L^2[0,\infty)$ satisfying*

$$f(x) = \text{l.i.m.}_{\omega\to\infty} \int_{-\omega}^{\omega} g(\sigma)\phi(x,\sigma)d\rho(\sigma)$$

and by means of the function f we may represent the function g in the form

$$g(\sigma) = \text{l.i.m.}_{a\to\infty} \int_0^a f(x)\phi(x,\sigma)dx.$$

Proof: see [5].

3.5 The functions $m(\lambda)$

In this section we shall be concerned mainly with the complex function m which is the coefficient function in the solution ψ denoted by

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda)$$

where θ, ϕ satisfy

$$\begin{cases} \phi(0,\lambda) = \cos\alpha, \ \ \phi'(0,\lambda) = \sin\alpha \\ \theta(0,\lambda) = \sin\alpha, \ \ \theta'(0,\lambda) = -\cos\alpha. \end{cases}$$

Note that with θ, ϕ defined in this way, $\Im m(\lambda) > 0$ whenever $\Im\lambda > 0$. Proceeding as in Section 3.3, we start with the function m_b which is given by the formula

$$m_b(\lambda) = -\frac{\theta(b,\lambda)\cos\beta + \theta'(b,\lambda)\sin\beta}{\phi(b,\lambda)\cos\beta + \phi'(b,\lambda)\sin\beta}$$

The function $m_b(\lambda,\beta)$ is a meromorphic function of λ with simple real poles $\lambda_n(b)$, the zeros of $\phi(b,\lambda)\cos\beta + \phi'(b,\lambda)\sin\beta$. Since the differential operator T is in the limit-point case at infinity the choice of β is immaterial, so we may take the value of β to be zero, which gives

$$m_b(\lambda) = -\frac{\theta(b,\lambda)}{\phi(b,\lambda)}$$

and the residue r_n of $m_b(\lambda)$ at $\lambda_n(b)$ is given by

$$r_n(b) = -[\int_0^b |\phi(x,\lambda_n(b))|^2 dx]^{-1} \tag{3.5.24}$$

[20] Lemma 10.2.1.

That the zeros of $\phi(x,\lambda)$ are all real follows from (3.3.16). To prove that the poles of $m_b(\lambda)$ are simple we use

$$\Im m_b(\lambda) = \Im\lambda \int_0^b |\psi(x,\lambda)|^2 dx$$

which says that

$$\Im m_b(\lambda) = \Im\lambda \int_0^b |\theta(x,\lambda) + m_b(\lambda)\phi(x,\lambda)|^2 dx \tag{3.5.25}$$

Suppose the poles of $m_b(\lambda)$ are not simple but of degree k, and set $\lambda = \lambda_{n,b} + i\nu$. Then as $\nu \to 0$ the right side of the above equality becomes infinite as a constant multiple of ν^{1-2k} while the left hand side becomes infinite at most as ν^{-k}. This forces k to be 1. Multiplying both side of (3.5.25) by ν and taking the limit as $\nu \to 0$. we obtain

$$-r_n(b) = (r_n(b))^2 \int_0^b \phi^2(x,\lambda)dx$$

and since $r_{n,b} \neq 0$ this gives (3.5.24). Also the function $m_b(\lambda, \beta)$ for a constant β is a meromorphic function where the poles are the zeros of denominator of m_b and all are real. Also on the circle C_b we have

$$\int_0^b |\theta + m_b\phi|^2 dx = \frac{\Im m_b}{\nu} \leq \frac{|m_b|}{\nu}$$

On the other hand

$$\int_0^b |\theta + m_b\phi|^2 dx \geq \frac{1}{2}|m_b|^2 \int_0^b |\phi|^2 dx - \int_0^b |\theta|^2 dx$$

If we combine the two last inequalities we find m_b in terms of ν such that

$$|m_b| \leq \frac{1}{\nu \int_0^b |\phi|^2 dx} + \left\{ \frac{2\int_0^b |\theta|^2 dx}{\int_0^b |\phi|^2 dx} + \frac{1}{\nu^2 (\int_0^b |\phi|^2 dx)^2} \right\}^{\frac{1}{2}} \tag{3.5.26}$$

For a given λ the above inequality shows that the circles C_b decrease as b increases, and hence $m_b(\lambda)$ converges boundedly in any region in the upper or (lower) half of the λ-plane. Hence if τ is in the limit-point case the limit $m(\lambda)$ of $m_b(\lambda)$ is analytic in either half-plane. Since the right hand side of (3.5.26) is $O(\frac{1}{\nu})$ as $\nu \to 0$ for any fixed b, it follows that the left hand side satisfies $m_b(\lambda) = O(\frac{1}{\nu})$. This implies that if the limit function m has poles they are all simple poles.

Related to Theorem 3.3.1, if τ is in the limit-point case and λ and λ' are non-real, we have the following theorems.

Theorem 3.5.1 *For any fixed and non-real λ and λ' we have*

$$\lim_{b \to \infty} W[\psi(x,\lambda), \psi(x,\lambda')] = 0$$

Proof: see [46] §2.3

Corollary 3.5.1 *Suppose that λ, λ' are non-real, then*

$$(\lambda' - \lambda) \int_0^\infty \psi(x,\lambda)\psi(x,\lambda')dx = m(\lambda) - m(\lambda').$$

Proof: The result follows from Green's formula and Theorem 3.5.1.

If $\phi(x, \lambda_n)$ is an eigenfunction corresponding to the eigenvalue λ_n then we have

Corollary 3.5.2 *For any fixed non-real λ we have*

$$\int_0^\infty \psi(x,\lambda)\phi(x,\lambda_n)dx = \frac{1}{\lambda - \lambda_n}$$

Proof: see [46] §2.5.

Corollary 3.5.3 *If r_n is the residue of $m(\lambda)$ at simple pole λ_n then*

$$\int_0^\infty \phi(x,\lambda_n)^2 dx = \frac{-1}{r_n}$$

Proof: [46] Section 2.5.

In the real context there is an important result regarding the m-function which was obtained by Titchmarsh around 1940,

$$\rho(\lambda) - \rho(\mu) = \lim_{y \downarrow 0} \frac{1}{\pi} \int_\mu^\lambda \Im m(x+iy)dx$$

Note that for thirty years after the workof Weyl, due to much emphasis on abstract functional analysis, there had been no remarkable work in this area. If $\alpha \neq 0$ the related formula for the m-function is

$$m(\lambda) = \int_{-\infty}^\infty \frac{1}{t-\lambda} d\rho(t) + \cot\alpha \tag{3.5.27}$$

for non-real λ. This is derived from the analyticity property of the function $m(\lambda)$ in the half-planes $\Im z > 0, \Im z < 0$, which was proved by H.Weyl in 1935 [51].

Chapter 4

Non-selfadjoint Differential Operators.

4.1 Introduction.

This section is devoted to an investigation of the properties of some non-selfadjoint one dimensional operators on the positive half line, and includes generalization of the limit-point and limit-circle theory and the extension of some results on the spectrum to the non-selfadjoint case.

4.2 Lagrange's Identity

In this section we first recall some relevant results from the selfadjoint case and then proceed to study the modified Lagrange's identity in the non-selfadjoint case.

Definition 4.2.1 Let $\mathcal{H} = L^2[0,\infty)$ and consider the ordinary differential expression

$$\tau \equiv -\frac{d^2}{dx^2} + q(x)$$

on the half-line $x \geq 0$ where q is a real-valued continuous function on $[0, X]$, for all $X > 0$. Let $T_{\max}$ be the *maximal operator* defined in §3.3 with domain $\mathcal{D}$. It is well known that $T_{\max}$ is densely defined and closed so that its adjoint $T^*_{\max}$ exists [35] p.72. A selfadjoint realisation T_α of $T_{\max}$ can be defined on $\mathcal{H}$ with domain

$$\mathcal{D}_\alpha = \{f \in \mathcal{D} : f(0)\cos\alpha + f'(0)\sin\alpha = 0\}, \quad \alpha \in [0, \pi)$$

by

$$T_\alpha f = \tau f \quad \forall f \in \mathcal{D}_\alpha \tag{4.2.1}$$

iff τ is regular at 0 and limit-point at ∞. Hence every selfadjoint differential operator generated by τ is either a restriction of the maximal operator $T_{\max}$ or an extension of the minimal operator $T_{\min}$ where $T_{\min}$ is defined as in §3.3.

Consider the differential equation

$$\tau f = \lambda f \tag{4.2.2}$$

where λ is a complex parameter and let $\phi(x, \lambda)$ and $\theta(x, \lambda)$ be two linearly independent solutions of (4.2.2) satisfying

$$\phi(0, \lambda) = \sin\alpha \quad \phi'(0, \lambda) = -\cos\alpha \tag{4.2.3}$$

$$\theta(0, \lambda) = \cos\alpha \quad \theta'(0, \lambda) = \sin\alpha \tag{4.2.4}$$

where the Wronskian of ϕ and θ at the point $x = 0$ is $W[\phi, \theta](0) = 1$. Note that the boundary conditions here are different from those in (3.3.15); this change is not important, but for convenience they are now consistent with the usage in [44] [46].

Theorem 4.2.1 *For every non-real value of λ there exists a non-trivial square integrable solution $\psi(x, \lambda)$ of equation (4.2.2) on the interval $[0, \infty)$ where*

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda)$$

and the corresponding function $m(\lambda)$ is analytic in the upper and lower half planes.

Proof: see [50].

Let f and g be two functions for which the expression

$$\tau f = -\frac{d^2 f}{dx^2} + q(x) f$$

makes sense. If $[f, g](x) = W[f, \bar{g}](x)$, and if $q(x)$ is real, then we have

$$\tau f \bar{g} - f \overline{\tau g} = \frac{d}{dx}(f \bar{g}' - f' \bar{g})(x) = \frac{d}{dx}[f, g](x) \tag{4.2.5}$$

which is called the *Lagrange identity.* Integrating both sides of (4.2.5) on the finite interval $[0, X]$ we obtain *Green's formula*

$$\int_0^X (\tau f \bar{g} - f \overline{\tau g}) dx = (f \bar{g}' - f' \bar{g})|_0^X = [f, g]_0^X \tag{4.2.6}$$

where $[f, g]_0^X$ is the difference of the values of $[f, g](x)$ for $x = X$ and $x = 0$. Hence an operator $\tilde{T}$ associated with τ is selfadjoint if and only if $\lim_{X \to \infty} [f, g]_0^X = 0$ for all f, g in the domain of $\tilde{T}$. However if the function q in the expression $-\frac{d^2}{dx^2} + q(x)$ is a complex function then (4.2.5) and (4.2.6) no longer hold and we have

$$\tau f \bar{g} - f \overline{\tau g} = \frac{d}{dx}(f \bar{g}' - f' \bar{g})(x) + f \bar{g}(q - \bar{q}) = \frac{d}{dx}[f, g] + 2 i q_2 f \bar{g}$$

$$\int_0^X (\tau f \bar{g} - f \overline{\tau g}) dx = [f, g]_0^X + 2i \int_0^X q_2 f \bar{g} dx \tag{4.2.7}$$

where $q_2 = \Im q$. So (4.2.7) implies that if q is complex the operators associated with τ cannot be symmetric, since $\int_0^X q_2 f \bar{g} dx$ cannot vanish for all $f, g \in D_\alpha$ and all $X > 0$, and hence are non-selfadjoint. In the case where f and g are two solutions of (4.2.2) we obtain the following formula

$$\int_0^X (\tau f \bar{g} - f \overline{\tau g}) dx = [f, g]_0^X + 2i \int_0^X f \bar{g} q_2 dx = 2 i \nu \int_0^X f \bar{g} dx$$

since

$$[f, g](X) = 2i \int_0^X (\nu - q_2) f \bar{g} dx + [f, g](0) \tag{4.2.8}$$

where $[f, \bar{g}]$ is the Wronskian of solutions f and g, and $\nu = \Im \lambda$. Note that (4.2.8) follows from

$$\begin{aligned} \int_0^X (f'' \bar{g} - f \bar{g}'') dx &= \int_0^X [(q - \lambda) f \bar{g} - f \overline{(q - \lambda) g}] dx \\ &= 2i \int_0^X (q_2 - \nu) f \bar{g} dx \end{aligned}$$

4.3 Limit-point, limit-circle

Consider a differential operator T_α defined on $\mathcal{D}_\alpha$ as in (4.2.1), but where α is now a complex number and the coefficient function q is a complex function. Then T_α is densely defined, closed and non-selfadjoint, and the corresponding differential equation is

$$\tau f = -f'' + qf = \lambda f \tag{4.3.9}$$

where $q = q_1 + iq_2$ and $\lambda = \mu + i\nu$. With $\theta(x, \lambda)$ and $\phi(x, \lambda)$ defined as in (4.2.3) and (4.2.4), where $\alpha = \alpha_1 + i\alpha_2$ we have the following theorem.

Theorem 4.3.1 (Sims) *Let $q_2 \leq 0$, in equation (4.3.9), and $\alpha_2 \leq 0$. Then for $\nu > 0$ there exists a square integrable solution $\psi(x, \lambda)$ and a complex function m satisfying*

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda).$$

and the corresponding m-function, $m(\lambda)$ is analytic in the upper half-plane.

Proof: Let $\nu > 0$ and for $b \in [0, \infty)$, let $m_b(\lambda)$ be defined by the condition that

$$\psi_b(b, \lambda) \cos\beta + \psi_b'(b, \lambda) \sin\beta = 0$$

for some fixed $\beta \in \mathbf{C}$, where

$$\psi_b(x, \lambda) = \theta(x, \lambda) + m_b(\lambda)\phi(x, \lambda) \tag{4.3.10}$$

Since $\psi_b(x, \lambda)$ satisfies the given boundary condition it follows that

$$m_b(\lambda, \beta) = -\frac{\theta(b, \lambda)\cos\beta + \theta'(b, \lambda)\sin\beta}{\phi(b, \lambda)\cos\beta + \phi'(b, \lambda)\sin\beta}$$

Substituting $\cot\beta = z$ gives

$$m_b(\lambda, z) = -\frac{\theta(b, \lambda)z + \theta'(b, \lambda)}{\phi(b, \lambda)z + \phi'(b, \lambda)} \tag{4.3.11}$$

which is a linear fractional transformation from the z-plane to the m_b-plane. Since ϕ and θ are linearly independent solutions and ϕ and ϕ' do not both vanish at b, lines and circles of the z-plane are mapped onto lines and circles of the m_b-plane.

The critical point $z = -\frac{\phi'(b,\lambda)}{\phi(b,\lambda)}$ on the z-plane corresponds to the point at infinity on the m_b-plane. We now show that the image of the real line $\Im z = 0$ in the z-plane is a circle C_b. Using the inverse transformation,

$$z(b,\lambda) = -\frac{\phi'(b,\lambda)m_b + \theta'(b,\lambda)}{\phi(b,\lambda)m_b + \theta(b,\lambda)} \tag{4.3.12}$$

and setting $z = \bar{z}$ gives

$$\frac{\phi'(b,\lambda)m_b + \theta'(b,\lambda)}{\phi(b,\lambda)m_b + \theta(b,\lambda)} = \frac{\bar{\phi}'(b,\lambda)\bar{m}_b + \bar{\theta}'(b,\lambda)}{\bar{\phi}(b,\lambda)\bar{m}_b + \bar{\theta}(b,\lambda)}$$

from which we have

$$\begin{aligned}
|m_b|^2(\phi'\bar{\phi} - \phi\bar{\phi}') \;&-\; m_b(\phi\bar{\theta}' - \phi'\bar{\theta}) - \bar{m}_b(\bar{\phi}'\theta - \bar{\phi}\theta') + (\bar{\theta}\theta' - \theta\bar{\theta}') \qquad (4.3.13)\\
&= |m_b|^2 W_b[\bar{\phi},\phi] - m_b W_b[\phi,\bar{\theta}] - \bar{m}_b W_b[\theta,\bar{\phi}] + W_b[\bar{\theta},\theta]\\
&= W_b[\bar{\phi},\phi]\left\{|m_b - \frac{W_b[\theta,\bar{\phi}]}{W_b[\bar{\phi},\phi]}|^2 - |\frac{W_b[\phi,\bar{\theta}]}{W_b[\bar{\phi},\phi]}|^2 + \frac{W_b[\bar{\theta},\theta]}{W_b[\bar{\phi},\phi]}\right\}\\
&= 0.
\end{aligned}$$

where $W_b[\bar{\phi},\phi] \neq 0$. We see that by (4.3.13), m_b lies on a circle C_b with centre

$$p_b(\lambda) = \frac{W_b[\theta,\bar{\phi}]}{W_b[\bar{\phi},\phi]}, \qquad W_b[\phi,\theta] = 1,$$

and radius

$$r_b(\lambda) = \frac{1}{|W_b[\bar{\phi},\phi]|}$$

since $W_b[\bar{\theta},\phi] = 1$ and

$$r_b(\lambda)^2 = -\frac{W_b[\bar{\theta},\theta]}{W_b[\bar{\phi},\phi]} + |\frac{W_b[\bar{\theta},\phi]}{W_b[\bar{\phi},\phi]}|^2 = |\frac{W_b[\theta,\phi]}{W_b[\bar{\phi},\phi]}|^2 = \frac{1}{|W_b[\bar{\phi},\phi]|^2}$$

The images of conjugates under a conformal mapping are the conjugates of images. Hence the image of the conjugate of the critical point $z = -\frac{\phi'(b,\lambda)}{\phi(b,\lambda)}$ is the centre $p_b(\lambda)$ of the circle C_b, since the centre of C_b is the conjugate with respect to C_b of the point at infinity in the m_b plane. Thus we have by (4.2.8)

$$r_b(\lambda) = \left|W_b[\bar{\phi},\phi]\right|^{-1} = |\int_0^b 2(\nu - q_2)|\phi|^2 - \sinh 2\alpha_2|^{-1} \tag{4.3.14}$$

On the other hand the point at infinity in the m_b-plane is outside the circle C_b and is the image of $z = -\frac{\bar{\phi}'(b,\lambda)}{\phi(b,\lambda)}$. Since we have $q_2 \le 0,\ \alpha_2 \le 0,\ \nu > 0$,

$$\begin{aligned}
\Im(-\frac{\phi'}{\phi}) &= \frac{1}{2i}(-\frac{\phi'}{\phi} + \frac{\bar{\phi}'}{\bar{\phi}}) = \frac{1}{2i}(\phi\bar{\phi}' - \bar{\phi}\phi')\frac{1}{|\phi|^2} \\
&= \frac{1}{2i|\phi|^2}W_b[\phi, \bar{\phi}] = \frac{1}{2i|\phi|^2}[\int_0^b 2i(\nu - q_2)|\phi|^2 - i\sinh 2\alpha_2] \\
&= \frac{1}{2|\phi|^2}[\int_0^b 2(\nu - q_2)|\phi|^2 - \sinh 2\alpha_2] > 0
\end{aligned}$$

so the upper half z-plane corresponds to the points outside C_b and the lower half z-plane corresponds to the points inside C_b. Also from (4.3.12)

$$\begin{aligned}
\Im z &= \Im z(b,\lambda) = \frac{1}{2i}(z - \bar{z}) \\
&= \frac{1}{2i}[-\frac{\phi' m_b + \theta'}{\phi m_b + \theta} + \frac{\bar{\phi}'\bar{m}_b + \bar{\theta}'}{\bar{\phi}\bar{m}_b + \bar{\theta}}] \\
&= \frac{1}{2i}[\frac{W_b[(\phi m_b + \theta), (\bar{\phi}\bar{m}_b + \bar{\theta})]}{|\phi m_b + \theta|^2}] \\
&= \frac{W_b[\psi_b, \bar{\psi}_b]}{2i|\psi_b|^2}
\end{aligned}$$

and using (4.2.8) we have

$$W_b[\psi_b, \bar{\psi}_b] = 2i\int_0^b (\nu - q_2)|\psi_b|^2 + W_0[\psi_b, \bar{\psi}_b]$$

or

$$W_b[\psi_b, \bar{\psi}_b] = 2i\int_0^b (\nu - q_2)|\psi_b|^2 - i(1 + |m_b|^2)\sinh 2\alpha_2 + 2i\Im m_b \cosh 2\alpha_2$$

Hence if $\Im z \le 0$ then

$$\begin{aligned}
\Im z &= \frac{1}{2i|\psi_b|^2}[2i\int_0^b (\nu - q_2)|\psi_b|^2 - i(1 + |m_b|^2)\sinh 2\alpha_2 + 2i\Im m_b \cosh 2\alpha_2] \\
&= \frac{1}{2|\psi_b|^2}(\int_0^b 2(\nu - q_2)|\psi_b|^2 - (1 + |m_b|^2)\sinh 2\alpha_2 + 2\Im m_b \cosh 2\alpha_2) \le 0
\end{aligned}$$

so that for $\Im z \le 0$ we have the inequality

$$\int_0^b 2(\nu - q_2)|\psi_b|^2 \le (1 + |m_b|^2)\sinh 2\alpha_2 - 2\Im m_b \cosh 2\alpha_2. \qquad (4.3.15)$$

Hence the point m_b is inside C_b iff the inequality in (4.3.15) holds and m_b is on C_b iff the equality in (4.3.15) holds.

Analysis of the limit-point and the limit-circle cases depends on an investigation of the behaviour of the circles C_b as b tends to ∞, and this can be done using the fact that the circles C_b are decreasingly nested as b tends to ∞. To see this, note that if $b' > b$ then $r_{b'} \leq r_b$ since

$$r_{b'} = |\int_0^{b'} 2(\nu - q_2)|\phi|^2 - \sinh 2\alpha_2|^{-1} \leq |\int_0^{b} 2(\nu - q_2)|\phi|^2 - \sinh 2\alpha_2|^{-1} = r_b$$

and $C_{b'}$ is inside C_b, because if $m_{b'}$ is a point in the circle $C_{b'}$ we then have

$$\begin{aligned}\int_0^{b} 2(\nu - q_2)|\psi_b(x,\lambda)|^2 dx &< \int_0^{b'} 2(\nu - q_2)|\psi_{b'}(x,\lambda)|^2 dx \\ &\leq (1 + |m_{b'}|^2)\sinh 2\alpha_2 - 2\Im m_{b'} \cosh 2\alpha_2\end{aligned}$$

or

$$\int_0^{b} 2(\nu - q_2)|\psi_b(x,\lambda)|^2 dx < (1 + |m_{b'}|^2)\sinh 2\alpha_2 - 2\Im m_{b'} \cosh 2\alpha_2$$

since $\nu - q_2 > 0$. This inequality shows that $m_{b'}$ is a point in the circle C_b. But the inequality

$$\int_0^{b'} (\nu - q_2)|\psi_b|^2 \geq \int_0^{b} (\nu - q_2)|\psi_b|^2$$

implies that as $b \to \infty$ then C_b tends to a limit, which is either a point (limit-point), or a circle (limit-circle). In the limit-point case $r_b \to 0$ as $b \to \infty$ and consequently $m_b(\lambda) \to m(\lambda)$, $\psi_b(x,\lambda) \to \psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda)$ and so by (4.3.15)

$$\int_0^{b} (\nu - q_2)|\psi_b|^2 \to \int_0^{\infty} (\nu - q_2)|\psi|^2 dx < \infty. \tag{4.3.16}$$

The inequalities (4.3.15) and (4.3.16) show that the differential equation (4.3.9) has a square integrable solution, ψ, which satisfies $\psi\sqrt{\nu - q_2} \in L^2[0,\infty)$. Note that in Sims' case, $\nu - q_2 > 0$, so that

$$\int_0^{\infty} (\nu - q_2)|\psi|^2 dx, \infty \iff \psi\sqrt{\nu - q_2} \in L^2[0,\infty)$$

This is not the end of the limit-point story, since from (4.3.15) if $r_b \to 0$ as $b \to \infty$, the integral $\int_0^{\infty}(\nu - q_2)|\phi|^2$ tends to ∞ but it is possible for $\int_0^{\infty} |\phi|^2$ to be convergent, in which case we have another linearly independent square integrable solution. This is also a limit-point case, but with two square integrable solutions.

In the remaining case when $r_b \to r_\infty(>0)$ as $b \to \infty$, then $\int_0^\infty(\nu-q_2)|\phi|^2 < \infty$ by (4.3.15) and we have two linearly independent solutions in $L^2[0,\infty)$ with weight function $\nu - q_2$.

We can therefore distinguish three cases:

Case I. There exists one and only one $L^2[0,\infty)$-solution, ψ, with the property $\int_0^\infty(\nu-q_2)|\psi|^2 < \infty$, a limit-point case.

Case II. There exist two linearly independent solutions ψ and ϕ in $L^2[0,\infty)$, with the properties $\int_0^\infty(\nu-q_2)|\psi|^2 < \infty$ and $\int_0^\infty(\nu-q_2)|\phi|^2 = \infty$, This is a limit-point case too.

Case III. There exist two solutions ψ and ϕ which are square integrable in $[0,\infty)$ and satisfy $\int_0^\infty(\nu-q_2)|\psi|^2 < \infty$ and $\int_0^\infty(\nu-q_2)|\phi|^2 < \infty$. This is the limit-circle case.

Note that in Sims' case, $\nu - q_2 > 0$, so that

$$\int_0^\infty (\nu-q_2)|\psi|^2 dx < \infty \quad \text{if and only if} \quad \psi\sqrt{\nu-q_2} \in L^2[0,\infty)$$

We omit the proof of the analyticity of $m(\lambda)$ in the upper half plane; this can be found in [44], Thm. 3.

The cases I, II and III are dependent only on q as the following theorem asserts.

Theorem 4.3.2 *If for a particular value λ_0 of the parameter λ all solutions of equation* (4.3.9) *are in $L^2([0,\infty);\nu-q_2)$ respectively $L^2[0,\infty)$, then for every other value of the parameter λ all solutions of the equation* (4.3.9) *are in $L^2([0,\infty);\nu-q_2)$, respectively $L^2[0,\infty)$.*

Proof: See [44] §3, Theorem 2.

We can see by the above argument that case II has no analogue in the classical Weyl theory, but this case exists as the examples of Section 4.4.2 show.

Theorem 4.3.3 *Let equation* (4.3.9) *and boundary conditions* (4.2.3-4) *be given for some $\alpha \in \mathbb{C}$ and suppose α, q and λ are as in Theorem* 4.3.1. *Then if $\beta \in \mathbb{C}$*

is such that $\alpha - \beta \neq k\pi, k \in \mathbf{Z}$, *and* $\Im\beta \leq 0$, *we have*

$$m_\alpha(\lambda) = \frac{-\sin(\alpha - \beta) + m_\beta(\lambda)\cos(\alpha - \beta)}{\cos(\alpha - \beta) + m_\beta(\lambda)\sin(\alpha - \beta)} \tag{4.3.17}$$

Proof: By Theorem 4.3.1, $m_\alpha(\lambda)$ and $m_\beta(\lambda)$ are analytic in the upper half-plane. The proof now proceeds as in [10].

4.4 Normal solutions.

We noted in the previous section that Case II has no analogue in the theory of Weyl. In order to show that this case actually occurs we give three examples. The statements of the first two examples are taken from Sims [44] p. 257 and the third one is new. First we give some standard material relating to series solutions of second order differential equations. From these observations we acheive that existence of the Case II is apparent.

4.4.1 Classification of singularities.

We shall discuss the differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{4.4.18}$$

when x is a real variable ranging over $[0, \infty)$ and p, q are single valued analytic functions on $[0, \infty)$, which may or may not have singularities at infinity. We briefly review the well-known classification of isolated singulatities of (4.4.18) (see eg [21]).

Let y_1 and y_2 be a fundamental system of solutions of (4.4.18).

1) The point x_0 is called an ordinary point of (4.4.18) iff p and q are analytic at x_0, otherwise we say x_0 is a singular point of (4.4.18).

2) The point x_0 is said to be a regular singular point of (4.4.18) iff $(x - x_0)p(x)$

and $(x - x_0)^2 q(x)$ are analytic at x_0, otherwise x_0 is an irregular singularity for equation (4.4.18)

3) We speak of an irregular singular point x_0 of finite rank $r - 1$ iff $(x - x_0)^r p(x)$ and $(x - x_0)^{2r} q(x)$ are analytic at x_0, where r is the least number satisfying the above statement. Note that a regular singularity has rank zero.

4.4.2 Normal solutions.

Let $x = \infty$ be an irregular singularity of (4.4.18) of finite rank r. It was discovered by Thomé (see: [9]) that certain formal solutions of the form

$$y = \exp[P(x)] \sum_{n=0}^{\infty} c_n x^{-\rho-n}$$

exist, where $P(x)$ is a polynomial, and ρ is an integer. Here y is called a normal solution.

We are going to give the construction of a normal solution in the case of an irregular singularity of rank 1. Using the change of variable

$$y = z \exp(-\frac{1}{2} \int p(x)dx) \tag{4.4.19}$$

transforms (4.4.18) into a differential equation without the coefficient p, so there is no loss of generality if we consider the differential equation

$$y'' + q(x)y = 0 \tag{4.4.20}$$

in which

$$q(x) = \sum_{n=0}^{\infty} q_n x^{-n}$$

A normal solution then has the form

$$y = e^{\omega x} \sum_{n=0}^{\infty} c_n x^{-\rho-n}, \qquad c_0 \neq 0 \tag{4.4.21}$$

More details of normal solutions are given in the following examples, which illustrate Case II.

Example 4.4.1 *In the differential equation* (4.3.9) *suppose* $q(x) = -x^6 - \frac{3}{2}ix^2$ *where* $x \in [1, \infty)$.

If we take $\lambda = 0$ we have

$$-y'' + (-x^6 - \frac{3}{2}ix^2)y = 0$$

Infinity is a singular point and in the terminology of Section 4.4.1, it is an irregular singular point of rank 4, since if we substitute $x = \frac{1}{X}$ in the equation we obtain

$$\frac{d^2y}{dX^2} + 2X^{-1}\frac{dy}{dX} + (X^{-10} + \frac{3}{2}iX^{-6})y = 0$$

so $p(X) = 2X^{-1}$ and $q(X) = X^{-10}(1 + \frac{3}{2}iX^4)$, and we see that $X^{10}q(X)$ and $X^5p(X)$ are analytic at point $X = 0$, i.e. the point at infinity is an irregular singular point of rank 4. Hence there is a normal solution $y = e^{Q(x)}u(x)$ where $Q(x)$ is a polynomial and $u(x) = x^{\rho}(c_0 + c_1x^{-1} + c_2x^{-2} + \ldots)$. Substituting for y in the first equation we have

$$e^{Q(x)}(u'' + 2Q'u' + (Q'' + Q'^2)u + qu) = 0$$

which gives, since $e^{Q(x)} \neq 0$,

$$u'' + 2Q'u' + (Q'' + Q'^2 + q)u = 0 \tag{4.4.22}$$

The normal solution exists subject to the existence of a regular solution of this equation at infinity. Let the degree of Q be $s \geq 1$ so that $Q'' + Q'^2$ has degree at most $2s - 2$ and Q' has degree $s - 1$; hence $2s - 2 \leq 6$ or $s \leq 4$ so that $Q(x) = a_0 + a_1x + a_2x^2 + \ldots + a_4x^4$. After doing the relevant calculations we obtain $a_4 = \pm\frac{1}{4}i$ and $a_0 = a_1 = a_2 = a_3 = 0$ so $Q_1(x) = \frac{1}{4}ix^4, Q_2(x) = \frac{-1}{4}ix^4$ and $\rho_1 = \frac{-9}{4}$, $\rho_2 = \frac{-3}{4}$ which give

$$u_1(x) = x^{\frac{-9}{4}}(c_0 + c_1x^{-1} + c_2x^{-2} + \ldots) = x^{\frac{-9}{4}}v_1(x)$$

and

$$u_2(x) = x^{\frac{-3}{4}}(d_0 + d_1x^{-1} + d_2x^{-2} + \ldots) = x^{\frac{-3}{4}}v_2(x)$$

where v_1 and v_2 are bounded at infinity. We have therefore obtained two solutions y_1 and y_2 which are square integrable, viz.

$$y_1(x) = e^{\frac{1}{4}ix^4} x^{\frac{-9}{4}} v_1(x)$$

$$y_2(x) = e^{\frac{-1}{4}ix^4} x^{\frac{-3}{4}} v_2(x)$$

However y_1 and y_2 are two solutions such that

$$y_1(\nu - q_2)^{\frac{1}{2}} = y_1(0 + \frac{3}{2}x^2)^{\frac{1}{2}} = \sqrt{\frac{3}{2}} x y_1 = \sqrt{\frac{3}{2}} e^{\frac{1}{4}ix^4} x^{\frac{-5}{4}} v_1(x)$$

so that

$$\int_1^\infty |(\nu - q_2)^{\frac{1}{2}} y_2|^2 = \int_1^\infty \frac{3}{2} x^{\frac{-5}{2}} |v_1|^2 < \infty$$

and

$$y_2(\nu - q_2)^{\frac{1}{2}} = y_2(0 + \frac{3}{2}x^2)^{\frac{1}{2}} = \sqrt{\frac{3}{2}} x y_2 = \sqrt{\frac{3}{2}} e^{\frac{-1}{4}ix^4} x^{\frac{1}{4}} v_2(x)$$

so that

$$\int_1^\infty |(\nu - q_2)^{\frac{1}{2}} y_2|^2 = \int_1^\infty \frac{3}{2} x^{\frac{1}{2}} |v_2|^2 = \infty$$

although

$$\int_1^\infty |y_2|^2 = \int_1^\infty x^{\frac{-3}{2}} |v_2|^2 < \infty$$

Example 4.4.2 *In the differential equation* (4.3.9) *suppose* $q(x) = \frac{5}{16}x^{-2} - ix^{-1}$ *and let* $(0, 1]$ *be the corresponding interval.*

Clearly $x = 0$ is a regular singular point since $x^2 q(x) = \frac{5}{16} - ix$ is analytic at $x = 0$. Using the method of Frobenius ([43] Chap. 5) at the regular singular point, the formal power series solution $y = x^m \sum_{n=0}^\infty a_n x^n$ satisfies the equation $y'' + (\frac{5}{16}x^{-2} - ix^{-1})y = 0$, and the indicial equation $m(m-1) + \frac{5}{16} = 0$ has the roots $m_1 = \frac{5}{4}$ and $m_2 = \frac{-1}{4}$, hence

$$y_1 = x^{\frac{5}{4}} \sum_{n=0}^\infty a_n x^n, \quad y_2 = x^{\frac{-1}{4}} \sum_{n=0}^\infty b_n x^n$$

Using the recursion formula we get

$$y_1(x) = x^{\frac{5}{4}}(a_0 - i\frac{65}{16}x - i\frac{241}{16}x^2 - \ldots) \sim x^{\frac{5}{4}}$$

$$y_2(x) = x^{\frac{-1}{4}}(b_0 + b_1x + \dots) \sim x^{\frac{-1}{4}}$$

as $x \to 0$, so that y_1 and y_2 are in $L^2(0,1]$.

Since $q_2(x) = -x^{-1}$ and

$$|y_1|^2 = x^{\frac{5}{2}}(a_0^2 - ia_0\frac{65}{16}x - ((\frac{65}{16})^2 + a_0i\frac{241}{16})x^2 + \dots)$$

we have $\int_0^1 x^{-1}|y_1|^2 = \int_0^1 x^{\frac{3}{2}}(a_0^2 - \dots)dx < \infty$. However

$$|y_2|^2 = x^{\frac{-1}{2}}(a_0^2 + a_0a_1x + (a_1^2 + a_2)x^2 + \dots)$$

we have $\int_0^1 x^{-1}|y_2|^2 = \int_0^1 x^{\frac{-3}{2}}(a_0^2 + \dots)dx = \infty$ although

$$\int_0^1 |y_2|^2 = \int_0^1 x^{\frac{-1}{2}}(a_0^2 + \dots) < \infty.$$

Example 4.4.3 *In the differential equation* (4.3.9) *let* $q(x) = -x^4 - ix$ *and let* $[1,\infty)$ *be the corresponding interval.*

We have

$$-y'' + (-x^4 - ix)y = \lambda y$$

To solve the problem for the particular value $\lambda = 0$, we see that infinity is an irregular singular point of rank 3 for the equation $-y'' + (-x^4 - ix)y = 0$, since if we substitute $x = t^{-1}$ in it then the point $t = 0$ is an irregular singular point of the corresponding equation which is

$$y'' + 2t^{-1}y' + (t^{-8} + it^{-5})y = 0.$$

Hence we can use the method of normal solutions to find the required solutions. Let a normal solution be of the form

$$y = e^{Q(x)}x^{\rho}(c_0 + c_1x^{-1} + c_2x^{-2} + \dots)$$

Substituting into the second equation we proceed as in Example 4.4.1 and first determine the degree s of $Q(x)$ which is less than or equal to 3, so that $Q(x) = ax^3 + bx^2 + cx$. Equating coefficients we get $a = \pm\frac{2i}{3}$, $b = 0$ and $c = 0$. If we denote two values of a by $a_1 = \frac{2i}{3}$ and $a_2 = \frac{-2i}{3}$ and the corresponding polynomials by $Q_1(x) = \frac{2i}{3}x^3, \rho_1 = -\frac{5}{4}$ and $Q_2(x) = \frac{-2i}{3}x^3, \rho_2 = -\frac{3}{4}$ then we obtain two independent solutions

$$y_1(x) = e^{\frac{2i}{3}x^3} x^{\frac{-5}{4}} (c_0 + c_1 x^{-1} + c_2 x^{-2} + \ldots)$$

$$y_2(x) = e^{\frac{-2i}{3}x^3} x^{\frac{-3}{4}} (d_0 + d_1 x^{-1} + d_2 x^{-2} + \ldots)$$

Since $\nu = 0$ we see that $(-q_2)^{\frac{1}{2}} y_1 \in L^2[1,\infty)$ and $y_1 \in L^2[1,\infty)$; on the other hand y_2 is an $L^2[1,\infty)$ solution but it is not an $L^2(-q_2;[1,\infty))$ solution, so we conclude that this is Case II in Sims'classification.

4.5 McLeod's results

Results about non-selfadjoint problems in the interval $[0,\infty)$ can be obtained under more general conditions than those considered by Sims. For example, Theorem 4.3.1 remains valid if $\nu < 0$ and $q_2 \geq 0$. To see this, note that since we can write $\overline{\tau f} = \overline{\lambda f}$ then

$$-\bar{f}'' + (q_1(x) - iq_2(x))\bar{f} = (\mu - i\nu)\bar{f}$$

Also if $\nu < 0$ then $-\nu > 0$ and if $q_2 \geq 0$ then $-q_2 \leq 0$ so there exists a solution $\bar{\psi}(x,\lambda)$ which is the conjugate of the Sims' square integrable solution $\psi(x,\lambda)$, and the corresponding m-function is the complex conjugate of the m-function given by Theorem 4.3.1. Moreover if we change the coordinates for $\lambda = \mu + i\nu$ and $q(x) = q_1(x) + iq_2(x)$ simultaneously then the same result will be obtained if

$$\nu > \sup q_2(x) \quad \text{or} \quad \nu < \inf q_2(x)$$

for, if we transform the origin for both λ and $q(x)$ to an arbitrary point (u,v), then the differential equation (4.3.9) remains unchanged, since

$$-f'' + ((q_1(x) - u) + i(q_2(x) - v))f = ((\mu - u) + i(\nu - v))f$$

and (4.3.9) is equivalent to the following equation

$$-f'' + (q_1(x) + iq_2(x))f = (\mu + i\nu)f.$$

The full generalization of Sims' results given by McLeod [30] is stated in the following theorem and subsequent corollaries, where we suppose that

$$\limsup_{x\to\infty} q_2(x) = L \quad \text{and} \quad \liminf_{x\to\infty} q_2(x) = l$$

Before going further we recall that $\phi(x,\lambda)$ and $\theta(x,\lambda)$ are two linearly independent solutions of (4.3.9) satisfying

$$\phi(0,\lambda) = \sin\alpha, \quad \phi'(0,\lambda) = -\cos\alpha \tag{4.5.23}$$

$$\theta(0,\lambda) = \cos\alpha, \quad \theta'(0,\lambda) = \sin\alpha \tag{4.5.24}$$

where $\alpha = \alpha_1 + i\alpha_2, \alpha_1, \alpha_2 \in \mathbb{R}$, and the result of Sims that, if $\nu > 0$, $q_2 \le 0$, $\alpha_2 \le 0$, then

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda) \tag{4.5.25}$$

is an L^2-solution.

Theorem 4.5.1 (McLeod) *There is a square integrable solution for* (4.3.9) *if* $\nu > L$, *or* $\nu < l$ *and the corresponding m-functions are meromorphic on the regions* $\nu > L, \nu < l$ *for any* $\alpha \in \mathbb{C}$.

Proof: See [30].

Corollary 4.5.1 *If* $L = l = \lim_{x\to\infty} q_2(x)$ *and if* $\nu \neq L$, *then* (4.3.9) *has a square integrable solution and the corresponding m-function* $m_\alpha(\lambda)$ *is meromorphic on the half planes* $\nu < L$ *and* $\nu > L$.

Proof: see [30].

Corollary 4.5.2 *If* $\lim_{x\to\infty} q_2(x) = L$ *and if* $\nu \neq L$, *then the m-function of the Corollary* 4.5.1 *at* λ *is regular or has a pole. If the m-function is regular at* λ, *and* $\nu \neq L$, *then*

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda)$$

is in $L^2[0,\infty)$.

Proof: see [30].

Corollary 4.5.3 *If $L = l = +\infty$ or $L = l = -\infty$ then there is a square integrable solution for all λ and the corresponding m-function is meromorphic on the whole plane.*

Proof: see [30].

Corollary 4.5.4 *If there is no $L^2[0,\infty)$ solution of (4.3.9) for any value of λ, then $l = -\infty, L = \infty$.*

Proof: see [30].

Remark 4.5.1 All cases I, II and III of Sims classification remain unchanged under McLeod's extension, since the condition

$$\nu > L = \limsup_{x\to\infty} q_2(x)$$

implies that for each $\epsilon = \nu - L > 0$ there exists $\delta > 0$ such that $\nu - q_2(x) > 0$ for all $x > \delta > 0$. Hence by Glazman's decomposition method and the inequality (4.3.15) there exists functions $\psi_{b,\delta}(x,\lambda)$ and $m_{b,\delta}(\lambda)$ satisfying

$$\int_\delta^b 2(\nu - q_2)|\psi_{b,\delta}|^2 \le (1 + |m_{b,\delta}|^2)\sinh 2\alpha_2 - 2\Im m_{b,\delta}\cosh 2\alpha_2.$$

where

$$\psi_{b,\delta}(x,\lambda) = \theta_\delta(x,\lambda) + m_{b,\delta}(\lambda)\phi_{b,\delta}(x,\lambda)$$

and $\{\theta_\delta, \phi_\delta\}$ is a fundamental set of linearly independent solutions of (4.3.9) on the interval $[\delta,\infty)$ which satisfy the following boundary conditions at the point $x = \delta$

$$\theta_\delta(\delta,\lambda) = 0, \quad \theta'_\delta(\delta,\lambda) = 1$$

$$\phi_\delta(\delta,\lambda) = -1, \quad \phi'_\delta(\delta,\lambda) = 0$$

Taking the limit as $b \to \infty$ we then have $\psi_\delta\sqrt{\nu - q_2} \in L^2[\delta,\infty)$, and hence $\psi_\delta\sqrt{\nu - q_2} \in L^2[0,\infty)$, so by recalling the arguments which established cases I, II and III in Sims' theorem we obtain the result. In the case $\nu < \liminf_{x\to\infty} q_2(x)$ the result follows by a similar argument.

Example 4.5.1 (McLeod) *Let* $q(x) = -2ie^{2(1+i)x} = 2e^{2x}\sin(2x) - 2ie^{2x}\cos(2x)$ *in equation* (4.3.9).

We have

$$\limsup_{x\to\infty} -2e^{2x}\cos(2x) = \infty, \quad \liminf_{x\to\infty} -2e^{2x}\cos(2x) = -\infty$$

and

$$\frac{d^2y}{dx^2} + (\lambda + 2ie^{2(1+i)x})y = 0$$

Making the change of variable $\xi = e^{(1+i)x}$ we obtain the Bessel equation

$$\frac{d^2y}{d\xi^2} + \frac{1}{\xi}\frac{dy}{d\xi} + (1 + \frac{\lambda}{2i\xi^2})y = 0 \tag{4.5.26}$$

Two linearly independent solutions are $J_\alpha(\xi)$ and $J_{-\alpha}(\xi)$ where $\alpha = \sqrt{\frac{-\lambda}{2i}}$ and $|\xi| \to \infty$ as $x \to \infty$. Since $\xi = e^x(\cos x + i\sin x)$, then $\arg\xi = x$, so using Watson's argument on solutions of the Bessel equation ([48] p.202), we have the asymptotic expansion

$$J_\alpha(\xi) \sim c_1\sqrt{\frac{2}{\pi\xi}}\exp\{i(\xi - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi)\} + c_2\sqrt{\frac{2}{\pi\xi}}\exp\{-i(\xi - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi)\}$$

where

$$c_1 = c_2 = \frac{1}{2}e^{2k\pi i(\alpha+\frac{1}{2})}$$

if $(2k-1)\pi < x < (2k+1)\pi$, and k a positive or negative integer

$$c_1 = \frac{1}{2}e^{2(k+1)\pi i(\alpha+\frac{1}{2})}, \quad c_2 = \frac{1}{2}e^{2k\pi i(\alpha+\frac{1}{2})}$$

if $2k\pi < x < (2k+2)\pi$, and k a positive or negative integer. Now let x satisfy the inequalities

$$(2k - \frac{1}{2})\pi \le x \le (2k - \frac{1}{6})\pi, \qquad k \in Z$$

Then

$$|e^{i\xi}| = \exp(-e^x\sin x) \ge \exp(\frac{1}{2}e^x)$$

and so $|J_\alpha(\xi)| \to \infty$ as $x \to \infty$ since

$$|J_\alpha(\xi)| \ge \sqrt{\frac{2}{\pi}}|e^{-\frac{1}{2}\alpha\pi i}||c_1|\exp(\frac{1}{2}e^x - \frac{1}{2}x)$$

Hence $J_\alpha(\xi)$ is not in $L^2(0,\infty)$, and similarly $J_{-\alpha}(\xi)$ is not in $L^2(0,\infty)$.

If one has a linear combination of $J_\alpha(\xi)$ and $J_{-\alpha}(\xi)$ in $L^2(0,\infty)$,we know that this linear combination should be chosen so that the terms in $e^{i\xi}$ and $e^{-i\xi}$ respectively in the intervals

$$(2k-\frac{1}{2})\pi \le x \le (2k-\frac{1}{6})\pi$$
$$(2k+\frac{1}{6})\pi \le x \le (2k+\frac{1}{2})\pi$$

are eliminated for each integer k. However no linear combination can eliminate both $e^{i\xi}$ and $e^{-i\xi}$ so there is no $L^2(0,\infty)$ solution for this equation.

4.5.1 Extension of the theory to the interval $(-\infty,\infty)$

We consider the non-selfadjoint differential expression $\tau \equiv -\frac{d^2}{dx^2} + q(x)$ on the interval $(-\infty,\infty)$ where $q(x)$ is a complex function integrable over any compact subinterval $[a,b]$ of $(-\infty,\infty)$. Let $\lim_{x\to\infty} q_2(x) = L$ and $\lim_{x\to-\infty} q_2(x) = M$ where $-\infty \le M \le L \le \infty$. By the decomposition method [15] we can extend the preceding results obtained in §4.5 to the subintervals $(-\infty,r]$ and $[r,\infty)$ where $r \in (-\infty,\infty)$ and $q(x)$ is regular at the point r. We suppose that the boundary condition α is real and define the operator T_∞ by

$$T_\infty f = \tau f$$

for all $f \in D_\infty$ where

$$D_\infty = \{f : f \in D, \lim_{b\to\infty} W[f,\psi](b) = \lim_{a\to\infty} W[f,\eta](a) = 0\}$$

and D denotes the set of all functions $f \in L^2(-\infty,\infty)$ for which f' exists and is absolutely continuous over every subinterval $[a,b]$ and $\tau f \in L^2(-\infty,\infty)$ and ψ and η are independent solutions in $L^2[r,\infty)$ and $L^2(-\infty,r]$ respectively.

Theorem 4.5.2 *Let $\nu = \Im\lambda \ne L$. Then there exists a solution $\psi(x,\lambda)$ of equation $\tau f = \lambda f$ which is $L^2[r,\infty)$.*

Proof: see [26]

Theorem 4.5.3 *Let $\nu = \Im\lambda \neq M$. Then there exists a solution $\eta(x,\lambda)$ of equation $\tau f = \lambda f$ which is $L^2(-\infty, r]$.*

Proof: see [26]

Theorem 4.5.4 *Let L and M be finite. Then the essential spectrum σ_{ess} of T_∞ is contained on and in between the lines $\nu = L$, $\nu = M$ and contains the zeros of $W[\psi,\eta]$. The zeros of $W[\psi,\eta]$ are in the point spectrum of T_∞.*

Proof: see [26]

4.6 Extension of Titchmarsh's results

In this section we establish some extensions to the non-selfadjoint case of some of Titchmarsh's results for the selfadjoint case. The first result is an analogue of [46] Chap.II, Lemma 2.3. We begin by recalling the differential equation (4.3.9) in which the coefficient function q is in Case I in the sense of Sims, under the assumption that $\lim_{x\to\infty} q_2(x) = L$, and that θ, ϕ are linearly independent solutions of (4.3.9) satisfying (4.2.3-4) for some $\alpha \in \mathbb{C}$.

Lemma 4.6.1 *Let f and g satisfy equation* (4.3.9), *and the corresponding equation with λ' instead of λ, respectively. Then*

$$(\lambda' - \lambda) \int_0^X f(x)g(x)dx = W[f,g](0) - W[f,g](X)$$

where $X > 0$.

Proof: For each $X > 0$ we can write

$$\int_0^X f\tau g - g\tau f = \int_0^X (\lambda' - \lambda) fg = \int_0^X -fg'' + gf'' =$$

$$-\int_0^X \frac{d}{dx}(fg' - gf') = W[f,g](0) - W[f,g](X)$$

and the result follows.

Theorem 4.6.1 *Under the conditions of Theorem* 4.3.1, *let* $\lambda \neq \lambda'$ *be two fixed points in the upper half plane which are not poles of* $m(\lambda)$. *Then*

$$\lim_{x\to\infty} W_x[\psi(x,\lambda),\psi(x,\lambda')] = 0$$

where $\psi(x,\lambda)$ *is as in* (4.5.25).

Proof: Take $b \in (0,\infty)$ and let $\Im\lambda > 0$ and $\Im\lambda' > 0$; then we have

$$W_b[\psi_b(x,\lambda),\psi_b(x,\lambda')] = 0$$

where $\psi_b(x,\lambda)$ is defined by (4.3.10), and satisfies the boundary condition

$$\psi_b(b,\lambda)\cos\beta + \psi_b'(b,\lambda)\sin\beta = 0$$

at $x = b$, which is independent of λ. Hence

$$W_b[\psi(x,\lambda) + (m_b(\lambda) - m(\lambda))\phi(x,\lambda), \psi(x,\lambda') + (m_b(\lambda') - m(\lambda'))\phi(x,\lambda')] = 0$$

so that

$$\begin{aligned} W_b[\psi(x,\lambda),\psi(x,\lambda')] &+ (m_b(\lambda) - m(\lambda))(m_b(\lambda') - m(\lambda'))W_b[\phi(x,\lambda),\phi(x,\lambda')] \\ &+ (m_b(\lambda) - m(\lambda))W_b[\phi(x,\lambda),\psi(x,\lambda')] \\ &+ (m_b(\lambda') - m(\lambda'))W_b[\psi(x,\lambda),\phi(x,\lambda')] \\ &= 0 \end{aligned}$$

But by Lemma 4.6.1 we obtain

$$W_b[\phi(x,\lambda),\psi(x,\lambda')] = (\lambda - \lambda')\int_0^b \phi(x,\lambda)\psi(x,\lambda')dx + W_0[\phi(x,\lambda),\psi(x,\lambda')]$$

and we have

$$\begin{aligned} |\int_0^b \phi(x,\lambda)\psi(x,\lambda')dx| &\leq \int_0^b |\phi(x,\lambda)\psi(x,\lambda')|dx \\ &\leq (\int_0^b |\phi(x,\lambda)|^2 \int_0^b |\psi(x,\lambda')|^2 dx)^{\frac{1}{2}} \\ &\leq M(\int_0^b |\phi(x,\lambda)|^2 dx)^{\frac{1}{2}} \end{aligned}$$

since $\psi(x,\lambda') \in L^2[0,\infty)$. Hence if $\|\phi\|_b = (\int_0^b |\phi(x,\lambda)|^2 dx)^{\frac{1}{2}}$ then

$$\int_0^b \phi(x,\lambda)\psi(x,\lambda')dx = O(\|\phi\|_b)$$

as $b \to \infty$ and since also $W_0[\phi(x,\lambda), \psi(x,\lambda')] = O(1)$ we have

$$W_b[\phi(x,\lambda), \psi(x,\lambda')] = O(\|\phi\|_b) + O(1) \quad \text{as} \quad b \to \infty$$

Since there is at most one square integrable solution in Case I and (4.3.14) holds for $\nu > 0$

$$|m_b(\lambda) - m(\lambda)| \leq 2r_b(\lambda) = |\int_0^b (\nu - q_2)|\phi(x,\lambda)|^2 dx - \frac{1}{2}\sinh 2\alpha_2|^{-1}$$

we have

$$(m_b(\lambda) - m(\lambda))W_b[\phi(x,\lambda), \psi(x,\lambda)] = O(\int_0^b (\nu - q_2)|\phi(x,\lambda)|^2 dx - \frac{1}{2}\sinh 2\alpha_2)^{-\frac{1}{2}}$$

and since $\nu - q_2$ is non-negative for all $x \in [0,\infty)$

$$(m_b(\lambda) - m(\lambda))W_b[\phi(x,\lambda), \psi(x,\lambda)] = O(\|\phi\|_b^{-1})$$

which implies that

$$\lim_{b\to\infty} (m_b(\lambda) - m(\lambda))W_b[\phi(x,\lambda), \psi(x,\lambda')] = 0$$

Similarly for $\nu' > 0$ we have

$$\lim_{b\to\infty} (m_b(\lambda') - m(\lambda'))W_b[\psi(x,\lambda), \phi(x,\lambda')] = 0$$

also

$$\lim_{b\to\infty} (m_b(\lambda) - m(\lambda))(m_b(\lambda') - m(\lambda'))W_b[\phi(x,\lambda), \phi(x,\lambda')] = 0$$

and hence

$$\lim_{b\to\infty} W_b[\psi(x,\lambda), \psi(x,\lambda')] = 0$$

thus the result follows. The argument is similar if $\nu < 0, \ \nu' < 0$.

Extension of the above theorem under the condition $\lim_{x\to\infty} q_2 = L$ is given by the following theorem.

Theorem 4.6.2 *For any fixed λ' and λ'' with $\lambda' \neq \lambda''$ which are not poles of $m(\lambda)$, and for which $\Im\lambda' > L$ and $\Im\lambda'' > L$*

$$\lim_{x\to\infty} W_x[\psi(x,\lambda'),\psi(x,\lambda'')] = 0$$

where $\psi(x,\lambda)$ is as in (4.5.25).

Proof: Let $\Im\lambda'$, $\Im\lambda'' > L$. Then since $\lim_{x\to\infty} q_2(x) = L$, there exist real numbers $s_1 > 0$ and $s_2 > 0$ such that

$$\nu' > q_2(x), \quad \text{for all } x \in [s_1,\infty) \tag{4.6.27}$$

and

$$\nu'' > q_2(x), \quad \text{for all } x \in [s_2,\infty) \tag{4.6.28}$$

where $\nu' = \Im\lambda'$, $\nu'' = \Im\lambda''$. Now let $s = \max\{s_1, s_2\}$; then the inequalities in (4.6.26) and (4.6.27) hold for all $x \in [s,\infty)$. Here we may use the decomposition method studied by Glazman [15]. Let the boundary condition at the point s be given by $\tilde{\alpha} = 0$ and $\{\tilde{\phi},\tilde{\theta}\}$ be a fundamental set of solutions of (4.3.9) satisfying

$$\tilde{\phi}(s,\lambda) = 0 \quad \tilde{\phi}'(s,\lambda) = -1 \tag{4.6.29}$$

$$\tilde{\theta}(s,\lambda) = 1 \quad \tilde{\theta}'(s,\lambda) = 0 \tag{4.6.30}$$

We now have $\nu', \nu'' > q_2$ on $[s,\infty)$ and $\Im\tilde{\alpha} \le 0$. Hence by making a simultaneous change of origin for $\Im\lambda, \Im q(x)$ (as in [30] p. 131), it follows from Sims' result (Theorem 4.3.1) that

$$\tilde{\psi}(x,\lambda) = \tilde{\theta}(x,\lambda) + \tilde{m}(\lambda)\tilde{\phi}(x,\lambda)$$

is an $L^2[s,\infty)$ solution for all λ such that $\Im\lambda \ge \min\{\nu',\nu''\}$, and that $\tilde{m}(\lambda)$ is analytic on the half-plane. It also follows that we may use the results of Theorem 4.6.1 to give

$$\lim_{x\to\infty} W[\tilde{\psi}(x,\lambda'),\tilde{\psi}(x,\lambda'')] = 0$$

But we are in Case I and the L^2 solutions are unique up to a constant coefficient, so there exist non-zero constants c, c' such that $\psi(x,\lambda') = c\tilde{\psi}(x,\lambda'), \psi(x,\lambda'') =$

$c'\tilde{\psi}(x,\lambda'')$. Hence

$$\lim_{x\to\infty} W[\psi(x,\lambda'),\psi(x,\lambda'')] = cc' \lim_{x\to\infty} W[\tilde{\psi}(x,\lambda'),\tilde{\psi}(x,\lambda'')] = 0$$

which completes the proof.

Corollary 4.6.1 *For any fixed λ and λ' satisfying the conditions of Theorem 4.6.2*

$$\int_0^\infty \psi(x,\lambda)\psi(x,\lambda') = \frac{m(\lambda)-m(\lambda')}{\lambda'-\lambda}$$

Proof: Consider the following identities (cf. Green's formula)

$$\int_0^\infty (\psi(x,\lambda)\tau\psi(x,\lambda') - \psi(x,\lambda')\tau\psi(x,\lambda))dx = (\lambda'-\lambda)\int_0^\infty \psi(x,\lambda')\psi(x,\lambda)dx =$$

$$\int_0^\infty \psi(x,\lambda)(-\psi''(x,\lambda') + q(x)\psi(x,\lambda')) - \psi(x,\lambda')(-\psi''(x,\lambda) + q(x)\psi(x,\lambda)) =$$

$$\int_0^\infty \frac{d}{dx}(\psi(x,\lambda')\psi'(x,\lambda) - \psi'(x,\lambda')\psi(x,\lambda))dx =$$

$$\lim_{b\to\infty}(W_b[\psi(x,\lambda'),\psi(x,\lambda)]) - W_0[\psi(x,\lambda),\psi(x,\lambda')]$$

By Theorem 4.6.2

$$\lim_{b\to\infty} W_b[\psi(x,\lambda),\psi(x,\lambda')] = 0$$

Also

$$W_0[\psi(x,\lambda),\psi(x,\lambda')] = m(\lambda) - m(\lambda')$$

so that from the identity

$$(\lambda'-\lambda)\int_0^\infty \psi(x,\lambda')\psi(x,\lambda)dx = m(\lambda) - m(\lambda')$$

from which the result follows.

Lemma 4.6.2 *Let $\{f_n\}$ be a sequence of functions converging in the mean to f on $[0,X]$ for all $X>0$ and suppose that for all n there is a constant M such that $\int_0^\infty |f_n(x)|^2dx \le M$. Then f is square integrable on $[0,\infty)$, and if g is in $L^2[0,\infty)$ then*

$$\lim_{n\to\infty}\int_0^\infty f_n g = \int_0^\infty fg.$$

Proof: See [46] §2.4.

Theorem 4.6.3 *Suppose that λ_0 is an eigenvalue of T_α with $\Im\lambda_0 > L$ and that $m(\lambda)$ has a simple pole at λ_0. Then*

$$\int_0^\infty \psi(x,\lambda)\phi(x,\lambda_0)dx = \frac{1}{\lambda-\lambda_0}$$

for each λ, $\Im\lambda > L$ not a pole of $m(\lambda)$

Proof: For any λ and λ' which are not poles of $m(\lambda)$ and for which $\Im\lambda$, $\Im\lambda' > L$, we have by Corollary 4.6.1

$$\int_0^\infty \psi(x,\lambda)\psi(x,\lambda')dx = \frac{m(\lambda)-m(\lambda')}{\lambda'-\lambda}$$

Since λ_0 is an eigenvalue of T_α, and $\Im\lambda_0 > L$, the simple pole of $m(\lambda)$ at λ_0 is isolated. Let r_0 be the residue of $m(\lambda)$ at λ_0. Since there is a deleted neighbourhood $\mathcal{N}(\lambda_0)$ of λ_0 on which $m(\lambda)$ is regular, let $\lambda_0 + i\nu \in \mathcal{N}(\lambda_0)$, then setting $\lambda' = \lambda_0 + i\nu$ with $\Im\lambda' > L$ gives

$$\int_0^\infty \psi(x,\lambda)\psi(x,\lambda_0+i\nu)\frac{i\nu}{r_0}dx = \frac{[m(\lambda)-m(\lambda_0+i\nu)]i\nu}{(\lambda_0+i\nu-\lambda)r_0}. \tag{4.6.31}$$

We are going to show that

$$\frac{[m(\lambda)-m(\lambda_0+i\nu)]i\nu}{(\lambda_0+i\nu-\lambda)r_0} \to \frac{1}{\lambda_0-\lambda} \tag{4.6.32}$$

and

$$\int_0^\infty \psi(x,\lambda)\psi(x,\lambda_0+i\nu)\frac{i\nu}{r_0}dx \to \int_0^\infty \psi(x,\lambda)\phi(x,\lambda_0)dx \tag{4.6.33}$$

as $\nu \to 0$.

Setting $\psi = \theta + m\phi$ at the point $\lambda' = \lambda_0 + i\nu$ in (4.6.31) we obtain

$$\int_0^\infty \psi(x,\lambda)[\theta(x,\lambda_0+i\nu) + m(\lambda_0+i\nu)\phi(x,\lambda_0+i\nu)]dx\frac{i\nu}{r_0} =$$

$$\frac{m(\lambda)i\nu}{(\lambda_0+i\nu-\lambda)r_0} - \frac{m(\lambda_0+i\nu)i\nu}{(\lambda_0-\lambda+i\nu)r_0}.$$

Since for $\lambda \neq \lambda_0$

$$\lim_{\nu\to 0}\frac{m(\lambda)i\nu}{(\lambda_0+i\nu-\lambda)r_0}=\lim_{\nu\to 0}i\nu(\frac{m(\lambda)}{(\lambda_0+i\nu-\lambda)r_0})=0$$

and

$$\lim_{\nu\to 0}\frac{m(\lambda_0+i\nu)i\nu}{(\lambda_0-\lambda+i\nu)r_0}=\lim_{\nu\to 0}\frac{i\nu(r_0(i\nu)^{-1}+\sum_{k=0}^{\infty}a_k(i\nu)^k)}{r_0(\lambda_0-\lambda+i\nu)}=\frac{1}{\lambda_0-\lambda}$$

for some complex valued sequence $\{a_k\}$, we have

$$\lim_{\nu\to 0}\frac{(m(\lambda)-m(\lambda_0+i\nu))i\nu}{(\lambda_0+i\nu-\lambda)r_0}=\frac{1}{\lambda_0-\lambda}$$

and (4.6.32) follows.

We now establish the limiting behaviour of (4.6.33) and first of all we prove that $i\nu\psi(x,\lambda_0+i\nu)$ tends to a limit $r_0\phi(x,\lambda_0)$ in the topology of $L^2[0,X]$ for all $X\in[0,\infty)$ as $\nu\to 0$.

Since

$$\int_0^X|i\nu\psi(x,\lambda_0+i\nu)-r_0\phi(x,\lambda_0)|^2dx=$$

$$\int_0^X|i\nu\theta(x,\lambda_0+i\nu)+[i\nu m(\lambda_0+i\nu)-r_0]\phi(x,\lambda_0+i\nu)+r_0[\phi(x,\lambda_0+i\nu)-\phi(x,\lambda_0)]|^2dx$$

using Minkowski's inequality we have

$$(\int_0^X|i\nu\psi(x,\lambda_0+i\nu)-r_0\phi(x,\lambda_0)|^2dx)^{1/2}\leq|\nu|(\int_0^X|\theta(x,\lambda_0+i\nu)|^2dx)^{\frac{1}{2}}+$$

$$|i\nu m(\lambda_0+i\nu)-r_0|(\int_0^X|\phi(x,\lambda_0+i\nu)|^2dx)^{\frac{1}{2}}+$$

$$|r_0|(\int_0^X|\phi(x,\lambda_0+i\nu)-\phi(x,\lambda_0)|^2dx)^{\frac{1}{2}}$$

Taking the limit as $\nu\to 0$ we then have

$$\lim_{\nu\to 0}|\nu|(\int_0^X|\theta(x,\lambda_0+i\nu)|^2dx)^{\frac{1}{2}}=0$$

and since r_0 is residue of $m(\lambda)$ at λ_0

$$\lim_{\nu\to 0}|i\nu m(x,\lambda_0+i\nu)-r_0|(\int_0^X|\phi(x,\lambda_0+i\nu)|^2dx)^{\frac{1}{2}}=0$$

and

$$\lim_{\nu\to 0} |r_0|^2 \int_0^X |\phi(x,\lambda_0+i\nu) - \phi(x,\lambda_0)|^2 = 0$$

so that

$$\lim_{\nu\to 0} (\int_0^X |i\nu\psi(x,\lambda_0+i\nu) - r_0\phi(x,\lambda_0)|^2 dx)^{1/2} = 0$$

and hence

$$\lim_{\nu\to 0} \int_0^X |i\nu\psi(x,\lambda_0+i\nu) - r_0\phi(x,\lambda_0)|^2 dx = 0 \tag{4.6.34}$$

for all $X > 0$.

On the other hand there exists a constant M such that

$$\int_0^\infty |i\nu\psi(x,\lambda_0+i\nu)|^2 dx \le M.$$

for all $\lambda' \in \mathcal{N}(\lambda_0)$. To show this, we see that since $\lim_{x\to\infty} q_2(x) = L$ and $\Im\lambda' > L$, then using the continuity of q_2 on $[0,\infty)$, there exists a point $s > 0$ in $[0,\infty)$ and a constant $K > 0$ such that $\frac{1}{2}(\Im\lambda' - L) < \Im\lambda' - q_2(x)$ for all $x > s$, and $|\Im\lambda' - q_2(x)| \le K$ for all $x \in [0,s]$. Hence we can write

$$\begin{aligned}|\nu|^2 \int_0^\infty |\psi(x,\lambda_0+i\nu)|^2 dx &= |\nu|^2 \int_0^\infty |\theta + m\phi|^2 dx \\ &= |\nu|^2 \int_0^s |\theta + m\phi|^2 dx + |\nu|^2 \int_s^\infty |\theta + m\phi|^2 dx\end{aligned}$$

and using (4.3.15) we obtain

$$\frac{1}{2}(\Im\lambda' - L)|\nu|^2 \int_s^\infty |\theta + m\phi|^2 dx \le |\nu|^2 \int_s^\infty (\Im\lambda' - q_2(x))|\theta + m\phi|^2 dx \le$$

$$\frac{1}{2}|\nu|^2(1+|m|^2)\sinh 2\alpha_2 - |\nu|^2 \Im m \cosh 2\alpha_2 - |\nu|^2 \int_0^s (\Im\lambda' - q_2(x))|\theta + m\phi|^2 dx \tag{4.6.35}$$

and

$$|\nu|^2 |\int_0^s (\Im\lambda' - q_2)|\theta + m\phi|^2 dx| \le K|\nu|^2 \int_0^s |\theta + m\phi|^2 dx \tag{4.6.36}$$

where $\theta = \theta(x,\lambda_0+i\nu)$, $\phi = \phi(x,\lambda_0+i\nu)$ and $m = m(\lambda_0+i\nu)$. Hence there exist constants K', $K'' > 0$ so that

$$\frac{1}{2}|\nu|^2(1+|m|^2)\sinh 2\alpha_2 - \Im m \cosh 2\alpha_2 \to \frac{1}{2}|r_0|^2 \sinh 2\alpha_2$$

with $\frac{1}{2}|r_0|^2|\sinh 2\alpha_2| \leq K'$, and

$$K|\nu|^2 \int_0^s |\theta + m\phi|^2 dx \longrightarrow K|r_0|^2 \int_0^s |\phi|^2 dx \leq K''$$

as $\nu \longrightarrow 0$. Therefore by combining (4.6.34) and (4.6.35) we have

$$\frac{1}{2}(\Im\lambda' - L)|\nu|^2 \int_0^\infty |\theta + m\phi|^2 dx \leq \frac{1}{2}(\Im\lambda' - L)|\nu|^2 \int_0^s |\theta + m\phi|^2 dx +$$

$$|\nu|^2 \int_s^\infty (\Im\lambda' - q_2)|\theta + m\phi|^2 dx \longrightarrow \frac{1}{2}(\Im\lambda' - L)|r_0|^2 \int_0^s |\phi|^2 dx +$$

$$\frac{1}{2}|r_0|^2 \sinh 2\alpha_2 \leq K' + K'' = M'$$

as $\nu \longrightarrow 0$, from which it follows by continuity arguments that for each $\lambda' = \lambda_0 + i\nu$ in $\mathcal{N}(\lambda_0)$ we have

$$\int_0^\infty |\nu|^2 |\psi(x, \lambda_0 + i\nu)|^2 dx \leq M.$$

Hence applying Lemma 4.6.2 to (4.6.33) we obtain $|r_0|^2 \int_0^\infty |\phi(x, \lambda_0)|^2 dx < \infty$, and

$$\lim_{\nu \to 0} \int_0^\infty \psi(x, \lambda)\psi(x, \lambda_0 + i\nu)\frac{i\nu}{r_0} dx = \int_0^\infty \psi(x, \lambda)\phi(x, \lambda_0) dx$$

i.e. (4.6.33) follows. In the case $\Im\lambda, \Im\lambda_0 < L$ same argument gives the result and this completes the proof.

Corollary 4.6.2 *Under the conditions of Theorem* 4.6.3

$$\int_0^\infty \phi^2(x, \lambda_0) dx = \frac{-1}{r_0}.$$

Proof: Setting $\lambda = \lambda_0 + i\nu$ in the following formula

$$\int_0^\infty \psi(x, \lambda)\phi(x, \lambda_0)\frac{i\nu}{r_0} dx = \frac{i\nu}{r_0(\lambda - \lambda_0)}$$

and taking the limit as $\nu \longrightarrow 0$ and using Lemma 4.6.2 as in Theorem 4.6.3 we have the result.

Theorem 4.6.4 *Let λ_1, λ_2 be two distinct poles of the m-function $m_\alpha(\lambda)$ which is given by Theorem* 4.6.2 *and let $\Im\lambda_1, \Im\lambda_2 > L$. Then*

$$\int_0^\infty \phi(x, \lambda_1)\phi(x, \lambda_2) dx = 0$$

Proof: Using the m-function connection formula (4.3.17) it is straightforward to show that there exist constants $K_1, K_2 \in \mathbb{C}$ and a boundary condition $\alpha' \neq \alpha (\bmod \pi)$ such that

$$\phi(x, \lambda_1, \alpha) = K_1 \psi(x, \lambda_1, \alpha')$$

and

$$\phi(x, \lambda_2, \alpha) = K_2 \psi(x, \lambda_2, \alpha').$$

Also by Lemma 4.6.1

$$\int_0^\infty [\tau\phi(x,\lambda_1)\phi(x,\lambda_2) - \tau\phi(x,\lambda_2)\phi(x,\lambda_1)]dx = (\lambda_1 - \lambda_2)\int_0^\infty \phi(x,\lambda_1)\phi(x,\lambda_2)dx =$$

$$\lim_{x\to\infty} W[\phi(x,\lambda_1), \phi(x,\lambda_2)](x) - W[\phi(x,\lambda_1), \phi(x,\lambda_2)](0).$$

However, using Theorem 4.6.1

$$\lim_{x\to\infty} W[\phi(x,\lambda_1,\alpha)\phi(x,\lambda_2,\alpha)](x) = K_1 K_2 \lim_{x\to\infty} W[\psi(x,\lambda_1,\alpha')\psi(x,\lambda_2,\alpha')](x) = 0.$$

Also

$$W[\phi(x,\lambda_1), \phi(x,\lambda_2)](0) = 0,$$

so that since $\lambda_1 \neq \lambda_2$

$$\int_0^\infty \phi(x,\lambda_1)\phi(x,\lambda_2)dx = 0$$

which completes the proof.

Remark 4.6.1 If we remove the condition $\lim_{x\to\infty} q_2(x) = L$ from Theorem 4.6.2, Corollary 4.6.1, Theorem 4.6.3 and Corollary 4.6.2 then these results also hold for λ, λ', λ'', and λ_0 in the region $\Im\lambda > \limsup_{x\to\infty} \Im q(x)$.

4.7 Spectral Theory of Non-Selfadjoint Differential Operators

The general theory of eigenfunction expansions for non-selfadjoint ordinary differential operators has not recieved much attention during the past forty years.

It seems that initial investigations into non-selfadjoint eigenfunction expansions need to address the problem under more restrictive conditions, adapting strategies from the selfadjoint case and using related results in the literature to extend the results to the non-selfadjoint case. Contributions to this area of research begin with work of a few researchers from Eastern Euorope and the former Soviet Union, such as I. M. Naimark and B. S. Pavlov. Investigation of the spectrum and the expansion of certain functions in terms of eigenfunctions of a non-selfadjoint ordinary differential operator of second order was initiated by M.A.Naimark in 1954. His results were obtained by considering a non-selfadjoint ordinary differential operator of order two from the point of view of analytic operator theory [33]. An important consideration is how far is a non-selfadjoint differential operator from selfadjointness There is a considerable body of results in the spectral theory of non-selfadjoint problems which are concerened with operators which are J-symmetric in the sense of Glazman and have J-selfadjoint extensions; for example David Race and Ian Knowles have contributed to this area. There is an analogy for much of the theory of differential operators associated with the expression τ, between selfadjoint extensions of the corresponding operator with real-valued coefficients and J-selfadjoint extensions of the operator with complex coefficients, there is also an analogue between the deficiency indices in the selfadjoint case and the defect number in the J-selfadjoint case, provided that the regularity field is not empty [37], [23]. There are some results in spectral theory by Allan M. Krall on the existence of solutions of a non-homogenous equation with complex coefficient [25], and we will describe these results in section 4.12. By considering non-selfadjoint problems from different points of view we can clarify aspects of the non-selfadjoint theory, and this will be demonstrated in the following sections.

4.8 J-Symmetricity and J-Selfadjointness

An operator J on a complex Hilbert space $\mathcal{H}$ is said to be a *conjugation* if it is an involution and

$$(Jx, Jy) = (y, x)$$

for all x and y in $\mathcal{H}$.

By *involution* we mean that there is a single valued linear map $x \to x^*$ on $\mathcal{H}$ with:

i)$(x + y)^* = x^* + y^*$

ii)$(\lambda x)^* = \bar{\lambda} x^*$

iii)$(x^*)^* = x$

for all x and y in $\mathcal{H}$ and $\lambda \in \mathbb{C}$.

Definition 4.8.1 A linear operator T with dense domain $D(T)$ defined on a Hilbert space $\mathcal{H}$ is said to be a J-symmetric operator if there exists a conjugation operator J on $\mathcal{H}$ so that

$$JTJ \subset T^*$$

and T is called a J-selfadjoint operator if

$$JTJ = T^*.$$

It is well known that symmetric linear operators on $\mathcal{H}$ need not possess self-adjoint extensions since such operators may have unequal deficiency indices ([8] XII.4.13). It follows from the definition that since T^* is closed, if T is J-symmetric then it is closeable.

Theorem 4.8.1 *Let T be a J-symmetric operator defined on a Hilbert space $\mathcal{H}$ Then T has a J-selfadjoint extension on $\mathcal{H}$*

Proof: see [23].

Let T_γ be a non-selfadjoint differential operator generated by a non-selfadjoint

Sturm-Liouville type differential expression τ given by

$$\tau f = (-pf')' + qf$$

where p and q are complex functions defined on $[0, \infty)$, with p locally absolutely continuous, and $\frac{1}{p}$, q locally integrable on $[0, \infty)$. Let the domain of definition $D(T_\gamma)$ of T_γ be such that $f \in D(T_\gamma)$ if and only if

i)$f \in L^2[0, \infty)$

ii)f and pf' are locally absolutely continuous

iii)$\gamma_1 f(0) + \gamma_2 f'(0) = 0$

iv)$\tau f \in L^2[0, \infty)$

and let T_γ be defined by

$$T_\gamma f = \tau f \qquad \forall f \in D(T_\gamma)$$

where $\gamma = (\gamma_1, \gamma_2)$ and $\gamma_1, \gamma_2 \in \mathbb{C}$. We see that although the spectrum of T_γ is not necessarily contained in the real line, some properties of the selfadjoint case are preserved such as the following,

Theorem 4.8.2 *If for some fixed value of parameter $\lambda = \lambda_0$, all solutions of the equation $\tau f = \lambda f$ are in $L^2[0, \infty)$, then all solutions of the equation are in $L^2[0, \infty)$ for all values of λ.*

Proof: see [23]; note that this is an analogue of Theorem 3.4.1 and 4.3.2.

There is a connection between square integrable solutions and the spectrum of T_γ which is given by the following theorem.

Theorem 4.8.3 *If there are no non-trivial square integrable solutions of $\tau f = \lambda f$ on the interval $[0, \infty)$, then $\lambda \in \sigma_c(T_\gamma)$ for all γ*

Proof: see [23].

This theorem will be proved later in Section 11 in the case of a more general class of operators

Definition 4.8.2 Let T be a closed densely defined linear operator in a Hilbert space $\mathcal{H}$. A complex number λ is called a point of regularity type of T if

$$\|(T-\lambda I)f\| \geq k_\lambda \|f\|$$

for all $f \in D(T)$ where k_λ is a positive constant independent of any f.

The set of such numbers λ is denoted by $\pi(T)$ and is called the *regularity field* of T. The resolvent set $\rho(T)$ of T is contained in $\pi(T)$, since if $\lambda \in \rho(T)$ then $(T-\lambda I)^{-1}$ exists and is bounded i.e. $\lambda \in \pi(T)$. However if $\lambda \in \pi(T)$ it is not necessarily in $\rho(T)$ since when $\|(T-\lambda I)f\| \geq k_\lambda \|f\|$ for some k_λ, this means that $(T-\lambda I)^{-1}$ exists but may be unbounded. We can see from definition of $\pi(T)$ that it is an open set, and the linear manifold $(T-\lambda I)D(T)$ is closed [15] Chap. 1, Sec. 1.

Definition 4.8.3 i) Let $T_{\max}$, $T'_{\min}$ and $T_{\min}$ be the linear operators in $\mathcal{H} = L^2[0,\infty)$ defined as in §3.3. We can see that $D(T'_{\min})$ and hence also $D(T_{\min})$ is dense in $L^2[0,\infty)$ [33] and by a method directly analogous to that of [34], §17, it may be seen that $f \in D(T_{\min})$ if and only if:

i)$f \in L^2[0,\infty)$

ii)pf' is locally absolutely continuous

iii)$f(0) = p(0)f'(0) = 0$

iv)$\tau f \in L^2[0,\infty)$

v)$\lim_{t\to\infty} p(t)(f(t)g'(t) - f'(t)g(t)) = 0$ for all $g \in D(T_{\max})$.

The following theorem shows that when explicit bounds on the coefficient function q are known, the location of the essential spectrum is known as well.

Theorem 4.8.4 *Let the coefficient function p be real and non-negative and let $\tilde{T}$ be a J-selfadjoint extension of $T_{\min}$. Then*

$$\sigma_e(\tilde{T}) \subset \{\lambda : \liminf_{t\to\infty} \Im(q(t)) \leq \Im(\lambda) \leq \limsup_{t\to\infty} \Im(q(t))\}$$

Proof: see [23].

Remark 4.8.1 The above results and the conclusion of [15] p. 29 assert that the essential spectrum is determined solely by the behaviour of the coefficients at infinity. On the other hand, the behaviour of the set $\sigma_p(\tilde{T})$ is affected by the behaviour of the coefficients at finite values of t. This explains why one cannot easily locate $\sigma_p(\tilde{T})$ by conditions involving the integral of $\Im(q)$. There is an important result on the location of the essential spectrum of a non-selfadjoint operator which is minimal in the sense of Definition 4.8.3. This result can be obtained under certain conditions on the coefficient functions p and q.

Theorem 4.8.5 *If $p \equiv 1$ and $\Re q$ satisfies one of the following :*
i) $\int_{M_\delta} (\Re q)^- < \infty$ for any $\delta > 0$ where $M_\delta = \{t \in [0,\infty) : (\Re q(t))^- \geq \delta\}$ and f^- is the negative part of f which is defined by $f^-(x) = \min\{f(x), 0\}$
ii) $\int_0^\infty \{(\Re q)^-\}^r < \infty$ for some $r \geq 1$
iii) $\lim_{x\to\infty} \int_x^{x+\omega} (\Re q)^- = 0$ for some $\omega \neq 0$ (and consequently for any $\omega \neq 0$)
iv) $\int_J \Re q \geq -C$ for some real number C, and for all intervals J of length ≤ 1 and if

$$\int_0^\infty |\Im q| < \infty$$

then

$$\sigma_e(T_{min}) \subset \Lambda,$$

where $\Lambda = [0,\infty)$ if i),ii), or iii) is satisfied and $\Lambda = [-2C^2, \infty)$ if iv) is satisfied

Proof: see [37].

4.9 The spectrum of an operator with integrable coefficient

In this section we continue the investigation of the spectrum of a non-selfsdjoint differential operator τ under the condition $p \equiv 1$ and report some results given by Naimark. Let $\tau \equiv -\frac{d}{dx^2} + q$ and let q be a complex valued function integrable on $[0,\infty)$. Naimark's results identify the location of the spectrum of the linear operator T_θ generated by the the expression τ on $L^2[0,\infty)$ in which the domain of definition D_θ is such that $f \in D_\theta$ if and only if

i) $f \in L^2[0,\infty)$,

ii) f and f' are absolutely continuous on $[0,X]$ for any $X>0$,

iii) $\tau f \in L^2[0,\infty)$,

iv) $f'(0)-\theta f(0)=0$ for some $\theta \in \mathbb{C}$ and $T_\theta f=\tau f$.

Consider the equation $\tau f=\lambda f$ where λ is a complex parameter and let $\sqrt{\lambda}=s=\sigma+i\delta,\ 0\le \arg s<\pi$, so that $\delta\ge 0$. The homogeneous equation $\tau f=0$ has two linearly independent solutions $f_1(x,s)=e^{isx},\ f_2(x,s)=e^{-isx}$ and hence, regarding

$$f''+\lambda f=qf \tag{4.9.37}$$

as a non-homogeneous equation, and using the method of variation of parameters, we find that

$$f(x,s)=c_1e^{isx}+c_2e^{-isx}+\frac{i}{2s}e^{-isx}\int_a^x e^{ist}q(t)f(t,s)dt-\frac{i}{2s}e^{isx}\int_b^x e^{-ist}q(t)f(t,s)dt \tag{4.9.38}$$

Under various choices of constants c_1,c_2,a and b one obtains the integral equations:

$$f_1(x,s)=e^{isx}+\frac{i}{2s}\int_x^\infty [e^{is(x-t)}-e^{-is(x-t)}]q(t)f_1(t,s)dt$$

where $f_1(x,s)$ is jointly continuous in x and s for $x\ge 0,\ \delta\ge 0,\ s\ne 0$ and for each $x\in[0,\infty)$ is an analytic function of s in the half-plane $\delta\ge 0$;

$$f_2(x,s)=e^{-isx}-\frac{i}{2s}\int_0^x e^{is(x-t)}q(t)f_2(t,s)dt-\frac{i}{2s}\int_x^\infty e^{-is(x-t)}q(t)f_2(t,s)dt$$

where $f_2(x,s)$ is jointly continuous in x and s for $x\ge 0,\ \delta\ge 0,\ |s|\ge r>0$ and for each $x\in[0,\infty)$ is analytic in s in the region $\delta\ge 0, |s|>r$;

$$\tilde f(x,s)=e^{-isx}+\frac{i}{2s}\int_x^\infty [e^{is(x-t)}-e^{-is(x-t)}]q(t)\tilde f(t,s)dt$$

where $\tilde f(x,s)$ is jointly continuous in x and s for each $x\in[0,\infty),\ s>0$.

We obtain $W[f_1,f_2]=W[f_1,\tilde f]=-2is$. The asymptotic behaviour of the solutions $f_1,\ f_2$ and $\tilde f$ when $s\to\infty$ is given by the following theorems.

Theorem 4.9.1 *i) For $\Im s > 0$ and $s \to \infty$, we have*

$$f_1(x,s) = e^{isx}[1 + O(\frac{1}{s})], \quad f_1'(x,s) = ise^{isx}[1 + O(\frac{1}{s})]$$

$$f_2(x,s) = e^{-isx}[1 + O(\frac{1}{s})], \quad f_2'(x,s) = -ise^{-isx}[1 + O(\frac{1}{s})]$$

ii) For $s > 0$ and $s \to +\infty$ we have

$$\tilde{f}(x,s) = e^{-isx}[1 + O(\frac{1}{s})], \quad \tilde{f}'(x,s) = -ise^{-isx}[1 + O(\frac{1}{s})]$$

uniformly with respect to x in the interval $[0, \infty)$.

Proof: [34]

Theorem 4.9.2 *i) For fixed $r > 0$ as $x \to \infty$, we have*

$$f_1(x,s) = e^{isx}[1 + o(1)], \quad f_1'(x,s) = e^{isx}[is + o(1)]$$

$$f_2(x,s) = e^{-isx}[1 + o(1)], \quad f_2'(x,s) = e^{-isx}[-is + o(1)]$$

uniformly for s in each region $|s| \geq r > 0$, $\Im s \geq 0$
ii) As $x \to +\infty$,

$$\tilde{f}(x,s) = e^{-isx}[1 + o(1)], \quad \tilde{f}'(x,s) = e^{-isx}[-is + o(1)]$$

uniformly with respect to s in every interval $[r, \infty)$, $r > 0$.

Proof: see [34]

These estimates lead to the main result concerning the spectrum of the operator T_θ which is obtained using the integral representation of the resolvent.

Theorem 4.9.3 *Every point of the positive semi axis $\lambda > 0$ is a point of the continuous spectrum of T_θ. With the exception of these points, the spectrum of T_θ is a bounded set consisting of a finite or countable number of eigenvalues whose accumulation points lie on the positive semi-axis $\lambda \geq 0$. For all the remaining points λ, not on the positive axis, the resolvent is a bounded integral operator with kernel $K(x,t,\lambda)$ satisfying*

$$\int_0^\infty |K(x,t,\lambda)|^2 dt < \infty, \quad \int_0^\infty |K(x,t,\lambda)|^2 dx < \infty.$$

Proof: see [34]

If we consider the specific case $\Im q(x) \leq 0$, $\Im\theta \leq 0$, then the function

$$F(s) = f_1'(0,s) - \theta f_1(0,s)$$

has no zero on the semi-axis $s > 0$ and hence we have the following result

Theorem 4.9.4 *Let $\Im q(x) \leq 0$, $\Im\theta \leq 0$, where x lies in the interval $[0,\infty)$. Then the eigenvalues of the operator T_θ may be contained only in the lower half-plane and on the negative real semi-axis. Here, if a single one of the eigenvalues of T_θ lies on the negative real semi-axis, then $\Im q(x) = 0, \Im\theta = 0$, so that T_θ is a selfadjoint operator. Consequently, in this case, all the eigenvalues of T_θ lie on the negative part of the real axis and only $\lambda = 0$ can be an accumulation point of the eigenvalues of T_θ.*

Proof: see [34]

Remark 4.9.1 An analogous theorem holds for the case $\Im q(x) \geq 0$, $\Im\theta \geq 0$, in the interval $[0,\infty)$. For a proof it is sufficient to replace T_θ and θ by $\bar{T}_\theta$ and $\bar{\theta}$, where the linear operator $\bar{T}_\theta$ on $L^2[0,\infty)$ is generated by the differential expression $\bar{\tau} \equiv -\frac{d}{dx^2} + \bar{q}$ with domain

$$\bar{D}_\theta = \{f \in L^2[0,\infty) : \bar{f} \in D_\theta\}$$

and $\bar{T}_\theta f = \bar{\tau} f$ for all $f \in \bar{D}_\theta$.

Theorem 4.9.5 *Suppose*

$$\int_0^\infty e^{\epsilon x}|q(x)|dx < \infty$$

for some $\epsilon > 0$. Then the operator T_θ has only a finite number of eigenvalues which do not lie on the semi-axis $\lambda \geq 0$.

Proof: see [34]

Remark 4.9.2 An interesting result in this section is that, although for non-selfadjoint differential operators generated by τ with integrable coefficient q and the more general case with continuous coefficient q, the distribution of the isolated points of the spectrum with boundary condition α is not the same as that with boundary condition β when $\alpha - \beta \neq k\pi$, the location of the essential spectrum σ_{ess} is independent of the variations of the boundary condition.

4.10 Eigenfunction expansion for operator with integrable coefficient

In the general theory of non-selfadjoint differential operators defined on the Hilbert space $\mathcal{H} = L^2(a,b)$, where $-\infty \leq a < b \leq \infty$, the set of eigenfunctions is not necessarily a complete set of orthogonal functions generating the corresponding Hilbert space, but under certain circumstances they can be, as shown by Naimark.

Let T_θ be as in section 4.9 and denote by E_θ the set of all functions g satisfying the following conditions:

i) $g \in L^1[0,\infty)$ on the interval $[0,\infty)$,

ii) g' exists and is absolutely continuous on every finite interval $[0,X]$,

iii) The function $\tau g \in L^1[0,\infty)$ on the interval $[0,\infty)$.

iv) $g'(0) - \theta g(0) = 0$.

Suppose also that

$$\int_0^\infty e^{\epsilon x}|q(x)|dx < \infty$$

for some $\epsilon > 0$, that

$$f_1'(0,s) - \theta f_1(0,s) \neq 0, \quad \tilde{f}'(0,s) - \theta \tilde{f}(0,s) \neq 0$$

for all $s \geq 0$ and that $s_1, s_2, \dots s_r$ are simple zeros of $f_1'(0,s) - \theta f_1(0,s)$. Then if $f(x,\lambda_1), f(x,\lambda_2), \dots f(x,\lambda_r)$ are the eigenfunctions corresponding to the eigenvalues $\lambda_1 = s_1^2, \lambda_2 = s_2^2, \dots \lambda_r = s_r^2$, we have the following identity

$$K(x,t,\lambda) = \sum_{i=1}^{r} \frac{f(x,\lambda_i)f(t,\lambda_i)}{(\lambda_i - \lambda)\int_0^\infty (f(x,\lambda_i))^2 dx} - \frac{1}{2\pi}\int_0^\infty \frac{\tilde{f}(x,s)\tilde{f}(t,s)}{(s^2-\lambda)A(s)\tilde{A}(s)} ds$$

where the integral converges absolutely and uniformly for $x,\ t$ in the region $0 \le x, t < \infty$ and

$$A(s) = f_1'(0,s) - \theta f_1(0,s),\ \ \tilde{A}(s) = \tilde{f}'(0,s) - \theta \tilde{f}(0,s)$$

and $K(x,t,\lambda)$ is the kernel of the resolvent of the operator T_θ.

Theorem 4.10.1 *Under the above conditions any function g in E_θ can be represented in the form*

$$g(x) = \sum_{i=1}^{r} \frac{a_i f(x,\lambda_i)}{\int_0^\infty (f(x,\lambda_i))^2 dx} - \frac{1}{2\pi}\int_0^\infty \frac{a(s)\tilde{f}(x,s)}{A(s)\tilde{A}(s)} ds$$

where

$$a_i = \int_0^\infty g(x) f(x,\lambda_i) dx$$

$$a(s) = \int_0^\infty g(x)\tilde{f}(x,s) dx$$

and the integral converges absolutely and uniformly with respect to x in the interval $[0,\infty)$,

Proof: see [34].

4.11 Essential spectrum

It is well known that in selfadjoint problems the essential spectrum, which is defined as the whole spectrum except the isolated point spectrum, is stable under change of boundary conditions cf [10], Thm. 2.5.2. We will show that this is also the case for the class of non-selfadjoint problems considered here. We recall that as in the selfadjoint case the essential spectrum of the non-selfadjoint operator T_α is the same as the full spectrum of T_α except for the isolated point spectrum of

T_α, and that in equation (4.2.3) the function $m(\lambda)$ is dependent on the boundary condition α. We can therefore write $m(\lambda) = m_\alpha(\lambda)$. It is not too hard to see that the m-function connection formula (Theorem 4.3.3) also holds in the particular complex case considered in this thesis.

Theorem 4.11.1 *Suppose that* $\lim_{x\to\infty} q_2(x) = L$ *and that* $\beta \in \mathbb{C}$ *is such that* λ *is not pole of* m_β *where* $\nu \neq L$. *Then we have*

$$m_\alpha(\lambda) = \frac{-\sin(\alpha-\beta) + m_\beta(\lambda)\cos(\alpha-\beta)}{\cos(\alpha-\beta) + m_\beta(\lambda)\sin(\alpha-\beta)} \tag{4.11.39}$$

for all $\alpha \in \mathbb{C}$ *such that* $\alpha - \beta \neq k\pi,\ k \in \mathbb{Z}$.

Proof: see [10].

Theorem 4.11.2 *If* T_α *is selfadjoint, the essential spectrum* $\sigma_{ess}(T_\alpha)$ *is independent of* α.

Proof: see [10]

The essential spectrum of a non-selfadjoint differential operator is also independent of the boundary condition (see [11], Cor. 1.8).

Theorem 4.11.3 *Let* T_α *be non-selfadjoint in the sense of* §4.3. *Then* $\sigma_{ess}(T_\alpha)$ *is independent of* α.

Proof: See [11].

According to the Hartman-Wintner paper [18], there is an oscillation theorem for continuous spectra. This concerns a Sturm-Liouville type differential operator T_α with real coefficients on the half-line $x \geq 0$ and which is defined on D^α by

$$T_\alpha f = (-pf')' + qf \quad \forall f \in D^\alpha$$

where $p > 0$ here D^α consists of all functions of $L^2[0,\infty)$ satisfying the following conditions:

i) f, pf' are locally absolutely continuous on $[0,\infty)$

ii) $(-pf')' + qf \in L^2[0,\infty)$

iii) $f(0)\cos\alpha + p(0)f'(0)\sin\alpha = 0.$

and the corresponding differential equation is

$$(-pf')' + qf = \lambda f. \tag{4.11.40}$$

If $\sigma'(\alpha)$ denotes the continuous spectrum of T_α, which includes the limit points of the point spectrum, then $\sigma'(\alpha)$ is independent of α, cf [6]. The theorem also states that for each λ which is not in $\sigma'(\alpha)$ there exists an α such that $\lambda \in \sigma_p(\alpha)$ where $\sigma_p(\alpha)$ is the point spectrum of T_α.

We can extend this result to a non-selfadjoint problem by assuming the coefficient functions p and q are complex and $\alpha \in \mathbb{C}$.

Theorem 4.11.4 *Let λ be such that the Sturm-Liouville type equation*

$$(-pf')' + qf = \lambda f \tag{4.11.41}$$

has no nontrivial square integrable solution, where p and q may be complex on $[0,\infty)$. Then λ belongs to the spectrum of the corresponding operator T_α of (4.11.40) for each $\alpha \in \mathbb{C}$.

Proof: Suppose λ satisfies conditions of the theorem. We will show that λ is not in the resolvent set of T_α for each $\alpha \in \mathbb{C}$. Let $\alpha \in [0,\pi)$ be fixed. Then equation (4.11.40) has a nontrivial solution g such that

$$g(x_0) \neq 0 \tag{4.11.42}$$

for some $x_0 \in (0,\infty)$, where g is not necessarily an $L^2[0,\infty)$-solution but still satisfies the boundary condition α cf [5].

Now consider the regular eigenvalue problem of Sturm-Liouville type

$$(T_\alpha - \lambda)f = (-pf')' + qf - \lambda f = \mu f, \quad f(x_0) = 0 \tag{4.11.43}$$

in the interval $[0, x_0]$. Then there exists an eigenvalue μ of this problem such that equation (4.11.43) has a nontrivial solution h so that $h(x_0) = 0$ but we have

$$h'(x_0) \neq 0 \tag{4.11.44}$$

because, if $h'(x_0) = 0$, then h would be identically zero. We will show that there exists a continuous function $u \in L^2[0,\infty)$ such that

$$-(pf')' + qf - \lambda f = u \tag{4.11.45}$$

has no nontrivial solution which is in $L^2[0,\infty)$ and satisfies the boundary cindition α and then λ is not in the resolvent set of T_α and so it will be in the spectrum of T_α.

Define

$$u(x) = \begin{cases} \mu h(x) & \text{if } 0 \leq x < x_0 \\ 0 & \text{if } x_0 \leq x < \infty \end{cases}$$

so that u is continuous in the interval $[0,\infty)$ and

$$\int_0^\infty |u(x)|^2 dx < \infty.$$

We now show that this function has the required property. Suppose that u fails to have the property, so that u has the property that equation (4.11.43) has a nontrivial solution v, satisfying $v(x_0) = 0$ which is square integrable in the interval $[0,\infty)$. We show that this assumption leads to a contradiction.

Since $u(x) = 0$ on $[x_0,\infty)$, equation (4.11.45) reduces to equation (4.11.40), for $x \in [x_0,\infty)$. Hence v satisfies (4.11.40) on $[x_0,\infty)$ and is therefore square integrable on $[0,\infty)$. Since equation (4.11.40) has no nontrivial square integrable solution, we obtain $v(x) = 0, \quad \forall x \in [x_0,\infty)$, so that $v'(x_0) = 0$. Since v is also a solution of (4.11.43) on the interval $[0,x_0]$, which satisfies the boundary condition α at $x = 0$, we must have

$$v(x) = h(x) + cg(x)$$

for each x in the interval $0 \leq x \leq x_0$ where g is the solution of (4.11.40) and (4.3.42) h is a solution of (4.11.43) and c is a constant. Since the solutions of (4.11.45) are differentiable it follows that

$$\begin{cases} v(x_0) = h(x_0) + cg(x_0) = 0 \\ v'(x_0) = h'(x_0) + cg'(x_0) = 0 \end{cases}$$

Since this is a linear homogeneous system in 1 and c, the determinant $hg' - gh'$ of the coefficients vanishes, so $(gh' - hg')(x_0) = 0$. This means that, since $h(x_0) = 0$, we get $g(x_0)h'(x_0) = 0$, which is a contradiction, by (4.11.42) and (4.11.44). Since α was chosen arbitrary, the result is independent of α.

Corollary 4.11.1 *If λ belongs to the resolvent set of T_α for any $\alpha \in \mathbf{C}$ then* (4.11.40) *has a square integrable solution.*

Proof: If there were no nontrivial L^2-solution of (4.11.40), then by Theorem 4.3.4, λ would be in the spectrum of T_α for all $\alpha \in \mathbf{C}$ and this is a contradiction.

It is known that for a self-adjoint differential operator the spectrum is real and that for each real number λ there is a real boundary condition α such that λ belongs to the spectrum. We now show that in the non-selfadjoint case it is true that for each complex number λ there exists a complex boundary condition α such that λ belongs to the spectrum.

Corollary 4.11.2 *Consider the differential operator defined by formula* (4.3.9) *and a boundary condition at* 0 *and let λ be any complex number. Then there exists a boundary condition α such that λ is in the spectrum of T_α.*

Proof: Let λ be a complex number. If there is an L^2 solution of (4.3.9) it certainly satisfies some boundary condition α at the point 0 since $x = 0$ is a regular point of (4.3.9) and so λ is a point of the spectrum of T_α. If there is no L^2 solution, then by Theorem 4.3.4, λ belongs to the spectrum of T_α for every $\alpha \in$ Cand the result follows.

4.12 Non-homogeneous Equations

In this section, we assume the Sims extension of the limit point-limit circle theory of Weyl-Titchmarsh on the interval (a, b), $-\infty \leq b < a \leq \infty$, which shows that a certain non-selfadjoint problem

$$-\psi'' + q(x)\psi = \lambda\psi$$

has a square integrable solution if $\nu = \Im\lambda > 0$. A. Krall [25] showed that the limit point-limit circle classification yields a natural proof that the nonhomogeneous equation

$$-\Phi'' + (q(x) - \lambda)\Phi = f(x) \tag{4.12.46}$$

also has a square integrable solution if $\nu > 0$ provided that $f(x) \in L^2(a,b)$, and derived a necessary and sufficient conditions under which this occurs. Let $r \in (a,b)$ be an interior point and recall the arguments in §4.3 and let $r < b' < b$. Then

Lemma 4.12.1 *Let $\nu > 0$, $m_{b'}(\lambda)$ be a point on the circle $C_{b'}$, $m_b(\lambda)$ be the limit point or a point on the limit circle C_b, s be fixed, $t_b(\lambda)$ be defined by*

$$t_b(\lambda) + sm_b(\lambda) = \int_r^b [\theta(x,\lambda) + m_b(\lambda)\phi(x,\lambda)]f(x)dx$$

and $t_{b'}(\lambda)$ be defined by

$$t_{b'}(\lambda) + sm_{b'}(\lambda) = \int_r^{b'} [\theta(x,\lambda) + m_{b'}\phi(x,\lambda)]f(x)dx$$

Then $\lim_{b'\to b} t_{b'}(\lambda) = t_b$

Proof: see [25], Lemma 1.

Lemma 4.12.2 *Let λ be a fixed parameter with $\nu = \Im\lambda > 0$ and $q_2(x) = \Im q(x) \leq 0$ and let $a = -\infty$ and $b = \infty$. Then*

$$\int_{-\infty}^{\infty} (\nu - q_2)|\psi|^2 dx < \infty \Longleftrightarrow \int_{-\infty}^{\infty} (1 - q_2)|\psi|^2 dx < \infty$$

Proof: Let $\int_{-\infty}^{\infty}(\nu - q_2)|\psi|^2 dx < \infty$ we can write

$$\int_{-\infty}^{\infty} (1 - q_2)|\psi|^2 dx = \frac{1}{\nu}\int_{-\infty}^{\infty} (\nu - \nu q_2)|\psi|^2 dx < \infty$$

since $\nu > 0$ and $\nu q_2(x) \leq 0$. Now let $\int_{-\infty}^{\infty}(1-q_2)|\psi|^2dx < \infty$ easily we have

$$\int_{-\infty}^{\infty}(\nu - q_2)|\psi|^2dx = \nu\int_{-\infty}^{\infty}(1-\frac{1}{\nu}q_2)|\psi|^2dx < \infty$$

since $\nu > 0$ and $\frac{1}{\nu}q_2(x) \leq 0$.

These lemmas are used in the proof of the following theorem and in all cases I, II and III we have

Theorem 4.12.1 (Krall) *Let $\nu > 0, m_b$ be the limit point or a point on the limit circle, s, t be arbitrary complex functions of λ and let $f(x)$ be $L^2[r, b)$. Then there is a square integrable solution $\Phi(x, \lambda; f)$ of (4.12.46). Let*

$$\Phi^*(x) = s\theta - t\phi + \int_r^x [\phi(x)\theta(t) - \theta(x)\phi(t)]f(t)dt$$

In case I $\Phi^* \in L^2(r, b; [1-q_2])$ and $\Phi^* \in L^2(r, b)$. if and only if

$$t + sm_b = \int_r^b [\theta(x) + m_b\phi(x)]f(x)dx.$$

In case II every solution of (4.12.45) is in $L^2(r, b)$ and $\Phi^* \in L^2(r, b; [1-q_2])$ if and only if

$$t + sm_b = \int_r^b [\theta(x) + m_b\phi(x)]f(x)dx.$$

In case III every solution of (4.12.45) is in both $L^2(r, b; [1-q_2])$ and $L^2(r, b)$.

Proof: see [25], Theorem 2.

Chapter 5

Complex Spectrum.

5.1 Introduction

This chapter is devoted to the extension of some results of Chaudhuri-Everitt [4] to the non-selfadjoint case. They showed that in the selfadjoint case there is a strong relationship between the m-function and the resolvent operator. We are going to prove that some aspects of this relationship can be extended to non-selfadjoint problems as well.

5.2 The m-function and the resolvent operator.

Throughout this chapter we consider the differential operator T_α, $(\alpha \in \mathbb{C})$, defined in Definition 4.2.1 under the condition $\lim_{x\to\infty} q_2(x) = L < \infty$ on the coefficient function q on the interval $[0,\infty)$, and recall that according to Theorem 4.3.1, if $\Im\lambda = \nu \neq L$, then there always exists an $L^2[0,\infty)$-solution $\psi(x,\lambda)$ of the equation $\tau f = \lambda f$ and a meromorphic function $m_\alpha(\lambda)$ satisfying

$$\psi(x,\lambda) = \theta(x,\lambda) + m_\alpha(\lambda)\phi(x,\lambda).$$

where λ is a regular point of $m_\alpha(\lambda)$. We also suppose that the potential function q is continuous and in the limit-point case at ∞ with just one $L^2[0,\infty)$-solution for $\nu \neq L$, i.e. that Sims' Case I holds. Under these assumptions we can prove a theorem concerning the resolvent operator. Before that we give a lemma.

Lemma 5.2.1 *Let $f \in L^2[0,\infty)$, and let $\nu \neq L$. Suppose that λ is a regular value of $m_\alpha(\lambda)$ and define the function $\Phi(x,\lambda;f)$ on $[0,\infty)$ by*

$$\Phi(x,\lambda;f) = \psi(x,\lambda)\int_0^x \phi(t,\lambda)f(t)dt + \phi(x,\lambda)\int_x^\infty \psi(t,\lambda)f(t)dt$$

where ϕ and ψ are solutions of the equation $\tau f = \lambda f$ satisfying (4.5.23) *and* (4.5.25) *respectively. Then $\Phi \in L^2[0,\infty)$ and there exists $K > 0$ such that*

$$\|\Phi\| \leq K\|f\|$$

for all $f \in L^2[0,\infty)$.

Proof: First we note that Φ is well defined, since f is by hypothesis and $\psi \in L^2[0,\infty)$ by Corollary 4.5.2. Also ϕ and f are square integrable on $[0,x]$ for all $x > 0$. Let $\nu > L$. Then there exists a real number $r > 0$ so that

$$\nu - q_2(x) > \frac{1}{2}(\nu - L) > 0, \tag{5.2.1}$$

for all $x \in [r,\infty)$, so proceeding as in [44], §5, there are square integrable solutions ψ_0 and ψ_1 and meromorphic functions m_0 and m_1 satisfying

$$\psi_0(x,\lambda) = \tilde{\theta}(x,\lambda) + m_0(\lambda)\tilde{\phi}(x,\lambda)$$

$$\psi_1(x,\lambda) = \tilde{\theta}(x,\lambda) + m_1(\lambda)\tilde{\phi}(x,\lambda)$$

where $\psi_0(x,\lambda) \in L^2[0,r]$ satisfies the boundary condition

$$f(0)\cos\alpha + f'(0)\sin\alpha = 0,$$

and $\psi_1(x,\lambda) \in L^2[r,\infty)$. Note that there is no loss of generality if we suppose that r has been choosen so that $m_0(\lambda)$ and $m_1(\lambda)$ are analytic at λ. The fundamental set $\{\tilde{\theta}, \tilde{\phi}\}$ is defined in the usual way in terms of the boundary condition $\tilde{\alpha} = 0$ at $x = r$ i.e.

$$\tilde{\phi}(r, \lambda) = 0,\ \tilde{\theta}(r, \lambda) = 1$$

$$\tilde{\phi}'(r, \lambda) = -1,\ \tilde{\theta}'(r, \lambda) = 0$$

Hence, since we are in case I, there are non-zero scalars $k_1(\lambda)$ and $k_2(\lambda)$ depending on λ such that

$$\psi_0(x, \lambda) = k_1(\lambda)\phi(x, \lambda)$$

$$\psi_1(x, \lambda) = k_2(\lambda)\psi(x, \lambda).$$

Now define the function f_b on the interval $[0, \infty)$ by

$$f_b(x) = \begin{cases} f(x) & \text{if } x \leq b \\ 0 & \text{if } x > b \end{cases}$$

for some $b > r$, and let

$$\begin{aligned} \Phi_b &= \Phi(x, \lambda; f_b) \qquad (5.2.2) \\ &= \frac{1}{W[\psi_0, \psi_1]} \left\{ \psi_1(x, \lambda) \int_0^x \psi_0(t, \lambda) f_b(t) dt + \psi_0(x, \lambda) \int_x^b \psi_1(t, \lambda) f_b(t) dt \right\} \end{aligned}$$

We can write

$$\begin{aligned} \int_0^b \bar{\Phi}\tau\Phi - \Phi\overline{\tau\Phi} &= \int_0^b \bar{\Phi}_b(-\Phi_b'' + q\Phi_b) - \Phi_b(-\bar{\Phi}_b'' + \bar{q}\bar{\Phi}_b) \\ &= \int_0^b (\Phi_b\bar{\Phi}_b' - \bar{\Phi}_b\Phi_b')' + \int_0^b 2iq_2|\Phi_b|^2 \\ &= W[\Phi_b, \bar{\Phi}_b]_0^b + 2i \int_0^b q_2|\Phi_b|^2. \qquad (5.2.3) \end{aligned}$$

On the other hand Φ_b satisfies the non-homogenous differential equation $\tau\Phi_b - \lambda\Phi_b = f$ on $[0, b]$ so

$$\begin{aligned} \int_0^b \bar{\Phi}_b\tau\Phi_b - \Phi_b\overline{\tau\Phi_b} &= \int_0^b \bar{\Phi}_b(\lambda\Phi_b + f_b) - \Phi_b(\bar{\lambda}\bar{\Phi}_b + \bar{f}_b) \\ &= \int_0^b 2i\nu|\Phi_b|^2 + \int_0^b 2i\Im(\bar{\Phi}_b f) \qquad (5.2.4) \end{aligned}$$

By (5.2.3) and (5.2.4) we have

$$2i \int_0^b (\nu - q_2)|\Phi_b|^2 = W[\Phi_b, \bar{\Phi}_b]_0^b - 2i \int_0^b \Im(\bar{\Phi}_b f) \qquad (5.2.5)$$

However, from (5.2.3)

$$W[\Phi_b,\bar{\Phi}_b](b) = \frac{1}{|W[\psi_0,\psi_1](b)|^2} W[\psi_1,\bar{\psi}_1](b)|\int_0^b \psi_0(t,\lambda)f_b(t)dt|^2$$

and

$$W[\Phi_b,\bar{\Phi}_b](0) = \frac{1}{|W[\psi_0,\psi_1](0)|^2} W[\psi_0,\bar{\psi}_0](0)|\int_0^b \psi_1(t,\lambda)f_b(t)dt|^2$$

Also from (4.2.8) and the fact that $W[\psi_1,\bar{\psi}_1](r) = 2i\Im m_1$, $W[\psi_0,\bar{\psi}_0](r) = 2i\Im m_0$, we have

$$W[\psi_1,\bar{\psi}_1](b) = 2i[\int_r^b (\nu - q_2)|\psi_1|^2 dx + \Im m_1]$$

$$W[\psi_0,\bar{\psi}_0](0) = 2i[\int_0^r (\nu - q_2)|\psi_0|^2 dx - \Im m_0]$$

Using these results in (5.2.5), we obtain

$$\int_0^b (\nu - q_2)|\Phi_b|^2 = \frac{1}{|W^2[\psi_0\psi_1]|^2}\{|\int_0^b \psi_0(t,\lambda)f_b(t)dt|^2[\int_r^b (\nu - q_2)|\psi_1|^2 dt + \Im m_1]+$$

$$|\int_0^b \psi_1(t,\lambda)f_b(t)dt|^2[\int_0^r (\nu - q_2)|\psi_0|^2 dt - \Im m_0]\} + \int_0^b \Im(\Phi_b\bar{f}_b)$$

It is straightforward to show that we may use inequality (5) of [44] to obtain

$$\int_r^b (\nu - q_2)|\psi_1|^2 < -\Im m_1$$

and

$$\int_0^r (\nu - q_2)|\psi_0|^2 < \Im m_0$$

Thus by the Cauchy-Schwartz inequality we have

$$\int_0^b (\nu - q_2)|\Phi_b|^2 \leq \int_0^b \Im(\Phi_b\bar{f}) \leq \int_0^b |\Phi_b\bar{f}| \leq (\int_0^b |\Phi_b|^2 \int_0^b |f|^2)^{\frac{1}{2}} \tag{5.2.6}$$

i.e.

$$\int_0^b (\nu - q_2)|\Phi_b|^2 \leq (\int_0^b |\Phi_b|^2 \int_0^b |f|^2)^{\frac{1}{2}}$$

or

$$\int_r^b (\nu - q_2)|\Phi_b|^2 \leq (\int_0^b |\Phi_b|^2 \int_0^b |f|^2)^{\frac{1}{2}} - \int_0^r (\nu - q_2)|\Phi_b|^2$$

By the continuity of $q(x)$ on $[0,r]$, there exists a positive constant K' such that $|\nu - q_2| < K'$. Hence, using also (5.2.1) we have

$$\frac{1}{2}(\nu - L)\int_r^b |\Phi_b|^2 \leq (\int_0^b |\Phi_b|^2 \int_0^b |f|^2)^{\frac{1}{2}} + K'(\int_0^r |\Phi_b|^2 \int_0^b |\Phi_b|^2)^{\frac{1}{2}}$$

On the other hand since $\nu > L$, we can write

$$\frac{1}{2}(\nu - L)\int_0^r |\Phi_b|^2 \leq \frac{1}{2}(\nu - L)(\int_0^r |\Phi_b|^2)^{\frac{1}{2}}(\int_0^b |\Phi_b|^2)^{\frac{1}{2}}$$

Hence, we may add the last two inequalities to obtain

$$\frac{1}{2}(\nu-L)\int_0^b |\Phi_b|^2 \leq (\int_0^b |\Phi_b|^2)^{\frac{1}{2}}(\int_r^b |f|^2)^{\frac{1}{2}}+(K'+\frac{1}{2}(\nu-L))(\int_0^r |\Phi_b|^2)^{\frac{1}{2}}(\int_0^b |\Phi_b|^2)^{\frac{1}{2}}$$

and dividing both sides of the above inequality by $\frac{1}{2}(\nu - L)(\int_0^b |\Phi_b|^2)^{\frac{1}{2}}$, gives

$$(\int_0^b |\Phi_b|^2)^{\frac{1}{2}} \leq \frac{2}{\nu - L}(\int_0^b |f|^2)^{\frac{1}{2}} + (\frac{2K'}{\nu - L} + 1)(\int_0^r |\Phi_b|^2)^{\frac{1}{2}}$$

But from (5.2.2), there exists a constant K'' such that

$$(\int_0^r |\Phi_b|^2)^{\frac{1}{2}} \leq K''(\int_0^r |f|^2)^{\frac{1}{2}} \leq K''(\int_0^b |f|^2)^{\frac{1}{2}}$$

since the functions $\psi_1(x,\lambda)$ and $f(x)$ are in $L^2[0,\infty)$, so using the above inequality in the previous one gives

$$(\int_0^b |\Phi_b|^2)^{\frac{1}{2}} \leq K(\int_0^b |f|^2)^{\frac{1}{2}} \tag{5.2.7}$$

where $K = \frac{2(K'K''+1)}{\nu-L} + K''$ is not dependent on f.

On the other hand

$$W[\psi_1, \psi_0](x) = k_1(\lambda)k_2(\lambda)W[\psi, \phi](x) = k_1(\lambda)k_2(\lambda)$$

so we have

$$\begin{aligned}\Phi(x,\lambda;f_b) &= \frac{1}{k_1(\lambda)k_2(\lambda)}[k_2(\lambda)\psi(x,\lambda)\int_0^x k_1(\lambda)\phi(t,\lambda)f_b(t)dt \\ &+ k_1(\lambda)\phi(x,\lambda)\int_x^b k_2(\lambda)\psi(t,\lambda)f_b(t)dt]\end{aligned}$$

or

$$\Phi_b = \Phi(x,\lambda;f_b) = \psi(x,\lambda)\int_0^x \phi(t,\lambda)f_b(t)dt + \phi(x,\lambda)\int_x^b \psi(t,\lambda)f_b(t)dt$$

Now let $b \to \infty$. Then $\Phi_b \to \Phi$ and by Fatou's theorem we conclude from (5.2.7) that

$$\int_0^\infty |\Phi|^2 \leq K^2 \int_0^\infty |f|^2.$$

so that

$$\|\Phi\| \leq K\|f\|$$

as required. If $\nu < L$ the proof is similar to the case $\nu > L$.

Theorem 5.2.1 *Consider the differential equation $\tau f = \lambda f$ generated by the non-selfadjoint differential expression τ on $[0, \infty)$ and let λ' be a complex parameter such that $\Im\lambda' \neq L$. Then λ' is in the resolvent set $\rho(T_\alpha)$ of T_α if and only if the corresponding m-function, $m_\alpha(\lambda)$, is regular at λ' and the resolvent operator $R_{\lambda'}(T_\alpha)$ is given by*

$$\Phi(x, \lambda'; f) = R_{\lambda'}(T_\alpha)(f)(x) = \int_0^\infty G(x, t, \lambda')f(t)dt \qquad (5.2.8)$$

where

$$G(x, t, \lambda') = \begin{cases} \psi(x, \lambda')\phi(t, \lambda') & \text{if } 0 \leq t < x < \infty \\ \psi(t, \lambda')\phi(x, \lambda') & \text{if } 0 \leq x < t < \infty \end{cases}$$

for all $f \in L^2[0, \infty)$

Proof: Suppose that $\lambda' = \mu' + i\nu'$ is a fixed point in $\rho(T_\alpha)$, where $\nu' > L$. Then there exists an L^2-solution of the equation

$$\tau f = \lambda' f \qquad (5.2.9)$$

and the corresponding m-function $m_\alpha(\lambda)$ in case I is meromorphic in the region $\nu' > L$ by Corollary 4.5.1. If $m_\alpha(\lambda)$ is regular at λ' then the solution $\psi(x, \lambda')$ is square integrable by Corollary 4.5.2, whereas if $m_\alpha(\lambda)$ has a singularity (a pole) at λ' the solution $\phi(x, \lambda')$ is square integrable, as will now be shown.

First suppose $\alpha \neq 0$ and let Θ and χ be two linearly independent solutions of (5.2.9) satisfying

$$\Theta(0, \lambda) = -1, \quad \Theta'(0, \lambda) = 0$$

$$\chi(0, \lambda) = 0, \quad \chi'(0, \lambda) = 1$$

Then by Corollaries 4.5.1, 4.5.2, and Theorem 4.11.1 there exists a meromorphic function $M(\lambda)$ such that

$$\Psi(x, \lambda') = \Theta(x, \lambda') + M(\lambda')\chi(x, \lambda')$$

is an L^2-solution whenever λ' is a point of regularity of $M(\lambda)$. Also there is a constant $k(\lambda)$ such that for all λ which are points of regularity of $m_\alpha(\lambda)$ and $M(\lambda)$, with $\Im\lambda > L$,

$$\Theta(x,\lambda) + M(\lambda)\chi(x,\lambda) = k(\lambda)[\theta(x,\lambda) + m_\alpha(\lambda)\phi(x,\lambda)] \tag{5.2.10}$$

because changing boundary conditions does not affect the existence of square integrable solutions. Applying (5.2.10) and its derivative at $x = 0$ we obtain

$$\Theta(0,\lambda) + M(\lambda)\chi(0,\lambda) = k(\lambda)(\theta(0,\lambda) + m_\alpha(\lambda)\phi(0,\lambda))$$

$$\Theta'(0,\lambda) + M(\lambda)\chi'(0,\lambda) = k(\lambda)(\theta'(0,\lambda) + m_\alpha(\lambda)\phi'(0,\lambda))$$

which gives

$$-1 = k(\lambda)(m_\alpha(\lambda)\sin\alpha + \cos\alpha)$$

$$M(\lambda) = k(\lambda)(-m_\alpha(\lambda)\cos\alpha + \sin\alpha)$$

and hence the m-function satisfies

$$m_\alpha(\lambda) = \frac{-\sin\alpha + M(\lambda)\cos\alpha}{\cos\alpha + M(\lambda)\sin\alpha}$$

It follows that $m_\alpha(\lambda)$ has a pole at $\lambda = \lambda'$ iff $\cos\alpha + M(\lambda)\sin\alpha$ has a zero at $\lambda = \lambda'$ but when $\cos\alpha + M(\lambda')\sin\alpha = 0$ we have

$$\cos\alpha(\Theta(0,\lambda') + M(\lambda')\chi(0,\lambda')) + \sin\alpha(\Theta'(0,\lambda') + M(\lambda')\chi'(0,\lambda') = 0$$

so the L^2-solution $\Psi(x,\lambda')$ satisfies the boundary condition α at $x = 0$, and hence $\phi(x,\lambda')$ is a scalar multiple of $\Psi(x,\lambda')$. Therefore $\phi(x,\lambda') \in L^2[0,\infty)$, so that λ' is an eigenvalue and $\lambda' \in \sigma(T_\alpha)$ whenever $m_\alpha(\lambda)$ has a pole at $\lambda = \lambda'$.

If $\alpha = 0$ the argument is similar; however, it is now necessary to choose the basis $\{\Theta, \chi\}$ so that

$$\chi(0,\lambda)\cos 0 + \chi'(0,\lambda)\sin 0 \neq 0$$

i.e. so that $\chi(x,\lambda)$ does not satisfy the boundary condition $\alpha = 0$ at $x = 0$.

Now suppose that $m_\alpha(\lambda)$ is regular at λ', where $\nu' > L$. Since $m_\alpha(\lambda)$ is meromorphic [30], λ' is not a pole of $m_\alpha(\lambda)$ and we will show that $\lambda' \in \rho(T_\alpha)$. To achieve this we first note that Φ is a bounded operator defined on $\mathcal{H}$ by Lemma 5.2.1, so that $\Phi(x,\lambda;f) \in L^2[0,\infty)$ wherever $f \in L^2[0,\infty)$. To complete the proof that $\Phi(x,\lambda';f) \in D_\alpha$ the domain of T_α we can show that Φ satisfies the boundary condition

$$\Phi(0,\lambda';f)\cos\alpha + \Phi'(0,\lambda';f)\sin\alpha = 0 \tag{5.2.11}$$

For since

$$\Phi(0,\lambda';f) = \phi(0,\lambda')\int_0^\infty \psi(t,\lambda')f(t)dt = \sin\alpha\int_0^\infty \psi(t,\lambda')f(t)dt$$

$$\Phi'(0,\lambda';f) = \phi'(0,\lambda')\int_0^\infty \psi(t,\lambda')f(t)dt = -\cos\alpha\int_0^\infty \psi(t,\lambda')f(t)dt$$

(5.2.11) follows immediately.

We also prove that $\Phi(x,\lambda';\ \cdot\ )$ is the inverse operator for the operator $T_\alpha - \lambda' I$. Obviously we have

$$\Phi(x,\lambda';(T_\alpha-\lambda' I)f) = \Phi(x,\lambda';(-f''+qf-\lambda' f)) =$$

$$\psi(x,\lambda')\int_0^x \phi(t,\lambda')(-f''+qf-\lambda' f)dt + \phi(x,\lambda')\int_x^\infty \psi(t,\lambda')(-f''+qf-\lambda' f)dt$$

Integrating by parts twice we obtain

$$\Phi(x,\lambda';(-f''+qf-\lambda' f)) = \psi(x,\lambda')\int_0^x (-\phi''+q\phi-\lambda'\phi)f +$$
$$\phi(x,\lambda')\int_x^\infty (-\psi''+q\psi-\lambda'\psi)f + \psi(x,\lambda')[(-f'\phi+f\phi')]_0^x +$$
$$\phi(x,\lambda')[(-f'\psi+f\psi')]_x^\infty$$

The last two terms are zero so

$$\begin{aligned}\Phi(x,\lambda';(T_\alpha-\lambda')f) &= \psi(x,\lambda')(f(x)\phi'(x)-f'(x)\phi(x)+f(0)\phi'(0)-f'(0)\phi(0)) \\ &+ \phi(x,\lambda')(W_\infty[f,\psi]-f(x)\psi'(x)+f'(x)\psi(x))\end{aligned}$$

Since $f \in D_\alpha$ then $W_0[f,\phi] = 0$, so that

$$\Phi(x,\lambda';(T_\alpha-\lambda' I)f) = f(x)W_x[\psi,\phi] + \phi(x,\lambda')W_\infty[f,\psi] = f(x)$$

since we are in case I and $D_\alpha = \{f \in D : W_\infty[f\psi] = 0\}$ ([44] p.267). On the other hand

$$\begin{aligned}
(T_\alpha - \lambda' I)\Phi &= -\Phi''(x,\lambda';f) + (q-\lambda')\Phi(x,\lambda';f) \\
&= -\psi''(x,\lambda')\int_0^x \phi(t,\lambda')f(t)dt - \phi''(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt \\
&+ f(x)W_x[\phi,\psi] + (q-\lambda')\Phi(x,\lambda';f) \\
&= f(x) + [-\psi''(x,\lambda') + (q-\lambda')\psi(x,\lambda')]\int_0^x \phi(t,\lambda')f(t)dt \\
&+ [-\phi''(x,\lambda') + (q-\lambda')\phi(x,\lambda')]\int_x^\infty \psi(t,\lambda')f(t)dt \\
&= f(x)
\end{aligned}$$

for all $f \in L^2[0,\infty)$. Hence we can write for each λ with the property $\Im\lambda > L$

$$(T_\alpha - \lambda I)^{-1} = \Phi(\,\cdot,\ \lambda;\ \cdot\,)$$

The operator $\Phi(\,\cdot,\ \lambda;\ \cdot\,)$ therefore has all of the properties that make it identically equal to the resolvent operator $R_\lambda(T_\alpha)$ for each λ with $\Im\lambda > L$, and we conclude that $\lambda' \in \rho(T_\alpha)$.

If $\Im\lambda' < L$ proof is the same as when $\Im\lambda' > L$. The special case of $\Im\lambda' = L$ is considered in the next theorem.

Theorem 5.2.2 *Let $\nu' = L$ and $\lambda' \in \rho(T_\alpha)$. Then $m_\alpha(\lambda)$ is regular at λ'.*

Proof: Let $\nu' = L$ and $\lambda' \in \rho(T_\alpha)$, Since $\rho(T_\alpha)$ is an open set in the complex plane, there is a disk $D_\delta(\lambda') = \{z : |z-\lambda'| < \delta\}$ around λ' such that $D_\delta(\lambda') \subseteq \rho(T_\alpha)$. Therefore, noting that there is no loss of generality if we take $L \neq 1$, by Corollary (4.6.1) we have

$$m_\alpha(\lambda) - m_\alpha(i) = (\lambda - i)\int_0^\infty \psi(x,\lambda)\psi(x,i)dx \tag{5.2.12}$$

for $\lambda \in D_\delta, \Im\lambda \neq L$. From the properties of Φ as a function we see that

$$(\tau - \lambda)(i-\lambda)\Phi(x,\lambda;\psi(t,i)) = (i-\lambda)\psi(x,i)$$

and since

$$(\tau - \lambda)\psi(x,i) = (i-\lambda)\psi(x,i)$$

it follows that $(i-\lambda)\Phi(x,\lambda;\psi(t,i))$ and $\psi(x,i)$ are solutions of the non-homogeneous equation $(\tau-\lambda)f = (i-\lambda)\psi(x,i)$, so their difference is a solution of the homogeneous equation $\tau f = \lambda f$. Hence

$$(i-\lambda)\Phi(x,\lambda;\psi(t,i)) - \psi(x,i) = c_1\phi(x,\lambda) + c_2\psi(x,\lambda) \tag{5.2.13}$$

where c_1 and c_2 are constants which can be determined, since if we set $x = 0$ in (5.2.13) and its derivative, and use (5.2.12), we obtain $c_1 = 0$ and $c_2 = -1$. We have therefore

$$\psi(x,\lambda) = \psi(x,i) + (\lambda - i)\Phi(x,\lambda;\psi(t,i))$$

Also we can use Theorem 5.2.1 and write $\Phi(x,\lambda;\psi(t,i)) = R_\lambda(T_\alpha)\psi(t,i)(x)$ for $\lambda \in D_\delta$, $\Im\lambda \neq L$ Then substituting for $\psi(x,\lambda)$ in (5.2.12) gives

$$m_\alpha(\lambda) = m_\alpha(i) + (\lambda - i)\int_0^\infty \psi(x,i)^2 dx + (\lambda-i)^2(R_\lambda(T_\alpha)\psi(t,i)(x), \bar{\psi}(x,i))$$

But this equation implies, since the function $\lambda \longmapsto (g, R_\lambda(T_\alpha)f)$ is an analytic function from $\rho(T_\alpha)$ to $\mathbb{C}$ for given fixed functions f and g in $\mathcal{H}$ [49] p.101., that $m_\alpha(\lambda)$ is analytic in the neighbourhood D_δ of λ', so that by analytic continuation m_α is regular on the resolvent set at λ'. We believe that the converse of Theorem 5.2.2 is also true, but have not been able to prove this. However, the following result provides a partial converse.

Theorem 5.2.3 *Suppose that q is eventually real and that $\nu' = L = 0$. Then if $m_\alpha(\lambda)$ is regular at λ', $\lambda' \in \rho(T_\alpha)$ and the resolvent operator is given by*

$$\Phi(x,\lambda';f) = \psi(x,\lambda')\int_0^x \phi(t,\lambda')f(t)dt + \phi(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt$$

Proof: Suppose firstly that $q_2(x) = 0$ for all $x > a$ for some real $a > 0$. Then we define two operators T_0 and T_∞ on $[0,a]$ and on $[a,\infty)$ respectively, with domains of definition D_0 and D_∞ respectively, where $f \in D_0$ if and only if

i)$f \in L^2[0,a]$,

ii)$\tau f \in L^2[0,a]$,

iii)f and f' are absolutely continuous on $[0,a]$, iv)$f(0)\cos\alpha + f'(0)\sin\alpha = 0,\ \alpha \in \mathbb{C}$,

v)$f(a)\cos\beta + f'(a)\sin\beta = 0,\ \beta \in [0,\pi)$,

and $f \in D_\infty$ if and only if

i)$f \in L^2[a,\infty)$,

ii)$\tau f \in L^2[a,\infty)$,

iii)f and f' are absolutely continuous on $[a, X]$, for all $X > a$,

iv)$f(a)\cos\beta + f'(a)\sin\beta = 0$.

Then T_0 and T_∞ are defined by $T_0 f = \tau f$ for $f \in D_0$ and $T_\infty f = \tau f$ for $f \in D_\infty$.

Let solutions $\tilde{\theta}$ and $\tilde{\phi}$ of $\tau f = \lambda f$ be defined in terms of the boundary condition $\beta \in [0,\pi)$ at $x = a$ in the usual way, viz.

$$\tilde{\phi}(a,\lambda) = \sin\beta \quad \tilde{\phi}'(a,\lambda) = -\cos\beta \tag{5.2.14}$$

$$\tilde{\theta}(a,\lambda) = \cos\beta \quad \tilde{\theta}'(a,\lambda) = \sin\beta \tag{5.2.15}$$

There then exist m-functions m_0 and m_∞ such that

$$\psi_0(x,\lambda) = \tilde{\theta}(x,\lambda) + m_0(\lambda)\tilde{\phi}(x,\lambda)$$

satisfies the boundary condition α at $x = 0$ and

$$\tilde{\psi}(x,\lambda) = \tilde{\theta}(x,\lambda) + m_\infty(\lambda)\tilde{\phi}(x,\lambda) \in L^2[a,\infty)$$

for each λ such that $\Im\lambda = \nu \neq 0$, apart from isolated poles of $m_0(\lambda)$, where $\tilde{\phi}(x,\lambda)$ satisfies the boundary condition at 0. Note that T_∞ is selfadjoint, so that $m_\infty(\lambda)$ has no poles off the real axis.

Since τ is in case I on the interval $[0,\infty)$ there exists a scalar $k(\lambda)$ depending on λ such that

$$\psi(x,\lambda) = k(\lambda)\tilde{\psi}(x,\lambda) \tag{5.2.16}$$

when $\nu \neq 0$, unless λ is a pole of $m_\alpha(\lambda)$, in which case $\tilde{\psi}(x,\lambda) = c(\lambda)\phi(x,\lambda)$ for some scalar $c(\lambda)$. From (5.2.16)

$$m_\infty(\lambda) = \frac{[\theta(a,\lambda) + m_\alpha\phi(a,\lambda)] - k(\lambda)\tilde{\theta}(a,\lambda)}{k(\lambda)\tilde{\phi}(a,\lambda)}$$

Now let $\lambda' \in \mathbb{R}$ and suppose that m_α is regular at λ'. There is no loss of generality if we assume $\beta \neq 0$, in which case, we see that if m_α is analytic in a neighbourhood $\mathcal{N}(\lambda')$ of λ' then $m_\infty(\lambda)$ is also analytic in a neighbourhood $\mathcal{N}'(\lambda')$ of λ'. Therefore by [4] § 5, Thm.(i) we deduce that since T_∞ is a selfadjoint operator $\tilde{\psi}(x,\lambda') \in L^2[a,\infty)$ and hence λ' belongs to the resolvent set $\rho(T_\infty)$ of T_∞ and the resolvent operator $R_{\lambda'}(T_\infty)$ is given by

$$R_{\lambda'}(T_\infty) = \tilde{\Phi}(x,\lambda';f) = \tilde{\psi}(x,\lambda')\int_a^x \tilde{\phi}(t,\lambda')f(t)dt + \tilde{\phi}(x,\lambda')\int_x^\infty \tilde{\psi}(t,\lambda')f(t)dt.$$

We now deduce that $\psi(x,\lambda') \in L^2[0,\infty)$ where $\Im\lambda' = \nu' = L = 0$. Using (5.2.16) and the fact that the solutions $\phi, \theta, \tilde{\phi}, \tilde{\theta}$ and m-functions m_∞, m_α are analytic in a neighbourhood $\mathcal{N}''(\lambda')$ of λ', we have by continuity and the uniqueness of solutions

$$\psi(x,\lambda') = \lim_{\lambda\to\lambda'} k(\lambda)\tilde{\psi}(x,\lambda) = k(\lambda')\tilde{\psi}(x,\lambda') \in L^2[a,\infty) \tag{5.2.17}$$

for all $x \geq a$, and hence $\psi(x,\lambda') \in L^2[0,\infty)$.

Now define the function $\Phi(x,\lambda';f)$ by

$$\Phi(x,\lambda';f) = \psi(x,\lambda')\int_0^x \phi(t,\lambda')f(t)dt + \phi(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt \tag{5.2.18}$$

for all $f \in L^2[0,\infty)$. Clearly Φ is well defined, since $\psi(x,\lambda'), f \in L^2[0,\infty)$ and Φ satisfies

$$\tau\Phi = -\Phi'' + q(x)\Phi = \lambda'\Phi + f$$

Also we have $\Phi(\,\cdot,\ \lambda'; T_\alpha f - \lambda' f) = f$ and $(T_\alpha - \lambda')\Phi(\,\cdot,\ \lambda';f) = f$ and Φ satisfies the boundary condition $\Phi(0,\lambda';f)\cos\alpha + \Phi'(0,\lambda';f)\sin\alpha = 0$. Thus Φ satisfies all requirements for the resolvent operator $R_{\lambda'}(T_\alpha)$ from $L^2[0,\infty)$ onto the range of $T_\alpha - \lambda'$, except that we still have to show that $\Phi(x,\lambda';f) \in L^2[0,\infty)$. To prove that for all f in $L^2[0,\infty)$ we have $\Phi \in L^2[0,\infty)$, and Φ is a bounded linear operator, we proceed as follows. Evidently

$$\|\Phi\|^2 = \int_0^\infty |\Phi|^2 = \int_0^a |\Phi|^2 + \int_a^\infty |\Phi|^2$$

On the other hand from (5.2.16) we have

$$\tilde{\psi}(x,\lambda') = \frac{1}{k(\lambda')}\psi(x,\lambda') \tag{5.2.19}$$

We may choose $\delta \in [0,\pi)$ such that the solutions $\phi(x,\lambda')$ and $\xi(x,\lambda')$ of the equation $\tau f = \lambda' f$, where $\xi(a,\lambda') = \cos\delta$, $\xi'(a,\lambda') = \sin\delta$ form a basis and the corresponding m-function which satisfies

$$p\phi(x,\lambda') + m_\infty(\lambda,\delta)\xi(x,\lambda') \in L^2[a,\infty)$$

where

$$W[p\phi(x,\lambda'),\xi(x,\lambda')] = 1$$

for some scalar p, is regular at λ'. There then exists scalars c and e such that

$$\tilde{\phi}(x,\lambda') = c\phi(x,\lambda') + e\xi(x,\lambda') \tag{5.2.20}$$

Substituting (5.2.19) and (5.2.20) into (5.2.18) we have

$$\tilde{\Phi}(x,\lambda';f) = \tilde{\Phi}(x,\lambda';f,\beta) = \frac{1}{k(\lambda')}\psi(x,\lambda')\int_a^x [c\phi(t,\lambda') + e\xi(t,\lambda')]f(t)dt +$$
$$[c\phi(x,\lambda') + e\xi(x,\lambda')]\int_x^\infty \frac{1}{k(\lambda')}\psi(t,\lambda')f(t)dt =$$
$$\frac{c}{k(\lambda')}[\psi(x,\lambda')\int_a^x \phi(t,\lambda')f(t)dt + \phi(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt] +$$
$$\frac{e}{k(\lambda')}[\psi(x,\lambda')\int_a^x \xi(t,\lambda')f(t)dt + \xi(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt]$$

Since the last term on the right hand side is $\frac{e}{k(\lambda')}\tilde{\Phi}(x,\lambda';f,\delta)$ we obtain by self-adjoint theory that

$$\int_a^\infty |\psi(x,\lambda')\int_a^x \phi(t,\lambda')f(t)dt \;+\; \phi(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt|^2dx$$
$$\leq\; K^2\int_a^\infty |f(t)|^2dt < \infty \tag{5.2.21}$$

for some positive constant K which is independent of f. But from (5.2.18)

$$||\Phi(x,\lambda';f)||^2 = \int_0^a |\psi(x,\lambda')\int_0^x \phi(t,\lambda')f(t)dt + \phi(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt|^2dx +$$
$$\int_a^\infty |\psi(x,\lambda')\int_0^x \phi(t,\lambda')f(t)dt + \phi(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt|^2dx$$

and

$$\int_a^\infty |\psi(x,\lambda')\int_0^x \phi(t,\lambda')f(t)dt + \phi(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt|^2dx =$$

$$\int_a^\infty |\psi(x,\lambda')(\int_0^a \phi(t,\lambda')f(t)dt + \int_a^x \phi(t,\lambda')f(t)dt) +$$
$$\phi(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt|^2 dx$$

By Minkowski's inequality

$$\int_a^\infty |\psi(x,\lambda')(\int_0^a \phi(t,\lambda')f(t)dt + \int_a^x \phi(t,\lambda')f(t)dt) +$$
$$\phi(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt|^2 dx \leq$$
$$[(\int_a^\infty |\psi(x,\lambda')\int_0^a \phi(t,\lambda')f(t)dt|^2)^{\frac{1}{2}} + (\int_a^\infty |\psi(x,\lambda')\int_a^x \phi(t,\lambda')f(t)dt +$$
$$\phi(x,\lambda')\int_x^\infty \psi(t,\lambda')f(t)dt|^2)^{\frac{1}{2}}]^2$$

The first square root in the above inequality is less than $K'\|f\|$ for some positive constant K' and the second one is less than $K\|f\|$ by (5.2.21). Hence the left hand side of the above inequality is finite; moreover it is easy to show that $\int_0^a |\Phi|^2 \leq C^2\|f\|^2$, and joining these results together we have $\|\Phi\| \leq C\|f\|$ for some $C \in \mathbb{R}$, so $\Phi \in L^2[0,\infty)$ also Φ is a linear operator from $(T_\alpha - \lambda' I)\mathcal{H}$ to $\mathcal{H}$ since

$$\Phi(x,\lambda';(cf + c'g)) = c\Phi(x,\lambda';f) + c'\Phi(x,\lambda';g)$$

Therefore Φ is a bounded linear operator and hence the theorem is proved.

5.3 The m-function and the continuous spectrum.

In this section we give a brief outline of the Chaudhuri-Everitt result on the continuous part of the spectrum of T_α which is characterised by properties of the m-function, and under certain conditions we prove the extension of the Chaudhuri-Everitt result to the non-selfadjoint case.

As in the previous section we consider the differential operator T_α where $\alpha \in \mathbb{C}$ and $\lim_{x\to\infty} q_2 = L$ on the interval $[0,\infty)$.

Definition 5.3.1 Let T be an operator defined on a Hilbert space. The complex number λ is in the continuous spectrum σ_c of T iff $(T-\lambda I)^{-1}$ exists and is a densely defined unbounded operator.

To see that the weather continuous part of the spectrum of T can exist in the non-selfadjoint case, consider the standard singular differential expression of Sturm-Liouville type $\tau \equiv -\frac{d^2}{dx^2}+q(x)$ on the interval $[0,\infty)$ with $q \equiv 0$. This generates the operator T_α as defined in Definition 4.2.1, where α is a real number and T_α is a selfadjoint operator. However we obtain a non-selfadjoint operator if α is a complex number, since for each $f, g \in D_\alpha$ we have

$$\begin{aligned}(T_\alpha f,g)-(f,T_\alpha g) &= \int_0^\infty (-f''(x)\bar{g}(x)+f(x)\bar{g}''(x))dx \\ &= \lim_{b\to\infty}(f\bar{g}'-f'\bar{g})(b)-(f\bar{g}'-f'\bar{g})(0) \\ &= 2if'(0)\bar{g}'(0)\Im(\tan\alpha) \\ &\neq 0\end{aligned}$$

since if $\Im\alpha \neq 0$ then $\Im\tan\alpha \neq 0$. Now take $\alpha = 0$; then by [10] we obtain that $(0,\infty)$ is the continuous part of the spectrum of T_0 (in fact, absolutely continuous) and also we see that T_α is a non-selfadjoint operator if we change the boundary condition from 0 to $\alpha \in \mathbf{C}\setminus\mathbf{R}$. Since changing the boundary condition is equivalent to a one dimentional perturbation [53], the continuous part of the spectrum of T_α exists and is the same as the absolutely continuous spectrum of T_0 i.e. $\sigma_c(T_\alpha)=(0,\infty)$ [15]

We now consider the differential operator T_α generated by the differential expression $\tau \equiv \frac{d^2}{dx^2}+q(x)$ on the interval $[0,\infty)$ where q is a real function and in the limit-point case at infinity. Let D_α be a linear manifold of the Hilbert space $\mathcal{H}$ consisting of all real functions f defined on $[0,\infty)$ such that:

i)$f \in \mathcal{H}$

ii)f and f' are absolutely continuous on the finite interval $[0,X]$

iii)$\cos\alpha f(0)+\sin\alpha f'(0)=0, \quad \alpha\in[0,\pi)$.

and define T_α by $T_\alpha f=\tau f$ for all $f\in D_\alpha$. We then have the following theorem

Theorem 5.3.1 (Chaudhuri-Everitt) *i) The number r is in the continuous spectrum of T_α iff the corresponding m-function $m_\alpha(\lambda)$ is not regular at r and*

$$\lim_{\nu \to 0} \nu m_\alpha(r + i\nu) = 0;$$

ii) If for all numbers γ and δ

$$\gamma\theta(x, r) + \delta\phi(x, r)$$

is not an L^2-solution then r belongs to the continuous spectrum of T_α.

Proof: see [4]

Since the following example shows Theorem 5.3.1 does not work generally for non-selfadjoint operators.

Example 5.3.1 Consider the simplest case of a boundary value problem

$$-y'' + q(x)y = \lambda y$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0$$

where $q(x) = 0$, $\forall x \in [0, \infty)$, and a fundamental set of solutions $\{\phi, \theta\}$ satisfying the boundary conditions

$$\phi(0, \lambda) = \sin\alpha, \quad \phi'(0, \lambda) = -\cos\alpha$$

$$\theta(0, \lambda) = \cos\alpha, \quad \theta'(0, \lambda) = \sin\alpha$$

where $\alpha \in \mathbb{C}$. Then

$$\theta(x, \lambda) = \cos\alpha\cos(x\sqrt{\lambda}) + \lambda^{-\frac{1}{2}}\sin\alpha\sin(x\sqrt{\lambda})$$

$$\phi(x, \lambda) = \sin\alpha\cos(x\sqrt{\lambda}) + -\lambda^{-\frac{1}{2}}\cos\alpha\sin(x\sqrt{\lambda})$$

and we obtain the m-function $m_\alpha(\lambda)$ explicitly as

$$m_\alpha(\lambda) = \frac{\sin\alpha - i\sqrt{\lambda}\cos\alpha}{\cos\alpha + i\sqrt{\lambda}\sin\alpha}$$

where $\Im\sqrt{\lambda} > 0$. We now show that m is regular on the whole complex plane except on the set

$$\{\lambda : \Im\lambda = 0,\ \Re\lambda > 0\}$$

and at poles of the m-function which satisfy the equation $i\cot\alpha = \sqrt{\lambda}$. To show that this is true, note that $m_0(\lambda) = -i\sqrt{\lambda}$ is the m-function for the selfadjoint problem with $q(x) = 0, \alpha = 0$, and is analytic on

$$S = \mathbb{C}\backslash\{\lambda : \Im\lambda = 0, \Re\lambda \geq 0\}$$

It then follows from the expression for $m_\alpha(\lambda)$ that for $\alpha \in \mathbb{C}\backslash\{0\}$, $m_\alpha(\lambda)$ is regular on S, apart from isolated poles at the zeros of $\cos\alpha + i\sqrt{\lambda}\sin\alpha$.

Let $\alpha = \alpha_1 + i\alpha_2$. Using complex trigonometry we have

$$\lambda = -\cot^2\alpha = \frac{\sin 2\alpha_1 - i\sinh 2\alpha_2}{2|\sin\alpha|^2}$$

where $\alpha_2 \neq 0$ and the condition $\Im\sqrt{\lambda} > 0$ is equivalent to

$$n\pi < \alpha_1 < (n + \frac{1}{2})\pi,\ n \in \mathbb{Z}.$$

Taking $\alpha_1 = 0, \alpha_2 = 1$ then m-function is

$$m(\lambda) = \frac{i\sinh 1 - i\sqrt{\lambda}\cosh 1}{\cosh 1 - \sqrt{\lambda}\sinh 1}$$

or

$$m(\lambda) = i\frac{1 - \sqrt{\lambda}\coth 1}{\coth 1 - \sqrt{\lambda}}$$

Note that the only pole of $m(\lambda)$ is at $\lambda = \lambda_0 = \coth^2 1$, so that $m(\lambda)$ is regular on S. To investigate whether Theorem 5.3.1(i) remains true in general in the non-selfadjoint case, we consider the behaviour of $\nu m(\mu' + i\nu)$ as $\nu \to 0$ for $\mu' = \lambda_0$. We have:

$$\lim_{\nu\to 0}\nu m(\mu' + i\nu) = \lim_{\nu\to 0}\frac{\nu[1 - \sqrt{\lambda_0 + i\nu}\coth 1]}{\coth 1 - \sqrt{\lambda_0 + i\nu}}$$

Using l'Hôpital rule we have

$$\lim_{\nu\to 0}\nu m(\mu' + i\nu) = \lim_{\nu\to 0} 2i(1 - \coth^2 1 - i\nu\coth 1)\coth 1 = 2i(1 - \coth^2 1)\coth 1 \neq 0$$

Since there is no $L^2[0,\infty)$ solution of the equation $-y'' = \lambda y$ for any $\lambda \geq 0$, S lies in the essential spectrum by Theorem 4.11.4 and λ_0 is not an eigenvalue. Hence λ_0 is a point of the continuous spectrum, but Theorem 5.3.1(i) is not satisfied for $\mu' = \lambda_0$. This shows that Theorem 5.3.1(i) is not generally true in the non-selfadjoint case.

5.4 Point spectrum and the m-functions.

Consider the differential expression τ generating the differential equation (4.3.9) on the interval $[0,\infty)$, in which the function q is limit point at infinity i.e. in case I in Sims' sense.

Let T be an unbounded operator defined on a Hilbert space $\mathcal{H}$. Then the complex number λ is said to be in the point spectrum (an eigenvalue) iff the operator $T-\lambda I$ is not one-to-one. This section is devoted to study of the relationship between the m-function and the point spectrum in terms of the square integrable solutions of the differential equation (4.3.9). Consider the formal adjoint $\bar{\tau}$ of the differential expression τ, where the formal adjoint operator $\overline{T}_\alpha$ of the non-selfadjoint operator T_α generated by $\bar{\tau}$ has domain of definition

$$\bar{D}_\alpha = \{f \in L^2[0,\infty) : \bar{f} \in D_\alpha\},$$

and let $\psi(x,\lambda)$ be the solution of the differential equation (4.3.9) satisfying (4.5.25) at point of regularity of $m_\alpha(\lambda)$. Then $\bar{\psi}(x,\lambda)$ is a solution of the formal adjoint differential equation

$$\bar{\tau} f = -f'' + \bar{q} f = \lambda f \tag{5.4.22}$$

We can see that in contrast to the selfadjoint context, in the non-selfadjoint context it is not necessarily true that $\bar{\psi}(x,\lambda) = \psi(x,\bar{\lambda})$, so if m_α and $\bar{m}_\alpha$ are the corresponding m-functions then they will not necessarily satisfy $\bar{m}_\alpha(\lambda) = m_\alpha(\bar{\lambda})$.

Lemma 5.4.1 *Let T_α and $m_\alpha(\lambda)$ be given as in §5.2, μ' be a simple pole of m_α with $\Im\mu' \neq L$, and suppose* $\limsup_{X\to\infty} |\int_0^X \Im[r\phi(x,\mu')\bar{\theta}(x,\mu')]dx| < \infty$. *Then* *the functions* $\theta(x,\mu') + r\phi_\lambda(x,\mu')$ *and* $\tau(\theta + r\phi_\lambda)$ *are in* $L^2[0,\infty)$, *where* r *is the residue of* m_α *at the point* μ' *and* $\phi_\lambda(x,\mu') = \frac{\partial\phi(x,\lambda)}{\partial\lambda}|_{\lambda=\mu'}$.

Proof: If μ' is a simple pole of m_α and $\Im\mu' > L$, then μ' is an isolated singularity of m_α, and hence there is a deleted neighbourhood $\mathcal{N}_\delta(\mu')$ of μ' so that $\mathcal{N}_\delta(\mu') \subseteq \rho(T_\alpha)$, i.e. we have for some δ with $0 < \delta < \Im\mu' - L$

$$\mathcal{N}_\delta(\mu') = \{\lambda \in \mathbf{C} : 0 < |\lambda - \mu'| < \delta\} \subseteq \rho(T_\alpha)$$

If we define the function $f(x,\nu) = \psi(x,\mu'+i\nu) - \frac{r}{i\nu}\phi(x,\mu')$, where $x \in [0,\infty)$ and $0 < \nu < \delta$, we can see that for all $0 < \nu < \delta$ we get the result $f(x,\nu) \in L^2[0,\infty)$; this is because $\mu' + i\nu \in \mathcal{N}_\delta(\mu') \subseteq \rho(T_\alpha)$ and μ' is a pole of m_α, which implies that $\psi(x,\mu'+i\nu), \phi(x,\mu') \in L^2[0,\infty)$ by Corollary 4.5.2 and the proof of Theorem 5.2.1.

Also we can show that $\tau f(x,\nu) \in L^2[0,\infty)$, since we have

$$\begin{aligned}
\tau[f(x,\nu)] &= \tau[\psi(x,\mu'+i\nu)] - \frac{r}{i\nu}\tau[\phi(x,\mu')] \\
&= (\mu'+i\nu)\psi(x,\mu'+i\nu) - \mu'\frac{r}{i\nu}\phi(x,\mu') \\
&= (\mu'+i\nu)(\psi(x,\mu'+i\nu) - \frac{r}{i\nu}\phi(x,\mu')) + r\phi(x,\mu') \\
&= (\mu'+i\nu)f(x,\nu) + r\phi(x,\mu') \qquad (5.4.23)
\end{aligned}$$

Thus, since $f(x,\nu), \phi(x,\mu') \in L^2[0,\infty)$ we have $\tau[f(x,\nu)] \in L^2[0,\infty)$ for all ν, where $0 < \nu < \delta$.

Applying the modified Green's formula (4.2.8) in the interval $[0,X]$, substituting $f = g = f(x,\nu)$ in it, and using (5.4.23) we obtain

$$\begin{aligned}
\int_0^X \bar{f}\tau f - f\overline{\tau f} &= \int_0^X \{\bar{f}(x,\nu)\tau f(x,\nu) - f(x,\nu)\overline{\tau f}(x,\nu)\}dx \\
&= W[f(x,\nu),\bar{f}(x,\nu)]_0^X + 2i\int_0^X q_2(x)|f(x,\nu)|^2dx \\
&= 2i\int_0^X \{(\Im\mu'+\nu)|f(x,\nu)|^2 + \Im(r\phi(x,\mu')\bar{f}(x,\nu))\}dx
\end{aligned}$$

and so, with obvious notation,

$$\int_0^X (\Im\mu' + \nu - q_2(x))|f_\nu(x)|^2dx = \frac{1}{2i}W[f_\nu(x),\bar{f}_\nu(x)]_0^X - \int_0^X \Im(r\phi\bar{f}_\nu)dx$$

$$= \frac{1}{2i}W[f_\nu(x),\bar{f}_\nu(x)]_0^X - \int_0^X \Im(r\phi\bar{\psi})dx$$
$$+ \frac{|r|^2}{\nu}\int_0^X |\phi(x,\mu')|^2 dx \tag{5.4.24}$$

Since $f_\nu, \tau f_\nu \in L^2[0,\infty)$, we deduce that $f_\nu \in D$ in Definition 4.2.1, so

$$\lim_{X\to\infty} W[f_\nu,\bar{f}_\nu](X) = 0$$

since $\lim_{X\to\infty} W[f,\psi(x,\mu'+i\nu)] = 0$ for all $f \in D$, (cf. [44] p. 252). On the other hand

$$\begin{aligned}
W[f_\nu,\bar{f}_\nu](0) &= W[\psi,\bar{\psi}](0) + \frac{|r|^2}{\nu^2}W[\phi,\bar{\phi}](0) + 2i\Im(\frac{r}{i\nu}W[\bar{\psi},\phi])(0) \\
&= i\{2(\Im m)\cosh 2\alpha_2 - (1+|m|^2)\sinh 2\alpha_2 - \frac{|r|^2}{\nu^2}\sinh 2\alpha_2 \\
&+ \frac{2}{\nu}[r_1\cosh 2\alpha_2 + (r_2 m_1 - r_1 m_2)\sinh 2\alpha_2]\} \\
&= i(\frac{2(r_2 m_1 - r_1 m_2)}{\nu} - \frac{|r|^2}{\nu^2} - |m|^2 - 1)\sinh 2\alpha_2 \\
&+ 2i(\Im m + \frac{r_1}{\nu})\cosh 2\alpha_2
\end{aligned}$$

where $r = r_1 + ir_2$, $m = m_1 + im_2$. Since m has a simple pole at λ, we have

$$m = \frac{r}{i\nu} + a + i\nu b + O(|\nu|^2) = \frac{r_2}{\nu} + a_1 - \nu b_2 + O(|\nu|^2) + i(\frac{-r_1}{\nu} + a_2 + \nu b_1),$$

where $a = a_1 + ia_2, b = b_1 + ib_2$. Hence the coefficient of $\sinh 2\alpha_2$ in the previous equation satisfies the following equality

$$\begin{aligned}
\frac{2(r_2 m_1 - r_1 m_2)}{\nu} &- \frac{|r|^2}{\nu^2} - |m|^2 - 1 = 2\nu(a_1 b_2 - a_2 b_1) - 1 - |a|^2 - \nu^2|b|^2 \\
&+ O(|\nu|^2)(-\frac{2r_2}{\nu} - 2a_1 + 2\nu b_1 + r_2 - O(|\nu|^2))
\end{aligned}$$

as $\nu \to 0$, and so we obtain

$$W[f_\nu,\bar{f}_\nu](0) = i\{2a_2\cosh 2\alpha_2 - \sinh 2\alpha_2(1+|a|^2) + O(|\nu|)\}$$

as $\nu \to 0$. Hence

$$W[f_\nu,\bar{f}_\nu](0) = K + O(\nu),\ K \text{ independent of } \nu$$

and so $W[f_\nu,\bar{f}_\nu]_0^X$ is bounded as $X \to \infty$ for all $0 < \nu < \delta$.

Now consider the behaviour of the second and the third terms on the right hand side of (5.4.24) together. Both of these terms individually are unbounded as $\nu \to 0$. However,

$$\Im(r\phi\bar{\psi}) = \frac{1}{2i}(r\phi\bar{\psi} - \bar{r}\bar{\phi}\psi) = \frac{1}{2i}[2i|\phi|^2\Im(r\bar{m}) + 2i\Im(r\phi\bar{\theta})]$$

and since $\Im r\bar{m} = \frac{|r|^2}{\nu} + \Im(r\bar{a} - i\nu r\bar{b}) + r_2 O(|\nu|^2)$ we have

$$-\Im(r\phi\bar{\psi}) + \frac{|r|^2}{\nu}|\phi|^2 = -\Im(r\bar{a} - i\nu r\bar{b})|\phi|^2 - \Im(r\phi\bar{\theta}) - r_2 O(|\nu|^2)|\phi|^2$$

Hence

$$\lim_{\nu\to 0}\{\int_0^X -\Im(r\phi\bar{\psi})dx + \frac{|r|^2}{\nu}\int_0^X |\phi(x,\mu')|^2 dx\} =$$

$$\lim_{\nu\to 0}\int_0^X [(\Im(-r\bar{a} + i\nu r\bar{b}) - r_2 O(|\nu|^2))|\phi|^2 - \Im(r\phi\bar{\theta})]dx =$$

$$\int_0^X -[\Im(r\bar{a})|\phi|^2 + \Im(r\phi\bar{\theta})]dx$$

But by assumption $\limsup_{X\to\infty} |\int_0^X \Im[r\phi(x,\mu')\bar{\theta}(x,\mu')]dx| < \infty$, and $\phi(x,\mu') \in L^2[0,\infty)$, so

$$|\int_0^X [\Im(r\bar{a})|\phi|^2 + \Im(r\phi\bar{\theta})]dx| < k$$

for all X where k is independent of X and ν. It now follows from (5.4.24) that

$$|\int_0^X (\Im\mu' + \nu - q_2)|f_\nu|^2 dx| \le k + K + O(\nu) \tag{5.4.25}$$

for all X, where the constants K and k are independent of ν. Now use the continuity of q_2 to find a real $s > 0$ such that $\Im\mu' - q_2 \ge K' > 0$ for all $x \ge s$ so that, by (5.4.25) we have using Minkowski's inequality

$$\lim_{\nu\to 0}\int_s^X (\Im\mu' + \nu - q_2)|f_\nu|^2 dx - \lim_{\nu\to 0}|\int_0^s (\Im\mu' + \nu - q_2)|f_\nu|^2 dx| \le K + k,$$

for all $X > s$. Hence

$$0 < \lim_{\nu\to 0}(K' + \nu)\int_s^X |f_\nu|^2 \le \lim_{\nu\to 0}\int_s^X (\Im\mu' + \nu - q_2)|f_\nu|^2 dx =$$

$$K + k + |\int_0^s \lim_{\nu\to 0}(\Im\mu' + \nu - q_2)|f_\nu|^2 dx| = K + k + |\int_0^s (\Im\mu' - q_2)|f_0|^2 dx| < K''$$

which gives

$$\lim_{\nu\to 0}\int_s^X |f_\nu|^2 dx \le K'''$$

where K''' is independent of ν, and so

$$\lim_{\nu\to 0}\int_s^X |f_\nu|^2 dx \le K''', \text{ for all } X > s$$

we deduce that

$$\int_s^X |f_0|^2 dx \le K'''$$

where $f_0 = \lim_{\nu\to 0} f_\nu$, since $f(x,\nu)$ satisfies

$$f(x,\nu) = \theta(x,\mu'+i\nu) + m(\lambda)\phi(x,\mu'+i\nu) - \frac{r}{i\nu}\phi(x,\mu') =$$

$$\theta(x,\mu'+i\nu) + r[\frac{\phi(x,\mu'+i\nu)-\phi(x,\mu')}{i\nu}] + \phi(x,\mu'+i\nu)(m(\lambda)-\frac{r}{i\nu})$$

or

$$f(x,\nu) = \theta(x,\mu'+i\nu) + r[\frac{\phi(x,\mu'+i\nu)-\phi(x,\mu')}{i\nu}] + \phi(x,\mu'+i\nu)(a_0 + O(|\nu|))$$

and then

$$\lim_{\nu\to 0} f(x,\nu) = f(x,0) = \theta(x,\mu') + r\phi_\lambda(x,\mu') + a_0\phi(x,\mu')$$

It follows that

$$\int_s^\infty |f_0|^2 dx \le K'''$$

and this together with $\int_0^s |f_0|^2 dx < \infty$ gives the result that $f_0 \in L^2[0,\infty)$ Hence $\theta(x,\mu') + r\phi_\lambda(x,\mu') \in L^2[0,\infty)$ since $\phi(x,\mu') \in L^2[0,\infty)$.

Moreover we see that $\tau(\theta(x,\mu') + r\phi_\lambda(x,\mu')) \in L^2[0,\infty)$. Since

$$\tau(\theta(x,\mu') + r\phi_\lambda(x,\mu')) = \tau\theta(x,\mu') + r\tau\phi_\lambda(x,\mu') =$$

$$\mu'\theta(x,\mu') - r\phi_\lambda'' + rq\phi_\lambda = \mu'(\theta(x,\mu') + r\phi_\lambda(x,\mu')) + r\phi(x,\mu')$$

To see this, note that since $\phi(x,\mu')$ is jointly continuous with respect to both variables x and λ we have

$$\frac{\partial^2}{\partial x^2}(\frac{\partial\phi}{\partial\lambda}) = \frac{\partial}{\partial\lambda}(\frac{\partial^2\phi}{\partial x^2})$$

$$\frac{\partial}{\partial\lambda}(\tau\phi) = \frac{\partial}{\partial\lambda}(\lambda\phi) = \phi + \lambda\phi_\lambda$$

However

$$\frac{\partial}{\partial\lambda}(\tau\phi) = \frac{\partial}{\partial\lambda}(-\phi'' + q\phi) = -\phi''_\lambda + q\phi_\lambda = \phi + \lambda\phi_\lambda$$

so taking the limit $\lambda \to \mu'$ we obtain the result

$$r\tau\phi_\lambda = r(\phi + \mu'\phi_\lambda)$$

$$\tau(\theta + r\phi_\lambda) = \mu'\theta + r\mu'\phi_\lambda + r\phi = \mu'(\theta + r\phi_\lambda) + r\phi.$$

Hence $\tau(\theta + r\phi_\lambda)$ is also in $L^2[0,\infty)$, so the proof of the lemma is complete. The case of $\Im\mu' < L$ is similar.

Let λ be an eigenvalue of T_α and let E_λ be the eigenmanifold corresponding to λ which is an invariant subspace of $\mathcal{H}$ and so reduces the operator T_α, i.e. if P_1 is the orthogonal projection on the subspace $\mathcal{H} \ominus E_\lambda = \mathcal{H}_1$ (say) and P_2 onto E_λ then $P_1 D(T) \subseteq D(T)$ and if T_1 and T_2 are defined by,

$$D(T_1) = P_1 D(T) \quad T_1 f = Tf \text{ for all } f \in D(T_1)$$

$$D(T_2) = P_2 D(T) \quad T_2 f = Tf \text{ for all } f \in D(T_2)$$

then we can write $D(T) = D(T_1) \oplus D(T_2)$ and $T = T_1 \oplus T_2$ i.e.

$$Tf = T_1 f_1 + T_2 f_2, \text{ where } f = f_1 + f_2 \quad f_1 = P_1 f, f_2 = P_2 f \text{for all } f \in D(T)$$

It can be shown that λ is not an eigenvalue of T_1 and for such values of λ we now define the resolvent operator $R_\lambda(T)$ as $(T_1 - \lambda)^{-1}$.

Lemma 5.4.2 *Suppose that under conditions of Lemma* 5.4.1 *an operator* Φ^* *on* $\mathcal{H}_1 = L^2[0,\infty) \ominus \{\phi(x,\mu')\}$ *is defined by*

$$\begin{aligned}\Phi^* f = \Phi^*(x,\mu'; f(\cdot)) &= \theta(x,\mu')\int_0^x \phi(t,\mu')f(t)dt + r\phi(x,\mu')\int_0^x \phi_\lambda(t,\mu')f(t)dt \\ &+ \phi(x,\mu')\int_x^\infty \{\theta(t,\mu') + r\phi_\lambda(x,\mu')\}f(t)dt\end{aligned}$$

for all $f \in \mathcal{H}_1$ *and all* $x \in [0,\infty)$. *Then* Φ^* *satisfies the following:*

i) Φ^* *is a linear operator*

ii) $(\tau - \mu')\Phi^*(x,\mu';f(\cdot)) = f(x), \quad \forall f \in \mathcal{H}_1, \quad \forall x \in [0,\infty)$

iii) $\Phi^*(x,\mu';(\tau - \mu')f(\cdot)) = f(x), \quad \forall f \in D_\alpha \cap \mathcal{H}_1, \quad \forall x \in [0,\infty)$

iv) $\Phi^*(x,\mu';f)$ *satisfies the boundary condition*

$$\Phi^*(0,\mu';f)\cos\alpha + \Phi^{*\prime}(0,\mu';f)\sin\alpha = 0$$

v) $\Phi^*(x,\mu';f) \in \mathcal{H}_1$.

Proof: Since $\theta + r\phi_\lambda \in L^2[0,\infty)$, by Lemma 5.4.1, the operator Φ^* is well defined, and then

i) since integration is a linear operator so the result follows.

ii) for each $f \in \mathcal{H}_1$ and all $x \in [0,\infty)$ we can write

$$\begin{aligned} -\Phi^{*\prime\prime} + q(x)\Phi^* &= -\theta''(x,\mu')\int_0^x \phi(t,\mu')f(t)dt - r\phi''(x,\mu')\int_0^x \phi_\lambda(t,\mu')f(t)dt \\ &- \phi''(x,\mu')\int_x^\infty (\theta(t,\mu') + r\phi_\lambda(t,\mu'))f(t)dt + q(x)\Phi^*(x,\mu';f) \\ &+ (\phi'(x,\mu')\theta(x,\mu') - \phi(x,\mu')\theta'(x,\mu'))f(x) \end{aligned}$$

If we substitute again for Φ^* in the above identity and use the fact that $W[\theta,\phi](x) = \phi'(x,\mu')\theta(x,\mu') - \phi(x,\mu')\theta'(x,\mu') = 1$, then

$$\begin{aligned} \tau\Phi^* &= (-\theta''(x,\mu') + q(x)\theta(x,\mu'))\int_0^x \phi(t,\mu')f(t)dt \\ &+ r(-\phi''(x,\mu') + q(x)\phi(x,\mu'))\int_0^x \phi_\lambda(t,\mu')f(t)dt \\ &+ (-\phi''(x,\mu') + q(x)\phi(x,\mu'))\int_x^\infty (\theta(t,\mu') + r\phi_\lambda(t,\mu'))f(t)dt \\ &+ f(x) \end{aligned}$$

Hence

$$\begin{aligned} \tau\Phi^* &= \mu'\theta(x,\mu')\int_0^x \phi(t,\mu')f(t)dt + r\mu'\phi(x,\mu')\int_0^x \phi_\lambda(t,\mu')f(t)dt \\ &+ \mu'\phi(x,\mu')\int_x^\infty (\theta(t,\mu') + r\phi_\lambda(t,\mu')f(t)dt + f(x) \\ &= \mu'\Phi^* + f(x) \end{aligned}$$

Thus we have shown that for all $f \in \mathcal{H}_1$ and for all $x \in [0,\infty)$

$$(\tau - \mu')\Phi^*(x,\mu';f(\cdot)) = f(x)$$

and ii) follows.

iii) If $f \in D_\alpha \cap \mathcal{H}_1$ and $x \in [0,\infty)$ then we have

$$\begin{aligned}\Phi^*(x,\mu';(\tau-\mu)f(t)) &= \theta(x,\mu')\int_0^x \phi(t,\mu')(-f''+qf-\mu' f)(t)dt \\ &+ r\phi(x,\mu')\int_0^x \phi_\lambda(t,\mu')(-f''+qf-\mu' f)(t)dt \\ &+ \phi(x,\mu')\int_x^\infty \{\theta(t)+r\phi_\lambda(t)\}(-f''+qf-\mu' f)(t)dt\end{aligned}$$

where $\theta(t) = \theta(t,\mu')$ and $\phi_\lambda(t) = \phi_\lambda(t,\mu')$. We now consider each of the three expressions on the right hand side of this identity separately. Firstly

$$\theta(x,\mu')\int_0^x \phi(t,\mu')(-f''(t)+q(t)f(t)-\mu' f(t))dt =$$

$$\theta(x,\mu')W[f,\phi]_0^x + \theta(x,\mu')\int_0^x (-\phi''+q(t)\phi-\mu'\phi)(t)f(t)dt =$$

$$\theta(x,\mu')W[f,\phi]_0^x = \theta(x,\mu')W[f,\phi](x) \tag{5.4.26}$$

since $f \in D(T_\alpha)$ and $W[f,\phi](0) = 0$. Secondly

$$r\phi(x,\mu')\int_0^x \phi_\lambda(t,\mu')(-f''+qf-\mu')(t)dt =$$

$$r\phi(x,\mu')W[f,\phi_\lambda]_0^x + r\phi(x,\mu')\int_0^x (-\phi_\lambda''+\phi_\lambda q(t)-\mu'\phi_\lambda)f(t)dt =$$

$$r\phi W[f,\phi_\lambda]_0^x + r\phi(x,\mu')\int_0^x \phi(t,\mu')f(t)dt \tag{5.4.27}$$

since $\tau\phi_\lambda = \phi + \mu'\phi_\lambda$, and finally

$$\phi(x,\mu')\int_x^\infty \{\theta(t,\mu')+r\phi_\lambda(t,\mu')\}(-f''+qf-\mu' f)(t)dt =$$

$$\phi(x,\mu')\int_x^\infty \{-(\theta+r\phi_\lambda)''+q(t)(\theta+r\phi_\lambda)-\mu'(\theta+r\phi_\lambda)\}f(t)dt + \phi W[f,\theta+r\phi_\lambda]_x^\infty =$$

$$-\phi W[f,\theta+r\phi_\lambda](x) + r\phi(x,\mu')\int_x^\infty \phi(t,\mu')f(t)dt \tag{5.4.28}$$

since $\tau(\theta+r\phi_\lambda) = \mu'(\theta+r\phi_\lambda)+r\phi$ and by the proof of Lemma 5.4.1
$W[f,\theta+r\phi_\lambda](\infty) = 0$.

Now let us add (5.4.26), (5.4.27) and (5.4.28) to give for the main identity

$$\begin{aligned}
\Phi^*(x,\mu';(\tau-\mu')f(\cdot)]) &= \theta W[f,\phi](x) + r\phi W[f,\phi_\lambda]_0^x \\
&+ r\phi(x,\mu')\int_0^x \phi(t,\mu')f(t)dt - \phi W[f,\theta+r\phi_\lambda](x) \\
&+ r\phi(x,\mu')\int_x^\infty \phi(t,\mu')f(t)dt \\
&= \theta(f\phi'-f'\phi) + r\phi W[f,\phi_\lambda](x) - r\phi W[f,\phi_\lambda](0) \\
&- \phi(f\theta'-f'\theta) - r\phi W[f,\phi_\lambda](x) + r\phi\int_0^\infty \phi(t)f(t)dt \\
&= f(x)W[\theta,\phi](x) - r\phi(x,\mu')W[f,\phi_\lambda](0) \\
&= f(x)
\end{aligned}$$

since $f\in\mathcal{H}_1$ and ϕ, θ satisfy boundary conditions (4.2.4-5). Hence iii) follows.

iv) We can write

$$\Phi^*(0,x,f)\cos\alpha + \Phi^{*\prime}(0,x;f)\sin\alpha =$$

$$(\cos\alpha)\phi(0,\mu')\int_0^\infty(\theta+r\phi_\lambda)f(t)dt + (\sin\alpha)\phi'(0,\mu')\int_0^\infty(\theta+r\phi_\lambda)f(t)dt =$$

$$[(\cos\alpha)\phi(0,\mu')+(\sin\alpha)\phi'(0,\mu')]\int_0^\infty(\theta(t,\mu')+r\phi_\lambda(t,\mu'))f(t)dt = 0$$

since $(\cos\alpha)\phi(0,\mu')+(\sin\alpha)\phi'(0,\mu')=0$.

v) Since $\mu'\in\sigma_p(T_\alpha)$ it follows from the definition of point spectrum of T_α that the range $(T_{\alpha,1}-\mu')D_{\alpha,1}$ of $T_{\alpha,1}-\mu'$ is $\mathcal{H}_1$ where $D_{\alpha,1}=D_\alpha\cap\mathcal{H}_1$ and for each $f\in\mathcal{H}_1$ there exists a function $g\in D_{\alpha,1}$ such that $(\tau-\mu')g=f$ and hence

$$\Phi^*(x,\mu';f) = \Phi^*(x,\mu';(\tau-\mu')g) = g$$

so $\Phi^*\in D_{\alpha,1}\subseteq\mathcal{H}_1$. Thus the lemma is proved.

Lemma 5.4.3 *Under the conditions of Lemma* 4.5.1, *the operator* Φ^* *defined in Lemma* 5.4.2 *may be identified with the operator* Φ *defined in Theorem* 5.2.1.

Proof: From Theorem 5.2.1, for each $f\in\mathcal{H}_1$, and for all $\lambda\in\mathcal{N}_\delta(\mu')\setminus\{\mu'\}$ we have the function

$$\Phi(x,\lambda;f) = \psi(x,\lambda)\int_0^x \phi(t,\lambda)f(t)dt + \phi(x,\lambda)\int_x^\infty \psi(t,\lambda)f(t)dt$$

where $x \in [0,\infty)$ and $\mathcal{N}_\delta(\mu')$ is a neighbourhood of μ'. We are going to show that Φ satisfies the following properties in all neighbourhoods $\mathcal{N}_{\epsilon\delta}(\mu')$ included in $\mathcal{N}_\delta(\mu')$ where $0 \le \epsilon < 1$.

1. Φ satisfies properties i)-iv) of Lemma 5.4.2.
2. Φ is bounded on $\mathcal{H}_1$
3. $\Phi \in \mathcal{H}_1$

1. By the linearity of integral operators 1-i) follows, except for $\lambda = \mu'$ in which Φ is not defined by the above equation. But since by [46] §2.6 we know that $\Phi(x,\lambda;f)$ is analytic in a deleted neighbourhood $\mathcal{N}_\delta(\mu') \setminus \{\mu'\}$ of μ', so by limiting process we have

$$0 = \lim_{\lambda\to\mu'}\{\Phi(x,\lambda;(\alpha f+\beta g)) - \alpha\Phi(x,\lambda;f) - \beta\Phi(x,\lambda;g)\} =$$
$$\Phi(x,\mu';(\alpha f+\beta g)) - \alpha\Phi(x,\mu';f) - \beta\Phi(x,\mu';g)$$

for all $f,g \in \mathcal{H}_1$ and all $\alpha,\beta \in \mathbb{C}$ and hence 1-i) follows at point $\lambda = \mu'$. We now proceed to justify 1-ii). By Lemma 5.4.1 we obtain $(\tau-\lambda)\Phi = f$ for all $\lambda \in \mathcal{N}_\delta(\mu') \setminus \{\mu'\}$ and each $f \in \mathcal{H}_1$. To show this property for $\lambda = \mu'$ we see that Φ satisfies $\tau\Phi = \lambda\Phi + f$ for all $\lambda \in \mathcal{N}_\delta(\mu') \setminus \{\mu'\}$ and this is equivalent to the following integral equation

$$\Phi(x,\lambda;f) = \Phi(0,\lambda;f) + x\Phi'(0,\lambda;f) + \int_0^x\int_0^t [(q(s)-\lambda)\Phi(s,\lambda;f) - f]dsdt \quad (5.4.29)$$

where $x \in [0,\infty)$. Now let $X > 0$ be fixed, then for each $x \in [0,X]$ we have

$$|\Phi(x,\lambda;f)| \le |\Phi(0,\lambda;f)| + X|\Phi'(0,\lambda;f)| + \int_0^X\{\int_0^t |q(s)\Phi(s,\lambda;f)|ds +$$
$$|\lambda|\int_0^t |\Phi(s,\lambda;f)|ds + \int_0^t |f(s)|ds\}dt$$

and hence by the Cauchy-Schwarz inequality

$$|\Phi(x,\lambda;f)| \le |\Phi(0,\lambda;f)| + X|\Phi'(0,\lambda;f)| +$$
$$\int_0^X\{(\int_0^X |q(s)|^2ds)^{\frac{1}{2}}(\int_0^X |\Phi(s,\lambda;f)|^2ds)^{\frac{1}{2}} +$$
$$|\lambda|X(\int_0^X |\Phi(s,\lambda;f)|^2ds)^{\frac{1}{2}} + X(\int_0^X |f(s)|^2ds)^{\frac{1}{2}}\}dt \quad (5.4.30)$$

So, using the fact that f and Φ are square integrable in $[0,\infty)$, we deduce that $\Phi(x,\lambda;f)$ is uniformly bounded for each $x\in[0,X]$ and each $\lambda\in\mathcal{N}_{\lambda\epsilon}(\mu')$ so that there exists a constant $K=K(\epsilon,X,f)$ which is independent of x and λ such that

$$|\Phi(x,\lambda;f)|\leq K<\infty \tag{5.4.31}$$

It follows from (5.4.30) and Theorem 1.2.2 (Lebesgue Bounded Convergence theorem) that we can take the limit as $\lambda\to\mu'$ on both sides of (5.4.29) and deduce that (5.4.29) holds if $\lambda=\mu'$. Then differentiating $\Phi(x,\mu';f)$ twice implies that Φ' is absolutely continuous on $[0,X]$ and that for all $f\in\mathcal{H}_1$, $x\in[0,X]$

$$\tau\Phi(x,\mu';f)=\mu'\Phi(x,\mu';f)+f(x) \tag{5.4.32}$$

and since x is arbitrary, this is true for all $x\in[0.\infty)$ and 1-ii) follows.
To prove 1-iii) we have by Lemma 5.4.1 that $\Phi(x,\lambda;(\tau-\lambda)f)=f$ for all $\lambda\neq\mu'$ and all $f\in D_\alpha\cap\mathcal{H}_1$, and since all terms in the identity

$$\Phi(x,\lambda;(\tau-\lambda)f(\cdot)=f(x)$$

are regular on $\mathcal{N}_\delta(\mu')\setminus\{\mu'\}$ we have by a limiting process

$$\lim_{\lambda\to\mu'}\Phi(x,\lambda;(\tau-\lambda)f(\cdot)=\Phi(x,\mu';(\tau-\mu')f(\cdot)=f(x)$$

for all $x\in[0,\infty)$ and all $f\in D_{\alpha,1}$, where $D_{\alpha,1}=D_\alpha\cap\mathcal{H}_1$.
To show the validity of 1-iv), we use the same argument as for 1-i) in Lemma 5.4.2 to give

$$\Phi(0,\lambda;f)\cos\alpha+\Phi'(0,\lambda;f)\sin\alpha=0$$

for all $\lambda\in\mathcal{N}_\delta(\mu')\setminus\{\mu'\}$ and all $f\in\mathcal{H}$. Then by a limiting process we have

$$0=\lim_{\lambda\to\mu'}[\cos\alpha\Phi(0,\lambda;f)+\sin\alpha\Phi(0,\lambda;f)]=\cos\alpha\Phi(0,\mu';f)+\sin\alpha\Phi(0,\mu';f)$$

Thus property 1 is proved.

2. To prove that Φ is bounded, we use Lemma 5.2.1 to show that there exists a constant K' such that

$$\int_0^\infty|\Phi(x,\lambda;f)|^2\leq K'\int_0^\infty|f(x)|^2dx \tag{5.4.33}$$

for all $f \in \mathcal{H}_1$ and all $\lambda \in \mathcal{N}_\delta(\mu')$, except for the singular point $\lambda = \mu'$ at which the residue R of Φ at μ' is

$$R = r\phi(x,\mu') \int_0^\infty \phi(t,\lambda) f(t) dt$$

since

$$R = \lim_{\lambda \to \mu'} [(\lambda - \mu')\psi(x,\lambda) \int_0^x \phi(t,\lambda) f(t) dt + (\lambda - \mu')\phi(x,\lambda) \int_x^\infty \psi(t,\lambda) f(t) dt] =$$

$$r\phi(x,\mu') \int_0^x \phi(t,\mu') f(t) dt + \lim_{\lambda \to \mu'} \phi(x,\lambda) \int_x^\infty (\lambda - \mu')\psi(t,\lambda) f(t) dt$$

and for any fixed $x > 0$ and all $X > x$ we have

$$\lim_{\lambda \to \mu'} \int_x^X |f_\lambda(t) - r\phi(t,\lambda)|^2 = 0$$

where $f_\lambda(t) = (\lambda - \mu')\psi(t,\lambda)$. Now using Lemma 4.6.2 we obtain

$$\lim_{\lambda \to \mu'} \phi(x,\lambda) \int_x^\infty (\lambda - \mu')\psi(t,\lambda) f(t) dt = r\phi(x,\lambda) \int_x^\infty \phi(t,\lambda) dt$$

where r is the residue of $m_\alpha(\lambda)$ at μ' and hence

$$R = r\phi(x,\lambda) \int_0^\infty \phi(t,\lambda) dt$$

So by a limiting process, i.e. using Theorem 1.2.2, we obtain the inequality (5.4.31) for $\lambda = \mu'$. Also we deduce by (5.4.31) for $\lambda = \mu'$ that $\Phi \in \mathcal{H}_1$ and hence 2 follows.

3. To prove $\Phi \in \mathcal{H}_1$ for all $\lambda \in \mathcal{N}_{\epsilon\delta}(\mu')$ we have to show that

$$< \Phi(x,\lambda; f), \phi(x,\mu') > = \int_0^\infty \Phi(x,\lambda; f) \bar{\phi}(x,\mu') dx = 0$$

Suppose it is not true i.e.

$$< \Phi(x,\lambda; f), \phi(x,\mu') > = \int_0^\infty \Phi(x,\lambda; f) \bar{\phi}(x,\mu') dx \neq 0$$

or equivalently suppose $\Phi(x,\lambda;\ \cdot)D_{\alpha,1} \not\subseteq \mathcal{H}_1$ Then there exists a function $g \in \mathcal{H}_1$ such that

$$\Phi(x,\lambda; g) = G \in \mathcal{H}$$

where $G = \tilde{f} + k\phi(x,\mu')$ with $\tilde{f} \in \mathcal{H}_1$ and some scalar $k \neq 0$, Hence

$$(\tau_\alpha - \lambda)\Phi(x,\lambda;g) = (\tau_\alpha - \lambda)\tilde{f} + k(\mu' - \lambda)\phi(x,\mu') \qquad (5.4.34)$$

where $\mu' \neq \lambda$. But arguing as in the proof of Lemma 5.4.2, v) we can write

$$(\tau_\alpha - \lambda)\tilde{f} = (\tau_\alpha - \mu')\tilde{f} + (\mu' - \lambda)\tilde{f} \in \mathcal{H}_1$$

so by (5.4.34) we have $(\tau_\alpha - \lambda)\Phi(x,\lambda;g) \notin \mathcal{H}_1$. On the other hand we obtain $(\tau_\alpha - \lambda)\Phi(x,\lambda;g) = g \in \mathcal{H}_1$, which is a contradiction. Thus $\Phi(x,\lambda;f) \in \mathcal{H}_1$ and the lemma follows.

Theorem 5.4.1 *The complex number $\mu', \Im\mu' \neq L$ belongs to the point spectrum $\sigma_p(T_\alpha)$ of T_α if and only if $m_\alpha(\lambda)$ has a pole at μ'. In this case $\phi(x,\mu') \in L^2[0,\infty)$ and if the pole of $m_\alpha(\lambda)$ is simple and*

$$\limsup_{X\to\infty} \left| \int_0^X \Im(r\phi(x,\mu')\bar{\theta}(x,\mu'))dx \right| < \infty$$

the resolvent operator at the point $\lambda = \mu'$ is a bounded linear operator from $\mathcal{H} \ominus E_{\mu'} = \mathcal{H}_1$ onto $D(T_\alpha) \cap \mathcal{H}_1$ which satisfies

$$\begin{aligned} R_{\mu'}(T_\alpha)(f(x)) &= \theta(x,\mu') \int_0^x \phi(t,\mu')f(t)dt + r\phi(x,\mu') \int_0^x \phi_\lambda(t,\mu')f(t)dt \\ &+ \phi(x,\mu') \int_x^\infty \{\theta(t,\mu') + r\phi_\lambda(x,\mu')\}f(t)dt \end{aligned}$$

where r is as in Lemma 5.4.1.

Proof: Let $\mu' \in \sigma_p(T_\alpha)$, $\Im\mu' > L$. Since $m_\alpha(\lambda)$ is a meromorphic function for $\Im\lambda \neq L$ by Corollary 4.5.1, μ' is either a point of regularity or an isolated pole of $m_\alpha(\lambda)$. But μ' cannot be a point of regularity of m_α, since then by Theorem 5.2.1 $\mu' \in \rho(T_\alpha)$ and this is a contradiction, hence μ' is a pole of m_α. If $m_\alpha(\lambda)$ has a pole at μ', then it follows from the argument in the proof of Theorem 5.2.1 that $\phi(x,\mu') \in L^2[0,\infty)$, and hence $\theta(x,\mu') \notin L^2[0,\infty)$ since q is in case I in the sense of Sims. If we denote by E_λ the eigenmanifold generated by the eigenfunctions corresponding to the eigenvalue λ, the set of all functions $f \in \mathcal{H}$ which are orthogonal to E_λ is given by $\mathcal{H}_1 = \mathcal{H} \ominus E_\lambda$, which is a subspace of $\mathcal{H}$. We will show that the restriction $(T_\alpha - \mu')|_{\mathcal{H}_1} = T_{\alpha,1} - \mu'$ of the operator $(T_\alpha - \mu' I)$ to $\mathcal{H}_1$ has a bounded inverse i.e. the resolvent operator $R_{\mu'}(T_{\alpha,1}) = (T_{\alpha,1} - \mu')^{-1}$ is a bounded operator from the whole of $\mathcal{H}_1$ to the closed linear manifold $D_{\alpha,1} = D_\alpha \cap \mathcal{H}_1$.

Since by Lemma 5.4.3 Φ has all required conditions for being the resolvent operator at μ' and by Lemma 5.4.2 Φ^* represents an explicit identity for the resolvent operator at μ', so if we prove that Φ and Φ^* which are defined on $\mathcal{H}_1$ into $D_{\alpha,1}$ are identical the theorem follows. To show this, for any $f \in \mathcal{H}_1$ we can write

$$\tau[\Phi(x, \mu'; f) - \Phi^*(x, \mu'; f)] = \mu'[\Phi(x, \mu'; f) - \Phi^*(x, \mu'; f)]$$

for all $x \in [0, \infty)$, since $\Phi,\ \Phi^* \in \mathcal{H}_1$ so $\Phi - \Phi^* \in \mathcal{H}_1$ and since there is no nontrivial solution of $\tau f = \mu' f$ in the subspace $\mathcal{H}_1$ so the solution $\Phi - \Phi^*$ is the trivial solution i.e. $\Phi = \Phi^*$ for all $f \in \mathcal{H}_1,\ x \in [0, \infty)$ This completes the proof.

Remark 5.4.1 It should be noted that the condition

$$\lim_{x \to \infty} \Im q(x) = L$$

can be removed in several of the results in Sections 5.2-4, namely Lemma 5.2.1, Theorem 5.2.1, Lemma 5.4.1-3 and Theorem 5.4.1. The requirement that $\nu \neq L$ in the statements of these results should then be replaced by

$$\nu \notin [\liminf_{x \to \infty} q_2(x), \limsup_{x \to \infty} q_2(x)].$$

5.5 Examples.

This section contains some worked examples for the non-selfadjoint case, where the spectrum includes non-real numbers. We also consider some properties of the spectrum (in the selfadjoint case) such as the relationship between the number of zeros of solutions and the number of eigenvalues located in the half-line $x < 0$, and investigate possible extensions of these result to non-selfadjoint problems.

Example 5.5.1 Consider the boundary value problem

$$y'' + \lambda y = 0, \qquad y(0,\lambda) = 0 \tag{5.5.35}$$

on the interval $[0, \infty)$, where λ is a real parameter.

This problem is limit-point at infinity in the sense of the Weyl classification, since for some $\lambda \in \mathbf{C}$ there is a solution of the equation $y'' + \lambda y = 0$ which is not square integrable. To see this note that for $\lambda \neq 0$ the function $y(x,\lambda) = \sin\sqrt{\lambda}x$ is a solution of the problem which is not square integrable. The essential spectrum is $[0, \infty)$, and finally no point spectrum exists [10] p. 30.

If we change the boundary condition to a more general one, say,

$$y(0,\lambda)\cos\alpha + y'(0,\lambda)\sin\alpha = 0 \qquad \alpha \in [0,\pi) \tag{5.5.36}$$

then a solution of the problem satisfying the boundary condition (5.5.36) for $\lambda = 0$ is

$$y(x,0) = x\cos\alpha - \sin\alpha \tag{5.5.37}$$

which is not square integrable, so zero is not an eigenvalue.

For $\lambda > 0$, a solution satisfying (5.5.35) is

$$y(x,\lambda) = \cos\alpha\sin x\sqrt{\lambda} - \sqrt{\lambda}\sin\alpha\cos x\sqrt{\lambda}$$

which is not square integrable, so there is no positive eigenvalue. If $\lambda < 0$, then $-\lambda = \mu > 0$ and a solution of the problem is of the form

$$y(x,\lambda) = \cos\alpha\sinh x\sqrt{-\lambda} - \sqrt{-\lambda}\sin\alpha\cosh x\sqrt{-\lambda}$$

or

$$y(x,\mu) = \cos\alpha\sinh x\sqrt{\mu} - \sqrt{\mu}\sin\alpha\cosh x\sqrt{\mu}$$

and the m-function is

$$m_\alpha(\mu) = \frac{\sqrt{\mu}\sin\alpha + \cos\alpha}{\sqrt{\mu}\sin\alpha - \cos\alpha}$$

Hence there is a pole of the m-function at $\lambda = -\mu = -\cot^2\alpha$ if and only if $\cot\alpha > 0$, and this is an eigenvalue of the problem.

Now let λ be a complex number; then a solution for the problem is

$$y(x,\alpha)=\cos\alpha\sin x\sqrt{\lambda}-\sqrt{\lambda}\sin\alpha\cos x\sqrt{\lambda}$$

and for $\Im\lambda>0$ the m-function is

$$m_\alpha(\lambda)=\frac{\sqrt{\lambda}\sin\alpha+i\cos\alpha}{\sqrt{\lambda}\sin\alpha-i\cos\alpha} \tag{5.5.38}$$

where $\sqrt{\lambda}$ satisfies $\Im\sqrt{\lambda}>0$. To obtain the poles of the m-function we use the following trigonometric formula for the complex case

$$\cot(x+iy)=\frac{\sin 2x-i\sinh 2y}{2|\sin(x+iy)|^2},\qquad y\neq 0$$

Let $\alpha=\alpha_1+i\alpha_2$, where $\alpha_1,\alpha_2\in\mathbf{R}$. Since the poles of $m_\alpha(\lambda)$ satisfy $\cot\alpha=-i\sqrt{\lambda}$

$$\cot^2\alpha=\frac{\sin^2 2\alpha_1-\sinh^2 2\alpha_2-2i\sin 2\alpha_1\sinh 2\alpha_2}{4|\sin\alpha|^4}=-\lambda\qquad \alpha_2\neq 0. \tag{5.5.39}$$

and the condition $\Im\sqrt{\lambda}>0$ is equivalent to $\Re\cot\alpha=\frac{\sin 2\alpha_1}{2|\sin\alpha|^2}>0$, or to $\sin 2\alpha_1>0$ or $n\pi<\alpha_1<(n+\frac{1}{2})\pi$, where $n\in\mathbf{Z}$.

We can check that if $\sin 2\alpha_1>0$, then this λ satisfies the following identity

$$\cos\alpha+i\sqrt{\lambda}\sin\alpha=0$$

and hence is a pole of $m_\alpha(\lambda)$, as given in (5.5.38). From (5.5.39), if $\sin 2\alpha_1>0$, then

$$\sqrt{\lambda}=\frac{1}{2|\sin\alpha|^2}(\sinh 2\alpha_2+i\sin 2\alpha_1)$$

since $\Im\sqrt{\lambda}>0$. Hence

$$\cos\alpha+i\sqrt{\lambda}\sin\alpha=\cos\alpha+\frac{i}{2|\sin\alpha|^2}(\sinh 2\alpha_2+i\sin 2\alpha_1)\sin\alpha=0.$$

as required, since

$$(2\cos\alpha\sin\alpha\sin\overline{\alpha}+i\sin\alpha(\sinh 2\alpha_2+i\sin 2\alpha_1))\frac{1}{2|\sin\alpha|^2}=$$

$$(2\cos\alpha\sin\overline{\alpha}+i\sinh 2\alpha_2-\sin 2\alpha_1)\frac{\sin\alpha}{2|\sin\alpha|^2}=$$

$$(\sin 2\alpha_1 - i\sinh 2\alpha_2 - \sin 2\alpha_1 + i\sinh 2\alpha_2)\frac{\sin\alpha}{2|\sin\alpha|^2} = 0$$

For example if $\alpha = \frac{\pi}{4} + i\frac{1}{2}$ then $4|\sin\alpha|^4 = \cosh^2 1$ and

$$\lambda = \frac{-1}{\cosh^2 1} + i\frac{2\sinh 1}{\cosh^2 1}$$

or

$$\lambda = \frac{4e}{(e^2+1)^2}(-e + i(e^2-1)).$$

As another example let $\alpha_1 = \frac{\pi}{4}$ and let $\alpha_2 = \ln\sqrt{2}$ then $\lambda = \frac{1}{25}(-7+24i)$ which gives $\Re\lambda, \Im\lambda \in \mathbb{Q}$.

If we put $\alpha = \alpha_1 + i\frac{1}{n}$, with $\sin 2\alpha_1 > 0$, then

$$\lambda = \frac{\sinh^2\frac{2}{n} - \sin^2 2\alpha_1}{4(\sinh^2\frac{1}{n} + \sin^2\alpha_1)^2} + i\frac{\sin 2\alpha_1 \sinh\frac{2}{n}}{2(\sinh^2\frac{1}{n} + \sin^2\alpha_1)^2}$$

and as $n \to \infty$ we have $\alpha \to \alpha_1$ and $\lambda \to -\cot^2\alpha_1$ as expected.

Example 5.5.2 We now consider a similar boundary value problem to the previous example, but with the potential function not identically zero.
Let $q(x) = i\epsilon,\ \ \epsilon > 0$; then with the boundary condition $\alpha = 0$, a solution $y(x,\lambda)$ is

$$y(x,\lambda) = \sin x\sqrt{\lambda - i\epsilon}$$

whenever $\lambda \neq 0$ and consequently there are no eigenvalues.
If we take a non-zero boundary condition α as in (5.5.35) then a solution will be

$$y(x,\lambda) = \cos\alpha\sin x\sqrt{\lambda - i\epsilon} - \sqrt{\lambda - i\epsilon}\sin\alpha\cos x\sqrt{\lambda - i\epsilon} \tag{5.5.40}$$

and with the condition $\Im\sqrt{\lambda - i\epsilon} > 0$ then (5.5.40) is square integrable if and only if

$$\cot\alpha = -i\sqrt{\lambda - i\epsilon}$$

Consequently (5.5.40) is square integrable iff $\Re\cot\alpha > 0$ or $n\pi < \alpha_1 < (n+\frac{1}{2})\pi$ where $n \in \mathbb{Z}$.

So if $\lambda = a + ib$ then

$$\begin{cases} a = \frac{\sinh^2 2\alpha_2 - \sin^2 2\alpha_1}{4|\sin\alpha|^4} \\ b = \frac{2\sin 2\alpha_1 \sinh 2\alpha_2}{4|\sin\alpha|^4} - \epsilon \end{cases}$$

and we obtain the eigenvalues in terms of α and ϵ. Obviously the eigenvalue λ is a continuous function of ϵ and α_2 such that as $\alpha_2, \epsilon \to 0$ then $a \to -\cot^2\alpha_1$, and $b \to 0$ as expected.

Example 5.5.3 Let T_α be the operator defined in Theorem 4.11.4 with $p \equiv 1$ and $q = -2ie^{2(1+i)x}$ as in Example 4.5.1. We can deduce that the equation $-y'' - 2ie^{2(1+i)x}y = \lambda y$ is in Case I in the sense of §4.5. Since there is no L^2 solution for each $\lambda \in \mathbb{C}$, it follows from Theorem 4.11.4 that $\lambda \in \sigma(T_\alpha)$ for all $\lambda \in \mathbb{C}$. Evidently there are no isolated points of the spectrum in the whole complex plane so that $\sigma(T_\alpha) = \sigma_{ess}(T_\alpha) = \mathbb{C}$.

Chapter 6

Applications

6.1 Introduction

It is well known that the one dimensional Schrodinger equation in the interval (a, b)

$$-\phi'' + q\phi = \lambda\phi$$

where $\infty \leq a < b \leq +\infty$ plays an important role in physical phenomena, such as classical physics, quantum mechanics and modern electromagnetic theory. It is associated with the singular real Weyl theory and the singular complex differential operator theory that we have discussed in Chapters 4 and 5. Differential equations with complex potentials arise in connection with electromagnetic theory as a result of separating variables. In this section we give a concrete example, which illustrates the type of difficulty which may arise in practice when applying the complex singular theory to physical problems.

6.2 Example

Generally we know that all time independent quantum mechanical problems are selfadjoint. However if we consider the time dependent problem, because of the complexity of the problem we need to restrict attention to the separable variable case which is in part non-selfadjoint. A simple physical illustration is Bohr's idea of the compound nucleus. Assuming that only one particle is involved in a central force field of finite extent, the wave equation is

$$-\frac{d^2\phi}{dr^2} + V(r)\phi = E\phi \tag{6.2.1}$$

where $\phi(0) = 0$, $V(r)$ is a potential energy with $V(r) = 0$ for all $r \geq r_0$ and E is the energy of the particle. This is a selfadjoint problem and for $r \geq r_0$ the wave equation has the form

$$\frac{d^2\phi}{dr^2} + E\phi = 0$$

and its solution $\phi(r, E)$ satisfies

$$\phi(r, E) = \frac{I}{\sqrt{E}} \sin r\sqrt{E} + Se^{ir\sqrt{E}}$$

where I, S are the amplitudes of the incident and scattered waves respectively. If we differentiate $\phi(r, E)$ with respect to r and set $k = \sqrt{E}$

$$\frac{d\phi}{dr} = I\cos kr + ikSe^{ikr},$$

from which we obtain easily that

$$\begin{aligned} \frac{d\phi}{dr} &= \frac{I}{2}(e^{ikr} + e^{-ikr}) + ikSe^{ikr} \\ &= Ie^{-ikr} + iI\sin kr + ikSe^{ikr} \\ &= Ie^{-ikr} + ik\phi(r, E) \end{aligned} \tag{6.2.2}$$

At the point $r = r_0$ we have

$$\frac{d\phi}{dr}|_{r=r_0} = Ie^{-ikr_0} + ik(\frac{I}{k}\sin kr_0 + Se^{ikr_0}) = Ie^{-ikr_0} + ik\phi(r_0, E)$$

or

$$I = e^{ikr_0}(\frac{d\phi}{dr}|_{r=r_0} - ik\phi(r_0, E)) \tag{6.2.3}$$

and by a similar argument

$$S = \phi(r_0, E)\cos kr_0 - \frac{d\phi}{dr}|_{r=r_0}\frac{1}{k}\sin kr_0$$

Therefore for $r \geq r_0$

$$\phi(r,E) = \frac{e^{kr_0}}{k}[\frac{d\phi}{dr}|_{r=r_0} - ik\phi(r_0,E)]\sin kr + e^{ikr}[\phi(r_0,E)\cos kr_0 - \frac{1}{k}\frac{d\phi}{dr}|_{r=r_0}\sin kr_0]$$

If there is no incident wave i.e. $I = 0$ we have from (6.2.2)

$$\frac{d\phi}{dr}|_{r=r_0} = ik\phi(r_0,E) \tag{6.2.4}$$

However, this condition is incompatible with the original one, since the solution would be non-real. To circumvent this difficulty, we consider the auxiliary problem

$$\frac{d^2\phi_n}{dr^2} + [\lambda_n - V(r)]\phi_n = 0, \quad \lambda_n \neq E$$

with the boundary conditions (6.2.3) and (6.2.4) respectively for $\phi = \phi_n(r,\lambda_n)$. The functions determined by these problems are real orthogonal and complex-valued and the corresponding eigenvalues are complex. Once these functions have been obtained, any solution ϕ_E of (6.2.1) satisfying the boundary conditions $\phi_E(0) = 0$ and (6.2.3) which is the true boundary condition at $r = r_0$ can be expanded in the form

$$\phi_E = \sum a_n\phi_n + X$$

where the ϕ_n satisfy the homogeneous condition (6.2.4) and X satisfies the inhomogeneous condition (6.2.3) [7] §3. Even though this example is non-rigorous from a mathematical point of view, it demonstrates the importance of the study of complex eigenvalue problems.

6.3 Conclusion

In this work we have developed the Weyl theory in three parts. The first part is concerned with Sims' extension of the limit point-limit circle geometry of Weyl in the case of complex coefficients. There were three cases here and in case II it was necessary to show that it can happen i.e. that the coefficient q can be limit point at infinity while there are two square integrable solutions of the corresponding equation. This was achieved by giving three worked examples; the statements of the first two examples can be found in the Sims' paper [44] but the third one is new. The second part was concerned with the extension of some of Titchmarsh's results which were needed for proving the main results on the m-functions and their relationship to the spectrum in the in the complex case. The third part was pretty long because a number of results on different parts of the spectrum were extended from the real case to the complex case. In Section 5.4 we proceeded to extend part (ii) of the theorem in Section 5 of Chaudhuri-Everitt's work [4]. We had to include an extra condition on the simplicity of the poles of the m-function which arise from the point spectrum of the corresponding operator. The existence of complex spectrum was shown at the end of Chapter 5 by some worked examples. We encountered an interesting question about the continuous, singular continuous and absolutely continuous parts of the spectrum in the complex case, that is, whether the results obtained by N. Aronszajn in his paper [2] in the real case are valid in the complex case. The matter is about finite dimensional perturbations that leave the absolutely continuous part of the spectrum unchanged, but there is no guarantee that this result is valid with respect to the continuous spectrum, since there is a one-dimensional perturbation which can transform a pure point spectrum into a pure continuous spectrum [17]. N. Aronszajn gave two sorts of examples: a Sturm-Liouville equation such that for one boundary condition it has a singular continuous spectrum concentrated on a closed set of Lebesgue measure zero, whereas for all other boundary conditions it has a pure point spectrum with all eigenvalues isolated, and a problem which for one boundary condition has a pure point spectrum dense on the whole real axis, whereas for all other boundary conditions the spectrum is purely continuous. The complex case can be obtained from a selfadjoint problem by considering a complex boundary condition or a complex coefficient or both; however, there is in general no known spectral function or resolution of the identity, and so the existence of a singular continuous part of the spectrum is also uncertain.

Recently an extension of Sims theory to the general Sturm-Liouville operator

$$\tau \equiv \frac{1}{w}\{-(py')' + qy\}$$

and corresponding equation

$$-(py')' + qy = \lambda wy$$

has been established under certain conditions on the complex coefficients p and q [3]. A corresponding analogue of the Titchmarsh-Weyl m-function is available to investigate the spectral properties of the operator τ using the three cases mentioned in §4.3. If we impose the condition of local integrability on q then our problem will be a special case of [3].

Subordinacy is an extension of the L^2 property of solutions of the Schrödinger equation in the real case, which appeared in [14]. It is a strong tool for characterising the spectrum with the aid of asymptotic properties of solutions of the Schrodinger equation. There is a close connection between subordinate solutions and the boundary behaviour of the Titchmarsh-Weyl m-function, which in turn depends on the spectral function. The characterisation of the absolutely continuous, singular continuous, and pure point spectrum of selfadjoint Schrodinger operators is probably extendable to the case of non-selfadjoint Schrödinger operators with complex boundary conditions.

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