Mediating Students' Mathematical Learning through Technology: The Role of the Graphical Calculator

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ABSTRACT

The aim of this study has been to investigate the potential of the graphical calculator for mediating students' learning of functions in mathematics at GCE Advanced level. In carrying out this investigation, the study has been primarily concerned with three inter-related themes:

- How does the way in which individual students behave affect the shared construction of meaning in a graphical calculator environment?
- How does the visual imagery provided by the graphical calculator mediate students' understanding of functions?
- What are the implications for the role of the teacher in graphical calculator environments?

In order to address these issues, the study has involved the development of materials and approaches that were subsequently trialled with Lower Sixth form students in a school and a college in the Local Education Authority of Sheffield. An ethnographic approach towards data collection and analysis was adopted, which entailed carrying out detailed *studies of singularities* in three key phases. The first phase consisted of the exploratory study and considered the learning experiences of novice graphical calculator users. The second phase involved experienced graphical calculator users and was concerned with identifying how knowledge construction might differ as a result of the longer-standing status of the graphical calculator as a tool for supporting mathematics learning. The third and final phase concentrated on the introduction of key function concepts to beginning Advanced level mathematics students and focused on the personal and social factors involved.

The findings of this study have served to illustrate both the complexity and interdependence of the individual, social and affective factors involved in students' acquisition of meaning with the graphical calculator. Evidence from the research suggests that the social context has direct bearing on the functioning of the graphical calculator as a *cognitive reorganiser*. The graphical calculator was found to mediate the development of the visual capabilities of individual students via more intensive interaction between the students themselves and with the teacher. In this respect, the pairing of *visualisers* and *non-visualisers* amongst the students was found to be especially conducive to successful collaborative learning with the technology. In this study the graphical calculator acted as both a medium for communication and also as a new *authority* in the classroom, which empowered students to act as autonomous and independent learners. The potential of the technology for inspiring confidence, even in instances where it is not the main source of answers was also highlighted. An important part of successfully introducing new function concepts to students was found to lie in the creation of *local communities of practice* in the classroom, where the graphical calculator was seen as a means of drawing students into these practices. In this way, some of the more reluctant participants were encouraged to act as peer tutors. The importance of the role of the teacher in *scaffolding* the students' learning was also continually emphasised throughout, especially in relation to the interpretation of unexpected results and instances of dependency on the technology, which were linked to individual work. In illuminating all of these factors, the study has demonstrated the strength and relevance of a Vygotskian socio-cultural perspective for exploring students' learning with graphical calculators.

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CONTENTS

APPENDICES

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1
INTRODUCTION

1.0 Aim and Objectives of the Study

The main aim of this study has been to investigate the potential of graphical calculators for mediating students' learning.

The basis for this research draws on the Vygotskian notion of mediation. Vygotsky (1978) proposed that the use of psychological and cultural tools 'mediate' the learning process, providing a link between the external social environment and internal mental processes. Accordingly all higher mental functions are products of mediated activity and thus technological tools, such as graphical calculators, are seen to fundamentally shape and define inner mental processes. In essence, the use of tools results in different kinds of knowledge than could be developed in their absence. This means that use of the graphical calculator in effect transforms the learning process, creating new learning opportunities that could not be achieved with pencil and paper alone.

The key objectives of the study have been to:

- investigate the process by which students acquire meaning for functions within a graphical calculator environment through (i) social interaction and (ii) individual working;

- investigate how the visual imagery provided by the graphical calculator mediates students' understanding of functions;

- investigate the role of the teacher in graphical calculator environments.

1.1 Background to the Research

This study has grown out of the researcher's interest in technology and, in particular, how widely available technological tools such as graphical calculators can be used effectively to further students' mathematical understanding. Practical experience both as a student and teacher of mathematics has highlighted the need for students' powers of visualisation to be supported and has pointed towards the benefits of using technology in this respect. As a consequence, this thesis has been conceptualised on the basis that graphical calculators potentially could assume a very powerful and influential role in stimulating and shaping students' visualisation capabilities and as such may prove to contribute significantly to the depth of student understanding.

Initially it was intended that this research would involve both graphical calculators and computers. However, as the study progressed, it was felt that focusing on a particular form of technology would allow for more in-depth analysis and for a richer illumination to emerge in relation to the use of a widely available tool. The Texas Instrument TI92 and TI82 were thus chosen for this investigation. However, the conclusions drawn from this study could be related to any type of graphical calculator.

Research that has been carried out surrounding computer use is seen as highly relevant to this study and has served to inform some of the interpretations that have been developed in this thesis. Overall, the findings of studies involving graphical calculators are very similar to those that utilised computer technology, although there are key differences between these two types of technologies that should not be overlooked. This is especially apparent with respect to the respective status of each technology as a cultural artifact and for this reason Berger (1998) maintains that there is a need for research that is specifically focused on the graphical calculator.

1.2 Focal Points of the Study

One area of particular interest in this research has been the way in which students are able to derive meaning, both individually and collectively, from the visual representations for functions that they produce using the graphical calculators. Most studies that have dealt with the impact of the graphical calculator on students' understanding of functions have focused on the way in which students construct meaning individually from their own explorations using the technology. The role of the social environment and interactions between students in the process of meaning making for functions with graphical calculators has largely been neglected. In recognising the need for research into this aspect, this study has investigated how meaning for the concept of function is mediated through the use of the graphical calculator and negotiated in the social context and how these meanings are internalised by individual students. This has involved examining the type of group discussions that arose in the graphical calculator environments that were created in each phase of the research and analysing students' individual work.

The relationship between an individual student's visualisation abilities, his/her understanding of functions and use of the graphical calculator has also been a focal point of the study. This interest in the visual aspects of students' reasoning has arisen from the fact that visualisation is now increasingly being recognised and accepted as an important aspect of mathematical reasoning. Numerous studies have been conducted which have found, along with Wheatley and Brown (1994), that activities that encourage the construction of images can significantly enhance mathematics learning. Consequently, this research has sought, in part, to identify and evaluate ways in which the graphical calculator can be utilised to further students' understanding of functions, through the visual imagery it provides. In this respect the impact of the graphical calculator on affective issues such as student confidence has also been explored.

As the study progressed, a further area of importance came to light in relation to the role of the teacher in mediating the use of the graphical calculators. The empirical data collected in the study illuminated the significance of the teacher's input into the student's learning processes with the technology, through negotiating and discussing meaning with the students. The teacher was seen as an essential element of the mediation process and this is reflected in the third objective of the study.

1.3 An Overview of the Thesis

This section provides a short summary of the contents of each of the nine chapters that comprise this study. The purpose of this overview is to clarify the relationship between the different stages of the research and to show how these are related to the main aim and objectives of the study.

1.3.1 Theoretical Perspective

1.3.1.1: Chapter 2 'Theoretical Perspectives on Learning Mathematics and Visualisation'

The first section of Chapter 2 is devoted to a discussion of theoretical perspectives that relate to students' learning of mathematics. In this discussion, these theoretical stances are considered in relation to issues that are of particular importance to this research. These issues include: the role of the social environment in students' learning and how this affects the way in which students derive meaning from interactions; the role of the teacher in creating supportive and active learning environments and how learning occurs in relation to technology environments. This discussion lays the foundation and provides some justification for the theoretical framework that has been adopted in this study, which is outlined further in the following section.

The final section of chapter 2 focuses on the role of visualisation in students' learning of mathematics. This begins with a review of various definitions that have been used to clarify what is meant by the term visualisation and what it means to visualise a concept. This is followed by a discussion of the perceived benefits to students of thinking visually and using visual representations. Particular attention is drawn to the literature concerning the effects of combining visual representations with other modes of representation. In addition, problems that have been associated with the use of visual thinking and visual representations are also discussed.

1.3.1.2: Chapter 3 'Learning Mathematics, Visualisation and Graphical Calculators'

Chapter 3 focuses on the theoretical ideas that have been developed in relation to learning mathematics with technology, in general, which are considered to be directly relevant to the overall theoretical perspective adopted in this study and to the objectives of the research. Initially, the discussion surrounds theories about learning with computers. These and other theories are then discussed in relation to graphical calculators.

The second part of this chapter considers the relationship between visualisation, students' learning of mathematics and the use of technology. Initially, literature concerning the role of computers and multi-representational software in relation to visualisation is outlined. Attention is then focused on the use of graphical calculators in particular and their impact on students' learning of functions. The issues that are discussed include the effect of graphical calculators on student understanding of functions, student confidence, classroom interaction and the difficulties that might be experienced in using technology to mediate students' comprehension of concepts through greater visual awareness.

1.3.2 Methodology

1.3.2.1: Chapter 4 'Research Methodology'

Chapter 4 elaborates the methodological approaches that have governed the collection, analysis and interpretation of the data in this study. This includes an outline of the structure of the study as a whole, the role of the researcher, and the type of data collected from each distinct phase of the research. Figure 1.1 gives a diagrammatic representation of these phases.

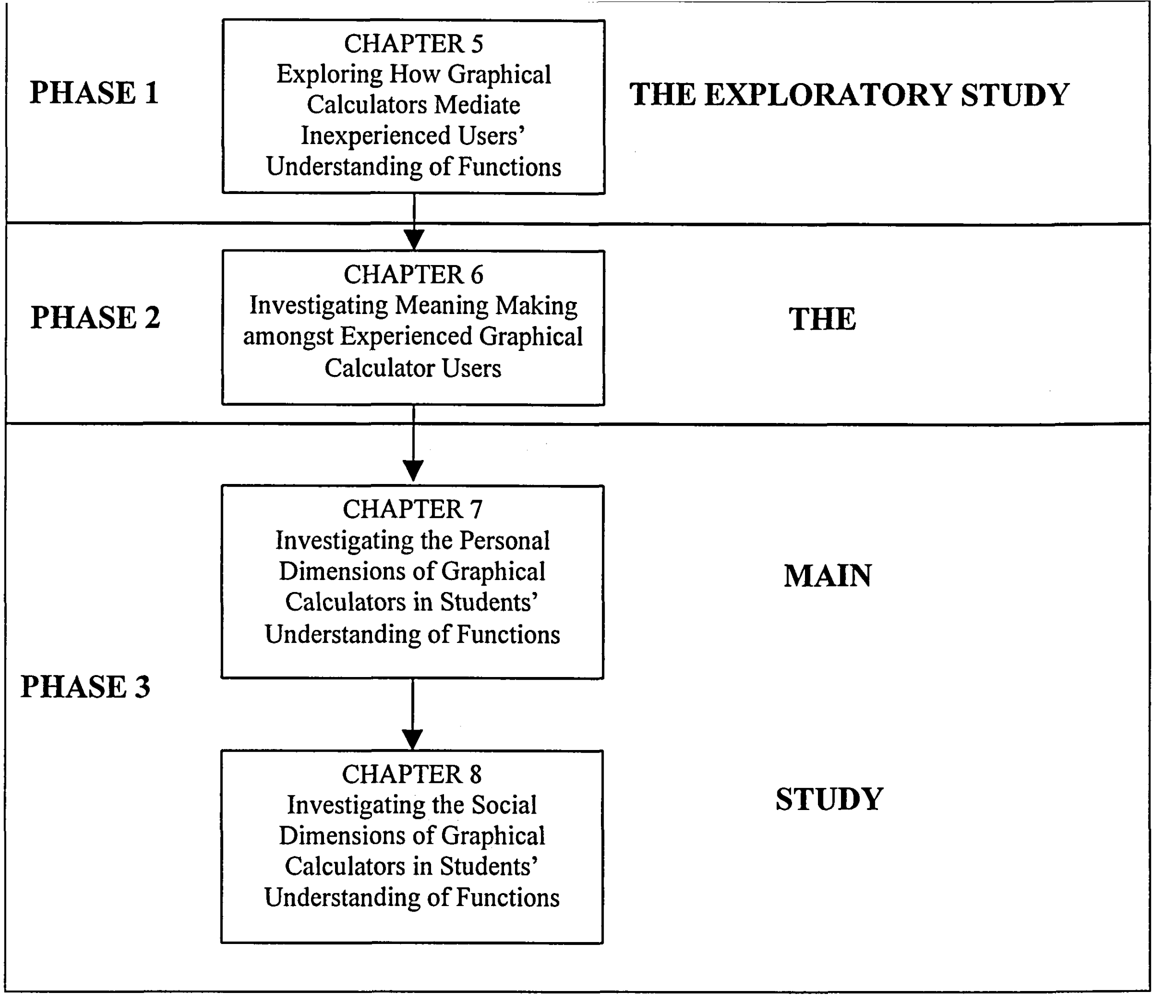


Figure 1.1 Structure of the study

1.3.3 The Exploratory Study

1.3.3.1: Chapter 5 'Exploring How Graphical Calculators Mediate Inexperienced Users' Understanding of Functions'

Chapter 5 describes the exploratory phase of the research. The main objective of this initial study was to explore whether students without prior experience of using graphical calculators would be able to use this technology to further their understanding of functions and how this might occur. It was also intended that the findings of this phase would highlight key issues for further exploration in the subsequent phases of the research. With respect to the study as a whole, this phase served to illustrate the importance of the social environment as an essential constituent of the students' meaning making with graphical calculators, especially through collaboration between peers and teacher intervention.

1.3.4 The Main Study

1.3.4.1: Chapter 6 'Investigating Meaning Making amongst Experienced Graphical Calculator Users'

Chapter 6 gives an account of the second phase of the research. This phase was conducted with the intention of focusing on a small group of regular graphical calculator users to try to establish how they constructed meaning when solving problems involving functions. Following on from the exploratory phase, the effects of the social environment, peer collaboration, peer tutoring and interactions with the teacher on the students' learning were of particular interest. A key research question lay in whether these particular students would use the graphical calculators differently. In particular, would they create knowledge differently to the students of the exploratory phase as a result of the accepted and well developed cultural status of the graphical calculator in their classroom? The data for this phase was obtained from interviews with the individual students, whole class discussions, and questionnaire responses.

The third and final phase of the research is recounted in chapters 7 and 8. This phase was primarily concerned with formally introducing the function concept to new GCE Advanced level Year 12 students using the graphical calculator and also with developing effective teaching strategies that would complement the use of the technology in furthering students' understanding of functions.

1.3.4.2: Chapter 7 'Investigating the Personal Dimensions of Graphical Calculators in Students' Understanding of Functions'

Chapter 7 details the individual aspects of learning about functions using the graphical calculator and how these aspects might be related to the social dimensions discussed in chapter 8. There is a discussion of how individual students use visualisation in problem solving and how the graphical calculator is seen to support this. Another theme that is considered is whether the students use the graphical calculator in the same way or differently when working alone as opposed to collaboratively. The data forming the basis for this chapter is comprised of individual interviews with students and questionnaire responses.

1.3.4.3: Chapter 8 'Investigating the Social Dimensions of Graphical Calculators in Students' Understanding of Functions'

In chapter 8, emphasis is placed on identifying the social aspects associated with the use of graphical calculators that contribute towards student sense making. The role of the teacher and interactions amongst students are explored in detail, as is the context in which these interactions have taken place. The data discussed in this chapter was obtained from whole class and small group discussions and questionnaire responses.

1.3.5 Conclusions and Discussion

1.3.5.1: Chapter 9 'Conclusions, Discussion and Implications'

The main conclusions and discussion surrounding the analysis and interpretation of the data for the study as a whole are presented in chapter 9. In this chapter the findings of each phase of the study are related to the theoretical positions that have been outlined in previous chapters. The implications for future research that have arisen from the study are subsequently detailed.

1.3.6 Dissemination of Findings

An important part of this research programme has involved active engagement with the mathematics education community in the form of seminars and articles for purposes of (i) feedback and (ii) dissemination. Copies of the papers that have been published in relation to the work in this thesis can be found in appendix D.

CHAPTER 2
THEORETICAL PERSPECTIVES ON LEARNING MATHEMATICS AND VISUALISATION

2.0 Overview

This chapter begins with a review of two of the major theoretical perspectives on learning mathematics and how these differ with respect to the way in which meaning is seen to be developed, the role of the teacher is perceived and the manner by which learning is believed to occur. This is followed by an outline of the underlying theoretical perspective that has been adopted in this thesis. The chapter is then concluded with a general review of the current literature on visualisation in mathematics. To begin with, various definitions of visualisation and related concepts are outlined in an attempt to answer the question what is visualisation? Following this there is a discussion of the status of visualisation and the problems that may be associated with its use.

2.1 Socio-Cultural Framework

2.1.1 Social Versus Radical Constructivism

Two of the major theoretical positions concerning the nature of students' learning of mathematics have influenced recent research in mathematics education. Of these two alternate theoretical orientations, up until a few years ago, radical constructivism had tended to be the dominant position. However, growing recognition of the social aspects and nature of learning have led to many researchers adopting varying social constructivist stances.

Radical constructivism is based on two underlying hypotheses (Lerman, 1989). The first of these suggests that knowledge is actively constructed by the cognising subject and is not passively received from the environment. The second proposes that coming to know is an adaptive process that organises one's experiential world and does not discover an independent, pre-existing world outside the mind of the knower. Central to this Piagetian approach is the individual cognising student and the way in which he or she actively constructs his or her own mathematical realities. This preoccupation with the individual has led some researchers to question whether under radical constructivism two or more persons' constructions of reality can be thought of as the same. In this way the radical constructivist position has been criticised for failing to account for intersubjectivity. In contrast, social constructivism has shifted the emphasis away from the individual to the social environment, having emerged out of an attempt to incorporate an explanation for intersubjectivity into an overall constructivist positon (Lerman, 1996). Yet, different researchers place differing amounts of emphasis on the social aspects of learning, thus giving rise to alternative types of social constructivism.

Ernest (1994) identifies two distinct forms of social constructivism, which are distinguished by the amount of emphasis that is placed on the social aspects of learning. The first type is grounded in a 'radical constructivist (Piagetian) theory of mind' and can be separated into two key standpoints. The first of these positions concentrates on the individual aspects of knowledge construction, whilst recognising the subsidiary role of social interaction. The second is referred to as a complimentarist position and involves the adoption of "two complementary and interacting but disparate theoretical frameworks" (p. 307). In this context the intra-individual and the inter-personal views of learning are explored together. The second kind of social constructivism is based on a Vygotskian theory of mind and is founded on the belief that all learning is inherently social. In support of this type of social constructivism, Ernest (ibid) outlines three assumptions as to why the human mind can be seen as social and conversational. Firstly he proposes that "individual thinking of any complexity originates with and is formed by internalised conversation". Secondly he suggests that "all subsequent individual thinking is structured and natured by this origin" and finally he observes that "some mental functioning is collective" (p. 310).

There are many researchers who in recognition of the need to incorporate the social dimension into the radical constructivist position are now adopting what could be regarded as a social constructivist stance based on a Piagetian theory of mind, as described by Ernest. However, there are others who question the legitimacy of such a hybrid approach. For example, when discussing the role of technology in reconceptualising functions and algebra, Confrey (1993) argues for integration of the Piagetian and Vygotskian frameworks for intellectual development. She maintains that Vygotsky's "dialectic of thought and language, and his recognition of the role of tools both physical and communication-orientated needs to be combined with Piaget's rich and varied examples of how children solve tasks to build conceptual operations to create a true dialectic" (p. 51). Conceptual development is thus seen as interplay between "grounded activity and systematic enquiry" (p. 51). In contrast, however, Lerman (1996) argues against integrating social constructivism into the radical constructivist view of learning. He suggests that merging these two opposing viewpoints leads to incoherence and involves some disengagement with their distinct interpretations of the action of the individual.

Lerman argues that the amount of emphasis that is placed on the social aspects of learning under Vygotskian and Piagetian frameworks has fundamentally different implications for the way in which meaning making and learning is thought to occur and for the role of the teacher. Lerman (1994) summarises these implications with respect to radical constructivism and social constructivism. Firstly, he attributes different interpretations of meaning making to the theories of radical and social constructivism. In the radical constructivism paradigm meaning is described as being 'construal'. In this context meaning is seen as being constituted by individual students. Social interactions are recognised as a feature of the learning process, although it is an individual's "personal construal of those interactions and experiences" which becomes the "essence of meaning for that individual" (p. 145). In contrast, meaning is seen as being acquired through 'positioning' in the social constructivism perspective. Meaning is regarded as socio-cultural in nature, a product of "discourse and discourse positions or regulates" (p. 145). Lerman argues that "individuals are acculturated into those meanings" and thus "the intersubjective becomes the intrasubjective" (p. 145). The individual student's input into meaning making is manifested "in a dialectic of the participants in discourse being changed by and changing that discourse" (p. 145). In this way the student derives meaning from his/her 'positioning' in social practices. Lerman is particularly critical of the radical constructivism model in principle for not accounting for the fact that different people might have the same knowledge at the same time.

The way in which learning is seen to occur from these two differing perspectives is characterised by Lerman (ibid) in the terms, 'constructing' and 'appropriating'. In the radical constructivism paradigm, the notion of 'constructing' embodies the relationship between the individual student and the sense in which his or her own knowledge is formed. Lerman (ibid) argues that from this perspective mathematics involves "internal mental operations and meaning is an association of mental operations with mathematical symbols" (p. 150). In contrast, the term appropriation from a social constructivism standpoint conveys a sense of an individual forming his or her "own something", which "already belongs to other people" (p. 150). The role of "communication" and the "cultural interpretation of consciousness" are acknowledged in this framework, as is the "internalisation of cultural life and experience" and the positioning of people through their involvement in this culture (p. 150). In addition, there is a sense that tools are required as a means by which appropriation occurs. Lerman (1996) refers to activity theory and argues that besides the cultural and social settings, goals, needs and purposes are also considered to be constituent of cognition. He suggests that whenever an action becomes significant to a learner, in terms of associated goals, aims, needs and purpose, it can be described as a social event.

Lerman (1994) similarly distinguishes between the radical and social constructivist perception of the role of the teacher. From the radical constructivist position, the teacher can be thought of as a 'facilitator', easing the process of the student's constructions. Lerman (ibid) argues that whilst this notion implies that the teacher has a particular function in the classroom, this is not an essential or indeed necessary one. Alternatively, the teacher can be regarded as a 'mediator' from a social constructivist perspective. This term is seen by Lerman to recognise the position that the teacher holds in 'apprenticing' the student into the particular discourse which constitutes the context, namely the mathematics classroom. He further stresses that as a mediator the teacher assumes an active and necessary role in the students' learning: "it emphasises the necessary function of the teacher and/or other mediators; the mediation is essential for the process to take place" (Lerman 1994, p. 149). The learning environment or cultural context is also recognised, as is the imbalance of power relations in the teaching task. Hence from a Vygotskian perspective, learning is believed to be constituted through mediation by materials, tools, peers and teachers and consequently, teaching and learning are regarded as inextricably integrated. To discuss teaching and learning separately, which is a feature of radical constructivism, would not make sense from this viewpoint.

Lerman (1996) clearly expresses and justifies the view that "radical constructivism does not offer enough as an explanation of children's learning of mathematics" (p. 133). In line with Lerman, Jones and Mercer (1993) also argue that individualistic models of learning fail to adequately address the social aspects of most learning, especially in technology environments. Jones and Mercer (ibid) contend that human problems are often solved by collaboration, stressing that "much learning, not least in relation to information technology, consists of sharing knowledge" (p. 20). From this social perspective, they also view successful classroom outcomes as a product of 'teaching and learning' as opposed to just 'learning' and as such the function of the teacher is regarded as an integral part of any learning situation within the classroom. Vygotskian theory is seen by Jones and Mercer to incorporate the role of the teacher as an "active, communicative participant in learning", as is also argued by Lerman (1994). Thus the teacher is seen as more than a mere provider of "rich learning environments' for children's own discoveries (a la Piaget)" or a reinforcer of "appropriate behaviour if and when it occurs (in the behaviourist mode)" (Jones and Mercer, 1993, p. 22).

Similarly, in seeking to interpret the role of technology in the classroom, Mercer and Scrimshaw (1993) stress the need for a theoretical framework that addresses the fact that learning is a socially and culturally grounded activity. They also emphasise that this framework needs to take into account the 'three-way' relationship between the student, the teacher and the technology. With this in mind, Mercer and Scrimshaw, like Jones and Mercer, propose that 'Socio-cultural' or 'communicative' theory is the most relevant and coherent theoretical framework. This is also the view that is taken in this thesis and a Vygotskian socio-cultural approach is seen as the most appropriate theoretical framework for discussing student's learning about mathematics with graphical calculators. Mercer and Scrimshaw (ibid) see the strengths of this theoretical perspective as being articulated in terms of the following three principles. Firstly they see 'knowledge' not as an abstract commodity, but rather as a state of understanding constructed by every 'knowledgeable' individual. Secondly they regard 'knowledge' construction as essentially social and cultural in nature and finally they emphasise that 'knowledge' construction is always mediated and facilitated by cultural practices and artifacts. Mercer and Scrimshaw regard language and computers as "two of the most important and powerful problem solving resources of our culture" and suggest that there is a great deal more to be discovered about the relationship between language, computers and education (p. 191). This could also be said of the relationship between graphical calculators, language and education and this thesis aims to shed light on the inter-related roles of each of these aspects in the student learning process.

2.1.2 Developing a Socio-Cultural Framework

The arguments proposed by Lerman (1994, 1996), Jones and Mercer (1993), and Mercer and Scrimshaw (1993) which were discussed above have served to illustrate the appropriateness and strengths of a socio-cultural perspective for investigating the way in which learning occurs in the classroom. Moreover, such a perspective has been argued and shown by these researchers to be particularly useful for interpreting the role of technology in student's learning. As such, the overall theoretical perspective on learning mathematics adopted in this study is based on a socio-cultural approach derived from Vygotskian psychology. This section outlines the key features of Vygotskian theory, which have informed the interpretation of the data collected in this thesis.

Vygotsky (1981) proposed that all individual mental processes are based on social interactions. From this position he developed the 'general genetic law of cultural development', theorising that learning proceeds from the interpsychological to the intrapsychological. In this manner the interactions experienced within the social context are gradually 'internalised' by the individual. The process of internalisation is seen as an important aspect of how "consciousness emerges out of human social life" (Wertsch and Stone, 1985, p.164). Vygotsky further proposed that the use of psychological and cultural tools mediate the learning process. Indeed Vygotsky argued that these tools fundamentally shape and define activity and as such provide an "essential key to understanding human social and psychological processes" (Wertsch, 1990, p. 113). He also emphasised that tools, such as speech, symbols, writing, mathematics and technology are social in origin. They are used firstly as a means of communicating with others, to "mediate contact with our social worlds", and eventually "these artifacts come to mediate our interactions with self; to help us think, we internalise their use" (Moll, 1990, p. 11-12). In particular, Vygotsky (1962) regarded language as the means through which thought is developed: "thought is not merely expressed in words; it comes to exist through them" (p. 125). He further described the relationship between thought and language as a "living process" (p. 153). Lerman (1996) argues that from a Vygotskian perspective, "language is not seen as giving structure to the already conscious cognising mind; rather the mind is constituted in discursive practices" (p. 137).

Wertsch and Stone (1985) comment that "Vygotsky's account of semiotic mechanisms" (especially language) that mediate social and individual functioning provides the bridge connecting the external and internal and the social and the individual (p.164). Indeed Diaz et al (1990) contend that 'internalisation', in a Vygotskian sense, does not refer to "mere mental image or mental representation of the external relation", rather this term encompasses a "new level of behavioural organisation" which was previously possible only with the "help of external signs and mediators" (p.134). In expressing his view of internalisation, Lerman (1996) quotes Leont'ev who proposes that: "the process of internalisation is not the transferral of an external to a pre-existing, internal 'plane of consciousness'; it is the process in which this plane is formed" (p.136).

Another important feature of Vygotsky's analysis is the distinction between spontaneous or everyday concepts and scientific concepts. Spontaneous concepts are formed from everyday experiences and are thus laden with meaning and dependant on the social context. However, these concepts are not connected to each other by "general systems of interrelated understandings" (Saxe, 1991, quoted in Noss and Hoyles, 1996, p. 131). Alternatively, scientific concepts develop through analytic procedures rather than concrete experiences (Panofsky et al, 1990). These abstract scientific concepts are initially empty, and meaning is derived from the interaction between the spontaneous and the scientific. Vygotsky (1962) stated that whilst these "two concepts differ in their functioning, these two variants of the process of concept formation influence each other's evolution" (p. 87). He proposed that "the rudiments of systematisation first enter the child's mind by way of his contact with scientific concepts and are then transferred to everyday concepts, changing their psychological structure from the top down" (p. 93). Thus the development of spontaneous concepts is seen to proceed upward, whilst the development of scientific concepts continues downward to a more elementary and concrete level (p.108). Saxe (1991, quoted in Noss and Hoyles, 1996, p. 131) emphasises that "in their interaction, spontaneous concepts enrich scientific concepts with meaning and scientific concepts offer generality to the development of spontaneous concepts".

The idea of the 'zone of proximal development' was also particularly significant in Vygotsky's work. This zone represents the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers. Lerman (1996) describes how learning occurs in the zone of proximal development, as "theoretical/scientific concepts 'ascend' from the abstract to the concrete in interaction with more knowledgeable others" (p. 138). Thus, peer tutoring and peer collaboration both play an important part in constituting the zone of proximal development, as do the mutual orientation of goals and desires (Lerman, 1998). Pairs of students can create "their own zones of proximal developments if they are motivated, taught how to share ways of working, have an appropriate personal relationship, and/or other factors" (ibid, p. 72). When a teacher introduces an activity into the classroom, differences in the answers formulated by students arise from their previous experiences, their 'zone of actual development', and these potentially pull other students, and possibly the teacher, into their zones of proximal development (ibid). Moll (1990) argues that "a major role for schooling is to create social contexts (zones of proximal development) for mastery of and conscious awareness in the use of cultural tools" (p. 12).

Vygotsky's ideas are further developed by Lerman (1998), in his argument for a 'discursive psychology' of mathematics teaching and learning. Such an approach is regarded as a useful framework for interpreting the data from this thesis and is concerned with the process of acquisition of meanings and sees mathematical concepts as social acts and tools, and words and symbols as mediators of thought. Learning is considered as being "predicated on one person learning from another, more knowledgeable, or desired, person" (p. 77) and is also developed through tool use. The mediation of cultural and metacognitive tools in mathematical meaning making is central to this approach. Lerman (ibid), also, argues that Vygotsky's zone of proximal development offers a 'sociogenetic mechanism' for interpreting learning (p. 73). Lerman (1996) stresses that as concepts are socially determined, they are socially acquired. In developing this position further he argues that as concepts derive their meaning from being used, the "acquisition of a concept, or 'understanding' can be interpreted as that of an individual coming to share in that meaning through negotiation and discussion" (p. 146). In this way, through discussion, dispute, cognitive conflict and sharing ideas in the classroom, "the intersubjective becomes internalised as the intrasubjective and the intrasubjective is offered to others, becoming intersubjective" (Lerman, 1992, p. 46). Jones and Mercer (1993) also see learning as occurring through social interactions and propose one possible measure of successful learning, which echoes the views expressed by Lerman. In their view successful learning is said to occur when "two or more people manage to share their knowledge and understanding, so that a new cultural resource is created which is greater than the knowledge and understanding that any of the individuals hitherto possessed" (p. 21).

Jones and Mercer (ibid) also stress that socio-cultural theory focuses on the way in which "talk and joint activity are used by teachers and learners to share knowledge" (p. 24). Indeed the teacher's role in promoting successful learning is another important part of Vygotskian theory, a fact that is emphasised by Moll (1990) who believes that the interdependence of teacher and student is central to Vygotskian analysis of instruction. As discussed previously, the teacher and students are seen to play a mutual and active part in creating the social environment (ibid). Lerman (1996) suggests that the mathematics teacher's objective could be interpreted as "assisting students to appropriate the culture of the community of mathematicians as a further social practice" (p. 146).

There is also an important role for the teacher in mediating the student's learning. Bruner (1985) emphasises that the teacher performs a "critical function" in 'scaffolding' the learning task, enabling the student to "internalise external knowledge and convert it into a tool for conscious control" (p.25). In this manner the assistance provided by the teacher enables the student to operate successfully within his or her zone of proximal development by helping them to bridge the gap between their actual and potential levels of development. Sutherland (1993) stresses that the notion of scaffolding does not signify simplification of the actual task by the teacher, rather he or she provides graduated assistance to the student in order to remove some of the cognitive demands. Scaffolding strategies used by the teacher may take on various different forms, such as directing the students' attention, motivating and encouraging perseverance, and/or acting as a memory bank for students to draw on (ibid). In time, however, the learner's participation gradually increases, depending on the needs and learning pace of the individual and support from the teacher is slowly faded (Noss and Hoyles, 1996). The notion of 'scaffolding' thus embodies the idea of an adjustable and temporary support which can be removed if no longer needed (Orhun, 1991).

Hoyles et al (1991) regard scaffolding as 'hooks' which are available to assist students in "overcoming significant obstacles in the generalisation process" (p. 219). In addition to the teacher's support, they (ibid) also see a role for technology and student interaction in scaffolding the learning task. They maintain that the problem solving tools which are available within certain computer environments and the nature of the interaction between students could serve as scaffolding that assists students in finding a starting point in problem solving and in progressing from the specific to the general. Clearly, Hoyles et al (ibid) regard the computer as a mediator of student learning, which will be discussed further in Chapter three.

2.2 Visualisation in Mathematics

2.2.1 Definitions of Visualisation and Related Concepts

Within the current literature there exist many differing notions of the key terms associated with the area of visualisation in the learning of mathematics, each developed with respect to a specific research purpose/focus, and each drawing on and sometimes expanding previous ideas.

Lean and Clements (1981) define *imagery* [following Hebb (1972)] as the occurrence of mental activity corresponding to the perception of an object, when the object is not physically seen. In the same vein they perceive *visual imagery* as imagery which occurs as a picture in 'the mind's eye'.

Other researchers in the field have preferred to incorporate wider definitions. For Presmeg's (1986) purposes, a *visual image* was considered as a "mental scheme depicting visual or spatial information" (p. 297). This definition was seen to incorporate the types of imagery that depict shape, pattern or form, and, also, verbal, numerical or mathematical symbols which may be arranged spatially to form the kind of numerical or algebraic imagery sometimes referred to as number forms. In this definition Presmeg dismisses the 'fixed image' notion of *visual imagery* developed by Lean and Clements, outlined above.

In the context of working with teachers of older students, aged sixteen and seventeen, Presmeg (ibid) distinguished between *visual* and *non-visual* methods of solution. The *visual* approach necessarily involves visual imagery, which may be explicit (in the form of a diagram) or implicit.

Reasoning and algebraic methods may, also, form part of the solution. In contrast, the *non-visual* mode excludes visual imagery, in preference of other methods. In furthering her analysis, she also made the distinction between *visualisers* and *non-visualisers*:

> ***Visualisers** are individuals who prefer to use visual methods when attempting mathematical problems that may be solved by both visual and non-visual methods.*
>
> ***Non-visualisers** are individuals who prefer not to use visual methods when attempting such problems (Presmeg, 1986, p. 298).*

These classifications have proven useful in the analysis of the data collected for this thesis.

Zimmerman and Cunningham (1991) regard *visualisation*, in general, as "the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated" (p. 1). More specifically, they consider *mathematical visualisation* to be the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding. In essence, "to visualise a diagram means simply to form a mental image of the diagram, but to visualise a problem means to understand the problem in terms of a diagram or a visual image" (Zimmerman and Cunningham, 1991, p. 3).

Mason (1992) regards *visualising* as "making the unseen visible", proposes that *imagery* has "the power to imagine the possible and the impossible" and suggests that *seeing* occurs "figuratively as well as literally" (p. 25). Mariottii and Pesci (1994) acknowledge *visualisation* occurring when "thinking is spontaneously accompanied and supported by images" (p. 22). Similarly, Presmeg (1995) recognises *visualisation* as "the process of constructing or using visual images, with or without diagrams, figures or graphics" (p. 60).

With particular emphasis on spatial visualisation, Solano and Presmeg (1995) define an *image* as a "mental construction of an object created by the mind through the use of one or more senses, where the mind plays an active role" (p. 67). Subsequently, they regard *visualisation* as the relationship between images – "in order to visualise, there is a need to create many images to construct relationships that will facilitate visualisation and reasoning" (p. 67). Following this, *imagery* is seen as a collection of one or more images, a dynamic process.

Hitt Espinosa (1997), also, highlights the possibility of using tools to obtain a physical representation of students' visual mathematical concepts:

> *Visualisation of mathematical concepts is not a trivial cognitive activity: to visualise is not the same as to see. To 'visualise' is the ability to create rich, mental images which the individual can manipulate in his mind, rehearse different representations of the concept and, if necessary, use paper or a computer screen to express the mathematical idea in question. (Hitt Espinosa, 1997, p. 697).*

Zazkis et al (1996), virtually regard all thinking as based on visualisation and their definition of visualisation as follows, is considered to be the most relevant to this study:

> *Visualisation is an act in which an individual establishes a strong connection between an internal construct and something to which access is gained through the senses. Such a connection can be made in either of two directions. An act of visualisation may consist of any mental construction of objects or processes that an individual associates with objects or events perceived by her or him as external. Alternatively, an act of visualisation may consist of the construction, on some external medium such as paper, chalkboard or computer screen, of objects*

or events that the individual identifies with object(s) or process(es) in her or his mind. (Zazkis et al, 1996, p. 441).

In this definition visualisation is not restricted to the learner's mind, or to some external medium, rather it is seen as the means for travelling between the two. In addition, the individual is the one who perceives these objects as internal or external, rather than the researcher (Nemirovsky and Noble, 1997).

2.2.2 The Status of Visualisation

Whilst questioning Pestalozzi's belief that sensory-perceptual observation (visualisation) is the 'absolute basis' for all cognition, Gutierrez (1996) acknowledges that visualisation is one of the essential components. This is a view that is increasingly being taken by mathematics researchers. For example, Mason (1992) regards mental imagery as fundamentally important "because it lies at the heart of meaning making, and is the means of preparing in the now, actions to take in the future" (p. 24). Breen (1997) also stresses the enormous potential in using images as a powerful starting point for providing rich learning situations. In a similar vein, Cunningham (1994) proposes that "some students can learn more effectively from visually based discussions and experiences than from symbolic and analytic work" and that adding images to words supplies students with a "richer set of ways to communicate their mathematics" (p. 84). Cunningham, also, acknowledges that "one of the most remarkable things about visualisation is the amount of mathematics students will learn and the amount of work students will do in order to create images describing a mathematical concept, especially when the computer is used as part of the process" (p. 83).

Cunningham (1991) describes several other advantages that are associated with the use of visualisation. These include "the ability to focus on specific components and details of very complex problems, to show the dynamics of systems and processes, and to increase the intuition and understanding of mathematical problems and processes" (1991, p. 70). In addition, he claims that the inclusion of visualisation in mathematics education, permits a broader coverage of mathematical topics and most importantly, allows students access to new ways to approach their own mathematics. Another such advantage resides in the student's ability to retain knowledge. Whilst purely algebraic proofs are fairly easy to remember in the short term, they are quickly forgotten in the longer term. A visual understanding of a given situation, however, is more likely to remain with the student and can be recalled when needed.

Many researchers maintain that visual arguments can prove to be extremely useful in helping students to conceptualise particular mathematical ideas. Diagrammatic proofs, offered as alternatives or as supplements to standard linguistic proofs, often enable students to develop a real appreciation of the meaning of a theorem and are more convincing than the standard arguments (Barwise and Etchemendy, 1991). Moreover, as Cunningham (1991) points out, the vocabulary used for communicating ideas is quite often visual, and thus a visual proof would constitute an appropriate form of argument. Conventionally, though, images were used almost exclusively in an illustrative manner, to enable students to make sense of symbolic processes. However, in 'real visual thinking', the students' visual understanding becomes the primary vehicle for delivering and developing concepts, which depends on interactive student experiences (Cunningham, 1994). Tall (1991a) highlights the importance of 'seeing' why certain theorems are true with the power of the 'inner eye' and has also stressed that the human brain is well adapted to process visual information (Tall, 1986). Yet, Davis (1993) stresses that what the eye reports must be interpreted properly and that this interpretation occurs as a result of experience and as such is expressed in natural language and action. Students' visualisations need adaptation and accommodation through common agreements and usage (ibid).

Whilst mental images may differ between individuals, there are likely to be "commonalities amongst people who share a common culture" (Presmeg, 1992, p. 597). Nevertheless, differences will occur in the processing of individual students, "even when they learn within the same class, with the same teacher, and within the social context of some experiences that may be taken to be shared" (Presmeg, 1992, p. 607). Images that underlie mathematical concepts are unique constructions of individuals, and are thus inaccessible to others. However, these images are based on a number of concrete images that are used mathematically in various ways by each individual learner. These concrete images may be shared to some extent, through for example gestures, diagrams, the use of technology or verbal descriptions, and are based in turn on mathematical experiences that may be shared and negotiated by others. Since these experiences occur in a particular social context, which may be variously interpreted by individual learners, Presmeg proposes that multiple worlds are created (ibid). Shared meanings are thus possible because of the commonalities in the students' experiences from which the concrete images arise.

Barwise and Etchemendy (1991) argue for the validity and acceptance of heterogeneous proofs (proofs which use multiple forms of representation). They contend that the main reason for the "low repute of diagrams and other forms of visual representation in logic is the awareness of a variety of ill-misunderstood mistakes that one can make using them" (p. 11). Yet, whilst they recognise that the potential for error in diagrammatic reasoning is real (as does Tall, 1991b), they highlight the fact that proofs without visual reasoning can, also, be equally flawed, and that often the construction of a simple diagram would reveal any such mistakes. They outline three ways in which visual reasoning can be considered as valid reasoning:

> *(i) visual information is part of the given information from which we reason, (ii) visual information can be integral to the reasoning itself, (iii) visual*

representations can play a role in the conclusion of a piece of reasoning. (Barwise and Etchemendy, 1991, p. 16).

Yet, visual aspects of a concept are often rated secondary or peripheral to the concept itself, and frequently the visual characteristics of a problem are not even considered (Eisenberg and Dreyfus, 1991). However, the visual component of mathematical reasoning needs to be included to enable students to develop more than merely a mechanical understanding of mathematical concepts, ideas and processes (ibid). Mason (1992) suggests that "imagery often forms a key for a rich network of connections and associations, and so has a crystallising effect" (p. 27). Visualisation in mathematics is not seen as an end in itself, but rather a means towards an end (Zimmerman and Cunningham, 1991). This end is understanding, and visualisation needs to be combined with other forms of mathematical representation in order to achieve this goal. Zimmerman and Cunningham (1991) insist that mathematical visualisation is not merely maths appreciation through pictures - a superficial substitute for understanding. Rather they maintain that visualisation supplies depth and meaning to understanding, serving as a reliable guide to problem solving, and inspiring creative discoveries. In order to achieve this level of understanding, however, they propose that visualisation cannot be isolated from the rest of mathematics (in line with Tall (1989)), implying that symbolical, numerical and visual representations of ideas must be formulated and connected. This thesis is conceptualised on the basis that visual thinking and graphical representation should be linked to other modes of mathematical thinking and other forms of representation. Thus, ultimately, this research has been concerned with enabling students to achieve a deeper, more meaningful and contextual understanding of certain mathematical concepts and ideas.

Knuth (2000) recommends that teachers emphasise the use of graphical representations whenever appropriate and give students the opportunity to share and discuss their different solution approaches and the relative merits of each. Yet, traditionally, a greater emphasis has been placed on algebraic or analytic proof (Cunningham, 1991) despite the proposed legitimacy of visual theorems (Davis, 1993; Barwise and Etchmendy, 1991). Tall (1989) recognises this problem and argues like Zimmerman and Cunningham that although traditional mathematics has emphasised the 'symbolic and sequential', algebraic symbolism, at the expense of the 'integrative and holistic', visual symbolism, both are necessary requirements in the study of mathematics. Whilst the proof of mathematical ideas involves algebraic symbolism, the construction of such ideas requires some form of visual symbolism. Thus, Tall (ibid) stresses the importance of the many facets of a student's 'concept image', which he defines as the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 37).

Tall (1991b) further develops his position by discussing the differences between intuitive thought processes and the logical thought required in formal mathematics. He professes that "intuition involves parallel processing quite distinct from the step by step sequential processing required by rigorous deduction. An intuition arrives whole in the mind and it may be difficult to separate its components into a logical deductive order" (p. 107). Visual information is processed simultaneously and, as a result, one could argue that an intuitive approach might not be entirely suitable in serving the logic of mathematics. Conversely, though, a purely logical view is similarly 'cognitively unsuitable' for students. Thus, according to Tall (ibid), both types of processing should be integrated, through "an approach that appeals to the intuition and yet can be given a rigorous formulation" (p. 108).

Zazkis et al. (1996) propose that visualisation and analysis are 'mutually dependent' in mathematical problem solving and reject the notion of an analyser/visualiser dichotomy or continuum. They assert that rather than clearly preferring either a visual or analytical strategy, the majority of the students in their study tended to use a combination of these approaches. Most visualisations contain some form of analysis (Presmeg, 1986; Zazkis et al., ibid) and conversely most analysis involves some form of visualisation (Zazkis et al., ibid). It is therefore proposed that both visual and analytic thinking need to be present and integrated in order for most students to be able to construct rich understanding of mathematical concepts (Zazkis et al., ibid). An important part of this thesis has been to clarify the students' perceptions of their visual orientations. Hoyles and Healy (1996) suggest that forging the links between the symbolic and visual is an important part of progressing towards the appropriation of a mathematical generalisation, which implies that justifications, like symbolic reasoning should be developed alongside the visual.

Presmeg's findings (1986), also, indicate that the ability to apply and interchange both visual and non-visual methods in problem solving is particularly advantageous for students, especially where one mode appears to be more appropriate. Clearly those students who are able to determine the most suitable approaches or where necessary use a combination of approaches in order to solve the given problem, to provide clarification and/or to check the validity of solutions, are likely to develop a deeper, more holistic understanding of mathematics. Indeed, Presmeg (1992) argues that "as many domains of experience as possible should be implicated in the mathematics learning process" (p. 607).

The connections between different modes of representation need to be made by students, the significance of particular links must be recognised and most importantly an appropriate balance of approaches should be introduced (Hughes Hallett, 1991). Kaput (1992) argues that each mode of representation reveals certain aspects of the idea more clearly than the other modes, whilst hiding some other aspects. In this way "the ability to link different representations helps reveal the different facets of a complex idea explicitly and dynamically" (ibid, p. 542). Hitt Espinosa (1997) stresses the importance of formalising the conjectures that have been generated by intuition and visualising a concept or idea, using analytical techniques/demonstrations. Indeed, many researchers support the view that whilst visualisation stimulates and reinforces conceptual understanding, this particular mode of representation is no more important than other modes. What is required is a multi-representational approach to mathematics, where each mode complements and strengthens the understanding the student acquires when operating in an alternative mode. Indeed all of the studies discussed thus far highlight the necessity of empowering students with multiple interchangeable approaches to problem solving. Yet encouraging visualisation and the use of multi-representational approaches within the classroom is unlikely to be without its problems.

2.2.3 Problems Associated with Visualisation

Eisenberg and Dreyfus (1991) postulate that "thinking visually makes higher cognitive demands than thinking algorithmically", and consequently "it is quite natural for students to gravitate away from visual thinking" (p. 25). They contend that often students (and teachers) are unable to answer problems based on concepts that have visual interpretations because "they have not learned to exploit the visual representations associated with these concepts" (p. 25). This means that often students do not know how to utilise diagrams they themselves draw in order to solve problems (Eisenberg and Dreyfus, 1990). To illustrate their point, they compare visual and algebraic processing in terms of cognitive efficiency:

> *It is relatively easy to present an argument that has already been ordered in a linear sequential manner. It is much more difficult to present a two-dimensional array of information with multiple links between various pieces of information and with implications in many different directions. (Eisenberg and Dreyfus, 1991, p. 33).*

Because of this, Eisenberg and Dreyfus (1990) claim that analytical arguments are more suitable for presenting mathematical ideas to students in school, through didactical transpositions, which also accounts for students' preferences for processing their mathematics symbolically.

Nevertheless, the information contained in a diagram tends to be explicit, whereas the same information will be implicit in the analytic representation (Eisenberg and Dreyfus, ibid). Thus, the information contained in the analytic expressions must firstly be extracted and conceptualised before problem solving can commence and as a result the diagrammatic representations, exhibiting important pieces of information and the conceptual links between them, are often more useful. However, "diagrammatic representations are not immediately intelligible to the uninitiated. It takes cognitive processing to make sense of diagrammatic representations" (Eisenberg and Dreyfus, ibid, p. 33). Diagrams use conventions, notations, generalisations and abstractions, and without these the diagram is unintelligible. Moreover, not all of the information contained in diagrams will necessarily be needed to solve the problem and as a consequence it may be difficult for the student to focus on the relevant information (Eisenberg and Dreyfus, 1990). Thus, if students have not been taught appropriately, they are unlikely to make use of diagrams in problem solving. Sometimes, the individual simply has inadequate experience of certain concepts to provide appropriate intuitions (Tall, 1991b).

Presmeg (1986) outlines four particular difficulties surrounding the use of imagery. Firstly, there is the possibility that a single image or diagram may be viewed inappropriately by students, and therefore lead to subsequent misconceptions. Secondly, an "image of a standard figure could induce inflexible thinking which prevents the recognition of a concept in a non-standard diagram" (p. 307). Similarly, students may already possess rigid 'uncontrollable' images that inhibit the development of further more significant images, especially if these persistent images are vivid. Alternatively, vague imagery should be combined with 'rigorous analytic thought' to be useful. These difficulties may contribute to the reluctance of students to adopt visual methods. Indeed, despite the current views of researchers regarding the importance of visualisation, some students, whilst possessing the ability to visualise mathematically, often opt for non-visual, more 'conventional' approaches to problem solving, which emanates, in the main, from the tendency for visualisation to be undervalued in mathematics classrooms (Presmeg, 1995). However, there are plenty of students who are not reluctant to visualise in mathematics and, when given the opportunity, do prefer to exercise their powers of visualisation (ibid). Hence, in order for all students to fully appreciate the "validity of visual thinking and reasoning in mathematics", the visual approach should be continually adopted and reinforced (Cunningham, 1994, p. 84). However, Eisenberg and Dreyfus (1991) observe that "students have a strong tendency to think algebraically rather than visually, even if they are explicitly and forcefully pushed towards visual processing" (p. 29). They maintain that students are still reluctant to visualise regardless of the way in which concepts are initially presented to them, whether this is either in a visual or an analytical way, or by using a combination of both (Eisenberg and Dreyfus, 1990). Furthermore, Eisenberg and Dreyfus (1986) found that students who tended to classify themselves as visual thinkers consistently used analytical approaches.

Knuth (2000) also found there was an overwhelming reliance on algebraic solution methods amongst the students in his study. Moreover, this often occurred at the expense of, and seemingly without awareness of, a simpler graphical solution method. The graphical approach was deemed to be unnecessary by the students and was only used as a means to support their algebraic solution methods rather than as a solution strategy in its own right. In general, students do not tend to make links between their visualisations and analytical thought readily (Presmeg, 1986).

Researchers have also considered the possibility that visual thinking eventually becomes unnecessary, uneconomical and/or exhausted and is replaced by symbolic thinking. Presmeg's findings (1985) and those of Eisenberg and Dreyfus (1986) support the representational-development hypothesis which suggests that less imagery is used with greater experience or learning. Presmeg, like Eisenberg and Dreyfus, argues that in mathematical thinking practice and habituation allow students to dispense with imagery which initially aids understanding and that frequently a formula or algorithm provides the quickest and most economical method of solving a problem. Thus, in her study facility led visualisers away from visual methods (ibid).

Presmeg's research (1986) in particular supported the view that mathematical giftedness is associated with a lack of visual reasoning. She suggests that "external features, such as the nature of mathematics" or "internal factors, such as cognitive preferences, confidence or mathematical abilities" might be responsible for this occurrence (p. 300). Moreover, the usage of visual techniques is comparatively time-intensive which suggests that tests and examinations will tend to favour the non-visual thinker (Presmeg, 1986). The students in Presmeg's study (ibid), who preferred to use visual approaches thus tended to be those with weaker mathematical abilities. Zazkis et al. (1996) suggest that this can be explained by the fact that thinking about a problem always begins with visualisation and therefore the weaker the student is, the less likely he or she is to progress very far. Lean and Clements (1981) also found that students who preferred to process mathematics analytically tended to out-perform more visual students in mathematical tests.

In general, visualisation techniques/skills tend to be employed by students privately to clarify, interpret and make sense of the given problem intuitively as tools for 'meaning-making' (Wheatly and Brown, 1994). Consequently, such processes are unlikely to be explicit in written arguments, as this is not the accepted norm (Presmeg, 1995). As such, Cunningham (1991) proposes that as visual processing is difficult to assess in standard ways, visually inclined students (and potentially all students) will continue to be disadvantaged in examinations, until a means to evaluate visual learning is developed and included as a routine part of all mathematics examinations.

These findings have important implications for the role of the teacher in encouraging and supporting students' use of visualisation. Indeed, Martin et al (1994) recognise that "the images held by the teacher will have a significant impact on the images constructed by the pupils and on their ability to understand these processes" (p. 249). They further emphasise that the teacher has a responsibility for creating an environment in which students can construct understanding based on appropriate images. However, merely presenting students with images does not guarantee that they will automatically relate to these images or perceive them in the same way (Mason, 1992). The internal image formulated by the student is unlikely to be an exact copy of the external image presented. It is questionable that a student can actually be given an image. Indeed there is a greater possibility that they will be prompted into forming an image for themselves (ibid). Students should be actively involved in creating their own mathematical visualisations, true visual understanding cannot be expected to be achieved from observation of the teacher's images alone (Cunningham, 1994). As a consequence of this, teachers need to allow students adequate time to develop their own images, and should anticipate and encourage diversity in visualisation (Bishop, 1989). Teachers also need to be aware that an image alone is unlikely to be an adequate stimulus to learning, particularly "if it does not challenge or surprise, or otherwise activate sense making" (Mason 1995b, p. 131).

Mason (1992) also warns of the danger of students being saturated by images. He emphasises that some images are more useful than others, provoking further awareness and thought and as such should be especially encouraged, whilst other less productive images should be discouraged (ibid). Bishop (1989) argues that abstraction is necessary if visual imagery is to serve a useful function in mathematical thinking, especially in transcending the rigidity and inflexibility associated with the use of specific concrete visual images. Presmeg (1992) also adopts this position, suggesting that the centrality of imagery in mathematical processing is possible only if imagery is used in the service of abstraction in mathematics.

Tall (1986) emphasises the importance of allowing students to "explore ideas to fill out their own imagery" (p. 8). He recognises that "visual ideas often considered intuitive by an experienced mathematician are not necessarily intuitive to an inexperienced student" (Tall, 1991b, p. 105). Mariotti and Pesci (1994) ask whether it is possible to intentionally intervene in the process of visualisation and, thus, are keen to investigate the relationship between 'external images' provided by the teachers and the 'internal images' of students. Certainly, as visualisation can be regarded as a 'subjective phenomenon', direct intervention to promote students' imagery may be "neglected or even discarded" by teachers (ibid, p.22). Mason (1992) raises some important questions concerning factors which affect the nature and degree of the robustness of images, their validity and the ability to communicate such images, which are often difficult to translate into words. He (ibid), also, recognises a difference between personal, idiosyncratic images, and conventional images as cultural tools and the role of the teacher in offering and supporting some images and not others.

Cunningham (1991) also highlights the role of the teacher for facilitating visual learning in the classroom, and suggests that an instructor using visualisation should:

- *determine exactly the critical mathematical details to be presented in an image and show these either by highlighting them or by removing conflicting information,*

- *determine the order in which material is to be demonstrated by the images and present this material in a logical and connected sequence,*

- *offer students options in ways that expand their mathematical knowledge without confusing or overwhelming them,*

- *look for opportunities to present dynamic or developing mathematical processes and give students appropriate opportunities to explore and control them,*

- *consider carefully how students will learn visually, how to evaluate such learning, and how to integrate this learning with other parts of their mathematical studies. (Cunningham, 1991, p. 74).*

In addition, Cunningham (ibid) emphasises the importance of considering which material is best introduced visually and which symbolically, and at what point, if ever, can the two approaches be combined. He further advises (Cunningham, 1994) that if and when a technology based visual approach is considered appropriate, symbolic approaches should not be introduced until the topic is understood visually and image based discussions and ideas are formed. Alternatively, Eisenberg and Dreyfus (1986) maintain that every topic should be developed in terms of its analytical as well as its visual aspects, thus enabling each student to "grasp the material in the way which is closer to his cognitive orientation" (p. 158). This thesis has combined these ideas so that all of the function concepts explored by the students have been considered in terms of their symbolic and visual aspects, whilst at the same time more visual based discussions have been encouraged.

Barwise and Etchemendy (1991) surmise that "much, if not most, reasoning makes use of some form of visual representation and that as the computer gives us ever richer tools for representing information, we must begin to study the logical aspects of reasoning that uses non-linguistic forms of representation" (p. 22). It is now seen to be appropriate to discuss the relationship between learning mathematics in general, visualisation and technology, which provides the basis for chapter three.

CHAPTER 3
LEARNING MATHEMATICS, VISUALISATION AND GRAPHICAL CALCULATORS

3.0 Overview

This chapter considers the theoretical perspectives surrounding the use of technology in students' learning of mathematics in general and more specifically with respect to visualisation and functions. In the first section, the socio-cultural framework that was developed in chapter two is extended to include the relevant theories on learning mathematics with technology in general and graphical calculators in particular. This is followed by further consideration of the current literature on visualisation in mathematics and how this relates generally to the use of technology and more specifically to graphical calculators. The role of technology in mediating students' use of visualisation and understanding of functions is then explored with particular emphasis on visual images for functions.

3.1 Socio-Cultural Framework: Theoretical Implications for the Use of Computers and Graphical Calculators

3.1.1 Technology and Reorganisation of the Classroom

In accepting the Vygotskian notion of learning as communicative, Jones and Mercer (1993) argue that there will be "significant implications" for the way in which the role of the computer in the learning process and the role of the teacher in relation to the use of computers are perceived. From this perspective, technology is viewed as a "medium through which a teacher and learner can communicate", rather than as a substitute for the teacher (p. 22). Jones and Mercer (ibid) develop this position by referring to the work of Cole and Griffin. Cole and Griffin distinguish between the computer viewed as a 'partner in dialogue' and as a 'medium'. As a 'partner in dialogue' the "student-computer system" is regarded as "an analogue to the student-teacher system with the computer replacing the teacher" (1987, cited in Jones and Mercer, 1993, p. 23). Alternatively the notion of the computer as a 'medium', does not intend to replace the teacher, and rather "reorganises" the interaction in the classroom, which creates "new learning environments". This view is also taken by Hoyles and Noss (1992), who propose that using the computer creates a shared language between teachers and students, in which they can talk about their activities and interact with the technology, and a shared medium for the teacher to communicate ideas to the students. The computer-based language thus supports inter-mental functioning, which from a Vygotskian perspective is a precursor of intra-mental functioning (Sutherland, 1995). In also adopting a Vygotskian stance, Lerman (1998) proposes that "powerful technologies can offer possibilities for novel ideas by children, which create zones of proximal development for other participants and change the social relations in the classroom" (p. 77). Similarly, Confrey (1993) argues that technology is necessarily seen to alter the character of knowledge.

Cole and Griffin (1987, cited in Jones and Mercer, 1993) regard teachers who successfully introduce technology into their classrooms as 'orchestrators' of student activities. The teacher has an important role to play in mediating the technology; the mediation between human-computer systems is a two way process. The teacher's input is an 'essential' element of any learning process and has a significant influence on the quality of learning that arises in a technological environment. This idea fits well within the Vygotskian theoretical framework adopted in this thesis, which sees mediation by more knowledgeable persons as a fundamental part of the way in which knowledge is transmitted through society. Similarly, Olive (1992) considers the role of the teacher to be crucial in promoting effective learning. Hoyles and Noss (1992) argue that teachers have an important part to play in encouraging conscious reflection amongst students, which is an integral constituent of higher mental functioning. They maintain that global mathematical understandings are developed and generated by the teacher, although at the same time the computer mediates this process.

Similar perspectives have also been elaborated with reference to the graphical calculator. Guin and Trouche (1999) argue that the reorganisation of the activity resulting from the introduction of graphical calculators affords new possibilities of action for the user, and that this may provide new conditions and new means of organising action. Borba (1996) proposes that the use of graphical calculators can "enhance mathematical discussions" and that, following Tikhomirov, this in turn 'reorganises' the way that knowledge is constructed. In his discussion of the effects of graphical calculators on human activity in the classroom Borba (ibid) draws on Tikhomirov's research with computers and describes three theories concerning the relationship between computers and human activity, which Tikhomirov had outlined and critiqued. The first of these theories, the theory of substitution, sees the computer as a substitute for human endeavour since the computer has the capacity to solve complex problems that were previously only solvable by humans. Tikhomirov rejected this theory on the grounds that the heuristic mechanisms used by computers and humans in solving problems differ significantly.

The second theory to be described by Tikhomirov concerns supplementation and sees the computer as a complement to, rather than a substitute for, human endeavour. In this way the computer increases the capability and speed of human beings to perform given tasks. Tikhomirov also rejects this theory, arguing that thinking involves more than the simple solution of problems. He stresses the importance of the formulation and attainment of goals in the thinking process and that there are other important characteristics of solving and formulating problems, such as human values. In addition, he suggests that this theory does not take account of the meanings that are given to manipulated symbols. Thus Tikhomirov argues that the supplementation theory does not adequately describe the relationship between computers and human activity and he puts forward the theory of re-organisation, which draws on Vygotskian ideas.

This theory sees computers as re-organisers of human activity rather than as a tool that is merely added to the human experience. Tikhomirov supports this stance and proposes that "as a result of using computers, a transformation of human activity occurs, and new forms of activity emerge" (1981, cited in Borba, 1996, p. 54). He stresses that emphasis should be placed on human-computer systems and to problems that can be solved by them. He also proposes that computers take on a role that is similar to that of language in Vygotskian theory, and thereby represent an alternative means of regulating human intellectual activity. He concludes that computers do not substitute or supplement the human experience but rather reorganise human activity, in the sense that "there are activities that cannot be performed by either humans or computers alone but only by human-computer systems" (1981, cited in Borba, 1996, p. 58). In addition, he argues that re-organisation occurs through mediation of the teacher-students relationships by computers. In this sense human-computers systems are seen to produce new forms of teacher-student relationships and thus can provide new ways of legitimating and justifying students' findings in the classroom (Borba and Villarreal, 1998).

In each of the theories outlined by Tikhomirov, there are associated implications for educational practices (Borba and Villarreal, 1998). If, for example, the computer or graphical calculator is viewed merely as a supplement, then tasks are likely to be set that are similar to those which can be solved without technology, thereby limiting the use of this tool to simple verification of results or illustration of a given topic. However, in accepting the view that technology re-organises the way in which knowledge is constructed, Borba and Villarreal (ibid) propose that a technology based experimental approach is a more fitting and productive. Using this type of approach Borba (1996) reported evidence that supported Tikhomirov's notion of re-organisation in the classroom with respect to the use of graphical calculators. The first feature of this re-organisation was manifested in the intensification of the discussion that occurred as a result of using the graphical calculators. In accounting for this occurrence, Borba (ibid) suggests that graphical calculators might facilitate more independent experimentation and generation of conjectures, thus contributing to a certain sense of ownership by the students. In addition, Borba (ibid) proposes that the graphical calculators represented a new 'authority' in the classroom, which was additional to that of the teacher, as the students "found strong support for their positions in the graphical results of their experimentation" (p. 59).

The second feature of re-organisation in the classroom was seen as the flexibility for students to pursue different paths of enquiry as afforded by the graphical calculators. In Borba's view "the more independent experimentation that is possible with such human-computer systems facilitates exploration of mathematical subjects that might not be ordinarily explored in the classroom" (p. 59). However, as this medium did not suppress the use of other media in the classroom, such as orality and pencil and paper, there were some instances where students used the graphical calculator merely as a means of checking results. This indicated that students might occasionally use technology in a way that resembles the supplementation theory (Borba and Villarreal, 1998).

Borba (1996) also suggests that the use of the graphical calculator had seemed to influence his students' understanding. Indeed, he argues that the graphical calculator has a central role in students' discussions and in the reorganisation of their thinking (Borba and Villarreal, 1998). Borba and Villarreal (ibid) suggest that the human-graphical calculator system can be thought of as being the actor of students' argumentation in terms of the metaphors that students use in discussing their ideas which are derived from their use of the graphical calculator. Moreover, they propose that even when the graphical calculators are not being used by all students at all times, "the graphical calculator, the tasks and the environment generated has led to such a reorganisation of thinking" (Borba and Villarreal, 1998, p. 138). In other words, using Tikhomirov's terminology, despite periods of non-calculator activity amongst their students, 'human-graphing-calculator systems' were still in action. Similarly, Hoyles and Noss (1992) also pointed towards the scaffolding provided by the computer in enabling the teacher to develop student understandings in non-computer settings. In relation to the research being conducted in this study, this is seen to be a key idea that has important consequences for the interpretation of data involving students' behaviour following the introduction and use of graphical calculators.

In Borba's (1996) research the graphical calculator was seen as a mediator in the Vygotskian sense, of both the teacher-student relationships and the interactions between individual students. Other researchers also refer to the mediational role of technology. For example, Pea (1987) argues that "social environments that establish an interactive social context for discussing, reflecting upon, and collaborating in the mathematical thinking necessary to solve a problem also motivate mathematical thinking" (p.104). In developing this position, he particularly emphasises the fact that technology can play a fundamental mediational role in promoting dialogue and collaboration in mathematical problem solving. Teasley and Roschelle (1993) also see the role of technology from a mediation perspective in which the use of external displays act as tools for negotiating meaning and bridge the gap between spontaneous and scientific concepts:

> *Rather than seeking a perfect denotational relationship between external sign and internal concept, the mediation perspective accepts that interpretation is inherently uncertain, especially to newcomers to a particular community (Teasley and Roschelle, 1993, p. 232).*

Borba (1996) reported that there was no evidence to suggest that graphical calculators differed from computers with respect to facilitating re-organisation of the mathematics classroom. Berger (1998) however questions whether an understanding of the interactions within the computer environment can automatically be transferred to the graphical calculator context. She highlights the fact that graphical calculators and computers differ in status as cultural artifacts, indicating that graphical calculators generally tend to have a lower socio-cultural status than computers. More pertinently, she stresses that as the graphical calculator does not have the same interactive capabilities as the computer, "the type of relationship that the learner forms with a graphical calculator is qualitatively different from most probable or possible relationships with a computer" (p. 14). This relationship between the learner and the technology will also be affected by the degree to which the user is able to modify graphical images. In general the computer provides the student with a greater opportunity for graphical manipulation. Berger (ibid) concludes that the learning experience afforded by the graphical calculator is significantly different to that which is made available through computer technology, and argues that as such this warrants separate research and interpretation into the effects of graphical calculators.

3.1.2 Amplification and Cognitive Reorganisation Effects of Technology

In seeking to interpret the influence of the graphical calculator on students' learning from a Vygotskian perspective, Berger (1998) regards the use of the graphical calculator as an "external activity (manipulating concepts via graphs or numbers) which is ultimately transformed into an internal activity (understanding maths)" (p. 15). She (ibid) provides a useful framework for identifying the way in which the graphical calculator mediates students' learning of mathematics, by drawing on the ideas of Pea. Pea distinguished between the 'amplification effects' and the 'cognitive reorganisation effects' associated with using technology. The 'amplification effects' refer to the speed and ease by which the student is able to operate whilst using the technology. This is similar to Tikhomirov's theory of 'supplementation'. In contrast the 'cognitive reorganisation effects' are described as qualitative changes which occur as a result of using the technology. In this way cognitive reorganisation effects could also be regarded as manifestations of Tikhomirov's theory of re-organisation. The benefits of amplification are regarded as short-term phenomenon, providing the student with immediate assistance during problem solving. Long-term changes in the quality of learning arise through cognitive re-organisation.

Berger (ibid) re-interprets these ideas with respect to a Vygotskian framework. The graphical calculator is seen to "amplify the zone of proximal development" by creating a situation where the student is able to complete more "conceptually demanding tasks" effectively and easily (p. 15). Cognitive reorganisation is defined as "a systematic change in consciousness of the learner as a result of interaction with a new and alternate semiotic system" (p. 16). It is proposed that if access to the graphical calculator enriches or alters the student's conceptions, then the technology can be seen as a "tool with which to think", and moreover as a "tool which helps thinking to develop" (p. 16). Berger (ibid) argues that the learner needs to "engage thoughtfully with the technology", if internalisation is to occur (p. 19). It is not sufficient for a student to be merely introduced to the technology. She further suggests that in order for the learner to "interact in such a mindful way" he or she needs to "use the technology actively and consciously in a socially or educationally significant way" (p. 19). Merely using the graphical calculator to provide support and verification, rather than as a tool in its own right may limit the type of relationship that the user can form with the technology. Indeed, Berger argues that the perceived and actual status of the graphical calculator in any mathematics course profoundly effects the influence that this technology has on the students. Berger's framework has been used in the analysis and interpretation of the data from this thesis. This has helped in identifying the ways in which technology mediates students' learning and the kinds of reorganisation that occurs in the classroom as a result of introducing graphical calculators.

3.1.3 Local Communities of Practice

Another useful framework that has further informed the interpretation of classroom interactions in this thesis is that developed by Winbourne and Watson (1998), based on the work of Lave. Winbourne and Watson's notion of 'local communities of mathematical practice' is regarded as a useful way of perceiving classroom situations from a social and interpretivist perspective. This is especially so in relation to active learning approaches involving the use of graphical calculators and in examining the roles that individual students occupy in the construction and negotiation of meaning and how these roles develop through access to technology. This framework also provides an indication of whether any re-organisation of the classroom occurs in terms of establishing and maintaining local communities of practice as a result of introducing graphical calculators.

Winbourne and Watson (ibid) propose that any classroom can be regarded as an intersection of a multiplicity of practices and trajectories, which occur locally in terms of time as well as space. Their notion of 'local' communities of practice has therefore been developed in recognition of the fact that particular practices "might 'appear' in the classroom only for a lesson and much time might elapse before they are reconstituted" (p. 178). The following key features are identified as necessary for initiating local communities of mathematical practice (ibid, p. 183):

1. Pupils see themselves as functioning mathematically within the lesson;
2. There is a public recognition of competence;
3. Learners see themselves as working together towards the achievement of a common understanding;
4. There are shared ways of behaving, language, habits, values and tool-use;
5. The shape of the lesson is dependent upon the active participation of the students;

6. Learners and teachers see themselves as engaged in the same activity.

Winbourne and Watson (ibid) argue that participation in the practice of asking questions can enable students to generate mathematical questions themselves. Similarly, participation in the practice of using graphical calculators can allow students to become 'masters' in the use of these tools. Ultimately, the students can come to operate masterfully, within the constraints of the social setting. The individual student's 'positioning' within the community of practice will determine their success as a learner.

The process by which the individual achieves his or her position within a community of practice is encapsulated by the notion of 'telos'. This notion presupposes a common direction of learning and Winbourne and Watson (ibid) broadly describe telos as "an unfulfilled potential to move or change in many different ways" (p. 182). They contend that "telos could be conceptualised as a set of constraints in some sense inherent in situations and in the individual's pre-dispositions to respond to situations as she does" (p. 182). In this sense the individual student's learning is both determinant of the common direction of learning and in part determined by the complex paths that the students have taken to be where they are. The students fulfil their ultimate positions within the community of practice through smaller-scale 'becomings' in which they join the practice and begin to assume their eventual position. From a rich layering of practices and becomings, local practices emerge and are defined by and require the active participation of those who together constitute the practices. Within such practices there is a strong social pull on all, including those who are more peripheral, to participate. The student's experiences at school are mediated by the images of themselves that they bring as learners.

To increase the likelihood of establishing local practices, within which learners regard themselves as "members of a mathematical community", Winborne and Watson (ibid) envisage an important role for the teacher. This entails "constraining the foci for attention and recognising and working with pre-dispositions, rather than ignoring them" (p. 183). Establishment of such communities is conducive to improving the mathematical experiences of students and provides an indicator of effective teaching.

If there is to be access to a community of practice, then according to Adler (1998) resources used in the practice, such as technology, should be transparent: both visible and invisible. Learners need to be aware of the technology (the visible aspect) in order to extend the practice and at the same time the technology needs to illuminate mathematics (the invisible aspect) so that they enable smooth entry into the practice. Transparency is a function of how the technology is used and understood in practice, rather than being an inherent feature and depending on the relative transparency of the technology, the use of this resource can either enable or impede access to mathematical knowledge. For access to be enabled a balance needs to be established between visibility and invisibility.

Whenever technology is drawn on, it becomes visible, and is the object of attention. It takes on specific and situated meanings in the practices and the context of the mathematics classroom. However, if the technology is to enhance and enable mathematics learning, then at some point it will have to become invisible. In other words, technology will no longer be the object of attention itself, but rather the means to mathematics. Hence, Adler (ibid) proposes that mathematical meaning is developed from the mediated use of resources and through their relative transparency.

Winbourne and Watson's framework provides a means of focusing on the individual student's participation within the whole classroom community and this is an area that has generally been neglected in recent research. Hershkowitz (1999) sees learning as a changing membership of communities of practice and identifies the need for focusing on the individual student's development as he or she participates in the collective construction of shared cognition in small groups or in the whole classroom community. She claims that socio-cultural studies focus mostly on the interaction or the interactional event itself and that the individual student is generally an anonymous participant in classroom episodes. Furthermore, this is especially so in relation to the use of technology in general and as such consideration of the individual student's development within the social setting forms the basis of much of the analysis and interpretation of the data from this thesis.

3.2 Visualisation, Functions and Technology

3.2.1 Technology and Mathematical Exploration of Visual and Symbolic Modes of Representation

Many studies advocate the usage of computer technology and graphics calculators in promoting and enhancing students' abilities to visualise (Hoyles and Healy, 1996; Souza and Borba, 1995; Smart, 1995b; Goforth, 1992). In particular though, the most extensive research has been carried out with respect to computers. Cunningham (1994) recognises two essential features that contribute to the success of the computer based visual approach in teaching mathematics: the motivational aspect and the opportunity to pursue an alternative and yet complementary mode of thought to the traditional symbolic approach. For learning to be successful, dynamic images are considered to be preferable to static ones. As such, Olive (1992) regards computing technologies as tools for effective learning, especially when used as a means for mathematical exploration. Tall (1991b) states that "the exploratory stage of mathematical thinking benefits from building up an overall picture of relationships and such a picture can benefit from visualisation" (p. 106).

Hoyles and Noss (1992) argue that the computer opens up a whole range of possible alternatives, or 'strategic apertures', through which students can gain access to approaches and solutions that are simply out of reach when using pencil and paper. Similarly, Pea (1987) postulates that because the use of computers is dynamic, it allows student interaction with mathematics in ways that would not be possible in non-computer environments. He maintains that through the use of technology "students can test out hypotheses, immediately see their effects, and shape their next hypotheses accordingly through many cycles, perhaps through many more cycles than they would with non-computer technologies" (p. 111). Thus technological forms are recognised as being mediums in which students can make and test predictions. The teachers interviewed by Furinghetti and Bottino (1996) were clearly aware of the potential of technology in this area, as one member of staff commented: "software is useful to enrich students' experience with mathematical concepts, especially students' capacity of visualisation" (p. 132). Thus, it was found (ibid) that technology was utilised in the majority of cases with the expectation that this would assist students in developing their visualisation skills. According to another of the teachers interviewed by Furinghetti and Bottino (1996), the success of technology in furthering the learning of individual students depends on the type of reasoning methods that students use (e.g. visualisation, abstract reasoning). In particular, 'good' students appeared to benefit from the technology and were able to exploit the capacity of using computers.

Tall (1989) suggests that "suitably programmed software can provide a tool which compensates for the human deficiency in visual communication" (p. 42). He also insists that:

> *generic organisers (environments that enable the learner to manipulate examples and non-examples of a specific mathematical concept or a related system of concepts) may be used to give a more overall holistic grasp of concepts, linking them together in a global, often visual way, as distinct from the accent on learning sequential processes of mathematics in the traditional curriculum. (Tall, 1989, p. 41).*

In such environments meaningful learning can be achieved by active participation of the learner during the reception of knowledge. However, he recognises that without some external organising agent, such as a teacher's guidance or textbook, cognitive obstacles could arise (Tall, 1986).

Technology makes external the intermediate products of thinking, which can then be analysed, reflected upon and discussed (Pea, 1987). In this way, technology externalises visual images and concurrently allows them to be manipulated, thus facilitating abstraction (Noss et al, 1997). On screen representations of visual objects and relationships can be acted on directly, enabling the user to observe the resulting changes in the represented relationships. Even more importantly, it becomes possible to investigate which actions will lead to a given change in the existing relationships, so that the situation can be inverted (ibid). Mason (1995b) proposes that the most productive and desirable use of electronic screens "requires exploiting not replacing the mental screen, which in turn involves seeing screens as things to look through rather than at, in order to contact potential generality through the particular screen images" (p. 119). The purpose of looking through a screen is to begin to see the screen image as an instance of a more general phenomenon, rather than merely seeing the particular in the image. He also stresses that screen images tend to require more work on the part of the student to be able to see through the particulars to the general, so that whilst these may enhance learning, they may not necessarily make this less time consuming. The very richness and complexity of diagrams increase the potential for ambiguity and multiplicity of interpretation, which means that it becomes important to "work at flexibility and multiple representations, so as not to be trapped in a single interpretation" (ibid, p. 129). He contends that with technology, the "notions of symbolic and imagistic begin to intertwine":

sometimes it is hard to distinguish between the mental experience and the physical screen; the latter acts as a window, and the mental screen becomes the world experienced through the window (Mason, 1995b, p. 130).

The "dynamic and interactive media provided by computer software make gaining an intuitive understanding of the interrelationships among graphic, equational and pictorial representations (traditionally the province of the professional mathematician) more accessible to the software user" (Pea, 1987, p. 96). Hoyles and Healy (1996) similarly argue that the fusion of action, visualisation and representation made possible in computer environments can provoke cognitive reorganisation. In recognition of this potential, Guin and Trouche (1999) advise that the teacher should present situations that lead to reflection on the various results arising from different calculation modes.

Tall (1987) envisages that the distinction between numerical, graphical and symbolic forms of representation will become 'more diffuse' as multi-representational software is developed. Confrey (1994) argues for a 'epistemology of multiple representations', in which the contrast between representations is recognised as significant in achieving the convergence of ideas necessary in establishing meaning. By identifying multiple representations, we can encourage students to find multiple ways to make sense of their results and to develop their sense of flexibility and elegance. Multiple approaches support the "diversity in students' preferences and provide alternative approaches to use when faced with cognitive obstacles" (p. 218). She, also, advises (Confrey, 1993) that "in a multi-representational tool, no representation should dominate others, and, in every representation there is both a loss and a gain" (p. 66).

Kaput (1992) identifies two key purposes of multiple linked notations: firstly, "to expose different aspects of a complex idea" and secondly, "to illuminate the meanings of actions in one notation by exhibiting their consequences in another notation" (p. 542). He contends that since "all aspects of a complex idea cannot be adequately represented within a single notation system", multiple systems are required for their full expression (p. 530). Multi-representational software, however, could contribute towards misunderstanding and confusion amongst students. According to O'Reilly et al (1997), "multiple representation software runs the risk that the difficulties of reading a representation are simply multiplied up by the number of modalities represented on the screen simultaneously. The student has to make sense of each modality in turn and the links between them" (p. 88).

3.2.2 The Use of Technology and Students' Understanding of Functions

The topic chosen for this investigation has been functions in recognition of the fact that this is a key area of the secondary mathematics curriculum, especially in relation to GCE Advanced level mathematics. Indeed, Confrey and Smith (1992) argue that the importance of the function concept in the secondary curriculum is virtually undisputed. They assert that calls for reform of mathematics place the function concept at the centre of the curriculum as an integrating concept and that its importance is located in its capacity for modelling. Wazir (1993) also comments on the significance of the function concept, suggesting similarly that functions are the centrepiece of mathematics instruction right across the curriculum. Wazir stresses the fact that "the function concept is one of the most important unifying concepts for arithmetic, algebra, transformational geometry, and calculus" (p. 475).

In the same vein Cuoco (1991) argues that the construction and use of functions is a central feature of most mathematical investigations. Yet, he also acknowledges that the function concept is notoriously difficult for students. He claims that thinking about functions requires students to reason about methods rather than about the particular objects themselves. Another problem arises from the fact that even when using the function concept for a specific purpose, people employ many different levels of abstraction when they define, describe and use functions. Thus, whilst the function concept is seen to be of central importance in students' learning of mathematics, it is also seen as an area that causes students considerable difficulties and is therefore an important research issue, especially in relation to new technologies. Leinhardt et al. (1990) suggest that the use of technology dramatically affects the teaching and learning of functions - perhaps more so than any other early mathematics topic.

In investigating the topic of functions, some researchers have chosen to adopt fairly broad definitions of functional thinking, whilst others have considered a much more limited concept of functions, as is the case in this thesis. Indeed, by focusing on fairly narrow definitions of functional thinking, as in this thesis, several researchers have found that by utilising graphical software that facilitates visualisation in the graphical context, a deeper understanding of functions, equations and inequalities can be fostered. For example, Confrey (1994) found that when students are given the opportunity to approach functions in a visual manner through the use of technology, they are more likely to develop an intuitive understanding of translations. Similarly, Bloom et al (1986) reported that students who were taught to recognise the graphs of functions as compositions of certain transformations of standard functions, using computer software, developed a greater understanding of functions and their graphs and in less time than those taught in the traditional manner.

Meissner and Mueller-Philipp (1993) also reported that students who had access to computer technology were able to develop better solution strategies for more problems in less time than traditionally taught students. This resulted in better overall performances and an improved understanding of functions in the computer students, who were seen to develop a relational, as opposed to a merely instrumental understanding for the concept of function. Fey (1989) suggests that "computer graphic tools can be used to revise the balance between conceptual and procedural knowledge in mathematics or create entirely new graphical orientated presentations of traditional mathematical topics" (p. 250).

Sivasubramaniam (2000) is another researcher who has found that students who explored graphs with computers performed significantly better than those using pencil and paper, suggesting that this was a consequence of the difference in the distribution of cognition over the tool and the individual. As the computer took over a large portion of the cognitive process for construction, students were freed to focus their attention on interpretation and the development of these skills, rather than viewing the construction process as an end in itself, as was a characteristic of the paper medium. Sivasubramaniam (ibid) points out that when students are confronted by a graphical situation, there are four possible alternatives with respect to their existing and possible schema:

- *the situation may reinforce their existing schema;*
- *the situation may cause an alteration of their existing schema to accommodate new information;*
- *the information provided by the situation is rejected and the old schema is retained;*
- *if a pupil does not have a schema then they construct a schema (p. 180).*

She proposes that the computer medium provides a means by which each of these possible actions can be realised, therefore providing the appropriate scaffolding for the development of a schema for graph interpretation. In contrast, the pencil and paper medium only supports the reinforcement of students' existing strategies, without direct interaction with the teacher or peers.

Hershkowitz and Schwarz (1997) similarly found that the cognitive development of students who were given access to technology whilst learning about function concepts was greatly enhanced. In particular they were enabled to (a) use many examples, (b) provide rich justifications of their answers, (c) show more flexibility within and among representations, (d) consider the acceptability of answers in light of the context and (e) better integrate prototypical elements of their function concept image with other examples. Furthermore, the process by which this conceptual change occurred was seen to be a product of social interaction and practices as well as individual activity. Computer transformations were internalised by students and argumentation played an important role in this internalisation. As Hoyles and Noss (1992) assert, the process by which pieces of mathematical knowledge are appropriated depends on the students' individual agendas, how they feel about their participation, the type of teacher intervention, and most significantly, on the setting in which the activities are undertaken.

Use of technology can also facilitate the process of generalisation and abstraction. Hoyles et al (1991) considered a fairly broad definition of functional thinking and found that students are more likely to formalise their arguments when working in technology environments, using theoretical descriptions, rather than empirical ones, which are much more characteristic of pencil and paper environments. Furthermore, in their study, the students' interaction with the computer was used as a means of checking relationships after they had been formalised. In contrast, students working in the pencil and paper environment tested their generalisation with a number of specific cases as a means of convincing themselves before any attempt to formalise. Whilst students need to deal with particular functions, Goldenberg (1991) proposes that most of the educational value is gained from the generalisations they abstract from the particulars.

The use of technology also has significant implications for the teaching and learning of calculus. Ubuz (1994) believes that visualisation in the graphical context can help students to understand the relationship between differentiation and integration. In general, students have "very weak visualisation skills in calculus" which subsequently results in "lack of meaning in the formalities of mathematical analysis" (Tall, 1991b, p. 105). Tall's (1986) findings indicated that students who used the programme, 'Graphic Calculus', were much more successful in visualising the derivative of a graph as a global function.

Technology is also seen by Smart (1995a) as a means of furthering students' understanding of functions and she stresses that as the advent of graphical calculators has made technology more widely available in the mathematics classroom, more students are now able to explore functions using technological tools. The following section will now focus primarily on research that has been carried out with graphical calculators.

3.2.2.1 The Use of Graphical Calculators and Students' Understanding of Functions

An important aspect of developing a robust understanding of the notion of function means not only knowing which representation is the most appropriate to use in different contexts but also to be able to move flexibly between different representations (Knuth, 2000). Use of the graphical calculator empowers students to explore, estimate, make discoveries graphically and approach problem from a multi-representational perspective (Hollar and Norwood, 1999). This in turn allows the student to progress from an operational to a structural understanding of function concepts (ibid). Technology enables the teacher to demonstrate effectively numerous functions and their graphs in a manner which could not possibly be achieved using relatively unsophisticated resources such as the blackboard and consequently gives the students a deeper insight into the relationship between functions and their graphs (Chola Nyondo, 1993). Technology therefore provides students with the opportunity to view many graphs alongside their corresponding equations, which allows them to begin to examine the relationship between graphical entailments and algebraic parameters (Leinhardt et al, 1990).

The ability of a student to recognise a given graph as belonging to, or resembling some member of a family of functions is a fundamental stage in the development of a solution (Ruthven, 1990). For, only when a student has been able to identify successfully the family of functions to which the graph belongs can the correct symbolisation be constructed (referred to as the 'process of refinement'). Goldenberg (1991) suggests that graphical exploration "provides valuable scaffolding for the required symbolic manipulations" (p. 85). Students must be visually aware of the effects of particular transformations and of the corresponding symbolic modifications. In this respect, graphical calculators could be utilised in enabling students to experiment with and investigate various graphs of functions prior to a more formal teaching approach (Cunningham, 1994). Students using graphical calculators are able to access and compare the graphs of many functions belonging to the same family, quickly and relatively easily, allowing them to see relationships between the graphs for themselves and to draw their own generalisations (Ruthven, 1990). In this way, graphs of a greater complexity could be explored, exposing students to more advanced and more challenging mathematics, perhaps beyond their current level, thus amplifying the zone of proximal development.

The graphical calculator can thus enable students to learn and understand function concepts at a higher level than previously available in the traditionally taught classroom (Shoaf, 1996). This occurs as conceptual understanding takes the place of rote understanding (ibid). Students who are normally passive become "actively involved in the discovery and understanding process, no longer viewing mathematics as simply the receiving and remembering of algorithms and methods of solution" (Shoaf-Grubbs, 1995, p. 227). Moreover, when students are more actively involved in the learning process through the use of technology, concepts that have been acquired are likely to be retained over a longer period of time (Guttenberger, 1992; Shoaf, 1996). A lack of familiarity or competence with algebraic techniques does not stop a student from exploring complicated graphical representations of functions generated by technology (Leinhardt et al, 1990). On the contrary, through the use of technology, it is possible to explore ideas of functionality independently of manipulative algebra (Ruthven, 1996).

Selinger and Pratt (1997) propose that the use of graphical calculators can support and develop students' mental images by making them more robust, suggesting that an experienced user is likely to form some sense of the characteristics of functions, through contrasting their images with the feedback provided on the screen. They suggest that technology can become a "vital medium for validating students' naïve attempts at expressing their mathematical ideas" (ibid, p. 39). As the student is the one who controls and formulates the input and then makes sense of the output, the graphical calculator can be used as a tool for mediating mathematical meaning. In addition, "the teacher can enter the world of a student's thinking through the screen as if it were a window on the students' mind, using this insight as a means to help scaffold the students' understanding" (ibid, p. 40). Noss et al (1997) also maintain that technology environments provide windows into students' meaning making.

Mesa (1997) found that the graphical calculator supported her students' abilities to solve problems involving functions in two distinctive ways, which differed depending on the students' prior experiences of functions. Firstly, the graphical calculator was used as a verification tool whenever the problems to be investigated by the students related to their prior knowledge of functions. In this way, the technology proved to be crucial for enabling students to detect any mismatches in their approaches. On the occasions where the problems posed did not relate to the students' previous knowledge of functions, however, the graphical calculator was alternatively used as a tool for exploration.

Doerr and Zangor (2000) also found that students used the graphical calculator in different ways. They reported the emergence of five rich patterns and modes of graphical calculator use in the classroom, which were influenced by the nature of the mathematical tasks and the role, knowledge and beliefs of the teacher. The graphical calculator was seen to take on the role of (i) a computational tool; (ii) a data collection and analysis tool; (iii) a visualising tool (iv) a tool for checking conjectures and (v) most significantly as a transformation tool, whereby tedious computational tasks were transformed into interpretation tasks. These patterns of graphical calculator use were not mutually exclusive and as the students' understanding grew certain modes were replaced by others. For example, the graphical calculator shifted from that of a visualising or graphing tool to a visual checking tool as the students grasped the ideas of transformations. This transformation in the way in which the graphical calculator was used supported students' thinking about the idea of the non-uniqueness of the algebraic representations for exponential or trigonometric graphs (Doerr and Zangor, 1999). However, Doerr and Zangor (2000) stress that it is important to make students aware that the graphical calculator cannot provide the authority for a mathematical argument by itself. The students in their study thus came to realise that there was also a need to check the absolute reliability of results produced by the graphical calculator based on their own mathematical understanding.

Shumway (1990) insists that the availability of graphical calculators trivialise mathematical computations, rather than the mathematical problems to be explored. As a consequence emphasis should be placed on developing conceptual understanding through technology, focusing on:

- *The meaning of a representational graph;*
- *The relationship between the function graphed and the actual problem;*
- *What to do when such methods produce erroneous solutions or no solutions at all;*
- *The proof that the result is correct (Shumway, 1990, p. 3).*

However, Ruthven (1996) points out that computation with the graphical calculator is not a completely trivial and automatic process. For example, to sketch the graph of some specified expression, the student must first translate this into a suitable format for entering into the machine and then choose an appropriate range and scale for the axes. Furthermore, using the technology does not imply that the students' own computational skills will be adversely affected (Hollar and Norwood, 1999).

In a review of research into the impact of graphical calculators on the teaching of functions, Dunham and Dick (1994) report that use of this technology can empower students to become better problem solvers, through:

> *(i)* *freeing time for instruction by reducing attention to algebraic manipulation;*
>
> *(ii)* *supplying more tools for problem solving, especially for students with weaker algebraic skills, and serving as a monitoring aid during the problem solving process;*
>
> (iii) *enabling students to perceive problem solving differently as they are freed from computational burdens, allowing them to concentrate on setting up the problem and analysing the solution (p. 442).*

They conclude that graphical calculators can facilitate changes in the roles of students and teachers in the classroom, resulting in the creation of more interactive and exploratory learning environments, where the technology is a catalyst rather than an obstacle to mathematics learning. Mesa (1997) recognises a need for teachers to provide students with "problems that can be solved either with or without the use of the graphical calculator, but such that if the graphical calculator is used the student can pursue different approaches, do more exploration and make more generalisations" (p. 246). She also emphasises the need for developing classroom environments where exploration is important, in which the teacher's role is to provide limits to that exploration. Selinger and Pratt (1997) recognise the need for mediation between the student, teacher and screen. They propose that there is an important role for the teacher in a graphical calculator environment in acting as an arbitrator in deciding whether a student's way of expressing mathematical thinking is valid. This role involves intervening where appropriate and pointing out the correct usage of acceptable mathematical notation, especially where graphical calculators or computers are directly involved. This thesis aims to build on the existing body of literature concerning the relationship between students' learning of functions, the teaching they experience and their use of graphical calculators.

3.2.3 Problems Associated with the Use of Technology

Carulla and Gomez (1997) appreciate that whilst the use of graphic calculators can enhance the learning of functions and graphing concepts, there may be associated problems. Their findings indicate that in certain circumstances students might misunderstand, misinterpret and, thus, misuse information provided by the graphic calculators. Similarly, Goldenberg (1987) found that students often misinterpreted what they saw in the graphical representations of functions that had been produced using technology. Moreover, he noted that when these students were left alone to experiment, without the input from a teacher, they could induce rules that were incorrect. Different students bring a variety of general strategies to their interpretations of graphs (Goldenberg, 1991). Interpretation is especially important when emphasis is placed on abstracting and relating features of several graphs, rather than on reading specific values off of the graph. However, for graphs to be interpreted correctly, specific mathematical knowledge and expectations are required. Students who lack this knowledge are likely to misinterpret graphical information and invent complex and misleading explanations of the inter-relatedness of graphs (ibid). Goldenberg proposes that mathematically rich problem situations that derive from ambiguities in graphical representations of functions can be generated and overcome by identifying patterns in students' misinterpretations. Chola Nyondo (1993) also stresses that the teacher plays a vital role in developing the background theory that is necessary for students' learning to be successful with technology.

The absence of a clearly labelled scale on the co-ordinate axes of graphs produced by the graphical calculator can also cause difficulties for students when trying to identify what part of a graph is being displayed (Doerr and Zangor, 1999; Ruthven, 1996). Doerr and Zangor (2000) found that this limitation increased the difficulties that students had in interpreting periodic graphs in particular. Leinhardt et al. (1990), with findings similar to Goldenberg (1987), stress that the issue of scale becomes much more fundamental when technology is used, especially as a graph cannot be interpreted fully without taking into account its scales. This makes it even more important for students to develop strategies to determine whether or not the portion of a graph visible on screen is reliable and representative of the behaviour of the graph as a whole. Leinhardt et al. (ibid) warn that if gone unchecked, incorrect images produced by technology may be remembered by students, which can cause misunderstandings to be perpetuated in future work.

Carulla and Gomez (1997) also reported that the use of graphical calculators could encourage students to concentrate on graphical representation systems at the expense of verbal and symbolic representations. In contrast, Penglase and Arnold (1996) found that graphic calculators could promote the transition between symbolic manipulation and graphical investigation and exploration of the different modes of representation associated with particular concepts. Ruthven's study (1990), also, supported the view that regular use of graphic calculators would probably 'strengthen' and 'rehearse' relationships between certain symbolic and graphic forms (p. 447).

Another potential problem associated with the use of graphical calculators or computers is the possibility that students might regard the solutions provided by the technology as irrefutable, and thus become too heavily reliant on the machines (Smart, 1995b, Guin and Trouche, 1999). Students using graphical calculators regularly should ideally be able to recognise when, due to scale factors, a particular portion of a given graph is not visible on the screen. However, a lack of understanding of the principles underlying graphing, facilitated and reinforced by over-reliance and blind faith in technology, can lead students into believing that the image on the screen represents the whole graph, rather than merely a window displaying part of the graph. Smart (ibid) and Leinhardt et al. (1990) refer to this as the 'magic' element of the technology, and insist that the teacher needs to be wary of this effect. Similarly, Doerr and Zangor (2000) claim that one of the major constraints and limitations of the graphical calculator resides in students' attempts to use it as a 'black box', without having a meaningful strategy for its use. Guin and Trouche (1999) found that the students' dependency on the graphical calculator was gradually eroded through co-ordinating the use of this technology with other media in the classroom, such as pencil and paper. This resulted in a decreasing trust of the results produced by the technology. In addressing the issue of dependency, Guin and Trouche (ibid) see a crucial role for the teacher, suggesting that an unaccompanied acquisition of the use of the graphical calculator may be dangerous to the conceptualisation process.

Zimmerman (1991) emphasises that if technology is used appropriately in the classroom, so that it is not merely used as a 'crutch', students can be given the power to master visual thinking skills, such as those associated with elementary function sketching. He proposes that students should be encouraged to work out the geometrical properties of a function by themselves before using the technology, wherever this is possible. In this way, the role of the technology is to verify calculations and conjectures about functions, to fill in quantitative details, and/or to plot functions of higher order complexity. Alternatively, computers or graphical calculators could be used to enable students to explore and experiment with the properties of graphs. However, Zimmerman (ibid) stresses that guidance, feedback and eventually a synthesis of important results must be built into the process. Zimmerman (ibid) further emphasises that the use of technology does not remove the need for the students to understand the properties of functions in order to graph them. Indeed, he sees the ability of students to recognise incorrect or misleading graphs produced by the technology, and then to make an appropriate interpretation as a component of visual thinking.

One should not assume that access to technology will automatically improve students' abilities to comprehend new concepts, to visualise, to learn more effectively or to retain knowledge. Simply interacting with technology is not sufficient to ensure that students will 'acquire' specified mathematical ideas (Hoyles and Noss, 1992). Zimmerman and Cunningham (1991) assert that without certain fundamental visualisation skills (such as the ability to construct, interpret, and use simple figures as aids to problem solving) it is doubtful that computer-based visualisation can be used efficiently or even meaningfully. Similarly, in the absence of a set of guiding principles to relate visualisation to the content and learning objectives of the course, any use of computer generated imagery is likely to be ineffective (Zimmerman, 1991). It is not sufficient merely to have access to technology, for sense-making does not follow automatically from direct manipulation, just as articulation does not necessarily arise from sense-making (Mason, 1995a). The awareness of an expert, a teacher, is required to encourage and support these transitions. The presence of teachers who are aware of their own awareness and can direct and stimulate students in sense making and articulating that sense is crucial (ibid).

Research suggests that students tend to find moving from a graph to an equation more difficult than moving from equation to graph, as there are different psychological processes involved (Leinhardt et al, 1990). Kaput (1992) suggests that it might be productive for the students to establish some idea of the links between different representation systems before technology is used, allowing them to gain some appreciation of the need for such connections to be made. Eisenberg and Dreyfus (1987) argue that students' tend to view graphs as being peripheral to the function itself, as an "additional load"; and as a consequence will avoid dealing with them wherever possible. They believe that teachers are responsible for:

- *Transmitting to the students a more well-rounded concept of what a function is, namely an abstract mathematical object having any of several concrete representations; one of the most useful of which is a graph,*

- *Teaching students to recognise those situations where graphical processing of functional relationships is more efficient than algebraic processing (Eisenberg and Dreyfus, 1987, p. 191).*

In their study the more able students were unexpectedly reluctant to experiment with unfamiliar functions using the technology, whilst the weaker students tended to experiment more with new formulae. However, the weaker students did not attempt to think them through beforehand (ibid).

Laridon's (1996) study also provides some interesting and yet rather unexpected results. His research concerns the effectiveness of the graphing calculator in the teaching and learning of function transformation concepts. As part of the teaching-learning process, particular emphasis was placed on enabling the learner to make generalisations, in view of the students being able to elicit a global view of common behaviour. Post-tests revealed that the group of students working with pencil and paper and ordinary scientific calculators, on average, outperformed those who used the graphing calculators. Moreover, the retention of the point-plotting group was higher. However, a "finer analysis which provided scores on specific conceptual elements within the test did not detect any qualitative differences across the two groups" (p. 182). This may have been a consequence of how the graphical calculator was used in this classroom. Many researchers (Berger, 1998; Gomez and Fernandez, 1997;

Penglase and Arnold, 1996) emphasise that the way in which new technology is integrated into the curriculum affects the learning outcomes.

Hewitt (1992) stresses that mathematical exploration in the classroom is often reduced to obtaining numerical results, finding patterns and generalising, rather than investigating particular situations in depth, which leads to more meaningful discovery. He questions whether "the diversity and richness of the mathematics curriculum is being reduced to a series of spotting number patterns from tables" (ibid, p. 7). Noss et al (1997) also recognise this as an undesirable, but common characteristic of school mathematics, especially in the UK, where students happily search for relationships by constructing tables of numerical data, without appreciating the need to understand the structures underpinning them. Cunningham (1991) argues that in order for visualisation to be introduced satisfactorily in the curriculum, the number of topics covered must necessarily be curtailed. He suggests that since manual symbolic manipulation and rote number crunching are "becoming less productive and appropriate" they are thus, reasonable choices to be replaced by visualisation in the curriculum (p. 72). However, to maintain mathematical rigour, some symbolic manipulation needs to be approached by hand. In other words, in keeping with the traditional view of learning mathematics and in preparing students for examinations, as well as enabling them to make sense of each representation system through exploring the links between them, there is a need for some symbolic manipulation to be attempted on paper. If students were completely reliant on technology to perform all symbolic operations, there is a possibility that these processes would become less understood, especially as there would no longer be the need for students to necessarily apply any in-depth thought to the algebraic problem solving process. The technology could possibly supply students with an answer to a given problem, without detailing the way in which this solution was derived. In this manner students miss out on seeing, following and reproducing the individual steps in the process for themselves. Moreover, these students would be denied the opportunity to explore an equally important area of mathematics. Hughes Hallett (1991), on the other hand, advocates replacing only the particularly "torturous analytical procedures with more graphical and numerical work", thus "emphasising the interpretation of results as well as computation", and instigating student thinking (p. 125).

3.2.4 Technology, Collaboration and Confidence

Several researchers have found that the use of technology can encourage collaborative learning (Hudson, 1996; Smart, 1992; Olive, 1992) and promote equal opportunities (Smart, 1992). Indeed, "introducing the graphic calculator as a tool for collaborative and investigative mathematics is the way forward to empowering *all* pupils" (ibid, p. 14). In particular, female students, especially, have welcomed and benefited from the opportunity to use graphic calculators, as they feel more comfortable using such personal and private forms of technology (ibid). Furthermore, Smart's study (1995b) of thirteen year old girls suggested that graphic calculators enabled these girls to "develop strong visual representations of functions given in the symbolic form" and in addition "allowed them to become unusually confident in talking about mathematics" (p. 195). Because of the ease with which the graphical calculator produces visual images of functions and the need for students to retain these pictures, Smart (1995a) found that these girls were prompted into talking about and describing their mathematics using more 'appropriate' mathematical language. In other words the graphical calculator provided an additional stimulus for meaningful talk (ibid).

Ruthven (1990) proposes that regular use of a graphic calculator can reduce student uncertainty and anxiety, and hence improve the confidence, competence and performance of all students. He argues that the graphical calculator provides the student with the facility to experiment with different ideas and methods, which leaves the student feeling more confident with complex and unfamiliar problems. Significantly, though, the relative attainment of the female students in his project group was greatly enhanced through the more extensive and accurate exposure to symbolised graphical images that the graphic calculator provided. The graphical calculator allows learning to occur on a more informal, as well as private basis, which can be an important factor in building confidence (Ruthven, 1996). Guin and Trouche (1999) also found that using the graphical calculator improved the self-confidence of students. Pea (1987) recognises the significance of collaboration with peers in helping to build individual students' confidence, suggesting that self-esteem can grow in collaborative computer environments where students view one of their peers as an expert.

Smart (1995b) began her study with the belief that the "presence of a visual image on the calculator screen pushes the learner into further investigations, enabling them to talk with confidence about mathematics" (p. 196). Her findings indicated that graphic calculators enabled these particular students to construct a "more fundamental visual image of equations given in the symbolic form" (p. 202). Consequently, these students chose to use visual reasoning when generalising and problem solving, rather than symbolic manipulations. Thus, her students became less reliant on symbolic manipulation as the predominant technique in problem solving. Indeed, students who are given the opportunity to develop graphical and numerical algorithms for understanding functions and are able to use these effectively could legitimately question the need for symbolic skills (O'Reilly et al, 1997).

Undoubtedly, the introduction of the graphical calculator into the classroom significantly changes the climate of the classroom (Dunham and Dick, 1994). This in turn however can, initially at least, lead to uncertainty. Gage (1999) argues that there is a disturbance to the normal social order in the classroom when students first use graphical calculators and that this may initially cause anxiety amongst students, "because nobody knows where they are in the 'pecking order' anymore" (p. 16). Students who are normally seen to be 'good at maths' can find themselves disadvantaged because they now need to work at their understanding of mathematical concepts, whereas this may not have been needed previously. Paradoxically, those students who have had to learn to "work at new things" can find themselves suddenly elevated to a position of much greater competence than they are used to. In Gage's study some of the students who would normally be expected to achieve very highly were either intimidated by the task or thought it was a waste of their time. However, the anxieties experienced by students were gradually overcome as their confidence in and experience of using the technology grew, along with the support of the teacher. The graphical calculator can ultimately empower students by giving them a private space in which to experiment and gain confidence (ibid).

Fey (1989) also acknowledges that there is a notable change in the roles and interactions of teachers and students accompanying the use of computer generated graphs as the focus of classroom discussion. Many students perceive traditional mathematics as a "formal game played according to arbitrary rules – a contest between teacher and student in which the challenge is to figure out secrets that the teacher keeps hidden" (ibid, p. 250). Within a computer environment, however, the classroom becomes a setting for student and teacher collaboration in which they attempt to make sense of the mathematics that is displayed before them. The teacher's role necessarily shifts from giving demonstrations of "how to" produce graphs, to providing explanations and asking questions of "what the graph is saying" about an algebraic expression or a situation it represents. In the same vein, the students' task is transformed from the plotting of points and drawing curves to developing explanations of key graph points and/or global features (ibid). Sutherland (1991) also emphasises that, through interacting with the computer, students are able to refine their own constructions, which contrasts with the traditional classroom situation in which students have to be told by a teacher that their constructions are incorrect. The computer provides an alternative source of explanation and validation (Hoyles and Noss, 1992). Moreover, as the students' sense-making occurs primarily as a result of the interaction between the student and the screen, rather than between the student and the teacher, the negotiation of meaning between the student and teacher becomes more equal (Selinger and Pratt, 1997).

The implementation of new technology also has implications with respect to the issue of control, as well as the social structure of classrooms (Kaput, 1992). Pea (1987) sees a technology as a means by which students can gain power and suggests that:

> *by infusing life into learning tools for mathematics, by integrating supports for the personal side of mathematical thinking with supports for knowledge, we can perhaps help each child realise how the powerful abstractions of mathematics confer personal power (Pea, 1987, p. 116).*

However, this source of power is not always exploited. Students can be reluctant to use the graphical calculator because they feel that doing so would sacrifice their intellectual autonomy, and ultimately they would be surrendering control of a mathematical argument (Ruthven, 1996). Povey and Ransom (2000) found that there was definite resistance, as well as enthusiasm, for the use of technology amongst the students in their study. In particular, many students were vehemently opposed to the idea of being taught to press buttons without sufficient understanding of the mathematical processes used by the technology to generate the output. There was a sense that these students associated conceptual 'understanding' with the ability to perform pen and paper algorithms and consequently, they assigned importance to the learning of equivalent pen and paper methods before technology is introduced. In their desire for understanding, the speed and opacity of the technology were seen to deprive them of their sense of control. Not knowing how the technology produces its results made the students feel out of control of their learning and they subsequently saw a need for these results to be checked by hand in order to regain control. Fears of dependency on the technology were expressed by several students, which they felt could result in lack of motivation for understanding and individual laziness. Thus, Povey and Ransom (ibid) conclude that for some students a strong sense of self can be placed in jeopardy by the use of technology. However, this does not mean that technology should not be used, rather they argue that teachers should "heed and work with students' concerns in ways which respect their roots in issues of personal worth and identity" (p. 61). Students' reluctance to use technology, however, can have beneficial consequences. Guin and Trouche (1999) propose that students who are more unwilling to use the graphical calculator can construct a more efficient relationship with the technology, whilst retaining a certain objectivity.

Technology is increasingly seen as a tool to support both the learning and teaching process and to foster interaction in the classroom (Furinghetti and Bottino, 1996). The computer draws the attention of the students and becomes a focus for discussion (Hoyles et al, 1991). Carulla and Gomez (1996) found that following the introduction of graphic calculators in a university course, mathematical knowledge appeared to be constructed socially rather than individually. Indeed, as discussed previously, technology mediates both teacher-student relationships and interactions between students, thus initiating some form of classroom re-organisation, the nature of which is dependant on the type of technology in use (Borba, 1996). Consequently, the computer and graphics calculator are recognised as mediums that promote communication (Hudson, 1997; Valero and Gomez, 1996). Pozzi et al (1993) maintain that introducing the computer into group settings changes particular aspects of the interaction, bringing a new dimension to group work, that is qualitatively different from group work occurring without technology. Working with the computer reduces the cognitive distance between students, creating the opportunity for all of the participants to gain a new perspective on the problem, or to co-ordinate different perspectives. In their study, there was found to be a high association between students who dominated the resulting interaction and those of high mathematical ability. They propose a scenario for optimal learning that arose from their research in which students first engage in mutual discussion with peers in the context of construction with the computer, then they come across the perspectives of others in whole class discussions. This allows students to develop understanding and strategies for solving problems in their small group discussions and explorations, so that they can make sense of any possible conflicting strategies from their peers. It also prevents students from remaining centred on their own way of understanding the problem, which might inhibit further learning, through discussion in the whole class context. However, whilst Furinghetti and Bottino's study (1996) showed that "software favours the communication between the students and the teacher" (p. 134), surprisingly few teachers assigned importance to the interactions between students.

Chronaki and Kynigos (1999) argue that collaboration in computer based learning environments can be used to augment ways of acting which generate common meanings with respect to activity. Indeed, Hoyles et al (1991) found that students were actively involved in negotiating goals and processes, brainstorming solution strategies, justifying ideas and actions, and developing a shared language for communicating actions, when collaborating within a computer environment.

Teasley and Roschelle (1993) also see technology as a resource that mediates collaboration, by providing an enriched background for students' talk and action. Through the use of computers students are enabled to "use the powerful resources of everyday conversation to converge on robust shared meanings for technical concepts" (ibid, p. 255). However, individuals must make a conscious, continuous effort to co-ordinate their language and activity with respect to shared knowledge. Collaboration does not necessarily happen because individuals are co-present (ibid). Indeed, Doerr and Zangor (2000) found that personal use of the graphical calculator served to break down group communications. In their study, whilst the use of the technology as a shared device supported mathematical learning in the whole class setting, the use of the graphical calculator as a personal device was seen to inhibit communication in a small group setting. This was particularly apparent when students that had individual difficulties would seek help from the teacher rather than peers.

Doerr and Zangor (2000) following Penglase and Arnold (1996) suggest that research into students' learning with graphical calculator should take account of the social context of the learning environment. Kaput (1992) argues that computers fundamentally alter the nature of work in the classroom, not merely its efficiency, accessibility or scope. Besides work being finished more quickly, the actions necessary to accomplish tasks change (Pea, 1987). Whilst the introduction of technology encourages innovation, the important changes that occur are a result of the corresponding changes in teachers' beliefs about mathematics, teaching, learning, students and appropriate use of classroom time (Kaput, 1992). Using technology therefore enables some students to challenge their existing understanding of the nature of mathematics and of mathematical knowledge, as their experience of mathematics changes (Povey, 1995). Knowledge and authority are shared (ibid).

Guin and Trouche (1999) stress that the teaching process needs to take account of the fact that the transformation of the graphical calculator into an efficient mathematical instrument varies from student to student, especially amongst their abilities to interpret and co-ordinate the results produced by the technology. They maintain that the "instrumentation process is slow and complex, because it requires sufficient time to achieve a reorganisation of procedures" (p. 223). In their view, the introduction of technology leads to renewal of teaching practices through taking over technical computations, which potentially promotes more conceptual understanding. In this context they see the new role of the teacher as one which involves organising and encouraging interaction within the computational environment and is essential in shaping the relationship between the computational media and mathematical knowledge (ibid).

Hoyles and Noss (1992) propose that a "combination of both on-computer activities and off-computer negotiation and discussion are critical in bridging the discursive disjuncture between the two practices – helping students to link intuitions derived from the computational interactions and their formal mathematics" (p. 54).

There has, however, been relatively little research carried out to date on visualising functions in a collaborative setting using graphical calculators and on the complex set of factors that are involved in enabling students to derive meaning in such an environment. In particular, as Hershkowitz (1999) suggests the role of the individual student in the collective construction of shared knowledge has been largely neglected. It is in seeking to explore these areas that it is hoped this thesis is distinguished from previous work.

CHAPTER 4
RESEARCH METHODOLOGY

4.0 Overview

This chapter initially describes the methodological approaches that have been influential in developing the overall research methodology in this study. These approaches are subsequently considered in relation to the aims and objectives of the research and the theoretical perspective that has been adopted. The corresponding methods of enquiry are then discussed and the distinct phases of the research outlined. The chapter is concluded with a discussion of data analysis.

4.1 Introduction

This research has consisted of three distinct phases, each of which could be characterised as a 'study of singularities', following Bassey (1995). The process of investigating a singularity involves describing, analysing, explaining, interpreting and possibly justifying, something which occurred at a particular place and at a particular time. According to Bassey at the point where it becomes the subject of study, "a singularity is a set of anecdotes about particular events occurring within a stated boundary, which are subjected to systematic and critical search for some truth" (p. 111). Furthermore, "this truth while pertaining to the inside of the boundary, may stimulate thinking about similar situations elsewhere" (p.111).

Bassey (ibid) proposes that the term study of a singularity embraces virtually every kind of empirical study and is preferable to the phrase 'case study' because this is often associated with generalisation. The findings of singularity studies are merely *related* to populations outside the boundary in space and time. As such studies of singularities tend to be undertaken on a small scale and are consequently very detailed. This approach is seen to be consistent with the objectives and theoretical perspective of the study. As such, the three distinct phases of this study have involved small scale and detailed enquiries surrounding singularities. In each case, the findings of the study that would be of interest to the mathematics educator community have been identified and consideration has been given to how the findings might be related to situations outside the immediate context of the research. Any further generalisations that could be drawn from these results lie in the readers' interpretations of the text in relation to their own experiences.

In conducting these studies of singularities, a qualitative interpretivist methodology has been adopted, which has governed the collection, analysis and interpretation of all data. The interpretivist perspective was considered to be the most appropriate philosophical position in relation to the goals of this thesis, since it is based on the assumption that "all human activity is fundamentally a social and meaning-making experience" (Eisenhart, 1988, p. 102). From the interpretivist perspective "meaning and action, context and situation are inextricably linked" and consequently make no sense when considered separately (p. 103). The main aim of interpretivist research is to enable the researcher to 'make sense' of the participants' world from their perspectives which resonates with the objectives of this research. To allow the interpretivist researcher to achieve this aim, ethnographic research methods were developed and have been used in this study to the same end.

4.2 Ethnography

The tenets of ethnography are derived from an interpretivist perspective (Hammersley and Atkinson, 1983; Eisenhart, 1988; Cohen and Manion, 1994; Woods, 1996). Goetz and LeCompte (1984) regard educational ethnography as an investigative means by which to study human behaviour. They argue that "broadly conceptualised" educational ethnography "includes studies of enculturation and acculturation from anthropology, studies of socialisation and institutionalised education from sociology and studies of sociocultural learning and cognition, of child and adult development from psychology" (p.13). The latter of these types of study encompassed by educational ethnography considers aspects of students' learning that reflect the objectives of this thesis. The purpose of educational ethnography is described as being "to provide rich, descriptive data about the contexts, activities, and beliefs of participants in educational settings", which can be used for evaluative studies, purely descriptive studies or for theoretical enquiries such as in this research (Goetz and LeCompte, 1984, p. 17). In conducting an ethnographic theoretical enquiry, the researcher repeatedly 'tests' an emergent theory of culture or social organisation by evaluating various questions, methods and interpretations. The ultimate goal of this critical analysis is to provide a theoretical explanation that encompasses all of the data and provides a comprehensive picture of the complex meanings and social activity (Eisenhart, 1988). The findings of such ethnographical studies may thus shed light on ways to improve practice (ibid).

Educational ethnographers are interested in the processes of teaching and learning; the intended and unintended outcomes of observed patterns of interaction; the relationship between certain educational actors, such as parents, teachers, and learners; and the sociocultural contexts within which 'nurturing', teaching and learning occur (Goetz and LeCompte, 1984). These areas of interest certainly resonate with the focal points of this research. In essence, the worlds of individual teachers, students and administrators are documented for "unique and common patterns of experience, outlook, and response" (p.31). Ball (1993) argues that the tradition of ethnographic research implies a need for the researcher to search consciously for meaning, to suspend any preconceptions and to be oriented towards discovery. The researcher has to interpret the situation and then vary his or her action accordingly. Ball believes that the process of ethnography is thus defined by the self-conscious engagement of the researcher within the world that is being studied. In the following diagram, figure 4.1, he offers a heuristic representation of the ethnographic research process:

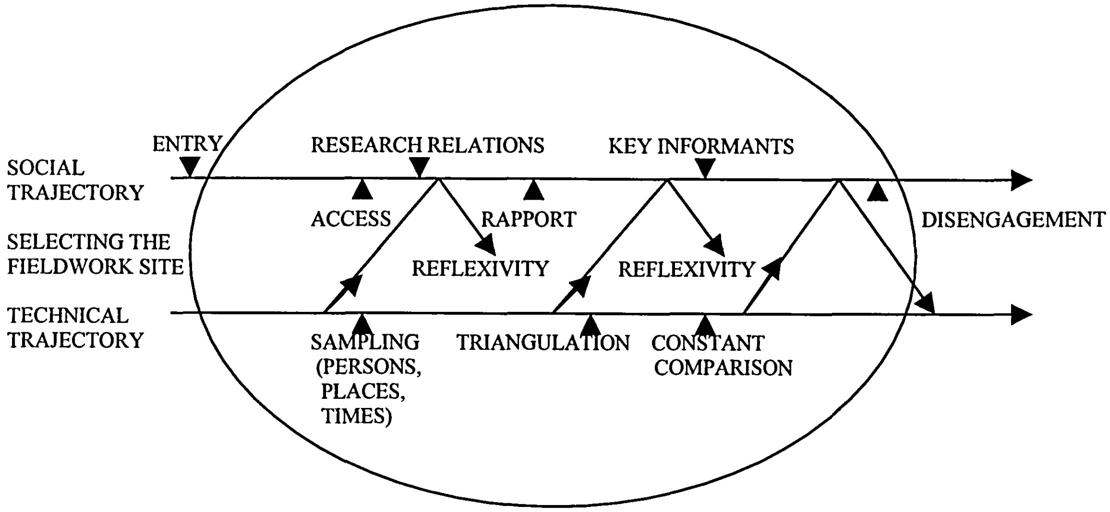


Figure 4.1: A Heuristic Representation of the Ethnographic Research Process (Ball, 1993, p. 34)

Goetz and LeCompte (1984) suggest that the results of educational ethnography contribute to the improvement in educational and school practice in several ways. In particular, ethnographical research can enable connections to be made between research activity, educational theory and pragmatic concerns. This occurs through the ethnographer's focus on the life world and perspective of those involved, which can confirm the reality experienced by educators and demonstrate concretely any relationship between theory and practice. Indeed, Hammersley and Atkinson (1983) suggest that "the value of ethnography is perhaps most obvious in relation to the development of theory" (p. 23). The flexibility of ethnographical research means that "the strategy and even directions of the research can be changed relatively easily, in line with changing assessments of what is required by the process of theory construction", leading to effective and economic development of emergent theories (p. 24).

Eisenhart (1988) argues that the ethnographic researcher does not merely adopt the views of those being studied. On the contrary, the researcher must also "be able to step back from the immediate scenes of activity and to reflect on what is occurring from the perspective of someone who is aware of other systems and of theoretical perspectives on socio-cultural systems" (p. 105). In general, the data obtained represent educational processes as they happen and the outcomes of these processes are reviewed within the "whole phenomenon" (ibid). Events are not isolated from one another. The ethnographer is seen as the "essential research instrument" (Wolcott, 1975, cited in Goetz and LeCompte, 1984, p.101) and his or her role as a data collector must 'mediate' all the other roles that are undertaken (ibid).

Doerr and Zangor (1999) propose that most of the existing studies on graphical calculators are quasi-experimental in design and give little insight into how and why students use graphical calculators. They believe, as is the view taken in this thesis, that the "psychological and sociological aspects of learning are co-ordinated as an active process in which students reorganise their thinking through their interactions in the social context" (p. 266). This social context includes the tools and representation systems that are shared amongst the students and teacher. Both the teacher and the students are seen to construct the meaning and the role of the graphing calculator as a tool for mathematics learning. This occurs through their "interactions, communications and shared use of the tool" (p. 266). Consequently in seeking to study the role of the teacher and the patterns and modes of graphing calculator use in the classroom, Doerr and Zangor advocate the use of qualitative research methods, which would certainly seem to fit within a socio-cultural framework. Ethnographic methods in particular were chosen in this thesis with the aim of illuminating the mediational role of graphical calculators in students' learning and to gradually develop a theoretical interpretation that accounts for all of the data.

Ethnographic methodology has produced some influential studies. However, there are potential weaknesses that have to be considered and addressed by the researcher when choosing to undertake any particular type of research. When compared with the results of scientific enquiries the findings of ethnographic studies have been considered by many researchers as unreliable, as lacking in validity and generalisability (Goetz and LeCompte, 1984). To counteract this criticism, some advocates of ethnography claim that such measures are irrelevant, since the goals of ethnographical research are essentially descriptive and generative, rather than concerned with verification and generalisation (ibid). Yet the fact remains that internal and external reliability and hence credibility can cause problems for ethnographers. Ethnographers are interested in the search for locally and personally relevant meaning and organisation within the settings and situations of the study and from its participants. In this way, they regard social scenes as being located in a particular time and space and their task is thus to communicate the specific characteristics of the scene being studied (Eisenhart, 1988). As a consequence, ethnographical studies "do not lend themselves easily to replication in other settings or by other researchers" (p. 108). However, Eisenhart (following Goetz and LeCompte, 1984) insists that "ethnography can and should be made replicable" (p. 108). She proposes that this could be achieved through researchers providing careful and thorough descriptions of:

1. the choice and use of settings and the people in the study;
2. the social conditions under which the study takes place;
3. the role and status of the researcher in the study;
4. the theoretical or analytical constructs used to guide data collection and analysis;
5. the data collection and analysis procedures.

In this manner, Eisenhart suggests that problems with external validity through comparability across groups can also be overcome by facilitating these comparisons to be made by other researchers. However, whilst every effort has been made to include such details in this thesis to enable other researchers to derive particular insights and transfer ideas from the context of this research, it is doubtful as to whether replication could actually be achieved and whether in fact this would be desirable.

On the other hand, internal validity is seen as a strong point of ethnographical research, since the researcher is thoroughly engaged in the world of the participants. In addition, ethnographers "carefully describe and account for factors that may affect the internal validity of their information" (Eisenhart, p. 109). These factors include the historical context, the selection of settings and people, the maturation and morality of informants, and observer reactive effects, which accounts for any personal bias. A particular area of concern is the "possibility of drawing erroneous conclusions from spurious relationships in the data" (p. 109). The ethnographer addresses this concern, as has been undertaken in this thesis, by attempting to trace all possible relationships within the data until he or she is satisfied that the conclusions are internally valid.

4.3 Methods for Data Collection

Eisenhart (1988) outlines four main methods of data collection that are used in ethnographic studies: participant observation, ethnographic interviewing, searching for artifacts and researcher introspection, each of which have been adopted in this research. Of these techniques, Eisenhart regards participant observation as the most important, as ethnographic research depends upon the active and personal involvement of the researcher in the collection and analysis of data. Distinct social groups are believed to "construct coherent systems of beliefs and actions from intersubjective meanings" (p. 103). This implies that outsiders may not be able to access the ways in which beliefs and actions make sense to insiders, since intersubjective meanings are implicit. As a consequence, in order for a researcher to understand human activity, he or she must firstly make a commitment to enter actively into the worlds of interacting individuals. In effect, the researcher needs to be involved in the activity as an insider and to be able to reflect on the observations as an outsider (ibid). The degree of participation and observation is dependent on the context of the research and the researcher's aims. Figure 4.2 shows the different roles that the researchers can adopt (Junker, 1960 from Hammersley and Atkinson, 1983, p. 93).

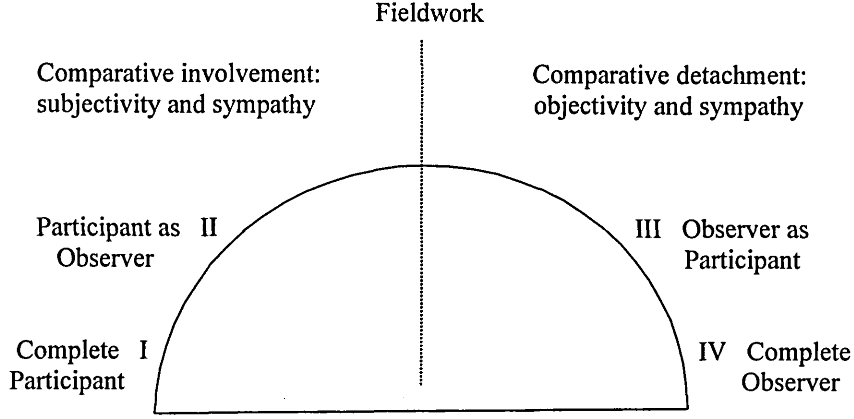


Figure 4.2: The Theoretical Social Roles for Fieldwork.

The principles of ethnography have governed the whole approach to carrying out the classroom-based research, from the choice of methods of enquiry, to the way in which each episode has been interpreted within the world of the participants. In this research, the role of participant as observer (type 2 in figure 4.2) or 'teacher-researcher' was assumed. This entailed outlining a plan of action for each lesson, teaching the students how to use the graphical calculators, helping the students with any problems that they experienced whilst using them, dealing with any conceptual difficulties, giving feedback to the students and general management of the classroom. In addition, time was spent recording field notes, administering the questionnaires, discussing the use of the technology informally with both staff and students, interviewing students and audio/video taping student interaction.

In addition to the four key means of obtaining primary data in ethnographic research, other research methods are often used, such as surveys, observation schedules, etc and these techniques provide a basis for triangulation. Previous studies indicate that research that combines qualitative interpretation with the quantitative experimental approach can make effective use of the most valuable features of each. For example, Cohen and Manion (1994) argue that by studying a social setting from more than one stand point, using both quantitative and qualitative data, the complexity and richness of human behaviour can be more fully explained.

Heid (quoted in Olive, 1992) suggests that such a combination is needed to characterise the changes in the delivery of school mathematics in a technology rich environment. Similarly, Kynigos and Argyris (1999) employed an ethnographic approach, combined with quantitative methods to trace the variety of roles and activities undertaken by teachers and students when using computers. In their study, the types of roles and interventions were allowed to emerge from the data. In other words, their data was not used to test a pre-determined hypothesis. With this in mind, quantitative methods have been adopted as appropriate and have been used to inform and support qualitative interpretations. These have consisted of simple quantitative analysis of the students' questionnaire responses and their work.

Denzin (1988) contends that since the social world is socially constructed, and its meanings are constantly changing, to both the observer and those observed, "no single research method will ever capture all of the changing features of the social world under study" (p. 512). Consequently, he recognises a need for triangulation and distinguishes between four basic types: data triangulation, investigator triangulation, theory triangulation and methodological triangulation.

Data triangulation encompasses three sub-categories of triangulation: time triangulation consisting of cross-sectional and longitudinal designs, space triangulation involving cross-cultural techniques and combined levels of triangulation which deal with the individual, the interactive (groups) and the collectives (organisational, cultural or societal). Investigator triangulation involves multiple, as opposed to single observers. Theory triangulation incorporates different theoretical perspectives in the interpretation of a set of data. Methodological triangulation consists of a combination of multiple observers, theoretical perspectives, sources of data and methodologies within a single investigation. This study has made use of theoretical triangulation and data triangulation in dealing with individual, interactive and cultural aspects. Throughout the whole process of data collection, emphasis was placed on achieving the triangulation of results through employing different methods of data collection and analysis and in gaining the perspectives of both the students and teachers.

Goetz and LeCompte (1984) argue that the choice of techniques used for ethnographic data collection involves the consideration of available alternatives, with continual re-examination and modification of decisions. Ethnographic data can be recorded through the use of tape recorders, film or videotapes, photographs, field notes, written records, questionnaires and diaries (ibid). Hammersley and Atkinson (1983) stress that the methods of enquiry must be selected according to the purposes of the research. Furthermore, Eisenhart (1988) proposes that the methods used to investigate social and meaning-making experience "must be modelled after or approximate it" (p. 102). The ongoing process of data collection and analysis may significantly influence the direction of the research and the future data collection procedures. The course of ethnography cannot be predetermined and the research design should be a reflexive process operating throughout every stage of a project (Hammersley and Atkinson, 1983). The design of ethnographic research is strengthened by the number of perspectives that are represented (Eisenhart, 1988).

The ethnographic methods used for data collection in this thesis are summarised in figure 4.3.

Data Collection Techniques

Participant as observer
Researcher introspection
Audio recordings and transcripts
Video recordings and transcripts
Assessment of students' work
Interviews
Questionnaires
Triangulation

Figure 4.3 Data Collection Techniques

The use of these research methods, however, varied from one phase of the research to the next. The data collection techniques used in phase one consisted of:

- participant as observer
- researcher introspection
- audio recordings and transcripts
- assessment of students' work
- questionnaires
- triangulation

In addition to these techniques that were developed during the first phase, phases two and three also made use of:

- video recordings and transcripts
- interviews

The first phase of the research consisted of the exploratory study and the initial development of the core classroom materials and approaches, which were designed to promote students' understanding of functions using graphical calculators. These materials were then trialled at Ashby school and feedback was obtained from the students via post-trial questionnaires, informal discussions, audio recording, and through the assessment of their written work. Members of staff were given the opportunity to offer their perspectives by means of post-trial questionnaires and informal discussions. This feedback from the students and staff and the observations made by myself during the lessons contributed towards achieving triangulation. The findings from phase one of the research are presented in chapter five.

Following preliminary analysis and interpretation of the data collected during the first phase and a period of reflection on the initial findings, the second phase began with refinement and further development of the classroom materials. Data was then collected from Anderson College and subsequently analysed and interpreted. During the second phase some additional techniques for data collection were used. Feedback was obtained from the students by means of a pre-trial assessment, pre-trial as well as post-trial questionnaires, semi-structured interviews and video recording. In addition audio recording focused on whole class rather than small group work. Feedback was obtained from staff using the same techniques as in the first phase. Chapter 6 details this phase of the study.

The third and most substantial phase of the research involved returning to Ashby school and was concerned with introducing the concept of functions to a new group of year twelve students, using the graphical calculator. This phase of the research led to further development of the theoretical framework adopted in this study and consequently, a greater emphasis was given to the theoretical implications of the findings than in the previous two phases. This third phase of data collection also involved more intensive use of video and audio recording of whole class and small group interaction, in addition to the continued use of the other data collection techniques developed in phase two. As in the previous trial the student interviews were audio taped and these along with all other audio and video recordings were consequently fully transcribed. In both of these phases, where appropriate, video and audio transcripts included descriptions of what was happening during the discourse in terms of accompanying student or teacher actions and their intended meanings. Chapters 7 and 8 represent the findings from this phase. Data collection and analysis proceeded together throughout these periods of study, as outlined by Eisenhart (1988). Table 4.1 summarises the background details of each phase.

Table 4.1 Background to the Three Phases of the Research

Background	**Phase 1**	**Phase 2**	**Phase 3**
Data collection period	February 1998	June 1998	October-November 1998
School/College	Ashby School	Anderson College	Ashby School
Number of students	13	6	17
Mathematics level studied	GCE Advanced	GCE Advanced Further	GCE Advanced
Graphical calculator used	TI-92	TI-92	TI-82
Prior experience of graphical calculators	Only 1 student	All 6 students regularly used graphical calculators	Only 2 students

4.4 Data Analysis

4.4.1 Overall Approach to Data Analysis

Eisenhart (1988) describes various systematic procedures that are available for analysing ethnographic data, which are each designed to identify the meanings held by the participants and researcher and to organise these meanings so that they make sense both internally and externally. In essence, ethnographic analysis consists of text-based procedures for assuring that the views of the participants and researcher remain distinct and that all aspects of material are accounted for. In general, these procedures involve defining 'meaningful' units of material which are either meaningful to the participants and/or researcher and comparing units with other units. Units that are alike are then grouped together in categories and these categories are compared with each other and the relationships between them espoused. The various categories and their interrelationships are then considered and reconsidered in light of old material and as new material is collected. As such, at each trial the focus of data collection and analysis is shifted slightly to allow different features to be addressed and different possible explanations to be considered. These themes are then considered in light of existing socio-cultural theories. This occurs once all the components of the data have been organised into plausible categories, or 'constitutive rules', that if used by an outsider would allow him or her to make sense of the participants' world in the same way they do.

The overall approach to data analysis in this study has followed the systematic procedures outlined above and as a consequence various categories have emerged from the data which are firmly grounded in the research. For example consideration of the data from all three phases of the research highlighted the fact that in each case there were some students who appeared to be overly reliant on the results produced by the technology. Thus, 'dependency' on the graphical calculators became a major category of analysis, which was further refined as the study progressed. The on-going process of defining and re-defining categories has subsequently formed an important part of the methodological approach to analysing data adopted in this study, which has also been influenced by the work of Erickson reported by Eisenhart (1988). Erickson describes a procedure for developing assertions about "what's going on". This is a process that involves searching the data for confirming and disproving evidence, thereby producing an 'evidentiary record' to warrant the acceptance of certain assertions. Assertions that are substantiated are then interpreted in relation to existing and emergent theories. Following Erickson, this research has involved the systematic search for an accumulation of evidence to support assertions that have been made and theories that have emerged.

4.4.2 Specific Data Analysis Techniques

4.4.2.1 Analysis of the Individual Aspects of Graphical Calculator Use

Berger (1998) recognises that despite the potential and importance of the graphical calculator, there is insufficient literature devoted to explaining and/or understanding how this tool functions in relation to the learner. To address this imbalance, she puts forward an interpretative framework for analysing the relationship between the learner and the graphical calculator based on Vygotskian psychology. The study that she reports has clear parallels with the research objectives of this thesis and is primarily concerned with interpreting how the graphical calculator and its sign systems mediate the learning process of a mathematics student. Of particular significance is Berger's interpretation of the notions of ‘amplification’ and ‘cognitive reorganisation’ effects arising from the use of technology. Within a Vygotskian framework, the amplification effects of the graphical calculator are seen to ‘amplify’ the zone of proximal development by removing cumbersome and time-consuming tasks from this zone, thereby creating more space for the user to perform conceptually demanding tasks with greater effectiveness and ease. Cognitive reorganisation effects are interpreted as systematic changes in the consciousness of the learner, which occur as a result of interaction with a new and alternate semiotic system. These ideas have been used to interpret individual aspects of the use of graphical calculators in this study and the results of this part of the analysis are presented in sections 5.2.1, 6.2.1, and 7.2.1 of chapters 5, 6 and 7 respectively. The analysis of data in this thesis has thus involved identifying and explaining evidence for amplification and cognitive reorganisation effects occurring and postulating possible relationships between the two. This has been undertaken in each phase in an attempt to ascertain, like Berger, how the graphical calculator might function in relation to a specific learner. For example, Robert’s learning with the graphical calculator in relation to the amplification and cognitive reorganisation effects will be considered in depth in section 6.2.1 of chapter 6.

A complementary framework that has provided an additional means of focusing on the individual student’s use of graphical calculators within the whole classroom community is the notion of local communities of practice developed by Winbourne and Watson (1998). Winbourne and Watson identify six necessary features of a local community of mathematical practice, which were outlined in chapter 3, section 3.1.3. These key features have been used to identify the existence of local communities of practice in the data from this thesis, which are regarded as environments that are conducive to learning mathematics with graphical calculators. According to Winbourne and Watson, individual students derive meaning from the positions that they occupy within the local community of practice and this in turn determines their success as learners and contributes towards the creation of shared knowledge. Their framework has thus been applied to the transcript data from the second and third phases to shed light on the way in which the behaviour of individual students may affect the shared construction of meaning and how their roles develop through use of technology. This is discussed in sections 6.3.2, 8.1.4 and 8.1.5 of chapters 6 and 8.

4.4.2.2 Analysis of the Social Aspects of Graphical Calculator Use

Notions developed by Teasley and Roschelle (1993) were used to analyse the interaction between the students and the teacher-researcher in phases two and three, the results of which are discussed in sections 6.3 and 8.1 of chapters 6 and 8. These ideas were developed in relation to a Vygotskian framework and their study was concerned with exemplifying the use of the computer as a cognitive tool for learning that occurs socially. They maintain that "cognitive representations are built through social interaction and activity, in addition to individual cognition" (ibid, p. 230).

Teasley and Roschelle propose that social interactions in the context of problem solving activity occur in relation to a Joint Problem Space (JPS). They maintain that the JPS is a shared knowledge structure that supports problem solving activity by integrating (a) goals, (b) descriptions of the current problem state, (c) awareness of available problem solving actions, and (d) associations that relate goals, features of the current problem state and available actions.

In their model collaborative problem solving consists of two concurrent activities, solving the problem together and building a JPS:

> *Conversation in the context of problem solving activity is the process by which collaborators construct and maintain a JPS. Simultaneously, the JPS is the structure that enables meaningful conversation about problem solving to occur. Students can use the structure of conversation to continually build, monitor and repair a JPS. (Teasley and Roschelle, 1993, p. 236)*

They do acknowledge, however, that the overlap of meaning in the collaborators' common conception of a problem is not necessarily complete or absolutely certain. Yet, this overlap is sufficient to allow students to gradually accumulate shared concepts and to permit convergence on certainty of meaning (ibid).

The analysis of the transcript data thus involved finding evidence for the construction and maintenance of a JPS as well as identifying certain 'categories of discourse events' that Teasley and Roschelle have outlined. These include 'turn taking', 'collaborative completions', 'repairs', 'narrations' and the combined use of language and action (see sections 6.3 and 8.1, chapters 6 and 8). These represent strategies that collaborators have for: (i) introducing and accepting knowledge into the JPS, (ii) monitoring ongoing activity to recognise any divergence in shared meaning, and (iii) rectifying misunderstandings that impede the process of collaboration.

Communication between individuals follows a well-specified form of 'turn taking', which is generally comprised of discourse units such as questions, acceptances, disagreements, and repairs. The flow, content and structure of turns provide an indication of whether the participants in a conversation understand each other. Teasley and Roschelle propose that during periods of successful activity, students' conversational turns build on each other and that the content of these turns contributes to the joint problem solving activity. Looking for evidence of successful collaboration between students using graphical calculators in this thesis therefore involved close examination of the quality of individual students' turns in the discourse. This in turn involved identifying instances of student initiation of the discourse, student acceptance of arguments and cases of students performing 'repairs'.

Teasley and Roschelle describe 'repairs' as strategies used by students in an attempt to reduce conflict by resolving misunderstandings. This conflict arises because the process of collaboration also involves periods of individual activity and consequently there are periods of conflict where individual ideas are negotiated with respect to shared work. Repairs are seen to be the main means of achieving and consolidating understanding and managing the mutual intelligibility of the collaborative problem solving activity. Without successful repairs, breakdowns in mutual intelligibility continue for longer periods.

Evidence was also sought for instances that involved 'collaborative completions' between students, where one partner's turn would begin a sentence and the other partner would use their turn to complete it. Teasley and Roschelle identify one particularly effective form of collaborative completion, which they refer to as a 'socially distributed production'. This consists of a compound sentence of the 'if-then' form, where the antecedent and consequent are produced in separate conversational turns, providing the opportunity for participants to accept and repair conditional knowledge. By collaborating in this way, multiple opportunities arise for partners to contribute towards the construction and verification of a new piece of shared knowledge.

A further category of discourse event that was identified by Teasley and Roschelle and was similarly used to classify interaction in this thesis is 'narration'. Narration serves as a verbal strategy that enables discourse partners to monitor each other's actions and interpretations. In this way narration informs one's partner of intentions which correspond to actions, which in turn enhances the partner's opportunities to recognise differences in shared understanding. Continued attention to narration and accompanying action can signal acceptances and shared understandings, whereas interruptions to narratives create an immediate opportunity to rectify misunderstandings.

Whilst there are numerous examples of narratives in collaborative activity, students are not wholly dependent on language to maintain shared understanding. Teasley and Roschelle propose that there is a major role for the computer in providing a context for the production of action and gesture, which in turn supports collaborative learning. Action and gestures can both serve as presentations and acceptances and their simultaneous production by separate partners can produce an effective division of labour. Analysis of the video recorded data in this thesis thus paid particular attention to the combined production of narrations, actions and gestures.

4.5 Summary

The aim of this study has been to identify and examine how the use of graphical calculators mediate student's learning of functions. In order to achieve this aim, the research has considered three inter-related aspects of this research question. Firstly, to investigate how students acquire meaning within a graphical calculator environment. Secondly, to examine the ways in which the visual representations provided by the graphical calculator acts as a tool in mediating the development of students' understanding of functions. Thirdly, to investigate how the teacher can effectively mediate the students' use of graphical calculators. The purpose of this chapter has been to outline and justify the methodological approaches and research methods that were chosen to carry out this study.

The research conducted in this study has consisted of three distinct phases, which have each entailed small scale (in terms of the number of students involved and the time frame) and detailed 'studies of singularities' (Bassey, 1995). The first phase consisted of the exploratory study in which inexperienced users' initial experiences with graphical calculators were explored. The aims of this phase were necessarily broad to allow areas of particular interest to emerge from the data, which then provided focal points for phase two. In addition, in order to extend, build on and make contrasts with the findings of phase one the second phase involved experienced graphical calculator users and the way in which they derived meaning for functions. In the final phase, the concept of functions was introduced to a beginning group of year twelve students using the graphical calculators and involved a microanalysis of the resulting classroom interaction. This was intended to shed light on the personal and social factors that contribute towards students' understanding of functions with graphical calculators.

CHAPTER 5
EXPLORING HOW GRAPHICAL CALCULATORS MEDIATE INEXPERIENCED USERS' UNDERSTANDING OF FUNCTIONS

5.0 Introduction

The main purpose of this research study has been to explore how graphical calculators mediate students' understanding of functions. However, within this aim, there were other subsidiary concerns. Some of these were related to cognitive factors such as the amplification and cognitive reorganisation effects of the technology and its impact on visualisation; others pertained to affective factors such as confidence and collaborative problem solving. There were also further issues concerned with the technology itself. The organisation of this chapter reflects these concerns.

The purposes of the initial study were to:

- explore whether students with no previous experience of using graphical calculators would be able to use this technology to further their understanding of functions and how this might occur,
- assess the suitability of early materials and approaches, designed to promote students' understanding of functions,
- elicit preliminary reactions to the use of technology,
- establish a framework for further data collection.

This chapter reports the findings of this exploratory study and is structured as follows:

- Background to the research (5.1).
- Cognitive factors in students' use of graphical calculators to understand functions (5.2).

- Affective factors which contribute towards students' learning of functions (5.3).
- Conclusions (5.4).
- Implications for subsequent phases of the research (5.5).

5.1 Background to the Research

The initial classroom trials were carried out at Ashby school, a Roman Catholic mixed comprehensive school in Sheffield educating approximately 1000 pupils, in the 11-18 age range. The school is situated in a socially advantaged area, but draws its pupils from a much wider geographical area. Prior to this research study, the researcher had spent two years teaching mathematics in this school. It was apparent that there was a growing interest in developing the use of technology, and particularly graphical calculators, within mathematics lessons. The familiarity with students, staff, departmental policies, syllabuses and existing technologies facilitated the integration of the researcher into the school and classroom.

5.1.1 Structure of the Exploratory Study

The exploratory study took place over a period of six hours (three two-hour sessions) during February 1998 and involved a Year 12 GCE Advanced level mathematics group. Table 5.1 outlines the classroom activities that took place during this trial and those who were involved.

Table 5.1 Structure of the First Research Session

Research Activities	Timing	Students	Staff
1. Teacher-researcher introducing the graphical calculator to students. This included how to graph and trace individual functions and exploration of the table menu, viewing window, zoom menu and maths menu.	1hr 15mins	(n = 13) Carl, Lea, May, Jan, Kurt, Guy, Pat, Don, Sally, Sue, Diana, Emma, Betty.	Teacher (Mr Doors) present as an observer. Research Colleague (James Green) audio recording five pairs of students as they attempt the introductory exercises together.
2. Students attempting the introductory exercises in small groups.	45mins		
3. Homework: introductory questions not completed in class.	N/A		

Table 5.2 Structure of the Second Research Session

Research Activities	Timing	Students	Staff
1. Further demonstration by the teacher-researcher of particular uses of the technology, namely graphing families of curves, performing series of transformations and drawing the inverses of functions.	35mins	(n =11) Carl, Lea, May, Jan, Kurt, Guy, Pat, Sally, Diana, Emma, Betty.	Teacher (Mr Irons) present as an observer.
2. Small group work on the main trial exercises: graphing functions using the TI-92.	1hr 25mins		
3. Homework: questions from the main exercises.	N/A		

Table 5.3 Structure of the Third Research Session

Research Activities	Timing	Students	Staff
1. Students continue working on the main trial exercises.	1hr 35mins	(n = 13) Carl, Lea, May, Jan, Kurt, Guy, Pat, Don, Sally, Sue, Diana, Emma, Betty.	Teacher (Mr Irons) present as an observer.
2. Post-trial questionnaires on the role of the technology completed by the students and two members of staff.	25mins		

5.1.2 Design of Research Materials

The classroom materials used in this phase were designed with the aim of promoting the development of students' understanding of functions using graphical calculators in the GCE Advanced level mathematics classroom.

These materials included an introduction to the graphical calculator and its applications and questions intended to familiarise students with the various functions of the calculator. These were followed by a sequence of main exercises designed to draw out their visual abilities and to develop key skills in understanding the concept of function (see appendix A). These exercises featured questions that involved:

- graphing functions,
- exploring and identifying the effects of transformations,
- finding inverse functions,
- solving equations - graphically and algebraically,
- investigating trigonometric and logarithmic identities.

The structure of each research session was influenced by the view that wherever possible different modes of representation should be combined in order to allow a more holistic view of functions to be developed. The majority of the questions required the students to use both visual and symbolic representations. The post trial questionnaires were devised to gauge preliminary reactions of both the students and the staff to the use of the graphical calculator and can be found in appendix A.

5.1.3 The Participants

The group of students who participated in this trial consisted of thirteen students, five male (Carl, Don, Kurt, Pat and Guy) and eight female (Betty, Sally, Diana, Emma, May, Jan, Lea and Sue). All of these students except one (Carl) had previous knowledge and experience of dealing with functions at A level, although only one of them had used a graphical calculator before (Emma).

Each student was given a graphical calculator, a Texas Instrument TI-92, although generally, they worked together in pairs (of their own choosing) sharing ideas during lesson time. Students were also regularly encouraged by the teacher-researcher to share their findings with the class by using the overhead projector set-up. After the first two sessions, they were given questions for homework and each took a graphical calculator home for this purpose.

The two teachers who shared the responsibility for this group of students observed each of the sessions in turn and frequently interjected with the students, discussing both the mathematics and the use of technology. The support of James Green, a research colleague was also gratefully received and he assisted in the collection of data by audio taping student interactions during the first lesson and in helping students to get to grips with using the technology. Teacher intervention occurred whenever the students experienced problems with the technology or in understanding particular questions.

5.2 Cognitive Factors in Students' Use of Graphical Calculators to Understand Functions

In order to determine the influence of graphical calculators on cognitive factors, data was analysed from three sources: (i) post-trial questionnaires administered to the students, (ii) individual students' work, and (iii) post-trial questionnaires completed by the staff.

Utilisation of post-trial student questionnaires provided substantial insight into the way in which use of the graphical calculator had influenced the students' thinking about functions. This was further elaborated through in-depth analysis of the students' work. In addition, the set of post-trial staff questionnaires shed light on the teachers' perceptions of how the graphical calculator affected students' thinking.

The findings that pertain to cognitive factors have been subdivided into the following themes:

- amplification and cognitive reorganisation,
- graphical calculators and dependency,
- graphical calculators and students' understanding of functions,
- graphical calculators and visualisation,
- the relationship between symbolic and visual modes.

5.2.1 Amplification and Cognitive Reorganisation

Analysis of the students' questionnaires revealed that ten of the students considered the speed, ease and accuracy by which the graphs of functions could be drawn as the main advantages of using the graphical calculator. Typical responses to the question "how important, in your opinion, is technology in the A level mathematics classroom?" were:

> **Guy:** *Technology is important because it saves time on menial tasks so more time may be spent on other areas.*
>
> **Lea:** *Technology is important as it allows you to investigate functions quickly and correctly. It saves you wasting time that could be spent on harder tasks.*

Similar responses were offered to the question "what do you consider to be the main advantages of using the graphical calculator?" as typified by Carl's comments:

> **Carl:** *Having immediate graphs and being able to work out intersections, along with maximum and minimum values with the press of a button.*

These factors are examples of what are known as the amplification effects of the technology (Berger, 1998) which are short-term consequences that are directly and immediately experienced by the student whilst using the technology. Thus, students who are using the technology for the first time are extremely aware of the instant and very visible benefits of amplification and consequently, as in this case, these are more likely to be seen as major advantages. Guy, Lea and Carl all refer to the amplification effects of the technology as factors that have contributed towards the development of their understanding in this area.

The staff questionnaires revealed that Mr Irons and Mr Doors also recognised the tangible benefits of the amplification effects of the technology. They regarded the main advantage of using the graphical calculator as the tremendous variety of functions that could be explored. In addition, Mr Doors also recognised the potential of using technology to introduce more generality to the topic being considered. When asked if he saw any potential for using the graphical calculator in his classroom, he responded:

> *Yes. I would aim to use it to quickly produce calculation diagrams, graphs and key features of functions, so that we could comment more generally on the mathematical structure of the topic being looked at.*

Berger (1998) interprets cognitive reorganisation effects as long term changes in the consciousness of the student, which may result in the student using mathematical concepts either more meaningfully or differently due to the use of the technology. Only three of the students referred to cognitive reorganisation effects in their questionnaire responses as main advantages of using the technology. One student, May, hinted that using the technology enabled her to look at mathematical concepts in a different way. In response to the question "how important, in your opinion, is technology in the A level mathematics classroom?" she replied:

> *I think it is quite important to have access to technology as it gives another way of seeing the ideas behind certain theories and rules in maths, some of which can be quite hard to picture yourself.*

However, the majority of these students appeared to be focusing on the short-term gains that arise as a result of amplification, which are more apparent, as the main benefits of using the technology.

In contrast, Mr Doors saw a connection between the amplification and cognitive reorganisation effects. This was clearly illustrated in his response to the question “what do you hope to gain by using technology in A level mathematics?”:

> *The freedom for students to see some of the structure of more complex mathematics more clearly. I feel that technology can provide a dynamic way of teaching, which can quickly do a lot more ‘technical’ areas of the subject without them interfering with the flow of the lesson, allowing a clearer view of more complex ideas.*

The potential of the technology in facilitating the consolidation of various concepts (particularly graphical work) and for introducing more complex ideas were considered by Mr Doors and Mr Irons to be the main gains of implementing technology use at Advanced level. Moreover, the speed by which students were able to perform procedural aspects using the graphical calculator was seen to provide a unique opportunity for thinking about complex mathematical ideas in a different, more meaningful way.

These findings highlight the potential for the speed and quality of learning to be increased through use of the technology. The amplification effects of the technology increase the speed of learning, as these effects create more space within each student’s zone of proximal development to attempt conceptually demanding tasks with greater effectiveness and ease. In addition to this the cognitive re-organisation effects of the technology can contribute to the depth of student understanding. For example, as students are able to see the visual graphical effects of transformations more clearly using the graphical calculator, they are more likely to develop an intuitive understanding of the relationship between different functions of the same family (see section 5.2.3).

5.2.2 Graphical Calculators and Dependency

The students’ solutions to each of the questions from the trial exercises were compared, evaluated and graded and the results of this analysis are displayed in Appendix A.

Close examination of the students' work from the introductory exercises revealed that a high proportion of them (10 students) assumed that the graphical calculator was displaying the whole graph without using the zoom facilities to see that this was not always the case. This led to misconceptions about the shapes of the graphs of particular functions. Evidence that students might be misled by their dependency on the graphical calculator's display is provided in the following examples.

The function $x^2 - x^3$ (question 9 in tables 1, 2 and 3, appendix A) caused particular problems, the nature of which is illustrated in the following two graphs (fig 5.1 and fig 5.2) produced by the graphical calculator.

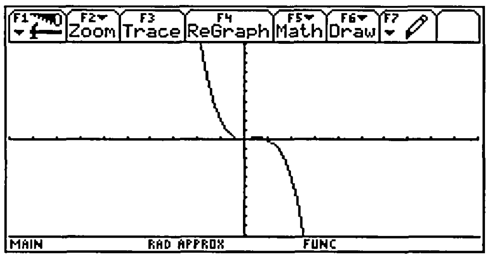


Fig 5.1 $y = x^2 - x^3$ in ZoomStd

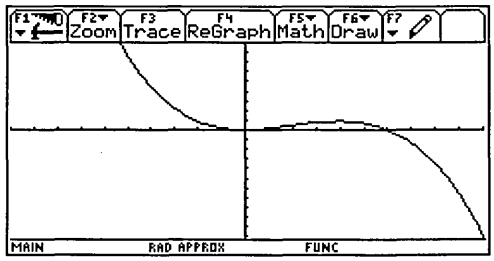
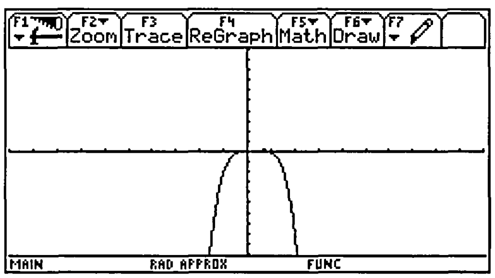


Fig 5.2 $y = x^2 - x^3$ ZoomIn (factor 6)

The students were asked to sketch the function and to determine the nature and co-ordinates of any turning points - a task that was well within their capabilities in light of their previous calculus experience. The first graph (fig 5.1) is drawn using the standard setting for the initial graphing of non-trigonometric functions, ZoomStd (where the x and y axes vary from -10 to 10, in divisions of 1 unit). The second graph (fig 5.2) results from zooming in on the first graph to a degree of factor six, centred on the origin.

Obviously, the second figure provides a much better picture of the actual shape of the graph. Yet, all of the students who attempted this question failed to use the zoom facilities and thus drew a sketch of the function, which resembled fig 5.1. Consequently, they mistook the point (0,0) as a point of inflection (clearly a local minimum in fig 5.2) and were unaware that a local maximum existed and as no student checked their results by differentiation these errors were undetected. Clearly, these turning points were missed because they were not initially visible on screen and the first graph was accepted without question. This suggested that the students were relying too heavily on the immediate results produced by the graphical calculator, without questioning their validity.

Similarly, some students were mistaken about the function $y = x^3(1 - x)$ (question 6 in tables 1, 2 and 3, appendix A). Figure 5.3 is the graph of the function drawn in ZoomStd and figure 5.4 is obtained by zooming in on figure 5.3, centred on the origin, by a scale factor of 6. Once again several students produced graphs in ZoomStd and did not explore their graphs further and thus missed the turning point visible in figure 5.4.



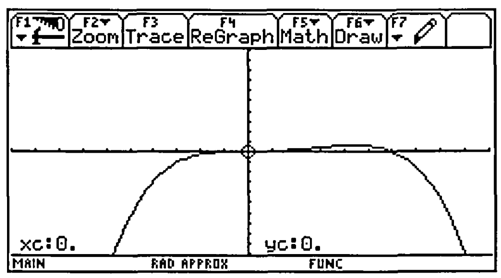
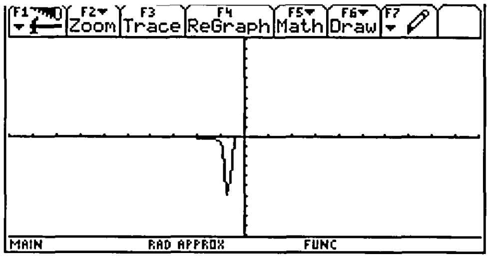


Fig 5.3 $y = x^3(1 - x)$ in ZoomStd   Fig 5.4 $y = x^3(1 - x)$ ZoomIn (factor 6)

In contrast, some students believed that the graph of $y = (x+1)/(x+2)^2$ had a minimum turning point at x = -2 (question 7 in tables 1, 2 and 3, appendix A). These students failed to realise that the function is undefined at this x value, as they completely misinterpreted the graphs displayed by the graphical calculator and neglected to inspect the equation (see fig 5.5 and fig 5.6 below).



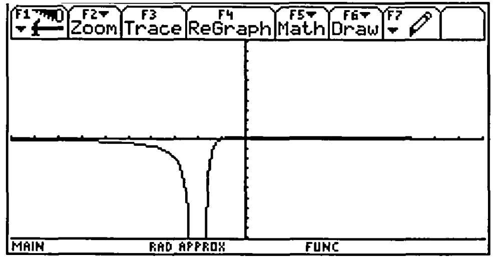


Fig 5.5 $y = (x+1)/(x+2)^2$ ZoomOut   Fig 5.6 $y = (x+1)/(x+2)^2$ in ZoomStd

Figure 5.5 shows that graph of the function $y = (x+1)/(x+2)^2$ produced by the graphical calculator by zooming out on the graph drawn in ZoomStd, centred on the origin, by a factor of three. Figure 5.6 displays the graph of the function in ZoomStd. Since only part of the graph appeared to be visible on the screen in ZoomStd mode, some students choose to zoom out, thus producing a graph resembling figure 5.5. The graph displayed in figure 5.5 looked to some of these students as though it had a minimum stationary point, and so these particular students (who this time made use of the zoom facilities) were misled.

As in the previous example, these students do not appear to have spent enough time thinking about the nature of the function or picturing what the function might look like for themselves - they seemed to assume that the technology always provided them with the correct answer. Smart's (1995b) research, also, emphasises this problem, referred to as the 'magic' element of the technology. The students appeared to be over-dependant on the technology.

These examples illustrate the possibility that under certain circumstances students might misunderstand, misinterpret and thus misuse the information provided by the graphical calculator. This is caused by their over-reliance on the technology. In order to prevent students from becoming too dependent on the graphical calculators, these instances suggest that students should be encouraged to question whether graphs produced by the technology display all the features of the function and to make more use of the zooming facilities offered by the technology.

Furthermore, the teacher needs to provide adequate examples and opportunities that enable students to use these facilities more effectively.

5.2.3 Graphical Calculators and Students' Understanding of Functions

In order to determine the impact of graphical calculators on students' understanding of functions, the student and staff questionnaires were analysed in conjunction with the students' work. Closer examination of individual student questionnaire responses revealed a number of potential ways in which use of the graphical calculator could support student learning of functions.

For example, Carl's responses suggest that students who have had little previous experience in dealing with functions may derive additional benefits from the opportunity to explore functions using a visual medium such as the graphical calculator. Carl was a mature student and had only recently joined this Advanced level mathematics group. As a consequence, whilst the other students had already covered the introductory Advanced level unit on functions, Carl was still catching up on the work that he had missed and was rather unfamiliar with the content of the exercises. When asked whether he had benefited from the opportunity to use the graphical calculator and if it had enabled him to picture functions more clearly, he responded:

> *Definitely. My understanding of graphs was quite limited previously. To have an immediate picture helped tremendously. I had no previous experience of functions and I think I've benefited by coming into contact with them for the first time whilst graphically seeing the results.*

In their responses to three different questions, Sally, Pat and May all commented on the way in which using the graphical calculator helped them gain a better understanding of transformations:

"Has using the graphical calculator enabled you to picture functions more clearly?"

> **Sally:** *It makes translations of functions quicker to learn and more clear since otherwise you either have to plot them yourself or use rules to translate them without understanding why.*

"Do you believe that using the graphical calculator has strengthened your understanding of functions?"

> **Pat:** *It helped me to understand them more, especially the transformations.*

"Do you feel that you have benefited from the opportunity to use the graphical calculator?"

> **May:** *Yes, concerning how certain functions produce graphs and [the graphical calculator] has helped with transformations of graphs a lot, as this is something I find difficult to remember. The main advantages are seeing how curves of different functions compare and how transformations affect the curves.*

The teachers also recognised that the technology had an affect on the students' ability to visualise transformations of functions, as Mr Irons' response to the following question demonstrates. "Do you feel that the graphical calculator has had any affect on students' abilities to visualise the graphs of functions?":

> *Certainly – the idea of transformations was very clear.*

There were, however, examples from the students' work which suggested that the potential of the graphical calculator for enabling students to gain a deeper insight into particular problems involving functions was not always realised. This was particularly apparent in their attempts at question 9 from the main trial exercises. This question, which involved logarithmic and trigonometric identities [use the TI-92 to show that a). $\ln x^a = a\ln x$ b). $\sin^2 x + \cos^2 x = 1$], caused some students problems, as they were uncertain how to proceed, having no prior experience of similar questions. For example, when asked to use the graphical calculator to show that $\ln x^a$

= $a$ln x, none of them provided sufficient evidence to show that this identity was true.

The majority of students firstly investigated the identity when $a = 2$, having presumably dismissed the case of $a = 1$ as arbitrary. Thus, the following two graphs were produced by the graphical calculator:

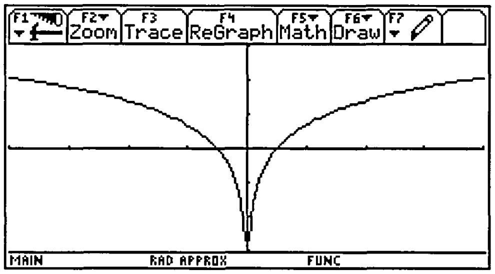


Fig 5.7 $y = \ln x^2$

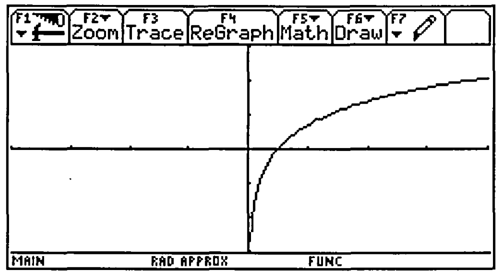


Fig 5.8 $y = 2\ln x$

Informal discussions with the students revealed that some of them were confused by these pictures as they had initially assumed that the two graphs would be identical for all x values. These students failed to realise that by definition ln x and, thus, $a$ln x is undefined for negative x values and so only in the domain of $a$ln x, are the two graphs equal. Consequently, only one student reported findings when $a = 3$, for which the two graphs are identical; others missed out the question entirely and another presumed that the identity was actually false for $a = 2$. Their written work suggested that they did not think carefully about why only part of the graphs were identical and either avoided the issue by missing out the question completely or questioned the validity of the identity in their solutions.

The number of cases examined was extremely low; very few of the students considered values of a other than 2, and even then they only looked at $a = 3$. Five students did state that if $a = 2$, $\ln x^a = a\ln x$ for positive real values of x. However, only one of these students looked at a further case when $a = 3$. Two students failed to reproduce any of the graphs that they had drawn using the graphical calculator and only described what they had seen. They claimed that no matter what value of a is chosen the graphs are the same, which is not strictly true for negative x values.

The problems experienced by the students when answering this particular question illustrated that they were not yet comfortable with the concepts of logarithmic functions, logarithmic identities and their graphs. They were able to draw the graphs using the graphical calculators but were unable to explain their shapes. Clearly, the class was in need of further exploration of the properties of logarithmic functions - an issue that would need to be addressed by their teachers. The remaining questions were completed satisfactorily overall (see tables 5, 6, and 7, appendix A, which display the students' performances in the main trial exercises).

Carl's comments provided some insight into how students who are inexperienced in dealing with functions in particular may benefit from access to technology. Whilst it is unlikely that Carl actually had no previous experience of functions, as he stated, he probably believed that this was the case because functions were not previously introduced to him in a formal manner. As such, he did not recall any prior work as being function-related. His lack of familiarity with functions was particularly apparent to him, as some time had passed since his previous schooling. Carl stressed the significance of the graphical mode of representation in contributing towards the development of his understanding of functions. Initially, he could not picture the graphs of the functions that were considered in class:

> *Before using the graphical calculator I had no comprehension of what they looked like. Now I have.*

In Carl's case, the introduction of the technology gave him immediate access to the graphs of numerous functions that he could not visualise himself. This enabled him to begin to appreciate the relationship between the symbolic, numerical and graphical representations of functions and to recognise the graphs of different families of functions. By experimenting with the graphical calculator, he was able to build substantially and quickly on his 'limited' existing knowledge of graphs.

The evidence presented in this section highlights the need for visual representations and examples to accompany the standard rules in learning about the effects of transformations. Rules alone may not easily be understood or remembered. The use of the graphical approach helps to make the actions of translations and the relationship between translated functions clearer to students. Indeed, Mr Irons believed that the effects of transformations were demonstrated clearly by the technology.

5.2.4 Graphical Calculators and Visualisation

In order to determine whether the graphical calculator was exerting influence on the students' abilities to visualise functions and how this might occur, data was again analysed from the student and staff questionnaires and the students' work.

From the perspective of the teachers (as indicated in their responses to the questionnaires) the graphical calculator was undoubtedly exerting some influence on the students' powers of visualisation. Furthermore, it was noted that prior to the introduction of the graphical calculator, the students had been reluctant to provide diagrammatic and graphical support for their work. Both teachers agreed that the ability of students to visualise and apply their mathematics at this level was extremely important and in this respect the graphical calculator was seen as a very powerful resource.

However, the student questionnaire responses indicated that individual students assigned different values to the use of visual images. For example, in response to the question: "would you consider yourself to be a person who forms and makes use of mental images when solving mathematical problems?" eight of the students claimed to form mental images:

Sue: *I prefer to.*

Lea: *It makes solving equations easier.*

Jan: *Sometimes, depending on how hard the problem is.*

Pat: *Kind of. It helps, but I only really use them when I get stuck, I don't use them first off.*

Kurt: *Increasingly now as the A level maths course has progressed, you need it.*

Sally: *I'm graphically minded.*

Diana: *Occasionally, only if the function is a simple one.*

Emma: *Yes, I use mental images.*

These students obviously used mental images to differing degrees, but their overall questionnaire responses indicated that they viewed the role of technology in very much the same way. Whilst these students were already more inclined to use a visual method of solution, they felt that the use of the technology enhanced their skills in this area.

In contrast, four of the remaining five students indicated that they tended not to form mental images.

May: *Not naturally, but I'm trying to get into the habit.*

Carl: *Not before using the TI-92.*

Guy: *I don't form mental images, so the TI-92 is useful.*

Betty: *No, I don't form mental images.*

The graphical calculator provided these students who were reluctant to use graphical methods with an opportunity to begin to explore a more visual approach to functions. Consequently, they were tending to use this kind of approach more often than they would have done previously.

However, some students failed to label the important x and y values in their sketches, which is probably because this particular form of technology does not include numbered axes. Indeed, Pat specifically commented on this factor when he was asked whether he felt that using the graphical calculator had enabled him to picture functions more clearly:

> *No, not really as the axes are not labelled clearly enough so sometimes it is confusing as the pixels are quite large so it doesn't provide a fantastic graph.*

Because of this aspect Pat stressed that he would prefer to use computers. In addition, this caused the students particular confusion over the graph of sin x + cos x (question 5b in tables 5, 6, and 7, appendix A), which they were asked to graph using the technology, taking a few moments beforehand trying to picture what it might look like and sketching their ideas. Several students were unsure about where the curve crossed the axes and the minimum and maximum values of the function and consequently the axes were just left blank. There was little evidence of any in-depth thought into the problem, except when they received help.

The study also revealed that few students attempted to visualise the graphs of functions or the effects of transformations before turning to the graphical calculator, even when they were specifically asked to do so (as in question 5, appendix A). Furthermore, those students who did so tended to require a certain amount of prompting. It appeared that the students' ability to visualise with technology depends on the nature of the support that the technology offers. In the case of the Texas Instrument TI-92, the lack of clear labelling on axes contributed directly towards the students' confusion about the graph of y = sin x + cos x.

5.2.5 The Relationship between Symbolic and Visual Modes

Analysis of the student questionnaires, the students' work and observations from the classroom also provided insight into the question of whether and how the use of the graphical calculator would clarify the relationship between symbolic and visual modes of representation.

In their questionnaire responses, some students commented on the way in which the graphical calculator made the relationship between the symbolic and visual modes of representation clearer. The following student responses illustrate this viewpoint:

"Has using the graphical calculator enabled you to picture functions more clearly?"

> **May:** *Yes, using the graphical calculator has definitely enabled me to picture functions more clearly, especially the way they interact with other functions.*
>
> **Don:** *The graphing ability has made it much easier to see the functions as a graph.*

"Do you feel that you have benefited from the opportunity to use the graphical calculator?"

> **Kurt:** *It is far less laborious than other methods and helps you get a feel of the objective of functions.*

Yet, contrary to their beliefs, the majority of these students tended to concentrate on graphical representation when both graphical and algebraic aspects were involved (see tables 5, 6 and 7, appendix A). Just over half the students actually attempted to specify the symbolic form of the graphs resulting from a series of successive transformations in question 4, even though this was requested in the question and only 12% of their answers contained the correct symbolic form. Question 4 asked the students to use the TI-92 to perform the following sequence of transformations on the graph of $f(x) = x^3$: i). f(x/2), ii). f((x/2) + 2), iii). f((x/2) + 2) – 3, iv).

$2(f((x/2) + 2) - 3)$. In each case they were asked to sketch the resulting graphs and write down the equation of the function in its simplest form.

In addition, only relatively few of the students successfully completed the algebraic components of questions 8 and 11. For example, only four students attempted to solve the equation in question 11b algebraically and just two students effectively manipulated the equations in question 11c and d. Question 8 involved identifying the transformations which, when applied to six given graphs, would produce a second set of six graphs and the symbolic form of each of the new functions. In question 11 they were asked to solve the following equations numerically, graphically and algebraically: a). $x^3 + 8x^2 + 4 = (x - 2)^2$, b). $\ln((2x+1)/(x-1)) = 2$,

c). $2^{2x+1} + 2 = 5(2^x)$ and d). $\ln(x+1) + \ln(x-1) = 3$.

However, for those students who completed both the symbolic and graphical components of these questions, the use of the graphical calculator exemplified the relationship between the symbolic form of a function and its graph. It enabled them to access and compare the graphs of a wide variety of functions quickly and with ease, allowing them to discover for themselves the relationships between graphs of the same family of functions and to see how the graphical forms of functions relate to the symbolic forms. This in turn led to the students gaining a more holistic understanding of the concept of function, which was evident in the way in which they could talk about functions in class and in their questionnaire responses, in the comments that they made about learning functions with graphical calculators.

The three examples from the students' work which were discussed in the section 5.2.2 also served to highlight the importance of encouraging students to use symbolic reasoning to verify the results of any graphical exploration using the graphical calculator. In each of these three questions, the majority of students had misinterpreted the graphs produced by the graphical calculators. This occurred as a result of their over reliance on the technology and because they did not verify their answers through the use of a symbolic approach. Combining use of visual and symbolic approaches in answering these questions might have enriched their understanding of these particular functions, as it did with some of the other functions that were explored.

What this study shows is that these particular students tended to concentrate on graphical representation when faced with questions involving both graphical and algebraic aspects. This suggests that if teachers want to discourage students from focusing solely on graphical representation as a consequence of using technology, this might be best achieved through structuring lessons in such a way that both graphical and symbolic aspects are explored together.

5.3 Affective Factors which Contribute Towards Students' Learning of Functions

In order to determine the influence of graphical calculators on affective factors, data was again analysed from three sources: (i) post-trial questionnaires administered to the students, (ii) transcript data and (iii) post-trial questionnaires completed by the staff.

The post-trial student questionnaires provided some insight into their perspectives regarding affective factors and the use of the graphical calculator. Furthermore, analysis of the transcript data highlighted the important role of collaboration in their learning with technology. In addition, the post-trial staff questionnaires served to ascertain the teachers' perspectives surrounding affective issues.

The affective issues that were apparent in the findings of this phase of the study have been categorised into five main areas:

- attitudes towards graphical calculators,
- graphical calculators and confidence,
- feelings surrounding dependency and graphical calculators,
- collaborative dimensions of graphical calculators,
- graphical calculators and the role of the teacher.

5.3.1 Attitudes towards Graphical Calculators

The student and teacher questionnaires provided insight into their attitudes towards graphical calculators. Although, technology was rarely used in Advanced level mathematics lessons in this school, questionnaire responses revealed that ten of the students viewed technology as an important addition to their mathematics classroom. As had been anticipated, there was an extremely positive response to the use of the graphical calculators. Indeed, twelve of the students believed that they had benefited from the opportunity to use the graphical calculator and that using the graphical calculator enabled them to picture functions more clearly. Similarly, twelve students felt that their experience with the graphical calculator had contributed towards strengthening their understanding of functions. In addition, all of the students except Sue would welcome further use of the technology. This was especially so in May's case, as she commented:

> **May:** *I feel it would help me develop my understanding of graphs, which is vital for A level.*

Other students were, however, more cautious about the benefits of using the graphical calculators:

> **Emma:** *Technology is very important and as it is relevant to maths I think it is good to use it. However, it should be used to aid other work, not as a separate thing.*

> **Kurt:** *You've got to think about what you're doing or otherwise the information that you gain from it will be minimal.*

Mr Irons and Mr Doors regarded the use of technology at Advanced level as increasingly important. Both recognised the need to continue utilising technology in the sixth form, following greater use in the lower school. In response to the question: "How important, in your opinion, is technology in A level mathematics?" Mr Doors responded:

> *I feel that it is sensible to develop the use of technology in A level maths as its use is increased within Key Stage 3 and Key Stage 4. In an age when students are expected to be technologically literate, it seems clear that they should relate these skills to their A level study. It is important, however, that technology should be used to take away the drudgery to allow very able students to take concepts much further.*

The amplification effects of the technology were thus seen to offer valuable scope for enabling students to take concepts further by removing 'drudgery'. Yet, technology was very rarely used in Advanced level mathematics lessons in this school. However, feedback suggested that given the opportunity, appropriate funding and training technology would be used more frequently with Advanced level mathematics students as a means of supporting learning.

5.3.2 Graphical Calculators and Confidence

The analysis of the student questionnaires also shed light on the role of the graphical calculator in terms of its impact on levels of student confidence, which constituted an important affective element of the learning process. This was apparent even though the students were not asked directly about this aspect and was illustrated in the fact that four of them had commented on the benefits of using the graphical calculator as a means of verification. Typical responses to the question: "What do you consider to be the main advantages of using the graphical calculator?" were:

> **Pat:** *The main advantages are the ability to save useful time on graph drawing and the ability to self-check work.*

> **Sally:** *It's quicker than drawing graphs and can be used to quickly check your thoughts during a question.*

When asked whether she had benefited from using the graphical calculator and how, Betty replied *yes, my graphing skills have improved.* As Emma explained in response to the question: "Do you feel that you have benefited from the opportunity to use the graphical calculator?"

> *I feel more confident with graphing and I have a graphical calculator which I should now be able to use more effectively.*

In addition, May commented that she found using the graphical calculator less intimidating than using computers:

> **May:** *I would prefer to use a graphical calculator over a computer as it seems less intimidating and easier to use.*

The following student responses highlight the significance of the role of technology in helping students to then make sense of new ideas or those which are not yet conceptualised, and in boosting their confidence, and ability, to visualise.

"Has using the graphical calculator enabled you to picture functions more clearly?"

> **Jan:** *It has helped me to understand the translations of graphs, which I had problems with before.*
>
> **Emma:** *Yes, using the graphical calculator has definitely enabled me to picture functions more clearly, especially less common ones or newer ones like e^x, ln x etc.*

"Do you feel that you have benefited from the opportunity to use the graphical calculator?"

> **Sue:** *It has helped me to picture graphs, as this has always been my weakness.*

"What do you consider to be the main advantages of using the graphical calculator?"

Kurt: *It helps you to visualise functions that would otherwise seem perplexing. It makes the situation clearer.*

"Do you believe that using the graphical calculator has strengthened your understanding of functions?"

May: *Yes, using the graphical calculator has strengthened my understanding of functions, especially the functions I find conceptually difficult such as ln x and transformations of it.*

The ability of students to use the graphical calculator in this manner allowed them to sense some ownership of their mathematics and contributed towards increased confidence in their solutions. In addition, each graph that was drawn using the graphical calculator had to be sketched on paper by the student. As a consequence, consideration had to be given to the scale which was being used, the co-ordinates of any distinctive features (e.g. zeros, stationary points) and the accuracy of the student's version. This meant that whilst the students were obtaining most of their graphs from the graphical calculator, they were also practising their individual graphing skills.

Betty's comments suggested that the use of the graphical calculator may also result in improvement in the student's own graphing skills. The ability of students to use the graphical calculator to produce accurate graphs of a whole range of functions and as a means of checking assumptions can lead to greater student confidence in the graphing of functions.

This study shows that use of the graphical calculator can provide scaffolding for students in areas that may be regarded as weaknesses or in areas where difficulty is experienced. In turn, this could result in improved confidence in these areas, as has been evident in this study. Introducing technology into the classroom can prove to be very motivating for students experiencing difficulty.

5.3.3 Feelings Surrounding Dependency and Graphical Calculators

One of the unexpected findings of this phase of the research was the feeling expressed by the students that they might become over-dependent on the technology. This was particularly apparent in the students' responses to the following questions taken from the student questionnaires: "What disadvantages do you perceive?"

> **Emma:** *It may mean that you don't think about what sketches of graphs look like and rely on it too much.*
>
> **Betty:** *They could be relied on a little too much if you put all graph work onto them.*
>
> **Kurt:** *Disadvantages, not thinking about functions and just relying on the machine.*
>
> **Sally:** *Technology is a useful tool especially for checking work but I find it harder to work for yourself, as you become lazy.*
>
> **Sue:** *It makes everything too easy. I think it is easier to learn something if you work it out for yourself.*

"How important in your opinion is technology in the A level mathematics classroom?"

> **Don:** *It is important to use technology to support mathematical skills but steps must be taken to make sure it does not replace them.*
>
> **Pat:** *It is important but shouldn't detract from the roots of maths, which could happen if over used.*

Mr Irons and Mr Doors also expressed some concerns surrounding over-reliance on the technology in their questionnaire responses and Mr Doors emphasised the need for careful structuring of activities involving the technology to prevent any "disengagement of brains":

> **Mr Doors:** *It would be essential to carefully structure and apply its use in class to prevent it from becoming an opportunity for students to disengage their brains.*

Overall, these students appeared to appreciate the opportunity to use the graphical calculator and the evidence suggests that they had benefited mathematically from the experience. The materials and exercises also seemed to have provided an adequate introduction to the graphical calculator. However, whilst most of the comments regarding the use of the graphical calculators were positive, as indicated in their questionnaires responses, eight of the students expressed concerns regarding over-dependency on technology. These concerns related to fears that the use of the technology could result in work being too easy and could replace basic mathematical skills. Moreover, this could facilitate laziness and in the process discourage individuals from thinking for themselves, especially about what the particular function may look like. The fact that the students in this study were aware of the potential dangers, would tend to imply that over-reliance was less likely to occur. However, as illustrated in section 5.2.2, consideration of the students' work indicated that this was not necessarily the case. The examples from this section provided evidence that there were occasions where the students were overly dependent on the technology and that this in turn led to misunderstandings and misinterpretations of the graphical information presented on screen.

5.3.4 Collaborative Dimensions of Graphical Calculators

As part of their introduction to the graphical calculator, the students were asked to use the technology to draw the graphs of the following functions:

1. $y = 4x^2 - 4x + 1$ 2. $y = (x + 3)^3$ 3. $y = 2(|x| - 1) / 3$ 4. $y = \cos(x/2)$

In each case, they were required to sketch the graph on paper and use the graphical calculator to find the value(s) of x when y = 0. These value(s) were to be checked by substituting y = 0 and solving in each case. In addition for questions 1 and 2, the students were asked to use the graphical calculator to determine the nature of any stationary points and to ascertain their co-ordinates. Again these values were to be confirmed by differentiation.

The way in which these questions were attempted was of interest. As previously stated, these students were not familiar with graphical calculators and following the initial teacher-led demonstration, this was the first time that they were able to experiment with the machines in groups by themselves. Subsequently, the discussions between certain students (see table 5.4) were audio taped whilst they worked on these problems. The discussions between May and Sue, and Diana, Jan, Guy and Lea, in particular, raise some relevant issues (the transcripts of the complete set of recorded discussions can be found in Appendix A).

Table 5.4 Students Audio Recorded

Names	Question Number(s)
Betty and Emma	2
May and Sue	2, 3
Kurt and Pat	2
Guy and Lea	3
Diana and Jan	3

May and Sue were working on the second question when recording was initiated. Sue considered herself to be a person who forms and makes use of mental images when solving mathematical problems, whilst May did not.

1 Sue: Let's do question 2.
2 May: Right ok.
3 Sue: Don't you have to put cubed in the bracket, or will it be all right?
4 May: Em, see what it looks like when we've done that.
5 Sue: Yes enter.

6 May: Yes, it's fine. When $y = 3x^3$, what do you get for the intersection of the x-axis for this?
7 Sue: I have 0.58, I think.
8 May: I didn't get that, not at all.
9 Sue: What did you get?
10 May: 5. It probably doesn't work.
11 Sue: Let's go into graph and draw it.
12 May: Oh it's one of those ones.
13 Sue: Oh yes.
14 May: Oh cool.

Initially, May and Sue chose to tackle the problem using a symbolic approach and worked independently for a couple of minutes trying to find the co-ordinates of the intersection point with the x-axis. However, their symbolic manipulations gave rise to different answers (lines 7 to 11) and so they decided to explore the problem graphically. From the moment that they began to use the graphical calculator for this purpose, they showed interest in the mathematics discussed and they worked together throughout, using the graphical calculator as a tool to help them think about the problem:

15 Sue: Hang on a minute what oh? It's going to have lots of turning points as well, isn't it?
16 May: Graph it. Graph it.
17 Sue: Do you reckon we should em see if…
18 May: Is there only one [turning point] when $y = 0$?
19 Sue: How do you know if there's more than one though?
20 May: I don't know I suppose you could zoom out. Are we in standard?
21 Sue: Yes we are, aren't we?
22 May: Yes.
23 Sue: Em no I'm not I don't think. Oh no.

24 May: What have you got?

25 Sue: Do you think we should zoom out to see a bit more?

26 May: Yes but we'll have to zoom out to get ourselves to get the same.

27 Sue: What do you mean the same centre as before?

28 May: Yes.

29 Sue: What do you reckon?

30 May: I think it would have shown.

Once both students had produced the graph, they appeared to be thinking along the same lines and each wondered if there were more turning points than were initially visible on screen in ZoomStd (lines 18 and 19). May suggested zooming out from ZoomStd, to enable them to see more of the graph, in an attempt to resolve their uncertainty (line 20). Since neither of the girl's graphs showed any more turning points, Sue questioned whether they should continue to zoom out (line 25). May agreed that they should, accepting Sue's proposed plan of action, and suggested that they both used the same 'zoom out' factor and centre to ensure both their graphs were the same (line 26). Following the second zoom out application, the girls' graphs still had the same shape and May concluded that if there were any additional turning points they would have shown up (line 30).

This part of the discussion illustrates the individual students' roles in the negotiation of meaning. Their use of the graphical calculator to structure their thinking about the problem was particularly evident when they started to speculate about the shape of the graph. When Sue asked May: "how do you know if there's more than one [turning point] though?" her response "I don't know I suppose you could zoom out" indicates that, whilst unsure, she was thinking about the problem in terms of the facilities offered by the graphical calculator. If they had drawn the graph on paper, zooming out would not be an option. Their use of the zooming function convinced them that there were no further turning points and they appeared to be confident in this assertion.

At this point James Green who had been audio recording and observing the interaction between the two students entered the conversation:

31 JG: You can just go to point on the graph – move it across and see what the co-ordinates are there. So you can say move it below the x-axis to about there, do you see?

32 Sue: Yes.

33 JG: That's quite a useful thing to do.

34 Sue: But do you have to go into the maths bit to work it?

35 JG: But if you want to do it, you know, yes go to maths. What are we on y = zero?

36 Sue: We're on $y = 0$.

37 JG: On the point of inflection?

38 Sue: Yes.

39 JG: So go down to inflection, press enter. Now then there could be – this is a nice simple curve with a single point of inflection.

40 Sue: Yes.

41 JG: You could have a wiggly curve with all sorts of points of inflections, minimums and maximums, and things.

42 Sue: Yes.

43 JG: So you've got to tell it that you're interested in the point between here and here.

44. Sue: Ok.

45 JG: So if you say –5 to - it doesn't actually matter as long as you cover this because there's only the one. But in general you'd have to estimate a point here and a point here, say –5.

46 Sue: Do you use the cursor?

47 May: The cursor at all?

48 JG: No I think you just enter or go down.

49 Sue: I used that cursor thing.
50 May: Yes. Enter down.
51 JG: You're making a box round it aren't you?
52 Sue: Yes.
53 JG: So…
54 Sue: There's an inflection at –6.
55 JG: Oh yes you just over type –5.
56 Sue: Ah and then enter.
57 JG: And then enter. Upper bound is 2 say enter.
58 May: There's an inflection at –3.
59 Sue: That's right yes. So that's a stationary point as well isn't it?
60 May: Yes.
61 Sue: Y = 0. [Working].

In light of the girls' lack of familiarity with the technology, James demonstrated how to use the maths menu to calculate the co-ordinates of the point of inflection, acting as a more knowledgeable person in Sue and May's zones of proximal development (lines 39-51). However, Sue incorrectly interpreted the information on the calculator screen which led her to assert that there was a point of inflection at –6 (line 54). James then explained what she needed to do in order to obtain the correct co-ordinates, in an attempt to repair her misunderstanding (lines 55 and 57). Following his instructions May was able to determine that (-3,0) were the co-ordinates of the point of inflection (line 58). Sue was also able to verify and thus accept this answer (line 59). Their written solutions also contained a symbolic proof of the result, which they were able to produce after their graphical exploration.

James showed the students how to use the technology effectively to solve the problem, whilst allowing the girls to make the discoveries by themselves. As a result, both students were happy with the solution offered by the graphical calculator and were able to re-examine and correct their symbolic manipulations in light of their graphical exploration. They both worked well together, taking it in turns to direct and reflect on their activity in order to develop a shared solution to the problem. Their joint work with the graphical calculator was certainly more successful than their initial individual attempts at symbolic manipulation. This episode would tend to support the assertions made by Jones and Mercer (1993), who maintain that a considerable amount of learning, not least in relation to information technology, consists of sharing knowledge. The technology appeared to promote collaboration and negotiation of meaning between May and Sue. They had to develop a joint strategy together for solving the problem using the graphical calculator.

May indicated that she had some difficulties when working with graphs in her questionnaire responses, which are reproduced below, and that although she did not naturally tend to form mental images, she was keen to try to do this more often.

> *"The technology gives another way of seeing the ideas behind certain theories and rules in maths, some of which can be quite hard to picture yourself."*

> *"The graphical calculator has helped with transformations of graphs a lot, as this is something that I find difficult to remember."*

> *"The use of technology has strengthened my understanding of functions, especially the functions, which I find conceptually difficult such as ln x and transformations of it."*

> *"I don't form mental images naturally but I am trying to get into the habit."*

The graphical calculator was, thus, seen by May as a means by which she could improve her skills in this area. On the other hand, whilst Sue "preferred to form mental images" whenever possible, she also stated that picturing graphs had always been a weakness:

> *I have benefited from the opportunity to use the graphical calculator because it has helped me to picture graphs as this has always been my weakness.*

In this respect, she found using the graphical calculator very beneficial. The pairing of these students was particularly successful - each student was able to support the other when facing any uncertainties and each played a crucial part in the outcome of the episode, taking turns to negotiate meaning. For example, Sue was the first to suggest using the graphical calculator, May was the one who suggested zooming out, Sue carried on the conversation with the additional researcher and May was the first to come up with the answer. Each of these actions contributed towards the development of a common understanding, where any misunderstandings were identified, challenged and eventually overcome.

The use of the graphical calculator would thus appear to benefit both visually and non-visually orientated students, particularly in creating a forum in which they could exchange ideas, test their assumptions and work together in order to obtain a joint solution to the problem under consideration. These two girls worked particularly well together. However, May who had previously shown enthusiasm for working on the problem with Sue (eg "oh cool") made fewer verbal contributions when James Green joined in their discussion and seemed to withdraw from the conversation to take on the role as an active listener. This suggested that May was probably less confident engaging with the researcher than she was with her peers and perhaps this could have been gender related. In her questionnaire responses, she commented that she found using the graphical calculator less intimidating than using computers. This could also have been a gender issue, having to use computers publicly in the presence of boys.

Interestingly, Sue was the only member of the group of students who would not welcome further use of the graphical calculator in future mathematics lessons and she stressed that in her opinion, "it is easier to learn something if you work it out for yourself". Thus, regardless of the benefits that she felt that she had experienced as a result of using the technology, her existing and more traditional view of learning mathematics was prevalent.

5.3.5 Graphical Calculators and the Role of the Teacher

The episode involving May and Sue and James Green, the fellow researcher, discussed in the previous section also served to illustrate the importance of the role of the teacher in scaffolding the students' use of the technology. In this episode James' contribution was crucial in steering May and Sue towards appropriate use of the graphical calculator and ultimately the correct answer. This can be seen in lines (34-45) where James talked the girls through how to determine the co-ordinates of the point of inflection using the maths menu of graphical calculator. It is also evident in lines (54-60) where Sue misinterpreted the display and his clarification of her error enabled both girls to obtain the co-ordinates.

The need for careful mediation of the students' use of the technology by the teacher was also illustrated in an episode involving Diana, Jan, Lea, Guy, the teacher-researcher (SE) and Mr Doors (SD). Diana and Jan were working together on question three and were confused by the answers given by their graphical calculators. This prompted two other students, Guy and Lea, who had already attempted this question and had experienced similar difficulties to be drawn into the conversation. As with the previous example, none of these students used the graphical calculator to try to differentiate the function directly via the derive programme.

1 Diana: What do you get for your minimum for question 3?
2 Jan: I get something really horrible, -1.56.
3 Diana: I get +2.16 to the –14.
4 Jan: You get what?
5 Diana: 2.16 x 10^{-14}.

6 Jan: I get that. I get that (showing her screen to Diana). They're the same line though.

7 Diana: Yes that's really odd. That's really odd for the same function.

8 Diana: Have you done number three? For question 3, we've got different answers. We've both got the same equation. But when we got to the middle it's given us different ones...

9 Guy: Yes.

10 Diana: It's given the same y value.

11 Guy: But it's given you what?

12 Lea: Are you trying to do the minimum. You can't do the minimum for number 3.

13 Diana: Why?

14 Lea: You just can't. No you can't.

15 Guy: Why not?

16 Lea: I've asked and she said you can't.

17 Jan: It's given us all different numbers. Is it something...?

18 Lea: Exactly, because I had about 20 different numbers.

19 Jan: So you're not supposed to do it. Oh right.

20 SE: The function actually goes to a point so it's not smooth like the others.

21 Jan: Right.

22 SD: You can't differentiate it because the gradient is not zero, it does get to a lowest point, but it's not a turning point.

Diana and Jan assumed that the function in question three had a minimum turning point. This led to much confusion for the girls, as both students' graphical calculators gave different approximations to the non-existent minimum turning point when they used the maths menu to calculate the co-ordinates of the presumed stationary point. Both girls had been able to calculate the turning points of the functions in the previous two questions and as they had as yet relatively little experience of calculus, they simply assumed that this particular modulus function was no different to these other functions. Their results seemed odd, and drew their attention to the fact that something was amiss, but they could not understand why. In search of clarity Diana turned to Guy for help (line 8). However, it was Lea who attempted to shed light onto the situation (line 12). She had been listening to the conversation and joined in voluntarily. However, her comments failed to really clear the confusion and it appeared that she did not follow the explanation given to her previously as to why this function did not have a minimum turning point (line 16). Having heard the students struggling to make sense of the question, the teacher-researcher and Mr Doors, the classroom teacher, offered an explanation as to why this was not a turning point.

This episode illustrates how in certain circumstances useful prior knowledge in one context may be misapplied to a new context. Furthermore, in this instance use of the graphical calculator exaggerated the confusion amongst the students, rather than resolving it. As a consequence, continued use of the technology without intervention by the teacher might perpetuate this kind of misunderstanding. None of the students from this example were able to make sense of the calculator's results until the teacher-researcher and classroom teacher intervened. The lack of clarity in the feedback provided by the graphical calculator could have led the students into making erroneous conclusions about the nature of this function. Thus, the teacher needs to be aware of possible misunderstandings that may arise as a consequence of using the technology and, as Hudson (1997) suggests, should monitor the interactions between students who are using the technology carefully.

The counter-intuitive results produced by the graphical calculator in this example also appeared to encourage greater interaction between students. Guy and Lea would probably not have entered the conversation if the technology had produced an answer that was not in standard index form.

Diana and Jan were not happy with the results that their graphical calculators produced and sought an explanation as to how and why these answers had arisen. They turned to their peers for clarification, although in this case this was not sufficient. The input of a more knowledgeable person, i.e. the teacher was required.

5.4 Conclusions

The chapter has considered the exploratory phase of the research and the ways in which students with no previous experience of using graphical calculators were able to use this technology to further their understanding of functions. This section summarises the findings of this phase of the research in relation to the cognitive and affective aspects surrounding the use of graphical calculators in the Advanced level mathematics classroom.

5.4.1 Amplification and Cognitive Reorganisation

This phase of the study has shown that the students were very aware of the amplification effects of the graphical calculator, i.e. the potential for the speed and ease of learning to be increased whilst using the technology. The restructuring of students' thinking which is made possible as a consequence of using the technology (the cognitive reorganisation effects) was recognised as a factor that influences the depth of student understanding. Yet, few students commented specifically on how their understanding had altered, or how they viewed functions differently as a result of using the technology.

5.4.2 Dependency

Analysis of the students' written work indicated that the solutions provided by the graphical calculators are sometimes regarded as irrefutable, as proposed by Smart (1995b), even when the answers are not feasible. This raised questions about their reasoning: were they actually thinking about the problems, comparing symbolic answers with the graphical, or were they just pushing buttons? The students' over reliance on the graphical calculators when solving certain problems meant that they misunderstood, misinterpreted and misused some of the information provided by the technology. These findings raise important issues for teachers who are using graphical calculators in their classrooms, for example how can they help students to develop a strategy which enables them to decide whether a graph is misleading or not? They also suggest that students should be encouraged to question the *absolute reliability* of the images that they produce using technology, to apply their own intuition and to make use of complementary problem solving techniques.

5.4.3 Visualisation

The findings of this phase suggested that use of the graphical calculator can encourage students to use and develop visual methods. Each student was able to use the graphical calculator effectively to reinforce and shape his/her visualisations. Moreover, those who did not tend to form mental images found the graphical calculator particularly useful in this respect. However, findings suggest that a student's ability to visualise is dependent on the nature of support offered by the technology. The absence of a numbered scale on the axes and the size of the screen caused some difficulties for students. These limitations highlighted the need for the teacher to encourage students to become familiar and confident with using the window application, which indicates the scale that is being used. In addition, another strategy that would allow the students to verify the scale for themselves would be to encourage them to calculate points on the graph manually. Students also tended to use the graphical calculator too readily to produce the graphs of functions, without taking the time to try to picture them for themselves beforehand.

5.4.4 The Relationship between Symbolic and Visual Modes

The results of this study show that many of the students were not using symbolic approaches to enrich their answers. Indeed, few students completed the algebraic components of some questions successfully, suggesting that they were concentrating on the graphical form of representation and neglecting the symbolic. The students' tendency to neglect the symbolic form of representation, whilst concentrating on the graphical emphasised the need for symbolic approaches to be presented alongside the visual (not in isolation), and the importance of using technology as a means of strengthening the links between the two approaches.

5.4.5 Scaffolding by the Technology

The study also highlighted a number of potential ways in which the graphical calculator could be used to support student learning of functions. In particular, it was found that the graphical calculator could be utilised as a means of strengthening the students' own graphing skills, to facilitate the introduction of the concept of functions for the first time and to help them to make connections between symbolic and graphical representations. In addition, use of the graphical calculator could provide a support mechanism for students who are either uncomfortable when working within a visual mode of representation, or who are experiencing difficulty. Students were also enabled to become more confident in their methods and solutions through the visual verification (or rejection) of their ideas provided by the technology. This clearly had a positive impact on their attitude towards their work. Confidence was thus a key affective issue that was tackled through use of the graphical calculator.

5.4.6 The Role of Technology

Overall examination of the students' work revealed that, whilst there are positive benefits in allowing them to experiment freely with technology, often there are associated problems. These observations were due possibly to the transparency effect described by Adler (1998). As these students were unfamiliar with the graphical calculators, they tended to concentrate on the technology rather than on the mathematics i.e. the graphical calculators were too visible, the object of attention. A longer period of study would be needed to discover whether the technology would eventually become invisible to students and the means to the mathematics.

The findings of this phase of the research also point towards the development of a more structured approach to promote students' learning with technology which takes account of:

(1) the type of technology being used and associated limitations,
(2) the timing of the introduction of the technology into the classroom,
(3) how the use of technology is combined with the use of other media in the classroom such as pencil and paper, oral communication,
(4) the role that the teacher has in mediating the use of the technology.

5.4.7 Attitudes

One of the most encouraging outcomes of this study was the positive response of the students towards the introduction of the technology and the ways in which they believed that the use of the graphical calculator had strengthened their understanding of functions. The students' comments suggested that by exploring the graphs of functions with the graphical calculators, in addition to using the more traditional symbolic approach, they were able to develop a more meaningful appreciation of the nature of functions.

5.4.8 Collaboration and Scaffolding by the Teacher

The two examples of student interactions provided insight into how they worked together using the technology and served to illustrate the value of collaboration amongst peers and the need for scaffolding by the teacher. May and Sue were able to overcome their lack of familiarity with the graphical calculator and work together towards the shared goal of finding the turning points of the function. Both girls were thoroughly engaged with the problem and the technology; they supported and questioned one another and made suggestions, which furthered the development of the problem solving process. The assistance of the additional researcher proved to be invaluable in guiding the students towards a solution, especially in light of the unfamiliar technology.

The second example involving Diana, Jan, Guy and Lea also highlighted the need for teachers to monitor interactions between students and to intervene when appropriate. The students' inability to transfer useful prior knowledge of differentiation to the context of the modulus function caused much confusion which was exaggerated in this case by use of the technology and as such warranted an explanation from a more knowledgeable person i.e. the teacher. In this respect the role of the teacher is an important aspect of this study and the results of this phase emphasise the need for careful mediation by the teacher over how the technology is introduced and of the students' use of the technology. These findings have general implications for the mathematics classroom and have influenced the approach to teaching adopted in phase two. It also raised the question as to whether the graphical calculator could be used as a means of enabling students to overcome their difficulties and to transfer knowledge between different contexts rather than creating more confusion and this was explored further in the second and third phases.

5.5 Implications for Subsequent Phases of the Research

The first phase of the research served to highlight a number of important issues regarding the cognitive and affective aspects of graphical calculator use. Consequently, these findings shed initial light on the first objective of the thesis as a whole, suggesting that there is a complex set of factors, both individual and collective, that had contributed towards the students' acquisition of meaning within the graphical calculator environment. The most striking of these, however, was the way in which the graphical calculator was seen to promote and scaffold discussion and collaboration amongst the students and also the necessary and supplementary role of the teacher in mediating the use of the graphical calculators. As such, these findings in particular significantly influenced the main focus of the following two phases of the research, in which subsequently a greater emphasis was placed on the social aspects of learning. Moreover, the results of this phase gave rise to the third objective of the research as a whole: to investigate the teachers' role in a graphical calculator environment. This in turn led to the search for a coherent framework to analyse the interactions between students and the teacher-researcher more carefully.

The data collected in this phase also provided the opportunity for the first evaluation of the graphical calculator as a tool for mediating the development of students' understanding of functions, through the visual imagery it supplies, which was the second objective of the research. In highlighting the factors involved in this process, this phase of the research pointed towards categories of analysis for subsequent phases.

CHAPTER 6
INVESTIGATING MEANING MAKING AMONGST EXPERIENCED GRAPHICAL CALCULATOR USERS

6.0 Introduction

This chapter represents the second phase of the research (see figure 1.1, page 6) and gives an account of the way in which a small group of experienced graphical calculator users derived meaning for functions from their interactions with each other, the teacher-researcher and their use of the graphical calculators.

More specifically the aims of this phase were to:

- explore how students who regularly use graphical calculators make use of this technology as a cultural artifact (Berger, 1998),

- investigate how these students approach problems involving functions, with and without the use of technology, together and individually,

- examine how these students use visualisation in problem solving, and in what ways the use of graphical calculators could facilitate this,

- promote the validity of visual approaches to students and thus greater use of visualisation and to encourage students to combine methods as much as possible,

- establish a 'local community of practice' within the classroom.

This phase of the research builds on the first phase through consideration of experienced, as opposed to totally inexperienced, graphical calculator users and how these students might use the technology differently to construct meaning for functions. The same categories of analysis that arose from the data in phase one were reapplied to the data collected in this phase, although there was a need to develop further sub-categories and modify existing ones in light of the new data.

6.1 Background to the Research

The second phase was conducted at Anderson College in Sheffield during June 1998. As in the previous trial, the investigation involved the teacher-researcher working with a small group of year 12 students, all aged 17, for a period of six hours (two three-hour sessions). In this case the group consisted of six students during the first session; Diane, Martin, Jason, Julie, Rachael and Robert, and three of these students during the second session; Martin, Julie and Robert. These students were studying GCE Advanced level Further Mathematics and were described by their teachers as being very capable mathematicians. Each student had purchased his or her own graphical calculator and was thus familiar with using this type of technology. The three members of staff, Mr. Pearson, Ms. Slater and Ms. Mooney, each taught one of the integral components of the course (pure mathematics, statistics and mechanics, respectively) and positively encouraged the use of the graphical calculators in lessons. Each individual student was given a Texas Instrument TI-92, although they generally worked together as a group sharing ideas. The lessons were designed with the aim that they would provide a supportive learning environment where 'local communities of practice' (Winbourne and Watson, 1998) could be established.

6.1.1 Methods of Data Collection

The second phase began with refinement and further development of the classroom materials and techniques for gathering data. The additional techniques for collecting data included a pre-trial assessment, pre-trial as well as post-trial questionnaires, semi-structured student interviews, audio taped class discussions and a video recording of a single student (Robert) working with a graphical calculator that was connected to the overhead projector. Overall this part of the study consisted of eight distinct stages of data collection, which are outlined in the following table.

Table 6.1: Data Collection Activities

Data Collection Activity	Timing of Activity	Students Involved	Staff Involved
(1) Student and staff questionnaires concerning visualisation.	Lesson 1, first 30 minutes.	Diane, Martin, Jason, Julie, Rachel, Robert.	Mr. Pearson, Ms. Slater, Ms. Mooney.
(2) Students' work on the pre-trial questions, solving functions without the use of technology.	Lesson 1, next 45 minutes.	Diane, Martin, Jason, Julie, Rachel, Robert.	N/A
(3) Introduction to the TI-92 and familiarisation questions.	Lesson 1, next 50 minutes.	Diane, Martin, Jason, Julie, Rachel, Robert.	N/A
(4) Students' work on the main pre-prepared exercises with the use of graphical calculators.	Remainder of lesson 1 and 2hrs 15mins lesson 2. (3hrs in total)	Martin, Julie, Rachel, Robert.	N/A
(5) Student interviews concerning functions.	Lesson 2, first 50 minutes. Conducted whilst the other two students worked.	Martin, Julie, Robert.	N/A
(6) Audio recording of classroom interaction.	Lesson 2, 25mins.	Martin, Julie, Robert.	N/A
(7) Video taping of individual students as they work on the main exercises.	Lesson 2, 10mins.	Robert.	N/A
(8) Student questionnaires on the role of technology.	Lesson 2, last 25 minutes.	Martin, Julie, Robert.	N/A

As illustrated in table 6.1, two questionnaires were administered to the students, one regarding the role of visualisation in A level mathematics and the other concerning their reactions to the technology. The questionnaires were devised to illuminate why, when and how they used imagery and whether they viewed technology as a resource that provides support for visual learning. The three members of staff who each taught this group of students were also given a questionnaire on visualisation. The staff questionnaire was intended to provide a background to the study, clarifying their views on visualisation and the extent to which visual methods are encouraged in the classroom. Each of these questionnaires can be found in appendix B.

The exercises were divided into two components: a pre-trial inquiry and the main exercises. The pre-trial inquiry exercises were designed to indicate how the students would attempt to solve standard problems involving functions initially without the use of technology (see appendix B). Each question was phrased in a manner that would be found in a traditional A level textbook. These exercises were followed by a brief introduction to the TI-92. The students then worked for the remainder of the lessons on the main pre-prepared exercises using the graphical calculators. Fewer introductory questions were given to the students than in the previous trial, as these particular students were already experienced graphical calculator users. The main exercises were also modified to take account of this, as well as the feedback received in phase one.

As a further means of data collection, one student, Robert, was filmed as he worked through certain questions from the main exercises. He was working by himself using a graphical calculator that was connected to the overhead projector. In addition, discussions between Robert, Martin, Julie and the teacher-researcher were recorded onto audio tape. These discussions surrounded possible solutions to one of the questions from the main exercises.

The student interviews were devised in order to illuminate how they would attempt to solve questions involving functions individually rather than collectively. Each question was selected to provide a range of different problems for them to consider. In addition, styles of questions that would possibly be unfamiliar to the students were chosen, especially question 5 (see Appendix B). The intention of this was to see if these students would tend to use more imagery with unfamiliar, non-typical problems.

The questions were phrased in a manner that would not necessarily promote a symbolic approach. For example, the first question was worded: for which values of x is the graph of $y = 3x^2 + 9x - 12$ below the x-axis? The same question could have been written: solve $3x^2 + 9x - 12 < 0$ or for which x values is the expression $3x^2 + 9x - 12$ less than zero? This type of wording could be considered as a precursor to a symbolic approach, and certainly as indicative of such a method.

In what follows, the data from each of these eight stages will be analysed in terms of cognitive factors (section 6.2) and affective factors (section 6.3).

6.2 Graphical Calculators and Cognitive Factors in Students' Knowledge of Functions

In order to determine the influence of graphical calculators on cognitive factors, data was analysed from the eight sources outlined in table 6.1. The findings that pertain to cognitive factors have been subdivided into the following themes, which build on those highlighted in phase one:

- Amplification, Cognitive Reorganisation and Students' Understanding of Functions
- Graphical Calculators and Dependency
- Graphical Calculators and Visualisation

In each of these three areas, the analysis of data was considered from a cognitive perspective and the graphical calculator was subsequently seen to have a significant impact on the students' cognitive development.

6.2.1 Amplification, Cognitive Reorganisation and Students' Understanding of Functions

The findings from the first phase of the research pointed to the potential for the range of students' learning to be increased as a consequence of the amplification effects of the technology. Even more significantly, the analysis also indicated that the cognitive reorganisation effects of the technology could have an impact on the depth of students' understanding. These findings raised the question of how the amplification and cognitive reorganisation effects might be interrelated and how these factors might influence the way in which students' thinking develops. Consequently, in this phase, evidence of the students' awareness of the amplification and cognitive reorganisation effects of the technology was sought from their questionnaire responses. The classroom interaction then provided a window into these effects in action and the nature of the relationship between them.

In the students' questionnaires, they were asked to specify what they considered to be the main advantages of using technology to study functions:

> **Martin:** *Graphical calculators/computers can be helpful and speed up calculating answers. The biggest advantage is the ability to quickly see how a function will change when certain things happen to it.*

> **Julie:** *Students are free to spend time manipulating the different graphs of functions to learn how they work.*

> **Robert:** *Technology will graph very quickly and help students to recognise and visualise characteristics of many functions. It is very useful in speeding up calculations.*

Martin also commented on the benefits of using the graphical calculator as a checking tool and the importance of being able to access a dynamic representation of transformations:

The TI-92 is useful because it can simplify equations, so can help check answers. Graphs are easy to understand when seen plotted and it has helped me to see how graphs can be manipulated.

The ability of students to verify or, in particular, disprove their solutions using the graphical calculator can lead to cognitive reorganisation (as can be seen in episodes 1 and 2 discussed below).

Through being able to see the visual effects of transformations, Martin felt that his use of the graphical calculator had strengthened his understanding of the relationships between functions:

It has definitely helped me to see how functions can be related and manipulated through translations and stretches.

In the eyes of these students, the main advantage of using graphical calculators to teach the concept of functions was the speed with which graphs could be drawn and manipulated, which left them free to explore functions further. Cumbersome and time-consuming tasks were removed by the technology, leaving them able to complete more conceptually demanding tasks with greater effectiveness and ease. The amplification effects of the technology were thus very visible to these students and appeared to be their main focus of attention, as was also the case in phase one.

Whilst none of the students except Martin commented specifically on the cognitive reorganisation effects of using the graphical calculators, these effects were clearly evident in practice, as is illustrated in the following episodes. The first of these involves Robert working individually on a problem from the main exercises using the graphical calculator. The second involves Martin, Julie and Robert discussing the solution to a different question from the main exercises, although it is Robert's utterances that are the focus of attention.

Episode 1 – Linking Amplification and Cognitive Reorganisation Effects

Robert's use of the graphical calculator was video taped whilst he attempted to show that ln $x^a = a\ln x$. It was anticipated that the resulting discussion between Robert and the teacher-researcher would shed light on Robert's thought processes and how these might have been affected by use of the graphical calculator. The initials SE are used to denote the teacher-researcher in this episode and in all subsequent episodes.

Initially, Robert used the graphical calculator to draw the graphs of ln x^2 and *2*ln x simultaneously. However, at this time he was unaware that this approach was unlikely to yield any discoveries, as one of the graphs would mask any differences between the two. When it was suggested that he draw them separately, Robert obtained the following two graphs (figures 6.1 and 6.2) and the dialogue below was initiated:

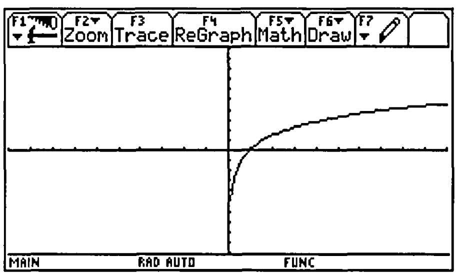


Figure 6.1 y = *2*ln x

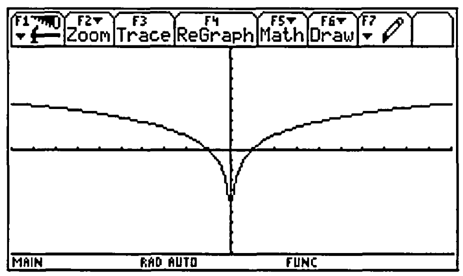


Figure 6.2 y = ln x^2

1. SE: Does the graph of *2*ln x surprise you?

2. R: Not really, you can't have logarithms of negative numbers.

3. SE: Exactly. So would you say that the two expressions were the same or not?

4. R: I'm hesitant to say. I would say that algebraically they were the same.

Robert recalled being taught during his previous experience of logarithms that these two expressions were equivalent (line 4). However, the graphs produced by the graphical calculator seemed to contradict this assumption, although he had established why the two graphs are not identical for negative x values (line 2).

5. SE: What will happen, do you think, for x^3: ln x^3 and *3*ln x?

6. R: I wouldn't have thought that there would have been any difference.

7. SE: Think about the graph of ln x^3. Why would that be different to ln x^2?

Robert drew the graph of ln x^3 on the TI-92 (figure 6.3):

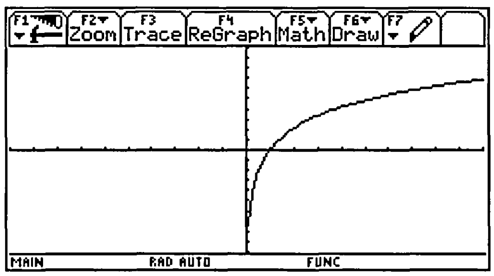


Figure 6.3 $y = \ln x^3$

8. SE: Why isn't there a part of that graph for negative x values?

9. R: For the same reason that there isn't a negative part for *2*ln x – you can't have a logarithm of a negative x value.

10. SE: So is *3*ln x going to be the same as ln x^3 then?

Robert plotted *3*ln x and agreed that they were the same (figure 6.4):

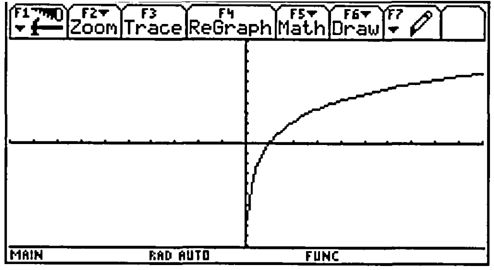


Figure 6.4 $y = 3\ln x$

When confronted by the question involving x^3, Robert's initial reaction was to assume that the same thing would happen (line 6). In an attempt to prompt him to think again, the teacher-researcher asked him to think about the graph of $\ln x^3$ and how this would be different from the graph of $\ln x^2$. In response to this question Robert drew the graph of $\ln x^3$ using the graphical calculator. By employing the technology, and being immediately able to produce the graphs of the functions he was considering, he was able to begin to formulate ideas as to why his original analysis was incorrect. In this way use of the technology induced cognitive reorganisation. Robert was now developing additional insight into the problem and was able to predict what would happen in the case of x^4:

11. SE: What do you think will happen with x^4?

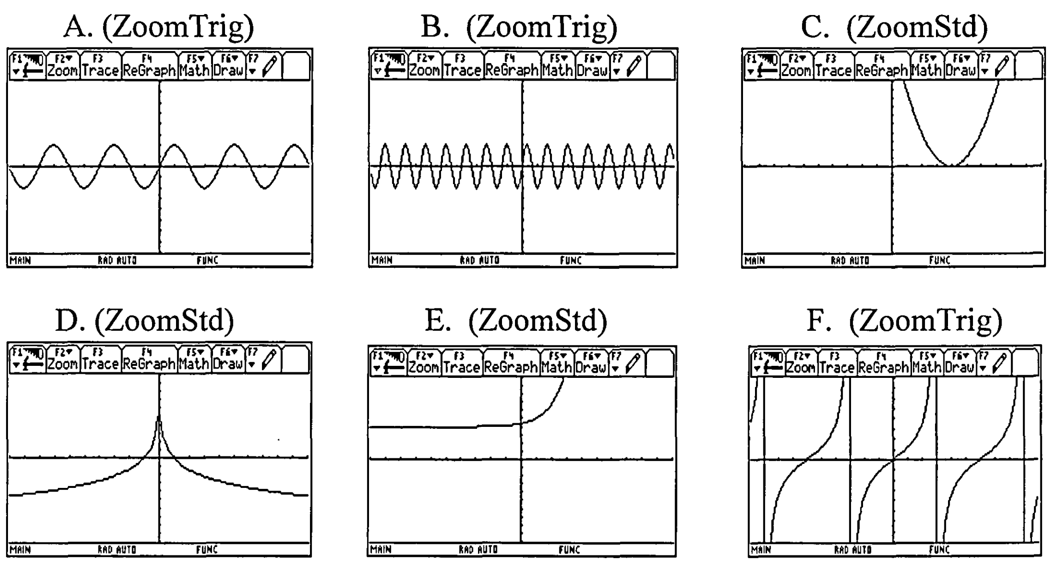
12. R: The same thing as with x^2, but it would become steeper, as in stretched.

At the end of this episode, Robert had established that the two graphs would only be identical for odd values of *a* and the reason why this is the case. This had occurred because Robert was able to make and test predictions using the graphical calculator, which allowed him to quickly produce the graphs of particular cases adding weight to his arguments. Ultimately, the graphical calculator enabled Robert to access graphical images of functions quickly and easily, which in turn allowed him to see the problem more clearly and to proceed towards the solution with confidence. As such, this example illustrates the way in which the amplification effects of the technology can contribute towards the cognitive reorganisation effects.

Episode 2 – The Role of Graphical Calculators in Enabling Students' Thinking to Develop

This example features an extract from a class discussion involving Julie, Martin and Robert. In this discussion, the students were attempting to solve a question from the main exercises together, in which they had to identify the correct symbolic forms of six graphed functions from a list containing several options (see figure 6.5).

Match up the six graphs with their corresponding functions, chosen from the list below:



1. $y = \sin(x/3)$	2. $y = \cos(x - \pi/2)$	3. $y = 3\sin x$	4. $y = \cos(x + \pi)$
5. $y = (x - 4)^2$	6. $y = \tan(x/3)$	7. $y = (4 - x)^2$	8. $y = \tan(x/6)$
9. $y = (x + 4)^2$	10. $y = \cos(x + \pi/2)$	11. $y = \sin 3x$	12. $y = \ln(1/x)$
13. $y = e^{x-1} + 4$	14. $y = \ln x^2$	15. $y = e^{-(x+1)} + 4$	16. $y = 2\ln x$
17. $y = -\ln x^2$	18. $y = -e^{x+1} + 4$	19. $y = (\tan x)/3$	20. $y = (\tan x)/6$

Figure 6.5 Class Activity: Identifying the Graphs of Functions

The episode that follows consists of extracts from the first part of the discussion where these students were trying to identify graph A. The entire transcript can be found in appendix B.

1	SE	Can anybody tell me which function represents the graph drawn in the first one?	
2	Martin	Is it cos (x + π/2)?	
~~3~~	~~SE~~	~~And why do you say that?~~	
4	Robert	It's a sine graph.	Robert was confident.
5	SE	Contradiction there. Explain your choice.	Directed at Martin.
6	Martin	Er well it looks – it's got to be like sine or cos and I think that cos starts at the top and each line on the scale is 90^0 which is π/2 radians, so it's been moved ...	Martin was motioning in the air, tracing the path of the graph of cos x with his finger.
7	SE	It's been moved across to the ...	
8	Martin	It's got to be -π/2 rads then because it's gone the other way, so it's number 2 [cos (x-π/2)].	

Initially, Robert quite confidently asserted that this was a sine graph (line 4). He recognised the distinctive shape of the graph as being of the form y = sin x and as such did not initially think of the graph in terms of a translation of the cosine function, as Martin had suggested. Martin then went on to justify his choice of function and in doing so recognised his initial error and consequently identified the correct form of the function from those specified: y = cos (x - π/2) (lines 5-8). Robert was then asked to explain his ideas:

9	SE	Why do you say that it might be a sine [graph]?	
10	Robert	Because sine of zero is zero and I'd say that that is in fact – because it seems that B is also a sine wave but that's more concentrated – I'd say that A is 1 [sin x/3].	
11	SE	You think that it's sin (x/3)?	
12	Robert	I wouldn't swear to it.	Robert clearly lacked confidence at this point.

Robert was somewhat confused by the fact that y = sin x was not one of the listed options. His initial image of this function as a sine graph was strong and he began to consider the other sine functions listed, focusing on sin (x/3) (line 10). Yet, he was still uncertain that this was the correct function (line 12). At this point Julie was drawn into the conversation by the teacher-researcher. She agreed with Martin's argument and offered some explanation for her choice [lines 13-20, see appendix B]. However, Robert seemed unaffected by the arguments proposed by Martin and Julie and in an attempt to repair his understanding he began using the graphical calculator:

21	SE	Yes. Ok so have you tried to actually graph on the TI-92 the first one that you thought it was?	Robert had just graphed the function y = sin (x/3).
22	Robert	Yes.	
23	SE	And what did you get?	
24	Martin	Isn't that cheating drawing the graph to see which?	
25	SE	No, no he is just convincing himself.	
26	Robert	To be honest I can't remember what I typed in.	
27	SE	Well, let's think about the first one y = sin (x/3). What is the graph of that going to look like?	
28	Robert	Wide, and wider than it is there.	Robert pointed to the graph to be identified.
29	SE	Yes. Ok, I'm going to say that you two are actually correct. Now it looks like a sine because it is sine of x, that is sin x.	
30	Robert	Yes.	
31	SE	But it can also be represented by y = cos (x - π/2) that's another...	Attempt to provide reassurance.
32	Robert	I see where that's coming from.	Robert regained his confidence. His tone of voice indicated that he understood this.

When asked to consider what the graph of sin (x/3) would look like in relation to graph A (line 27), Robert was able to recognise that the graph of sin (x/3) would be wider than graph A. Use of the technology and the teacher-researcher's question aimed at making him think about the relationship between the two graphs had helped Robert to perform a self-repair. He now realised that sin (x/3) was not the correct form of this function, and he started to question his initial thoughts and to eliminate the other sine functions listed. When it was explained that the graph could be represented symbolically by either $y = \cos(x - \pi/2)$ or $y = \sin x$, Robert remarked "I *see* where that's coming from" (line 32). This suggested that he could visualise the action of the transformation $f(x-\pi/2)$ on the graph of $f(x)=\cos x$ and how this would produce the graph of sin x. He appeared to have internalised the argument that was presented by the teacher-researcher. The use of the technology and the discussion in this example appeared to have resulted in some form of cognitive reorganisation for Robert. His thinking during the course of the episode had changed and by the end of this part of the discussion he was able to transfer his prior knowledge of trigonometric functions to this context. The concept of transformations became more meaningful to him, adding greater depth to his overall understanding of functions.

With regard to collaboration, the use of the graphical calculator also provided a means of furthering the discussion and preventing a breakdown in communication. When Robert was unable to move forward he turned to the graphical calculator in an attempt to clarify his thoughts, rather than merely accepting the arguments put forward by Martin and Julie without really understanding them. This example illustrates how use of the graphical calculator can help students' thinking to develop. However, in this episode the teacher-researcher also played a vital role in contributing towards Robert's understanding, which will be discussed in greater detail in section 6.3.3.

6.2.2 Graphical Calculators and Dependency

The findings from phase one suggested that students who are unfamiliar with graphical calculators might initially rely too heavily on this resource and regard the solutions produced by the technology as irrefutable. In light of such findings, evidence was also sought in this phase as to whether these particular students, all regular graphical calculator users, would show any similar signs of being over-reliant on the technology. The underlying supposition was that these students would tend not to rely as heavily (if at all) on the results produced by the graphical calculators because of their familiarity with this form of technology. In addition, these particular students were repeatedly advised to question whether the solutions produced by the graphical calculator made sense, and to check these results using symbolic methods throughout the trial. However, whilst instances of over-dependency on the graphical calculator were rarely observed in this phase, it was evident from the students' written work that there were a couple of occasions where over-reliance was still a problem, as the following examples illustrate.

In order to determine whether these students would experience some of the misunderstandings surrounding particular functions that had resulted from over-dependency on the graphical calculators in phase one, Robert, Martin and Julie were given similar questions for homework. In these questions they were asked to determine the nature and co-ordinates of any turning points of the following functions:

(1) $y = x^2 - x^3$, (2) $y = (x+1)/(x+2)^2$, (3) $y = (x-1) + 1/(x+1)$, (4) $y = (1+x^2)/(1+x+x^2)$.

In contrast to phase one, the function, $y = x^2 - x^3$ did not cause them any problems and all three students correctly identified the co-ordinates of both turning points. As had been expected, these experienced students did not automatically assume that the calculator display screen was necessarily showing the whole graph of a function or all the important features of that graph. Instead, they made proper use of the zooming facilities and zoomed in on the graph of this function to obtain the correct co-ordinates of both turning points. This was also the case with the fourth function.

However, zooming out proved to be less successful and the graph of the second function, $y = (x+1)/(x+2)^2$ was again misinterpreted. As was the case in phase one, Julie and Martin used the zoom menu to zoom out on the graph drawn in zoom standard (see figures 6.6 and 6.7) which produced a graph that appeared to have a minimum turning point. As a consequence, both students produced written answers which stated that there was a minimum turning point at $x = -2$.

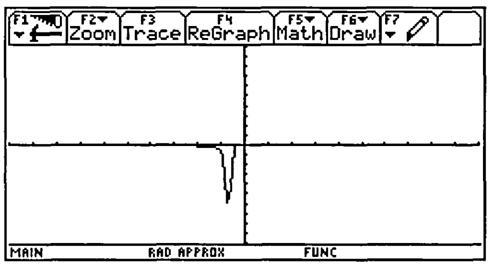


Fig 6.6 $y = (x+1)/(x+2)^2$ ZoomOut

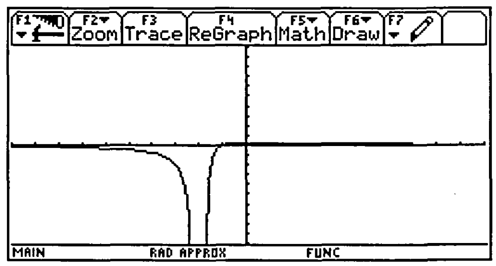


Fig 6.7 $y = (x+1)/(x+2)^2$ in ZoomStd

In exactly the same way as the students of the previous trial, these two students had misinterpreted the information provided by the technology.

What was even more surprising though was the fact that Julie's written solution to this problem contained approximations produced by the graphical calculator. She identified two stationary points, for which the proposed co-ordinates were $(-2, -6.44 \times 10^{12})$ and $(1.068 \times 10^{-38}, 0.25)$. This indicated that she had not thought carefully about how sensible these answers were, and this was further confirmed by the fact that there were no symbolic manipulations to accompany her answers.

Julie also made the same error in the next question, concerning the function $y = (x - 1) + 1/(x+1)$. She stated that there was a minimum turning point at ($-3.266x10^{-8}$, 0). By substituting x=0 into the equation she had already determined that (0,0) was a point on the curve. Consequently, if she had thought carefully about her two conflicting values of x when y was zero she may have realised that the graphical calculator was approximating zero as $-3.266x10^{-8}$ (because of the zoom factors). The fact that the actual co-ordinates of the turning point were (0,0) would also have become clear through differentiation of the function. However, as for the previous question, this was not attempted.

Martin, on the other hand, had tried to differentiate the function $y = (x+1)/(x+2)^2$. However, his attempt was unsuccessful and thus did not rectify his misunderstanding. He incorrectly applied the quotient rule for differentiation, obtaining: $dy/dx = (1- (2x+2))/(x+2)^4$ instead of $dy/dx = ((x+2)^2 - (2x+2)(x+2))/(x+2)^4$.

These examples illustrate that over-dependency on the technology may still occur even when students are regular graphical calculator uses. Moreover, since there was no further evidence of dependency on the graphical calculators within the classroom, this raises the question of whether dependency is more likely to occur when students are working individually with the technology, rather than collaboratively. Dependency may also be related to students' existing prior knowledge and whether they can check their answers by other means.

All of the students were made aware of the fact that the answers produced by the graphical calculator could be misleading. In spite of this warning however, Julie appeared to have complete confidence in the solutions produced by the graphical calculator, and as such did not question their legitimacy. She consequently did not feel the need to check her results by differentiation, as had been requested as part of the question.

Over-dependency could be considered as a negative consequence of the amplification effects of the technology. The graphical calculator provides students with the means of quickly and easily producing answers to questions. This saves the students considerable time and effort, and it becomes easier for them to accept the solutions produced by the technology, rather than thinking at any great depth about the problem themselves, especially if time is scarce. The use of graphical calculators can thus encourage a certain amount of 'laziness' amongst students. Ultimately, the use of the graphical calculator in these examples did not enhance the students' understanding of these particular functions because their graphical exploration was not supplemented by a symbolic approach.

These examples also highlighted a limitation of this particular form of technology with respect to resolution. Martin and Julie may have been misled by the lack of clarity on their graphical calculator screens, whereas this may not have been the case if they were using computers. In his questionnaire responses Martin acknowledged the "limited display resolution" of the graphical calculators and stressed that he would prefer to use computers in his A level mathematics classes because of the comparatively "better display" on the computer screen. The technology used needs to be precise or otherwise students can be misled.

These examples suggest that the environment in which the students use the graphical calculator may have some influence on their level of dependency on the technology. As they were given these particular questions for homework they were completed individually (as their different answers show) without support from each other or the teacher-researcher. When they worked collaboratively, however, there was no evidence to suggest that they were being overly reliant on the technology. Consequently, it is proposed that the learning environment that was established in the classroom and the interaction amongst the students and with the teacher discouraged students from being dependent on the technology. In this local community of practice they were encouraged to compare their answers and question each other and the teacher-researcher about the results obtained by use of the technology. They did not need to be dependent on the answers produced by the technology in this environment, as they could gain additional support from one another and the teacher-researcher. Dependency thus appeared to be related to (i) individual work, (ii) students' prior knowledge and (iii) display resolution.

6.2.3 Graphical Calculators and Visualisation

The findings of phase one had indicated that the graphical calculator could have a positive effect on students' abilities to visualise functions. In order to determine the impact of the graphical calculator on the students' use of visualisation in this phase, data was analysed from the student and staff questionnaires, the student interviews, the students' work and audio transcripts.

Analysis of the student questionnaires highlighted the fact that all of the students, except for Robert, regarded themselves as visualisers. However, closer examination of individual students' responses revealed a surprising lack of confidence surrounding the accuracy and validity of visual solutions and in the students' abilities to visualise functions themselves. This was accompanied by a definite orientation towards working symbolically. For example, when Martin was asked whether he had a preference for working symbolically or visually, he responded:

> *When I'm comfortable with knowing how the methods work symbolically I would then find it easier to visualise it, as I can check that I am visualising it right. Visual methods are encouraged but I would probably try to learn an algebraic method first until I am comfortable with my understanding of the methods, so I may not take as much notice of learning a more visual approach. When I am not used to the type of problem, it is not easy to relate it to a graph or system, so I would use algebra.*

Julie, Robert and Rachel similarly commented that they achieved greater success when using symbolic arguments and all believed the symbolic approach to be easier and more efficient. Consequently, these three students all had a clear preference for working symbolically, even though Julie and Rachel claimed to be more visually orientated, as their responses to the following question illustrate:

"Do you have a preference for working either symbolically or visually?"

> **Julie:** *I do prefer to work symbolically, although I visualise things more often. I tend to get the right answers when I work symbolically more often than when I visualise. It can be very difficult to work visually.*
>
> **Rachel:** *I find [working] symbolically easier. I'd like to do more maths visually but I can't apply [this approach] to some situations. For example when maths becomes abstract.*

This underlying preference for symbolic methods could have been perpetuated by the fact that their pure mathematics teacher, Mr Pearson, tended to concentrate on the symbolic aspects initially when teaching functions to his sixth form students. In response to the question "when teaching functions to lower sixth form students, do you tend to devote fairly equal amounts of time exploring the graphical, symbolic and numerical aspects, or does one particular approach predominate?" he stated:

> *The symbolic aspect predominates initially. Then the graphical and numerical aspects gain equal weighting.*

He also commented on the current emphasis that is placed on symbolic manipulation in Advanced level mathematics:

> *Even if a problem is visualised the technique for solving invariably involves algebra.*

The students' initial reluctance to explore visual methods was also reflected in their written solutions to questions from the pre-trial enquiry. These questions were solved without the use of technology and only 9 out of 32 of their solutions included graphical representations as well as symbolic manipulations. In particular, the only question that could not be solved directly using algebra (solve the equation $\sin x = 2x^2$) was attempted graphically by only one student, Diane. The others missed out this question completely. This suggested further that these students lacked confidence in their own graphing skills. The student interviews also indicated that Robert, Martin and Julie would only tend to draw graphs when working individually if their symbolic approaches failed or if the problem was completely new to them.

The following episode demonstrates how the graphical calculator was needed to support the students' visualisations.

<u>Episode 3 – The Role of the Graphical Calculator in Supporting Students' Visualisations</u>

In this episode Robert, Martin and Julie were trying to identify the function represented by graph E in figure 6.5. This graph, $y = e^{x-1} + 4$ caused the students particular problems since they were unfamiliar with the graphs of more complicated exponential functions.

1	Robert	It could involve an exponential this time.	
2	SE	Yes this is an exponential.	
3	Robert	It's obviously got +4 on the end, so it's either 15, or 18 or 13 even. [$y = e^{-(x+1)} + 4$, or $y = -e^{x+1} + 4$, or $y = e^{x-1} + 4$].	Robert was able to recognise the function as exponential and thus identify the possibilities.
4	Julie	It hasn't been reflected, so it's not 18 [$y = -e^{x+1} + 4$].	Correct assertion.
5	Robert	It's probably 13 actually. [$y = e^{x-1} + 4$]	
6	SE	Why do you say that one?	
7	Robert	Because the negative sign somehow has to fit that [the graph], although I can't explain how the minus sign affects it.	At this point Robert and Julie began to conjecture incorrectly about the effects of the functions on the shape of their graphs.
8	Julie	That's some sort of reflection, isn't it?	Referring to $y = e^{x-1} + 4$.

9	Robert	15 [$y = e^{-(x+1)} + 4$] would be a reflection.	
10	Julie	Why?	
11	Robert	It would be a reflection in x, wouldn't it?	
12	Julie	I don't know.	
13	Robert	18 [$y = -e^{x+1} + 4$] would be a reflection in y. This is like ignoring the transformation of +4, which I'd say is 13 [$y = e^{x-1} + 4$].	
14	SE	Yes you are correct and if you are not sure you can always draw the graphs of them to see which is a reflection in x and which is a reflection in y.	

In this episode, each of the students clearly had difficulty in visualising the effects of different transformations on the graph of $y = e^x$. Out of all the students' contributions the only correct argument that was presented was the one proposed by Julie in line 4, which allowed her to eliminate $y = -e^{x+1} + 4$ from the list of possible functions.

The discussion had suggested that these students would need additional support to enable them to visualise the effects of certain transformations on exponential functions correctly, despite their knowledge of how different types of transformation affect other, less complex, functions. This is an occasion where technology was particularly effective in mediating the students' visualisation powers. The students needed to test their conjectures and to investigate the visual connections between the various forms of the given exponential functions. In this way, they were able to study the visual effects of different transformations on exponential functions and develop a clearer understanding of these processes, following the discussion of the final graph.

The teachers also recognised the role of technology in the development of students' powers of visualisation, as is evident in their responses to the following question: "how important, in your opinion, is technology in supplementing and enriching students' visual capabilities?"

Ms Mooney: *Very - familiarity obviously helps and using technology means it is easy to become very familiar.*

Mr Pearson: *Of growing importance in the teaching and enriching of mathematics in general – visual capabilities are supplemented and learned by use of technology.*

Triangulation of the students' and teachers' questionnaire responses suggested that, in line with their beliefs, these teachers did indeed encourage students to visualise in lessons and that this involved the use of the graphical calculators.

In light of the research exercise, Robert indicated that he may "use graphical methods more", in the future, "when solving the more involved problems". This was an especially promising outcome as Robert had initially regarded visualisation solely as a "basis on which to use an algebraic method". Despite the fact that visual solutions were usually encouraged by their teachers, Robert stated that he "generally attempts to ignore such suggestions". He claimed that he only used visualisation as "a last resort". In his opinion, symbolic methods were of paramount importance:

A visual approach is most effective as a foundation for a symbolic solution. Technology is useful as an aid for analysis, yet understanding is best developed through algebraic methods. The concept of functions is best taught in a more traditional manner, so students might gain a more profound understanding of the field.

Following the trial, Robert appeared to have recognised additional benefits of using graphical approaches, through his exploration with the graphical calculator and seemed to be more confident in using them. Martin's confidence had also improved and he commented:

I think that I will be more comfortable in using a visual method such as plotting points and drawing sketches when solving problems.

Since Martin had expressed reservations about the accuracy of visual methods during the early stages of the trial, this outcome was particularly encouraging.

The students' written solutions to the main exercises [see Appendix B] and the class discussions that were audio recorded also provided evidence concerning the effect of the graphical calculator on their understanding of the relationship between symbolic and visual modes of representation. In particular, their attempts at questions from the main exercises that involved symbolic and graphical representations and required solutions incorporating both of these aspects were very encouraging. This was especially so considering their apparent scepticism surrounding the use of visual methods. For each of these problems, they provided valid solutions in which symbolic and graphical approaches were effectively combined. For example, all parts of question four (see appendix B), both graphical and symbolic, were answered correctly by those who attempted this question. In this case they did not appear to concentrate more on the graphical mode of representation as a result of using the technology, which was a feature of phase one. On the contrary, use of the technology encouraged them to use a combination of symbolic and graphical techniques and to explore the links between these two modes of representation. This was illustrated in episode one, in which Robert investigated the identity $\ln x^a = a\ln x$ (see section 6.2.1).

Analysis of the transcript data and the students' work provided a means of establishing the effect that the graphical calculator had on their abilities to move confidently amongst different modes of representation and to establish connections between them. One of the most promising outcomes of this phase of the research was the fact that each student was able to use the graphical calculator effectively to graph and translate functions and to check their own visualisations.

6.3 Affective Factors which Contribute Towards Students' Learning of Functions

In order to determine the relationship between the graphical calculator and affective issues, data was analysed from each of the eight sources outlined in Table 6.1. The affective issues that were highlighted in the findings of this phase of the research have been categorised into three main areas, which again build on those identified in phase one:

- Graphical Calculators and Effective Collaboration
- The Role of the Learning Environment
- The Role of the Teacher in Promoting Collaboration and Meaning Making in a Graphical Calculator Environment

As argued by Mcleod (1992), the relationship between affective factors and mathematics learning is "always influenced by the social context" (p. 587). All aspects of the social environment are seen to have an effect on students' emotions, attitudes, beliefs and behaviour. The three categories of analysis involving collaboration, the learning environment and the role of the teacher have thus been classified as affective issues. It is also recognised, however, that these affective social factors have interrelated implications for cognitive development.

6.3.1 Graphical Calculators and Effective Collaboration

Ideas developed by Teasley and Roschelle (1993, see section 4.4.2.2, chapter 4) were used to analyse the interaction between Martin, Robert and Julie and the teacher-researcher, as these students attempted the question from the main exercises, which involved identifying the graphs of the six functions (see figure 6.5). Teasley and Roschelle propose that social interactions in the context of problem solving activity occur in relation to a Joint Problem Space (JPS). They maintain that the JPS is a shared knowledge structure that supports problem solving activity by integrating (a) goals, (b) descriptions of the current problem state, (c) awareness of available problem solving actions, and (d) associations that relate goals, features of the current problem state and available actions.

In this model, collaborative problem solving consists of two concurrent activities, solving the problem together and building a JPS. The analysis of the data thus involved finding evidence for the construction of a JPS as well as identifying student 'initiation' of the discourse, student 'acceptance' of arguments and cases of students 'repairing' misunderstandings. Evidence was also sought for instances that involved 'collaborative completions' between students, where one partner's turn would begin a sentence and the other partner would use their turn to complete it. The entire set of transcript data can be found in Appendix B.

The analysis of the transcript data revealed that the use of graphical calculators could lead to particularly effective collaboration amongst the students, as the following episode demonstrates.

<u>Episode 4 – The Role of the Graphical Calculator in Collaborative Meaning Making</u>

1	SE	Finally F.	
2	Robert	It's a tangent.	There was a pause.
3	SE	Think about the scale the TI-92 uses.	
4	Robert	To see if it was increasing I could just draw the normal graph.	
5	SE	Ok, if it helps you can draw the, you can all draw the tan x graph and see what happens on your machine and then from there you can hopefully deduce what the function is.	
6	Robert	It's a stretch of factor 3.	
7	Martin	It's tan of x over 3.	
8	Robert	Yes.	
9	SE	Is that number 6 or number 19 [y= tan (x/3) or y =(tan x)/3], because there are two of them?	
10	Martin	Number 6 [y = tan (x/3)].	
11	SE	Number 6 and what do you think? Have you managed to get the tan?	Directed at Julie

12	Julie	Yes. That's the whole thing.	Pointing to the tan x in (tan x)/3
13	SE	That's tan of x all divided by 3.	
14	Julie	So yes number 6.	
15	SE	Number 6, yes well done you are right.	

After using the graphical calculator to graph tan x, Robert compared this with graph F and deduced that this was obtained using a stretch of factor three. To complete Robert's statement, Martin added that the correct function was "tan of x over three" and Robert immediately agreed, indicating that he had internalised Martin's response. Robert and Martin having attempted to construct shared knowledge had effectively produced a collaborative completion. However, as there were two functions which could be verbalised as 'tan of x over three', the teacher-researcher sought confirmation that Martin had identified the function correctly and was quickly satisfied that he had. Up until this point Julie had not contributed to the discussion and the teacher-researcher drew her into the conversation by questioning her to see if she was following the arguments being presented. Julie agreed with the choice of function offered by Martin and provided some evidence that she had understood why this was the correct function (line 12), which was then confirmed by the teacher-researcher.

The students had thus been able to develop some shared understanding of the transformations used in this example through the creation of a joint problem space. Moreover, the use of the graphical calculator was an important part of this process. The interaction between the students was constructed in relation to the graphs produced by the graphical calculator and it was this factor that led directly to the collaborative completion between Martin and Robert. This occurred because the students were able to establish a shared visual interpretation of the function using the graphical calculator. In other words, in this episode use of the technology provided Julie, Martin and Robert with a common starting point from which they were able to think about the problem in the same visual terms.

From this position they were each able to contribute towards correctly identifying the symbolic form of the function.

The graphical calculator can also be thought of as a catalyst in the collaboration process and in furthering individual students' thinking. The students interact with the feedback provided by the graphical calculator. In the first episode that was discussed in this chapter, the use of the graphical calculator resulted in cognitive conflict for Robert. This occurred when he drew the graphs of $y = 2\ln x$ and $y = \ln x^2$ using the graphical calculator and saw that they were not identical:

> 3. SE: So would you say that the two expressions were the same or not?
> 4. R: I'm hesitant to say. I would say that algebraically they were the same.

In contrast, in the second episode, it was the arguments that were posed by Martin and Julie that initially made Robert question his understanding, as these seemed to contradict his assertions. Thus, when a student works individually with the graphical calculator, and does not have the benefit of interacting with peers, the technology provides another way of viewing the problem that needs to be further explored and explained.

In this case, the graphical calculator provided an authoritative means by which Robert could investigate the ideas being discussed and modify his own visual images of the graphs accordingly. Robert began to have more confidence in the arguments being posed by his peers following his graphical exploration with the technology. Initially, Robert appeared to dismiss the arguments being presented by Martin and Julie, even though he lacked confidence in his own ideas as a consequence of the apparent contrast between them.

In episode one, the graphical calculator became the focus of attention of this discussion and its use encouraged Robert to think more carefully about the problem and to generate conjectures. The graphical calculator also provided a means by which the teacher-researcher could guide Robert's thinking.

6.3.2 The Role of the Learning Environment

With reference to the framework offered by Winbourne and Watson (1998, see section 3.1.3, chapter 3), it can be seen that during this phase of the research 'local communities of practice' were established. Winbourne and Watson (ibid) identify six key features of classroom behaviour, which are viewed as indicators that local communities of practice have been created:

- pupils see themselves as functioning mathematically within the classroom,
- there is public recognition of competence,
- learners see themselves as working together towards the achievement of a common understanding,
- there are shared ways of behaving, language, habits, values and tool-use,
- the shape of the lesson is dependent on the active participation of the students,
- learners and teachers see themselves as engaged in the same activity.

Analysis of the transcript data from this trial has provided evidence that each of these factors was observable in the classroom environment. Table 6.2 summarises the types of interaction used by the students and the teacher-researcher.

Table 6.2 Types of Interaction Used by the Individual Students and the Teacher-Researcher

	Robert	**Julie**	**Martin**	**SE**
Presenting ideas	4	2	1	1
Explaining ideas	3	3	0	3
Making assertions	7	2	3	1
Making statements	6	1	2	5
Showing acceptance	4	4	3	6
Repairing ideas	2	0	0	3
Self repairing ideas	1	0	1	0
Questioning	1	2	2	23
Performing collaborative completions	4	0	2	2
Number of interactions involving natural language	18	9	8	25
Number of interactions involving scientific language	14	5	6	19
Total number of interactions	**32**	**14**	**14**	**44**

Firstly, as shown by table 6.2, the students each showed willingness to explore and explain ideas to one another. They clearly saw themselves as functioning mathematically within the lesson, as they were each offering suggestions as to which functions represented the given graphs, based on some mathematical reasoning which enabled them to obtain the correct form of the function in each case.

Secondly, the teacher-researcher ensured that the students received public recognition of their competence. This was achieved through the teacher-researcher's acceptance of the students' ideas ("yes, well done you are right", "yes, you are all right, it's sin 3x").

Thirdly and most significantly, as the discussions progressed, the students began actively working together towards achieving a common understanding of each problem, through the sharing of ideas and questioning of one another. This led to successful collaboration in the form of collaborative completions between the students themselves and with the teacher-researcher.

Fourthly, the students each shared behavioural traits, such as presenting and justifying their own arguments and listening to, accepting and questioning the arguments of others. The language used by the students was both scientific and natural, and they appeared to have shared conceptions of the scientific language that was used. The students also used the graphical calculator together as a group in their attempts to identify the fifth and sixth graphs.

The shape of the lesson depended on the active participation of the students. Each student created his or her own role in the practice, which varied accordingly. During the discussions the patterns of interactions between the students were continually changing as each new graph was considered. In each case, the individual students appeared to occupy different positions within the discussion, modifying their roles depending on their needs. Martin initiated the discussion around the first two graphs, and Robert took over this role for the discussions concerning the remaining four graphs. Robert also began to act as a more capable peer in the Vygotskian zone of proximal development. He continually made verbal contributions to the discussions and at times took control of the discussion, whilst Julie and Martin spent some time actively listening and thinking rather than speaking. In most cases, Julie did not contribute voluntarily to the discussions and needed to be drawn in by the teacher-researcher.

Both the students and the teacher-researcher regarded themselves as being involved in the same activity. The teacher-researcher attempted to initiate each student into the discussion with the aim of encouraging the construction of shared knowledge. Table 6.2 also highlights that questioning formed an extremely important part of the teacher-researcher strategy for encouraging participation and the construction and maintenance of a joint problem space amongst the students. It also illustrates the quality and frequency of interactions made by Robert in comparison to Martin and Julie. Robert performed more successfully overall in the class activities in this trial and this may have been the result of his additional willingness to share ideas and difficulties with the other students and the teacher-researcher.

This type of environment in which local communities of practice are established has been found in this study to be conducive to successful collaborative work involving graphical calculators. Within this supportive environment, students are able to establish an effective means of operating with graphical calculators, in which the knowledge generated is shared amongst the participants. This may not necessarily involve working with graphical calculators all of the time. For example, if the students are able to visualise the effects of a particular transformation on the graph of a function effectively without the aid of technology, then they may choose not to use the graphical calculator in that instance. However, the way in which they approach the problem is likely to reflect their prior use of the technology. As proposed by Borba and Villarreal (1998), the human-graphical calculator system will still be in action. This was the case when Martin, Robert and Julie tried to identify the function represented by graph B in figure 6.5.

Episode 5 – The Role of the Graphical Calculator Environment in Effective Collaboration

1	SE	Can anybody think of a function for B?	
2	Martin	I reckon it's sin 3x.	
3	SE	Sin 3x.	Seeking acceptance.
4	All	Yes.	Confident and firm responses.
5	SE	You seem to agree on that one. So how did you come up with that conclusion?	Question directed at the group.
6	Robert	There don't seem to be any sneaky cosine tricks.	Robert was wary of the existence of equivalent symbolic forms.

7	SE	Not this time.	
8	Martin	It's a sine wave and it's been er…	Martin paused.
9	Robert	Three times x would condense it.	
10	Martin	It's got a stretch parallel to the x-axis of a third, because it got closer together, or so…	
11	SE	Yes, you're all right it's sin 3x.	

Martin initiated the discussion by asserting that this was the graph of sin 3x (line 2). The other two students immediately accepted that this was the correct form of the function and when asked to give reasons why, both Martin and Robert took turns to give an explanation, each building and elaborating on the previous utterances, thereby producing a collaborative completion (lines 8,9,10). Rather than concentrating on developing their own arguments separately, Martin and Robert produced a joint explanation of why sin 3x was the correct symbolic representation of graph B. Together they provided a convincing argument for their choice of function. The students were thus all confident that they had identified the function correctly.

Martin and Robert were able to perform a collaborative completion together because their visual images of the function were strong and corresponded to one another. In this case, each of the students appeared to be able to clearly visualise the effects of the transformation, without using the technology. Yet, there was evidence that the graphical calculator was having an impact on the way in which these students were thinking about the problem. In particular, Robert was now actively looking for alternative symbolic forms for the graphed functions following his exploration of the function represented by graph A with the graphical calculator (line 6). The use of the graphical calculator can be seen to restructure the way in which students think about problems, and this is most productive when used as part of a local community of practice.

6.3.3 The Role of the Teacher in Promoting Collaboration and Meaning Making in a Graphical Calculator Environment

Analysis of the transcript data served to highlight the important function of the teacher in maintaining and encouraging the discourse between students, in prompting the use of the graphical calculators, in verifying students' assertions and in providing clarification and explanation of solutions, where needed.

For example in episode two (in section 6.2.1) the teacher-researcher was almost entirely responsible for steering the discussion and encouraging participation and explanations of ideas. As such, during the discussion of this graph, the teacher-researcher purposefully and actively engaged in the act of asking questions. This was particularly effective in encouraging the students to share ideas, to justify their suggestions and to question and consequently begin to repair their understanding. In Martin's case, by being asked to reflect on why he had chosen cos $(x + \pi/2)$, he was prompted to think about the problem more carefully and in the act of justifying his response he realised that he had made a mistake. Thus he was able to identify the correct form of the function and consequently had self-repaired his initial understanding. Similarly, when Robert was asked to explain the reasoning behind his assumption that this was a sine graph (line 9), he began to develop an argument as to why the function might be sin $(x/3)$. The use of questions also encouraged Julie to elaborate on her initial explanation of why she agreed with Martin's choice, thus showing that she had made sense of the problem:

13	SE	And what do you think? Have you got any ideas about this one?
14	Julie	I think it's number 2 [cos $(x - \pi/2)$].
15	SE	And why do you think that it's number 2?
16	Julie	It's been moved.
17	SE	It's been moved?
18	Julie	Yes it's a translation.
19	SE	And in which direction is it moved?
20	Julie	Er $\pi/2$ in the x-axis.

Following Julie's contribution, the teacher-researcher re-focused attention on Robert and through continually questioning him about his ideas and by asking a specific question about the form of the graphed function, he was eventually able to begin to repair his initial ideas about the problem.

Only at the end of the discussion was it revealed by the teacher-researcher that Martin's and Julie's arguments were correct and that Robert's recognition of this as a sine function was also valid. Thus, at this point, which seemed an appropriate time for the solution to this problem to be revealed and explained, the teacher-researcher assumed the role of a more knowledgeable person in the Vygotskian zone of proximal development and helped the students to make sense of the apparent contradictions. This use of questioning which allowed the students to make discoveries for themselves with some guidance and reinforcement of solutions, where relevant, was considered to be an appropriate strategy for effective learning. Indeed throughout the discussion of all six graphs the teacher-researcher was instrumental in convincing students of the validity of arguments. The teacher-researcher also presented ideas, explained ideas, made statements, repaired ideas and performed collaborative completions (see table 6.2), as well as actively encouraging oral participation. Julie in particular needed to be prompted to share her ideas with the group.

In episode two, Martin regarded the use of the graphical calculator as a means of cheating (line 24) rather than as a tool that could help the students to clarify their inner thoughts and hence move forward to a different level of understanding. Indeed Martin appeared to be reluctant to use the technology for all parts of this question. The teacher-researcher therefore played an important part in encouraging Martin and the other students to make use of the technology when it was evident that this would be particularly beneficial. The students' reluctance to use the technology as a starting point for solving these problems raised questions about the long-term status of the graphical calculator in this classroom. The students only appeared to turn to the technology when experiencing difficulty and not spontaneously, as a means of initially tackling these problems, even though this would have given them a stronger focal point for discussion.

In addition Martin had commented that using the graphical calculator could sometimes confuse rather than clarify a problem:

> *For some things that I don't understand fully it can be confusing when it gives an unexpected result.*

This re-emphasised the fact that the use of technology alone may not be sufficient to help students to overcome misunderstandings or to make mathematical ideas clearer to them. There is an important role for the teacher in initiating discussion of the results obtained by the graphical calculator. This could be on a one-to-one basis, in small groups or as a whole class. The findings of this phase of the study combined with that of the exploratory phase suggest that the teacher is needed to ensure that the technology is being used to the greatest effect - that it is being used appropriately and that results are being interpreted correctly.

6.4 Conclusions

This chapter has considered the second phase of the research and the ways in which students who were experienced graphical calculator users were able to apply this technology to further their understanding of functions. This section summarises the findings of this phase in relation to cognitive and affective factors.

6.4.1 Amplification, Cognitive Reorganisation and Students' Understanding of Functions

This study has shown that the ability of students to use the graphical calculator as a means of immediately verifying or, in particular, disproving their ideas can lead to them gaining a more meaningful and general understanding of functions. In other words, a major finding of this study is that the amplification effects of the technology are seen to contribute towards the cognitive reorganisation effects, indicating that the short-term and long-term consequences of using the technology are interrelated. This provides evidence to back up Berger's (1998) speculation that these might indeed be interconnected.

6.4.2 Graphical Calculators and Dependency

The data from this phase of the study, when contrasted with that from phase one, suggested that students who are regular users of graphical calculators were less likely to be overly dependent on the technology than students who have comparatively very little experience. Moreover, a significant and previously unreported finding to emerge from this study is that dependency is influenced by three key factors; (i) individualistic ways of working, (ii) students' prior knowledge and (iii) display resolution.

6.4.3 Graphical Calculators and Visualisation

The results from this phase have illuminated ways in which the technology can have a positive impact on students' visualisation capabilities, enabling students to derive richer meaning from given problems. In this study, use of the graphical calculator encouraged students to make greater use of visual methods, and helped students to visualise functions more clearly. The class discussions suggested that these students had not previously made the visual connections between sine and cosine graphs and translations. Use of the technology in this case provided a means by which these relationships could be explored and conceptualised.

The potential use of the graphical calculator in mediating the development of students' visual capacities through student-student and student teacher interaction was also highlighted in this phase of the study. Findings suggest that individual students' images are modified through their interaction with the co-participants, as they share use of the technology. The results of this phase also suggests that ultimately technology could be used to begin to change students', such as Robert's, views surrounding the nature of mathematics, in which the symbolic mode of representation is seen as paramount.

6.4.4 The Relationship between Symbolic and Visual Modes

The students from this phase combined symbolic and visual approaches more frequently than was observed in the previous phase. The graphical calculator enabled the students to make connections between visual and symbolic modes of representation more easily. In this way they were able to move confidently between representations and were given the flexibility to be able to make sense of, and formulate solutions to, unfamiliar and challenging problems. In turn they were able to achieve a level of understanding which may not have been available to them if they concentrated on one mode of representation alone.

6.4.5 Graphical Calculators and Effective Collaboration

The use of the graphical calculator in this study enabled the students to build and elaborate on each other's arguments and thereby created the opportunity for effective collaboration. This was achieved because the students were able to produce a shared visual representation of the problem using the graphical calculator. This study also found that successful collaboration was promoted by the ability of students to reinforce their arguments to one another through use of the graphical calculator. This discouraged periods of little or no interaction. The graphical calculator also acted as a medium for communication between the teacher and student, and a means by which the teacher was able to guide the students' learning.

6.4.6 The Role of the Learning Environment

The findings of this phase suggest that the establishment of 'local communities of practice' in the classroom was conducive to successful collaborative work involving graphical calculators. In this type of environment it was found that students shared ownership of their use of the technology. Findings indicate that it was not essential for the students to use graphical calculators all of the time in order for learning to be successful in their community of practice. However, the way in which the students operated whilst using the graphical calculators was seen to influence the way in which they approached problems without use of the technology.

6.4.7 The Role of the Teacher in Promoting Collaboration and Meaning Making in a Graphical Calculator Environment

Analysis of the data in this phase pointed to the centrality of the teacher's role in maintaining and encouraging discussion between the students, especially in relation to the results produced by the graphical calculators and in providing additional verification of these results and the students' assertions. A further important function of the teacher lay in providing clarity and explanation of the results of the students' exploration with the technology, especially when the students are unable to reach a common understanding of their findings by themselves. The teacher needs to scaffold the students' learning with the graphical calculators to ensure that the technology is used effectively and results are interpreted correctly by the students, so that any misunderstandings are not perpetuated.

6.4.8 Graphical Calculators and Confidence

One of the unexpected findings of this phase was the fact that the students in this study were very reluctant to use visual approaches initially, due to the perceived inaccuracy of this mode of representation. At the outset of this phase of the study, it was assumed that these students would be confident in using this type of technology and in working in the visual mode of representation, as graphical calculators were firmly embedded in the culture of the classroom. The results of this phase were therefore very surprising and contrary to expectations. Martin and Julie's lack of confidence in visual representations was especially surprising considering that they both classified themselves as visualisers. All of the students generally found working symbolically easier than working visually. Moreover, this may have been related to the kind of teaching they had experienced.

The findings of this phase of the study suggest, however, that if technology is utilised purposefully, it could encourage students to have greater confidence in visualisation and help them overcome their initial difficulties. If students are experiencing visualisation problems with particular areas of mathematics, such as translations of functions, then the graphical calculator can be used to provide effective scaffolding in these fields. This phase found that student visualisations could be supported and enhanced by frequent access to technology and that this can lead to greater confidence in visual approaches. The graphical calculator appeared to influence the students' perceptions in a positive way towards the validity of visual methods in mathematics. This was especially so in Robert's case.

6.5 Emerging Areas of Interest and Implications for Phase Three

The preceding analysis and interpretation of the data collected during phase two has revealed a rich picture of the kinds of interaction that occur between students and teacher in a graphical calculator environment and how these types of interaction affect meaning making. It has also provided useful data regarding the teachers' role in scaffolding the learning task. The audio taped class discussions revealed a complex pattern of interactions between the students and teacher, which was continually changing. This raised some interesting questions particularly concerning the nature of shared knowledge, which had important implications for phase three. For example, when can knowledge be taken as shared? What role does each individual play in constructing such knowledge and how is this then ‘appropriated’ (Lerman, 1994)? How can the teacher, or use of the technology, facilitate the appropriation process? How does the individual student make further use of shared knowledge? Another aspect that this phase suggested might be further researched in phase three, was the role of the graphical calculator in relation to peer tutoring.

One of the goals of this research has been to interpret the relationship between the social and personal dimensions of meaning making and graphical calculator use. Thus phase three further considers the role of the graphical calculator in individual problem solving and how this might differ from its role in collaborative activity.

CHAPTER 7
INVESTIGATING THE PERSONAL DIMENSIONS OF GRAPHICAL CALCULATORS IN STUDENTS' UNDERSTANDING OF FUNCTIONS

7.0 Introduction

This chapter consists of an analysis of the individual aspects of learning about functions using the graphical calculator that arose during the third and final phase of the research. The overall aims of this phase of the research were to:

- explore how students with no previous experience of using graphical calculators and only limited experience of functions (at GCSE level), learned about the concept of function through the use of this technology,
- investigate the role of technology in providing a basis for the development of students' visualisation skills and in improving student confidence,
- determine whether and how the use of graphical calculators affected students' preferred problem-solving strategies,
- investigate how the behaviour of individual students affected the shared construction of meaning, and how this behaviour was influenced by the use of technology.

The first three of these aims are considered in this chapter and chapter 8 is concerned with the latter.

The exploratory study (chapter 5) indicated that students who were being introduced to functions for the first time at Advanced level might benefit more concretely from being able to use graphical calculators in the long term, especially with respect to visualising functions and linking different modes of representation. Evidence from the second phase (chapter 6), also implied that students who had been introduced to graphical calculators at an early stage in their A level course would be less likely to be dependent on the technology. This phase of the research was aimed at looking at the personal and social factors involved in meaning making with graphical calculators and considers the effect of the timing of their introduction.

7.1 Background to the Research

Data was collected for the third phase of the research during October and November 1998 from the same school that was visited during phase one, i.e. Ashby School. This time the trials involved the teacher-researcher working with a group of seventeen Year 12 GCE Advanced level mathematics students (13 male, 4 female) aged between sixteen and seventeen for a period of nine hours (six lessons). The students were initially taught the function topics which were covered in their textbook (Mannall and Kenwood, 1994), using this resource, the TI-82 and supplementary exercises. These topics consisted of: mappings and functions, function notation, graphing functions, composite functions, inverse functions, the modulus function, even and odd functions, sketching graphs and transforming graphs. Following this, the students worked on the main trial exercises (see appendix C).

During phase one, the feedback from the staff had indicated that in their opinion using graphical calculators to help introduce the concept of functions to a new group of year twelve Advanced level mathematics students would be particularly beneficial for these students. They felt that use of the graphical calculators could enhance the students' learning of the material covered in the students' textbooks. Thus, it was arranged that the graphical calculators would be introduced to the new intake of Advanced level mathematics students at the beginning of the new school year. In addition, one of the members of staff had also commented that a less powerful machine than the TI-92, which was used during phase one, might be more appropriate, especially if graphical calculators were to be introduced in the lower school. It was stressed that it would be unlikely that the mathematics department would purchase a classroom set of TI-92s and it was felt that a more practical, compact and affordable graphical calculator, such as the TI-82 should be trialled.

Not surprisingly, as these students were just beginning their Advanced level course, they were inexperienced in dealing with functions. Only two students, Roy and Julian, had purchased their own graphical calculators and were familiar with the applications needed. The rest of the students had never used a graphical calculator before. Subsequently, as with the previous phases, part of the first lesson was spent familiarising students with the technology.

7.1.1 The Sequence of Lessons

For each lesson a plan of action was decided upon in advance which would allow plenty of time for the students to explore functions as a whole class, in small groups and individually. Figure 7.1 shows how these lessons were structured.

PLAN OF ACTION

Lesson One

1. Whole class introduction to the research, followed by the distribution of the visualisation questionnaires (30 mins).
2. Whole class demonstration of how to use the TI-82, encouraging the students' use of the OHP to share ideas with the rest of the class and involving examples from the students' textbook chapter 3 section 3.1 Mappings and functions (30 mins).
3. Small groups work on exercise 3A (30 mins).
4. Homework: the student interview questions.

Research Aim: To collect data on individual students' views surrounding visualisation and visual orientations using the student questionnaires.

Lesson Two

1. Whole class introduction to composite functions (section 3.2) and small group work on exercise 3B (1 hr).
2. Commencement of individual student interviews (30 mins).

Research Aim: To collect data on individual students' use of graphical calculators through the student interviews.

Lesson Three

1. Continued small group work on exercise 3B and completion of individual student interviews (30 mins).
2. Whole class introduction to inverse functions (section 3.3) and small group work on exercise 3C (1 hr 30 mins).
3. Homework: completion of exercises 3B and 3C.

Research Aim: To collect further data concerning personal use of the technology through the completion of the student interviews.

Lesson Four

1. Whole class introduction to the modulus function, odd and even functions and transformations (sections 3.4, 3.5 & 3.6) and small group work on exercise 3D (1 hr).
2. Homework: completion of exercise 3D.

Research Aim: To collect data on how the students interact whilst using the graphical calculators as a whole class, through audio and video recording of these discussions.

Lesson Five

1. Whole class exploration of some of the additional uses of the TI-82 not yet introduced, namely, how to graph families of functions simultaneously and the maths menu (20 mins).
2. Small group work on the trial exercises: graphing functions using the TI-82 and identifying the graphs of functions (40 mins).

Research Aim: To collect data on the way in which students interact whilst using graphical calculators in small groups, through audio and video recording of particular groups of students.

Lesson Six

1. Completion of small group work on the trial exercises (1 hr 40 mins).
2. Closing remarks and distribution of the technology questionnaires (20 mins).

Research Aim: To collect data on individual students' views surrounding their use of the graphical calculators using the technology questionnaires.

Figure 7.1 Plan of Action

7.2 The Personal Dimensions of Graphical Calculators and Students' Understanding of Functions

In order to determine the influence of the personal dimensions of graphical calculator use on the individual students' understanding of functions, data was analysed from seven sources:

(i) pre-trial student questionnaires,
(ii) individual student interviews,
(iii) the students' written work,
(iv) audio transcripts,
(v) video transcripts,
(vi) post-trial student questionnaires,

(vii) post-trial staff questionnaires.

The findings that pertain to the personal dimensions of graphical calculator use have been subdivided into the following categories:

- Amplification, cognitive reorganisation and students' understanding of functions
- Analysing the individual dimensions of visualisation and the use of graphical calculators
- Graphical calculators, motivation and confidence
- Graphical calculators, dependency and misinterpretation of results

7.2.1 Amplification, Cognitive Reorganisation and Students' Understanding of Functions

When asked as part of their post-trial questionnaires (see appendix C) to specify what they considered to be the main advantages of using technology, nine of the students referred to the speed and ease by which problem solving could be achieved. Claire and Marie's responses to the question "what do you consider the main advantages of using technology to teach the concept of functions to students such as yourselves?" capture the general feeling amongst the students:

> **Claire:** *Using technology such as calculators is more appealing to people of my age than writing everything down and working everything out. Getting through work is a lot easier and a lot faster.*
>
> **Marie:** *I think that the main advantage is that solving the problems becomes much quicker.*

Thus, these students were highlighting the amplification effects of the technology. Nine students also commented on cognitive factors. The following students' responses to the same question indicated that they saw the impact of the technology on related cognitive factors as the main advantages of using the graphical calculators:

Marty: *It removes the need for drawing graphs and makes it easier to draw sketch graphs. It helps when understanding translations and stretches.*

Roy: *It helps make the transformation of functions easier to understand.*

Julian: *The fact that it gives everyone a chance to get the right answer, and from that understanding can be developed.*

Paul and Jim saw the main advantages of using the graphical calculators ~~as the cognitive changes that stem from their ability to produce graphical~~ solutions to problems quickly:
"What do you consider the main advantages of using technology to teach the concept of functions to students such as yourselves?"

Paul: *Technology gives students a quick and clear solution, which can avoid confusion.*

Jim: *Technology makes things faster, that is why we continue to refine things all the time, the use of graphical calculators increases peoples' knowledge much quicker.*

This was a view that was also shared by Marie and Julian. Marie's response to the question: "do you feel that you have benefited from the opportunity to use the graphical calculator?" suggested that she could also see a link between the amplification and cognitive reorganisation effects of the technology.

Marie: *Yes, it has increased my understanding of graphs. Time was not wasted doing easy things such as plotting graphs. It allowed me to concentrate on the harder equations.*

Similarly, Julian's response to the question "do you believe that using the graphical calculator has strengthened your understanding of functions?" indicated that he partially attributed the cognitive reorganisation effects of the technology to the amplification effects.

Julian: *Probably, as being able to visualise the function quicker has given me more time to explore further into the subject.*

All four of these students emphasised that as graphs could be drawn more easily using the technology, problem solving became quicker and more in depth work could subsequently be done. This in turn allowed the students to develop a better understanding of functions.

Thus, use of the graphical calculators was believed to minimise confusion and to help students to visualise the forms of different types of functions and the effects of transformations. This was achieved through the clear and accurate graphs that were produced by the technology and the easy access provided by the technology to many more examples of functions than could be produced by hand. The students believed that the graphical calculator enhanced their learning of functions, through speedier calculations, a reduction in errors and by providing the opportunity to explore more graphs in the same space of time (i.e. amplification effects). The latter aspect, in particular, was viewed as especially beneficial for learning about functions and their graphs, by enabling several functions to be drawn simultaneously and allowing the students to explore and establish the patterns amongst different families of functions. Being able to access the graphs of functions quickly allowed further exploration of the subject and in turn this had a positive impact on student understanding. Some of the students clearly recognised this connection between the amplification and cognitive reorganisation effects of the technology.

These students thus appeared to have recognised the graphical calculator as a tool for cognitive growth more than students in phase one, who were also from this particular school. This could possibly be attributed to the fact that the students from phase three had used the technology for a longer period in the trial and were encountering more formal definitions of functions than they had met previously. Their total lack of experience with some of the types of functions introduced may have influenced the way in which the technology was perceived to be beneficial. However, for these students, using the technology had a more significant and longer term effect in helping them to appropriate the concept of functions.

7.2.2 Analysing the Individual Dimensions of Visualisation and the use of Graphical Calculators

The students' visualisation questionnaire responses revealed that ten students classified themselves essentially as visualisers. Table 7.1 illustrates who these students were and which of the seventeen students alternatively perceived themselves as non-visualisers.

Table 7.1 Visualisers and Non-visualisers: Individual Students' Self-Classifications

Visualisers	Non-visualisers
Carol, Claire, Fay, Jake, Julian, Justin, Marvin, Nigel, Perry, Roy	Jim, Kirk, Marie, Marty, Mick, Paul, Pierce

To ascertain this information the students were asked to indicate where they would place themselves if a continuum existed between that of pure visualiser and pure non-visualiser and this formed the basis of their self-classification. Figure 7.2 shows the students' relative positions on this hypothetical continuum.

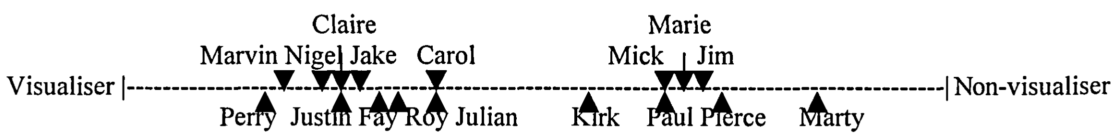


Figure 7.2 Students' Perceptions of Their Visual Orientation

Using the students' relative positions in figure 7.2, table 7.2 orders the students from the most visually orientated (V1) to the least visually orientated (V17). This provides some indication of how the students viewed themselves in comparison with the other members of the class.

Table 7.2 Students' Perceptions of How Visually Orientated They Are in Rank Order.

V1	V2	V3	V4	V5	V6	V7	V8	V9
Perry	Marvin	Nigel	Justin	Claire	Jake	Fay	Roy	Carol
V10	V11	V12	V13	V14	V15	V16	V17	
Julian	Kirk	Mick	Paul	Marie	Jim	Pierce	Marty	

Eleven students, including both visualisers and non-visualisers, indicated that the reason that they would tend to construct visual images was in an attempt to clarify their thoughts about a problem. This in turn could enable the students to develop solution strategies based on their visual perceptions and allow them to check the reasonableness of symbolic results. Those who classified themselves as visualisers were on the whole particularly positive about the usefulness of images:

> **Roy:** *I believe that constructing mental images helps greatly when understanding a problem, this can help because it makes a solution method clearer to see.*
>
> **Julian:** *I feel that images help me as they let me see clearly what the question is asking, if you like they clear any uncertainty about the problem up.*
>
> **Fay:** *Images help me to understand the problems and often give me an idea of how to solve them.*
>
> **Claire:** *By using images I know what the problem looks like and whether the answer that I got looks right or not.*

Students who classified themselves as non-visualisers had also recognised the potential benefits of using visual images:

> **Jim:** *Images can give you a better feel for the question. A couple of numbers on a piece of paper does not always mean very much to you. But if you visualise the numbers in some way then you may be able to answer a question better.*
>
> **Mick:** *Images help me to set out the problem and to get all the information from the question into a format which I can use.*
>
> **Kirk:** *Images give me a basis to work from. Something to see how the situation may occur.*

The main stages during problem solving when these students would tend to formulate visual images would be at the beginning of the problem (8 students), when experiencing difficulty finding a solution (5 students), and/or if the problem seemed complex (4 students). The students who classified themselves as visualisers, in particular, were keen to begin problem solving by formulating mental images, as illustrated by the following comments:

> **Roy:** *I like to have an idea of a problem in my head at the beginning of a question so that as I work through the problem, I can follow it visually in my head.*
>
> **Julian:** *If I am tackling an area of mathematics I find testing I will try to use tools like mental imagery at the beginning of calculation. Then as I become more proficient I will tend not to use it as much.*
>
> **Claire:** *At the beginning so that I have got everything sorted out in my mind before I really begin to tackle the problem.*
>
> **Carol:** *Either at the beginning so I can understand the task asked or when I hit a problem. The mental images help me to see where I have gone wrong.*

Whilst some non-visualisers also appreciated the value of visualising the problem to begin with, others were more reluctant as Marty's comment reveals:

> **Jim:** *I find that formulating an image at the start of solving a problem is usually the best idea, so that you get an idea of what it looks like from the start.*
>
> **Mick:** *This depends on the complexity of the question, but normally as soon as necessary if I can't solve the problem without them.*
>
> **Marty:** *When I start to get stuck and this is usually the last resort before looking at an example.*

Perry, a visualiser, highlighted the usefulness of visualisation for dealing with new or complicated problems:

> *If I am struggling with a problem, or I can see it is going to be complicated, I would draw a quick sketch and write on it what I know and what I can work out. With more complex problems and new problems, visualising helps me.*

Despite all of these positive comments about the usefulness of visual imagery, during the course of the study it became clear that the introduction of the graphical calculator definitely had an influence on individual students' willingness to use visual approaches. This was particularly striking where visualisers were concerned. In the following extract Marvin, a visualiser, was asked to describe how he intended to approach question 8a from the main exercises (solve the equation $x^2+2x-8=0$):

Extract 1

1. SE: So how are you going to tackle this problem then?
2. Marvin: Well first of all I'm going to put the formula into the Y1.
3. SE: Yes.
4. Marvin: And work out where it crosses using the table.
5. SE: Right.
6. Marvin: And then do it, putting it in brackets, actually solving it algebraically.
7. SE: If you were doing this ordinarily and didn't have the graphical calculator, would you still do the same thing, do a quick sketch and then…
8. Marvin: No. I wouldn't do the sketch.
9. SE: You wouldn't do the sketch. You'd do the algebra?
10. Marvin: Yes.

Jake and Julian, both visualisers, also hinted that without the graphical calculator they would be less likely to use a graphical approach when discussing how they would attempt question 8b (solve the equation $8x^2+4=(x-2)^2$). They were intending to use both graphical and symbolic approaches to solve the problem.

Extract 2

1. SE: If you didn't have the graphical calculator would you actually draw these?
2. Jake: No. We'd rely on algebra probably.
3. Julian: We tend to just battle on with the algebra.

Similarly Nigel, also a visualiser, tended only to use visual approaches when experiencing difficulty with questions, or if he did not know how to tackle a question algebraically. This was particularly apparent when he was interviewed about how he would attempt to find the values of x where the graph of $y = x^2 - x + 4$ lies above the graph of $y = 4x - 2$ (question 3 in the student interviews).

Extract 3

1. SE: What about question three?
2. Nigel: I wasn't really sure about this one. What I would have to do is I'd have to plot it onto a pair of axes and then I could actually picture what it would look like and it would help me to understand what I've actually got to find out.
3. SE: So you would have two graphs and they would intersect at some points.
4. Nigel: Yes.
5. SE: So you would be able to see whereabouts one lies above the other.
6. Nigel: Yes.
7. SE: Do you think that there is an algebraic way of doing it?
8. Nigel: There probably is but I don't know what it is.
9. SE: If you had the knowledge would you use a graphical approach or would you...?
10. Nigel: If I knew how to do it algebraically, I would always do it algebraically.

Julian, another visualiser, was encouraged to answer the first interview question (For which values of x is the graph of $y = 3x^2 + 9x - 12$ below the x-axis?) using a graphical approach because the graphical calculator facilitated this, whereas without the technology his approach would have been symbolic.

Extract 4

1. SE: If you didn't have the graphical calculator what would you have done?
2. Julian: If I didn't have the graphical calculator I would probably have attempted it algebraically, but as I have the calculator it was easier to do it that way, so I chose that method.

The amplification effects of the technology made visual representations as accessible as standard symbolic techniques to these students, and so encouraged them to make greater use of graphical approaches.

Mick, a non-visualiser, explained how his use of the graphical calculator had enabled him to use a graphical approach to solve question 8a (solve the equation $x^2+2x-8=0$), when otherwise this would not have been the case:

Extract 5

1. Mick: I've done the graph first.
2. Perry: So you know what you're working out before you work it out.
3. Mick: Yes. I think I find that way easier than... well using the calculator it would be, but if we didn't have the calculators then we would *have* to do it algebraically. We wouldn't probably consider drawing it.

Marie, another non-visualiser, described how she would use a graphical approach to solve the second student interview question, in which she was asked to find the x values where the graph of $y = 3x + 6$ intersects the graph of $y = 2x^2 + 5x$. However, without access to the graphical calculator, she would have been reluctant to use such an approach, as she did not have enough confidence in her own graphing skills:

Extract 6

1. Marie:	Ok number two, probably again I'd do a graph and read off the graph. If I couldn't, if it wasn't that clear, then I would do it algebraically.
2. SE:	So you would rather do the graphical approach first?
3. Marie:	Yes, to see if it was obvious – if it's an integer or not.
4. SE:	So is that because you've got the graphical calculator or would you do that even without the graphical calculator?
5. Marie:	What draw that by hand? No because I wouldn't trust myself being able to draw it well.

Mick and Marie's comments illustrate how the use of technology can encourage non-visualisers to become more open to using visual approaches. Marie, in particular, clearly knew how to approach these problems visually, and yet without the graphical calculator, she would not have felt able to use a graphical approach through lack of confidence in her own graphing skills. By using the technology, Marie had the confidence to adopt a predominantly graphical approach when solving all of the interview questions, which is something that she would normally do "quite rarely". Without access to the technology, she would only have considered drawing graphs if the problems were difficult to solve algebraically. The graphical calculator facilitated a graphical approach as graphing becomes easier, quicker and is more accurate than by hand. This feature clearly affected the way in which some of the students attempted to solve the questions.

Thus, it was apparent that whilst most of the students appeared to recognise the potential benefits of visualisation, this was not always transferred to their work with their usual mathematics teacher. Analysis of the pre-trial questionnaires indicated that overall only two of the students preferred to work visually, whilst six preferred to work symbolically and nine had no such preference. Of the self-identified non-visualisers five preferred to work symbolically, as might be expected and the remaining two had no preference. Of the visualisers, however, just two students, Claire and Jake preferred to work visually, whilst seven students had no particular preference and Julian actually preferred to work symbolically.

One might expect that students referring to themselves as visualisers would generally prefer to work visually, however as in phase two this did not appear to be the case here. These particular visualisers had adapted to the culture of the Advanced level mathematics environment and were used to using symbolic expressions more regularly. Indeed, the students' teacher, Mr Moore (a visualiser) commented that in order for his class to become successful at A level mathematics, it was more important for them to be able to perform symbolic manipulations than to be able to visualise their mathematics. There seemed to be an underlying predisposition amongst some of the students to work symbolically in the first instant, even amongst visualisers:

Extract 7

1. SE: How are you going to tackle this? You've started doing it with algebra again.
2. Jake: The same as we always do.

However, the student interviews revealed that following the introduction of the technology these students were much more willing to explore graphical approaches, as is illustrated in table 7.3 (n = 11).

Table 7.3 Approaches used by students to solve the interview questions

	Wholly Symbolic	Primarily Symbolic	Symbolic & Graphical Combined	Primarily Graphical	Wholly Graphical
Q1	0	0	6	0	5
Q2	3	0	0	1	7
Q3	0	0	3	1	7
Q4	0	0	2	0	8
Q5	0	0	0	0	9
Q6	0	0	0	0	4
Q7	0	0	3	0	0

For the different types of questions the students adopted different approaches. However the vast majority of these (95%) involved some graphical element. A graphical approach tended to be used in situations where the students were unfamiliar either with the type of question or with the functions to be graphed. Similarly, if a problem was regarded as complicated by the students or proved to be difficult to solve symbolically then a graphical approach would generally be used. In this way, since having access to the technology made the graphical approach more accessible to the students, they were able to tackle problems that were seemingly complex and completely new to them with a relatively high degree of success.

One of the students to perform particularly well in his interview was Julian. Interestingly, whilst Julian classified himself as a visualiser, he claimed to only use visualisation when experiencing difficulty with problems and generally preferred to work symbolically:

> *I prefer to work with symbols. This might be that I tend to only use visualisation when I'm struggling with a problem. The majority of the times I use such tools I usually use them in tandem, the visual methods as a way of simplifying and symbolic as a way of communicating that information.*

Clearly, part of the value of employing symbolic approaches in Julian's opinion lies in the comparative ease by which these approaches can be shared, followed and/or discussed with other members of the class when forming a written solution. This viewpoint also appeared to be shared by other students as the following responses to the question "do you have a preference for working either symbolically or visually?" demonstrate:

> **Roy (visualiser):** *I have no preference. I often use both in order to gain an answer. I find that visually helps me to understand a problem, which is very useful, whilst symbolically is often much quicker and more simple for others to follow.*
>
> **Marie (non-visualiser):** *I prefer symbolically because I can usually find the solution quicker that way and it is no more difficult. Sometimes a diagram helps to see and understand the problem but symbols are needed for a solution.*
>
> **Marty (non-visualiser):** *Symbolically, because it is easier to analyse and if you get stuck you can see more easily where you have gone wrong.*

Six of the students would aim to use symbolic arguments for written purposes, even if they had solved the problem visually. Paul's comment explains why this tendency may have persisted:

> **Paul (non-visualiser):** *I find it more difficult to include visual processes in my solutions especially when putting it into writing.*

What was significant, however, about the students' use of the graphical calculators in this phase, was that the technology enabled them to compare and contrast their graphical approaches much more easily than if they had been using pencil and paper alone. This in turn meant that they were using visual arguments to communicate ideas to one another. Moreover, individual students' justifications were strengthened by their ability to show others the basis of their argument using the graphical calculator, which is discussed in chapter 8. The clarity of the images produced by the technology also facilitated the ability of students to include sketches of the graphs in their written work.

Each of the students felt that by using the graphical calculator they had been able to visualise functions more clearly. Seven students said the graphical calculator had assisted them in this respect by enabling them to access the graphs of several functions (often simultaneously). This in turn promoted their abilities to recognise, reproduce and transform the graphs of different functions and allowed them to explore and establish the patterns amongst different families of functions. Jake believed that the main advantage of using the graphical calculator was that it "allows us to visualise the functions and is a change from writing all the time".

The graphical calculator was also regarded as a tool for helping students to make the connections between symbolic and graphical representations:

> **Perry (visualiser):** *At A level a lot of the algebra is learnt. By drawing the graph, you can see what you are doing to the graph when you are playing with algebra.*

> **Claire (visualiser):** *I feel that I have benefited from using the calculators. They have helped me to visualise what different types of graphs look like. It has helped me to imagine what graphs look like just by looking at the formula of the function. They make it easier to understand what happens to the graph when you apply a particular function and what the functions do to it.*

> **Jim (non-visualiser):** *I am starting to get better at seeing a graph and recognising what type of function it uses, though I am not an expert yet. The calculator definitely aids people to do this.*

The use of the graphical calculator had clearly had an impact on these students' problem solving strategies. Indeed, twelve of the students indicated that their experience of using the graphical calculator would influence their approach to solving problems, involving functions, in the future. Generally this would involve a greater use of visual approaches:

> **Claire (visualiser):** *I will try to visualise what I am trying to work out. I think I will find this easier to do now I have used the calculators.*

> **Pierce (non-visualiser):** *I will always consider how the calculator applied the functions and use this to try and help me.*

> **Mick (non-visualiser):** *I will now look to solve a problem graphically first rather than algebraically.*

> **Perry (visualiser):** *I would still attempt to solve them algebraically but it is always good to have a check by drawing it so you can see where you are wrong.*

Mick and Pierce classified themselves as non-visualisers and it was a very encouraging outcome that they were both willing to use visual approaches more readily at the end of the trial. This was especially so, considering the difficulty that Pierce claimed that he had in visualising his mathematics and Mick's usual tendency to avoid drawing graphs:

> **Pierce:** *I personally prefer symbolic working as I often find it difficult to visualise things in my head.*

> **Mick:** *I don't like graphs. I try to stay clear of them as much as possible.*

Mick felt that the main advantages of using the graphical calculator were in helping students, such as himself, who had difficulty in visualising:

"What do you consider to be the main advantages of using technology to teach the concepts of functions to students such as yourselves?"

> *For people who find it hard to visualise some things. It is a very visual method of explaining things.*

However Jake, like Perry claimed that he would still use algebraic methods in most cases, using visual approaches mainly to check answers, despite the fact that they both classified themselves as visualisers. Carol and Justin, also visualisers, and Marty a non-visualiser stated that they would only modify their approaches if they were given permanent access to graphical calculators.

The vast majority of students, fifteen in total, would welcome further use of graphical calculators in their mathematics lessons. The graphical calculators were seen as valuable visual aids to problem solving, a means of highlighting and eliminating mistakes, and exploring the effects of various operations on functions:

> **Perry (visualiser):** *For complicated functions it has allowed me to see what I am working with and helps show obvious mistakes. It has also helped greatly with things such as inverse functions and changing the shape. While you're getting the grasp of new ideas it's good to have a visual aid.*

Pierce (non-visualiser): *When drawing graphs the calculators aren't answering the question for you, but giving a helpful aid. They can cut down on some careless errors. I have benefited greatly from the use of the TI-82, as it has given me new methods of answering questions.*

However, several students felt that using the graphical calculator was most productive when they had developed some understanding of functions, indicating that they would have rather been introduced to the graphical calculator following a period where functions were explored without the technology. In contrasting these views with the clear benefits of using the technology to introduce functions to students, this raised an important pedagogical question, namely, when exactly is the most appropriate time to introduce the technology to the class as a whole, taking account of the needs of individual students?

7.2.3 Graphical Calculators, Motivation and Confidence

One of the most noticeable features of this phase of the research was the effect that the graphical calculator had on the students' enthusiasm towards learning about functions and on their confidence to use graphical approaches, talk about their findings and challenge one another's ideas.

Observations within the classroom suggested that these students were keen to use the graphical calculators and thus were highly motivated throughout the course of the investigation. This point was also raised in the student questionnaires, where several students commented on the use of technology as motivating, enjoyable and interesting. The following students' responses are typical of the general feeling towards the use of the technology:

Marvin (visualiser): *It has been great fun and an interesting approach.*

Fay (visualiser): *I enjoyed learning with the TI-82, a great opportunity.*

Marie (non-visualiser): *It makes the lessons more interesting as well as helping understanding.*

Jim (non-visualiser): *Using the graphical calculator has been very interesting and helpful.*

Marvin in particular felt that the main advantages for students in using the technology were "seeing accurate diagrams on the board and the fun and motivation of it all!"

Through using the technology, these students were encouraged to solve problems using techniques that they may not have considered otherwise, with keen enthusiasm. For example, in his interview Julian, a visualiser, described how having access to the graphical calculator had influenced the way he answered the first question (for which values of x is the graph of $y = 3x^2 + 9x - 12$ below the x-axis?):

Extract 8

1. Julian:	For question one I just typed it into the graphical calculator, and when it came up I used the trace facility on the calculator to highlight the first value that was below the x-axis and then wrote the answer out in an inequality.
2. SE:	Did you think at all about using a symbolic technique to solve the problem?
3. Julian:	I was going to use a symbolic technique to solve the problem, but then I thought this is a new toy, so I thought that I'd try that.

Similarly, when Carol, a visualiser, was interviewed and asked to describe how the graphs of $y = x^2 + 3x - 2$ and $y = x^2 + 3x + 2$ are related, she replied:

I did this on the graphical calculator and I found that they were the same graph except it's moved up by four. But I did actually know that already. I didn't really need to use the calculator, but I did seeing as it was there, because I quite like it - because it's fun.

The 'novelty' appeal of using the technology thus appeared to be a factor in motivating these students.

In addition to highlighting the motivational value, the students also made comments concerning how their confidence had improved as a result of using the technology:

> **Carol (visualiser):** *I think that I have been given the confidence to use graphical calculators and help me to understand functions better.*
>
> **Jim (non-visualiser):** *If I had previous knowledge of using a graphical calculator I would definitely have found the questions easier to complete and I would have been much more confident than I was.*
>
> **Marie (non-visualiser):** *Being more confident when using the calculator I think I would now use it more.*
>
> **Claire (visualiser):** *If I had been more used to using the graphical calculator I would have answered the student interview questions differently because I would have been more confident in what I was doing and I would have done a lot more with it.*
>
> **Mick (non-visualiser):** *I think that without it I would have struggled with this chapter. I find it hard to visualise shapes and graphs, so it has been a real help. The main advantages are for people who find it hard to visualise things.*

As was seen in the exploratory study, one of the major benefits of using the graphical calculator lies in its potential to promote individual students' overall confidence, which acts as an important affective element of the learning process. In his interview Julian, a visualiser, insisted that even when the graphical calculator is not used all the time it could still instil confidence in students because it plays an important part in verifying their answers:

> *Even if you don't want to use it to do the work, it's definitely a good back up, because it gives you confidence – you know you've got it right because you've seen it happen.*

He also indicated that this factor had influenced him to use the graphical calculator when he described how the slope of the function $y = 2x^2 - 3$ changes from $x = -3$ to $x = 3$:

<u>Extract 9</u>

1. Julian: It's going to go from negative to - it's going to stay negative, pretty steep then it will level off and it will get steep again and that will be the same as that [indicating the gradient at 3 and –3] but one will be minus.
2. SE: Right Ok yes that's fine. So you can see that in your head if you need to?
3. Julian: I didn't need the calculator no, but I just it's a way of checking because there's no problem of doing it. It's not against any exam law so you might as well make sure you are right instead of just hoping.

The graphical calculator represents a potential source of verification in the classroom, which is either used in addition to, or in conjunction with, that provided by the teacher and fellow students. Roy, a visualiser, indicated in his interview that his own graphing skills had improved as a consequence of using the technology and that this had given him a greater sense of confidence in his solutions and ownership of his mathematics.

<u>Extract 10</u>

1. SE: Do you feel like you're ordinary plotting skills have suffered at all because you've been using a graphical calculator, or because you don't always rely on it and do sketches yourself do you think that it hasn't made any difference, or has it helped?
2. Roy: I find that it's helped because I can plot the graph on my calculator and by hand and then I can check that I am getting it right, because if there's a problem I

know that there is a problem. I don't have to wait for it to be marked. I can see the problem immediately and try it again so that way I can make sure that I get it right. It's helped in that way.

The following episode demonstrates how use of the graphical calculator can motivate students in such a way that they become excited, as well as more confident, about their mathematics.

Episode 1 – Graphical Calculators and Student Enthusiasm for doing Mathematics

Perry, a visualiser, was asked to describe how he would solve the simultaneous equations: $x - 3y = 16$ and $x^2 - 4y^2 = 13$, both symbolically and graphically. Firstly, he explained how he had solved the problem via symbolic manipulation and had obtained the correct solution. Then he attempted to find the solution graphically using the graphical calculator. Following a discussion with the teacher-researcher and his classroom partner Mick, a non-visualiser (see Chapter 8), he was eventually able to use the technology to produce the right graphs of the two functions.

Once Perry had zoomed out on these two graphs and located the two intersection points he zoomed in on them in turn in order to obtain the co-ordinates, and was able to solve the problem:

1	SE	You can see the two crossing points there can't you?	Perry zoomed out again.
2	Perry	Take them one at a time. Zoom in.	Perry's narration referred to zooming in on the point of intersection on the right hand side.
3	SE	Right so if you trace that you can see what the co-ordinates are can't you?	
4	Perry	Work out the intersections.	
5	SE	Or you can calculate yes sure.	

By using the zooming facilities, Perry was able to successfully obtain the co-ordinates of the first point of intersection (see figure 7.3).

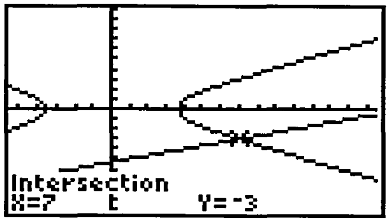


Figure 7.3 Identifying the first intersection point

This was very satisfying for Perry, and he was eager to use the technology to find the co-ordinates of the second intersection point:

6	Perry	Hurray!	Perry was jubilant.
7	SE	Yes and that's what we've got isn't it, yes.	
8	Perry	We did get that.	
9	Mick	I got that as well.	Mick added reassurance.
10	SE	And the other one is on the other side of the screen that you haven't got on at the moment, but you'd have to zoom around that wouldn't you?	Perry zoomed in to the left and calculated the intersection point.
11	Perry	You could do a dance whilst waiting for it! Which curve am I on now? The intersection point is –2.6, -16.2.	See fig 7.4
12	SE	Yes. Ok that's what we got yes.	
13	Perry	I got that, success!	Perry was really pleased.
14	SE	That's great. So that shows that you got the correct answers with your algebra.	
15	Perry	Yes, I did yes. It's very nice to know that I worked it out right for a change.	Perry was given confidence in his algebra and in the validity of the graphical approach.

Peter was so eager to obtain the co-ordinates of the second intersection point that he became a little impatient whilst waiting for the graphical calculator to perform its task (line 11). When the co-ordinates finally appeared on the screen, he appeared to be genuinely excited (line 13).

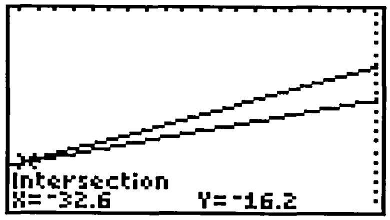


Figure 7.4 Identifying the second intersection point

He was particularly pleased that the graphical calculator had confirmed that he had answered the question correctly using his symbolic approach. Using the graphical calculator gave Perry greater confidence in his ability to solve these types of problems symbolically and in the validity of the graphical approach.

As was also demonstrated in this example, Perry was able to use the technology effectively to show the teacher-researcher and the students watching this demonstration how to solve the two simultaneous equations graphically. By using the technology, students were able to show one another, or the teacher, clear and accurate diagrams when working on problems together, which facilitated the sharing of ideas. Having confidence in the accuracy of the answers produced by the technology also encouraged individual students to be more self-assured when discussing or presenting their ideas to others. Confidence at the individual level is thus transferred to group work becoming a key affective element of the learning process, which is discussed further in chapter 8.

Consideration of individual students' written work also shed light on how student confidence (or lack of it) affected their overall performances in the set exercises. Table 7.4 illustrates their relative success in the exercises overall. This includes their performance in exercises 3A-3D from their textbook and the main trial exercises. Their success is compared with their self-classification as visualisers or non-visualisers. Table 7.5 further displays the mean scores and the standard deviation for the visualisers and non-visualisers in the class.

Table 7.4 Individual students' overall performances compared

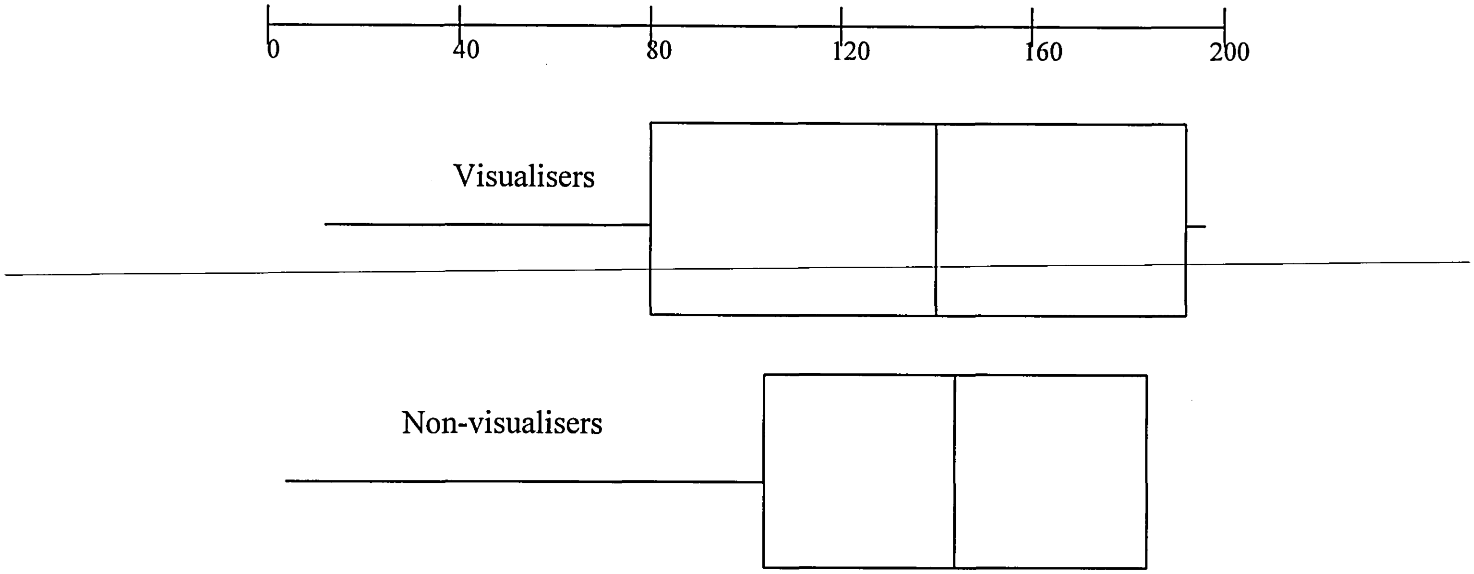
Position in Class	Student	Self Classification	Overall Score	Relation to Mean (123.4)
1	Julian	Visualiser (V10)	196	> mean
2	Paul	Non-Visualiser (V13)	176.5	> mean
3	Marty	Non-Visualiser (V17)	162.5	> mean
4	Claire	Visualiser (V5)	159	> mean
5	Perry	Visualiser (V1)	150	> mean
6	Roy	Visualiser (V8)	142	> mean
7	Jake	Visualiser (V6)	140.5	> mean
8	Fay	Visualiser (V7)	137	> mean
9	Mick	Non-Visualiser (V12)	136	> mean
10	Kirk	Non-Visualiser (V11)	132.5	> mean
11	Marie	Non-Visualiser (V14)	120	< mean
12=	Pierce	Non-Visualiser (V16)	116	< mean
12=	Carol	Visualiser (V9)	116	< mean
14	Marvin	Visualiser (V2)	109	< mean
15	Nigel	Visualiser (V3)	90	< mean
16	Justin	Visualiser (V4)	11.5	< mean
17	Jim	Non-Visualiser (V15)	3.5	< mean

The information presented in table 7.5 is represented pictorially by box and whisker plots in figure 7.5.

Table 7.5 Comparative mean scores and standard deviation for the visualisers and non-visualisers

Mean Overall Score for Visualisers	Mean Overall Score for Non-Visualisers
137.7	140.6
Standard Deviation	Standard Deviation
29.1	21.9

Figure 7.5 Box and whisker plots to show the comparative scores of visualisers and non-visualisers



The mean overall scores for the students who classified themselves as visualisers was slightly lower than that of the non-visualisers and the scores tended to be more spread out. As can be seen from table 7.4, Julian, the least visually orientated amongst the visualisers and a student who preferred to work symbolically in most cases, was the most successful student in the exercises overall. The next best scores belonged to Paul and Marty, both non-visualisers. To score highly in these exercises required the students to be able to combine symbolic and graphical approaches effectively in their solutions to these problems. The success of Julian, Paul and Marty, who all indicated a definite preference for working symbolically at the start of the trial, can thus be attributed to their ability to combine symbolic with graphical approaches. The graphical calculator gave them the opportunity to supplement their symbolic work with complementary visual images, which they were then able to link together to make more sense of the whole problem and to verify their symbolic answers. This gave them confidence in the accuracy of their symbolic answers. As they were encouraged to use both symbolic and graphical approaches throughout the series of exercises, they were thus more open to using graphical approaches in conjunction with symbolic manipulation.

Apart from Jim and Justin who failed to hand in the majority of their work, the three lowest scores belong to visualisers. Of these, Marvin and Nigel were two of the most visually orientated in the group. Their relative lack of success appeared from observations in the classroom to lie in part with the difficulties that they experienced whilst working in the visual mode of representation and in their lack of confidence in working in this way. These difficulties prevented them from progressing very far through the exercises. This was also evident in the student interviews (see section 7.2.4). They particularly had problems in integrating visual and symbolic approaches and clearly needed additional support in this area. These two students did derive benefits from using the graphical calculator, although they still struggled with some of the concepts that were introduced to them. Marvin emphasised that it was mainly through discussing the work with others that he was able to begin to make sense of the functions that he had explored using the technology. The use of the technology by itself was not sufficient to improve their understanding of functions, and input from the teacher and peers was necessary (see chapter 8).

7.2.4 Graphical Calculators, Dependency and Misinterpretation of Results

As was characteristic of the previous two phases, concerns were also raised in this phase about possible over-dependency on technology, students being drawn away from algebraic processes and theory, and technology replacing the need for in-depth thought processes. Overall thirteen of the students expressed reservations of this nature in their questionnaire responses. Comments made by Jim, Roy and Nigel typify these views:

> **Jim (non-visualiser):** *Technology such as graphical calculators is very helpful but they can become easy to use. People can become dependant on technology. I think that the use of technology should continue but personally I will try to learn the manual way of working things out first.*

> **Roy (visualiser):** *It can weaken the understanding of the symbolic side if graphs are used totally.*

Nigel (visualiser): *I feel that maths should be about using your brains to work out problems rather than a computer, but they are useful for speeding up diagrams.*

Mr Moore, a visualiser, was particularly wary that use of the graphical calculators might result in a "lack of understanding of how graphs are generated" amongst the students.

Jake a visualiser seemed to view using the graphical calculator as a means of 'cheating', as did Martin (also a visualiser) in phase two, although Jake recognised that technology could be used to support students' collective thinking, rather than replacing it:

The influx of graphical calculators into lessons, at first, seemed unfair and in someway a cheat, but in mathematics it is helpful to get every method of solving a problem within your group so I suppose it is ok.

Clearly this group of students was extremely wary of using technology too indiscriminately, despite recognising the potential cognitive and time saving benefits that use of the technology can afford. In particular, Mick and Pierce recognised that these factors have important implications for the teacher:

Mick (non-visualier): *It does take some of the understanding away from the algebra so the teacher must be careful.*

Pierce (non-visualiser): *A teacher may cut corners and not give an example for the reason that it is time consuming and a student may become confused. The technology means the teacher can quickly show the diagram.*

Indeed, the teacher plays a crucial part in scaffolding the students' learning, ensuring that they know enough about the various functions and limitations of the technology to avoid the potential misunderstandings which can surround its use, and in introducing the students to a combination of problem solving strategies. This would involve placing equal amount of emphasis on both visual and symbolic approaches, so that with the aid of the technology students would be able to work comfortably in and between both modes.

The findings from the second phase of the research suggested that dependency on technology might be a consequence of how it is used by students and at what stage it was introduced. In particular, it was proposed that dependency could be more likely to occur amongst students working individually rather than collaboratively with technology. As such, evidence was also sought from the data in this phase to further substantiate and elaborate on these claims.

For this purpose the student interview transcripts were analysed. This analysis revealed that in addition to over dependency on the graphical calculators, there were other more poignant problems associated with individual use of the technology. The following three episodes, involving Carol and Marvin, highlight the kind of problems that students may experience when using technology in isolation from the input of others.

The students had been given the interview questions for homework and these questions were based on the content of the previous chapter from their textbooks, which was covered by the class immediately prior to commencement of the trial.

Episode 2 – Amplification, Dependency and Missing Links between Representations

In this episode Carol, a visualiser, was asked to describe how she had attempted to solve the third interview question, in which she was asked to find the values of x where the graph of $y = x^2 - x + 4$ lies above the graph of $y = 4x - 2$:

1	SE	What about question three?
2	Carol	I typed in the two equations so that I could plot the graphs on the calculator and then I could look on the table to find the values of x er of where the graph was above the other. I used the calculator but I haven't got one myself but, so I would have done it drawing the graphs I find them a lot better though, the calculators, they're good.
3	SE	Mm. So would you have known how to do this algebraically?
4	Carol	Algebraically? How do you mean?
5	SE	Using the formulas and rearranging them in some way to find out what x is.
6	Carol	I don't think I could just say now what it would be, but I might be able to get my head round it if I sort of got stuck into it. But that would take longer really so …
7	SE	Is that why you prefer to do it graphically?
8	Carol	Yes, it's a lot quicker.

When Carol had originally attempted this question as part of her homework it would appear that she had not spent any time thinking about the way in which this problem might be solved symbolically. As such she could not offer any suggestions as to how to approach this problem algebraically and was reluctant to try (line 6). Her comments suggested that the reasoning behind her choice not to explore a symbolic method was that this would have required more in-depth thought about the problem and would thus have taken much longer than using the graphical calculator to obtain the solution (line 8). In this case the graphical calculator was seen to provide the correct solution to the problem with little effort on the part of the student.

Indeed Carol's written solution to this and other problems suggested that she had not thoroughly thought-out the implications of the results provided by the technology. Carol's written solution in this case consisted of a list of integer values of x: -1, 0, 1, 4, 5, 6, 7, 8, 9 and 10. Similarly, her solution to the first question (for which values of x is the graph of $y = 3x^2 + 9x - 12$ below the x-axis?) was also of this form and she gave x = -3, -2, -1, 0 as her answer. These written solutions reflected the fact that in both cases she had used the table facility to establish the values of x that satisfied the inequalities. Whilst she had also graphed both of the functions in each case, she did not consider her graphs in light of the table she had produced to check the validity of her solution. Thus, her written solutions implied that she was concentrating on the numerical representation of the problem provided by the graphical calculator, rather than linking the results of the tabular exploration to the graphical representation. She clearly was not seeing the solution as a continuous range of x values, corresponding to a particular region of the graph. She appeared to accept the tabular solution without questioning how these results were obtained or how they were related to the graphs.

Carol seemed primarily concerned with the speed by which problems could be solved by using the graphical calculators and gave little thought to the nature or validity of her answers. As a consequence she was overly dependent on the results produced by the technology. Her failure to try a symbolic approach or to consider the graphs she had drawn and what these meant in relation to the numerical data meant that she was not alerted to the fact that her answers were incomplete.

Marvin, another visualiser, also had difficulty in finding the solution to question three as the following episode demonstrates:

Episode 3 – Graphical Calculators and Misinterpretation

Marvin struggled to make the connections between the visual and numerical representations in this example. The graphical calculator enabled him to explore these different representations simultaneously, although he was not able to use the technology effectively to help him formulate the links between them:

1	Marvin	With question 3 what I did is I started by trying to do it algebraically. I did try doing it by doing it where x^2-x +4 is equal to 4x –2.
2	SE	Yes.
3	Marvin	I worked it out as an inequality but then I was really getting confused with that because it seemed like it was the wrong way to do it. So what I did was used the calculator and I drew them both and you get one straight line and one u shaped one…

4	SE	Yes.
5	Marvin	And it looked like all the values of the u shaped one were above but when you zoom in, some of them are on the line and some are just slightly below because I did that trace thing with it.
6	SE	Yes.
7	Marvin	I did the table, you know where you draw the table and you get x down one and then you get two sets of y values because you've got two graphs right?
8	SE	Yes.

~~Marvin had graphed both functions using the graphical calculator and~~ made use of the table facility to produce the screens reproduced in figure 7.6.

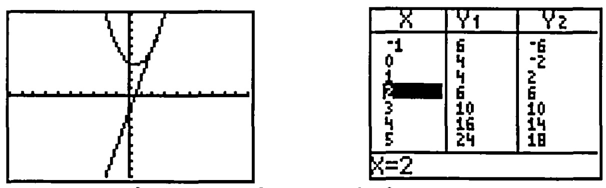


Figure 7.6 Marvin's screens

Marvin was initially confused by the symbolic approach he adopted. His lack of success and difficulty with applying this approach suggested to him that this might not be the right way to tackle the problem (line 3). Consequently, he began exploring the problem using the graphical calculator. His first action was to draw the graphs of the two functions and then trace the co-ordinates. However, this failed to make things any clearer (line 5).

His next step was numerical and he examined the table of values for the functions from which he was able to identify values that he believed fitted the inequality:

9	Marvin	Now what I was trying to do is work out the relationship between the two – where one graph was higher than the other graph. And there was a – it worked yes, some, a lot of the values were higher than the straight line graph, but the points where I think - there were two points where they were 6 and 10 both of the y values. I didn't know whether they would be classed as above the graph of 4x-2. So if it had of been it would have been – it would have either been all values minus 6 and 10 between 6 and 10 or the lot, every one. So, but I did get confused with that.

The two intersection points, however, caused more problems for Marvin, as he did not seem to recognise them as such. He questioned whether these points would be classed as above the graph of $y = 4x - 2$, rather than recognising them as points situated on both graphs. He concluded that, depending on how these points were interpreted the range of values that satisfied the inequality was either: $6 < y < 10$ or $6 < y < 10$. In addition to giving ranges of y values instead of a range of x values which satisfied the inequality, the suggested ranges of y values refer to the regions where $x^2 - x + 4 < 4x - 2$ or $x^2 - x + 4 < 4x - 2$ respectively. In other words these were the solutions to the wrong inequalities. What was required was the values of x for which $x^2 - x + 4 > 4x - 2$. Marvin had clearly misinterpreted the graphical information produced by the technology.

In this question, Marvin viewed the graphical representation of the two functions inappropriately and was thus under the impression that the intersection points could be classified as above the graph of $y = 4x + 2$. Marvin also had difficulty with other problems. In the second question, he could vaguely picture the graphs of the functions, however he could not combine his image with rigorous analytical thought:

Episode 4 – Graphical Calculators and Verification

In this question Marvin was asked to determine the x values for which the graph of $y = 3x + 6$ intersects the graph of $y = 2x^2 + 5x$. From his written work it was evident that he did not use any graphical images to help him find the solution to this problem.

1	SE	You didn't use any images in question two?
2	Marvin	No, no, not at all.
3	SE	Did you check it using the graphical calculator?
4	Marvin	I did - I did check it, yes. Em I mean the reason why I didn't use any imagery with that, I mean em I myself, I mean, I can imagine roughly what the graph would look like but I wouldn't be able to link the two together, if you see what I mean. Some people might be able to, some mathematical geniuses in the world might be able to, but I couldn't so em that way I tried – I tried, I don't know whether you can, but I tried – I went for the easy option and used the formula. I'm quite proud of that actually, I learned that off by heart.

This example illustrates that Marvin experienced difficulty in making connections between different modes of representation and that he saw the ability to make such connections as an almost unobtainable skill reserved for elite mathematicians ("some mathematical geniuses in the world may be able to"). He clearly lacked confidence in this area, and in this case applying a wholly symbolic approach was much easier for him than using a combination of approaches. Moreover, he derived satisfaction from successfully employing the quadratic formula. In this case, the graphical calculator occupied a mere verification role rather than a means of exploring possible solutions.

These episodes suggest that, at this stage, Marvin was not able to use the technology constructively to help him to clarify and amend his own visual ideas, or make connections between representations. The difficulties that he experienced in using visual approaches persisted despite his attempts to overcome them. As found in the previous two phases, the use of technology alone does not guarantee that individual students will automatically make connections between different modes of representation, and the teacher plays an important part in fostering this process. During the course of his interview Marvin was able to reformulate some of his ideas through discussion of these difficulties. This highlighted again the need for the teacher to scaffold the learning task and the importance of conversation in building and repairing an individual student's understanding.

In question six, Marvin's initial image of the quadratic function being of the form $ax^2 + c$ prevented him from progressing any further with the solution to the problem. He was, however, able to modify his image through discussion with the teacher-researcher.

Episode 5 – The Importance of Discussion

In this episode Marvin was attempting to solve question six:

When throwing a single biased dice, the probability of getting a 2 is 0.1, a 3 is 0.12 and a 6 is 0.3. All the probabilities can be worked out using a particular quadratic formula. Explain how the probabilities of getting 1, 4 and 5 can be deduced and how the quadratic formula could be obtained.

1	Marvin	A quadratic curve starts at zero and it accelerates up almost. So the probability of 1 would be below 0.1 more towards zero, between 0 and 0.1, I would say. Is that what you mean?
2	SE	Well it's on the right lines, but do all quadratics actually go through 0?
3	Marvin	No I don't mean - I mean between 0 and 0.1 on the y because that's at 0.1. So it would be somewhere between – see what I mean?
4	SE	I see what you mean, I think, yes.
5	Marvin	I mean if the graph is like x^2+3 it would go 3 up the y axis, so it will be between 3 and whatever the next value of x is because that's 0.1 and you don't know the first value, it's going to be between. Do you see what I mean? It might be below the x.
6	SE	It could be possible with a quadratic though that if you had x is 1 and x is 2…
7	Marvin	Right.
8	SE	And you've got a reflection in the curve, don't you, it goes like that [demonstrating the shape of the curve] so sometimes two x values can give the same y value.
9	Marvin	Right.
10	SE	So it could be that when x is one this could be 0.1 as well…
11	Marvin	Right
12	SE	And at 2 it could be 0.1 and it could have a minimum point in between, I mean there are different possibilities.
13	Marvin	Right I see what you mean, right yes, yes. Am I getting the right idea then that you're saying that the quadratic formula would be in the form of $ax^2 + bx + c$?
14	SE	Yes.
15	Marvin	Ah right, not just $ax^2 + c$?
16	SE	No.
17	Marvin	Right, I see.

Marvin assumed that if a quadratic curve joined the six points then this would go through (0,y) and be increasing (line 3), when in fact the value x = 0 was of no significance here and the nature of the curve was still undetermined. Through the discussion it became clear that Marvin had been thinking in terms of $y = ax^2 + c$, rather than $ax^2 + bx + c$, so that in his model the maximum or minimum point of the curve would always be located on the y-axis (0,c). Marvin was then able to think appropriately about the problem and was enabled to find the solution.

7.3 Conclusions

This chapter has concentrated on the third phase of the research and the ways in which the individual aspects of learning with graphical calculators influenced the students' understanding of functions. This section summarises the findings of this phase in relation to these factors.

7.3.1 Amplification, Cognitive Reorganisation and Students' Understanding of Functions

This phase of the study found that individual students assigned value to both the amplification and cognitive reorganisation effects of the technology, and saw amplification as a precursor for cognitive reorganisation. In other words they regarded the speed and facility by which they were able to operate whilst using the technology as a prerequisite for a more meaningful long-term understanding of functions. The use of the graphical calculator was also seen to have an effect on the way in which individual students communicated their mathematics to one another, thereby changing the way in which knowledge was created. This occurred as a result of the amplification effects, whereby the user was able to immediately and easily access accurate graphs of numerous functions, which made graphical representations as accessible to students as standard symbolic techniques. This allowed students to communicate their ideas to one another using visual rather than symbolic reasoning, which had previously tended to be the norm in this classroom.

7.3.2 Graphical Calculators and Visualisation

The findings of this phase of the research suggested that the existing mathematics culture that had been built up in this particular classroom emphasised the symbolic over the visual. As a consequence, students who classified themselves as visualisers had been enculturated into this environment and were therefore more used to using symbolic approaches.

The introduction of the graphical calculator was subsequently seen to have a significant impact on the visualisers', as well as the non-visualisers', willingness to use graphical approaches.

On the whole, the non-visualisers tended to achieve a greater degree of success in the class exercises than the visualisers. This was generally due to the non-visualisers' abilities to successfully combine symbolic and graphical approaches. In this way, whilst both types of students benefited from using the graphical calculators, the non-visualisers, in particular, seemed to benefit the most.

7.3.3 Graphical Calculators, Motivation and Confidence

This phase of the study indicated that by using the graphical calculators to introduce these students to the concept of functions, they were encouraged to take more of an interest in actively creating their mathematics. All of them were enthusiastic about using the technology. This was partially attributed to the 'novelty' factor and partially due to the way that the graphical calculator offered the opportunity for different avenues of exploration than were usually available.

Use of the graphical calculator also contributed towards improving student confidence in several different areas, which was seen to be a key affective element in the quality of the learning that took place. They became more confident in their ability to visualise functions, in their symbolic solutions, in their own graphing skills, in sharing and discussing ideas with peers, and in presenting ideas to the whole class. The graphical calculator played an important part in verifying individual students' ideas and in giving them a sense of ownership of their mathematics.

7.3.4 Graphical Calculators, Dependency and Misinterpretation of Results

One of the most significant findings of this phase is that the use of technology alone may not be sufficient to enable individual students to make the appropriate connections between complementary modes of representation. It is proposed that individual students need the opportunity to discuss their strategies and images, with either their peers or the teacher, in order to begin to address their problems, especially when they are exploring questions of a non-standard nature. It is also important to note that the visual images that are available to novices differ from those that are available to experts (Tall, 1991b). This is particularly so with respect to non-standard problems and consequently whilst the technology enables students to access these visual images, scaffolding by the teacher or more capable peers can enrich their understanding of them and discourage students from becoming dependent on the technology.

CHAPTER 8
INVESTIGATING THE SOCIAL DIMENSIONS OF GRAPHICAL CALCULATORS IN STUDENTS' UNDERSTANDING OF FUNCTIONS

8.0 Introduction

This chapter consists of an analysis of the social aspects of learning about functions using the graphical calculator that were observed during the third and final phase of the research. The overall aims of this phase of the research were presented in chapter 7. This chapter is primarily concerned with the fourth aim:

- To investigate how the behaviour of individual students affected the shared construction of meaning, and how this behaviour was influenced by the technology.

The exploratory study (chapter 5) highlighted the importance of the social environment in contributing towards students' meaning making with graphical calculators. The graphical calculator was seen to promote and scaffold discussion and collaboration amongst students and the teacher was seen as an essential part of the learning process. Evidence from the second phase (chapter 6) further suggested that the graphical calculator acted as a medium for communication between teacher and student, and a means by which the teacher could guide the students' learning. The establishment of local communities of practice in the classroom was seen to be conducive to successful collaborative work involving graphical calculators. Consequently, this phase of the research was concerned with further investigating the impact of social factors on students' learning of functions to build on these earlier findings. Data was collected from Ashby School with a class of seventeen year twelve GCE Advanced level mathematics students (13 male, 4 female), details of which can be found in chapter 7.

8.1 The Social Dimensions of Graphical Calculators and Students' Understanding of Functions

In order to determine the influence of the social dimensions of graphical calculator use on students' understanding of functions data was analysed from five sources:

(i) pre-trial student questionnaires,
(ii) the students' written work,
(iii) audio transcripts,
(iv) video transcripts,
(v) post-trial student questionnaires.

The findings that relate to the social dimensions of graphical calculator use have been subdivided into the following categories:

- Students' perceptions of graphical calculators and group dynamics and how these relate to practice
- Graphical calculators and collaboration between visualisers and non-visualisers
- The role of the teacher in scaffolding students' understanding of functions in a graphical calculator environment
- Graphical calculators and peer tutoring
- Creating an effective classroom environment for learning about functions with graphical calculators

8.1.1 Students' Perceptions of Graphical Calculators and Group Dynamics and how these relate to Practice

One of the underlying assumptions of this study as a whole was that the use of technology encourages interaction between the students and with the teacher. To ascertain whether these students believed that using the graphical calculators had altered the group dynamics of the classroom, and if so how this had affected their ability to solve problems, they were asked to reflect on this aspect as part of their post-trial questionnaire responses (see appendix C).

There were several themes surrounding the role of collaboration in relation to the students' use of the technology to emerge from their responses. Firstly, the graphical calculator was seen mainly to have encouraged greater discussion in small groups and pairs, rather than as a whole class, as Julian's typical response illustrates:

> *The graphical calculator has encouraged discussion, particularly in small groups, pairs and threes.*

The graphical calculator was also seen to promote collaboration in which the students were able to provide scaffolding for one another's understanding. Marvin and Fay both commented on this aspect:

> **Marvin:** *The graphical calculator has definitely encouraged group discussions and private discussions in pairs, to help and explain to each other. I found this a great benefit.*
>
> **Fay:** *The calculators did encourage group discussions, more in pairs/threes/ fours than the whole class. This was probably because it was relatively new methods to most of us and discussion helped overcome problems we had individually or as groups. It meant you had someone to discuss, help and support you.*

Another way in which the graphical calculator promoted discussion was through the students' use of this resource as a basis for debate. Mick commented on the potential of the technology in this respect:

> *I think that the TI-82 has promoted discussion in the classroom, mainly in pairs but also as a whole class. It gives people something to argue over.*

The technology also facilitated the comparison of results, which was in turn seen to encourage greater discussion, as the following responses illustrate:

Claire: *I think that the graphical calculators have encouraged group discussions in small groups because you ask if friends have got the same graph/answer and how they did it if you have different answers.*

Justin: *The graphical calculator has encouraged discussion more in pairs because you discuss what answers each other got and how they got it.*

Paul: *It has encouraged group discussions by us checking each other's graphs against our own and discussing how to use it.*

Jim: *Group discussions have definitely become more common. Instead of just asking someone what they have done, you can show what you have done and see how they have gone about a question, and compare the results much easier.*

The fact that the students' use of the graphical calculator created more intensive discussion through the unification of their methods was also apparent, as Pierce's response indicates:

I think that using the graphical calculators has encouraged group discussions rather than pair work. When working algebraically everybody has their own preferred methods and will use them accordingly. When using the calculators though, only one system is used to find the answer, meaning everyone answers the questions more or less the same and so group discussions result.

Marty, like Fay, indicated that the students' lack of familiarity with the graphical calculator and initial difficulties in getting to grips with this resource created a need for greater discussion:

Yes the graphical calculator has encouraged discussion because you may need to ask more because the calculator is new to me and at first is difficult to use.

Marvin also found the technology difficult to use at first and he in particular found the collaborative component essential to his success in using the technology effectively. When he was asked "do you believe that using the graphical calculator has strengthened your understanding of functions?" he responded:

No, because I struggled at first and the only way I could combat this was to work it out with someone else.

Thus Marvin's understanding of functions was strengthened through collaboration with other students and the teacher-researcher whilst using the technology. Marvin clearly felt that the use of the graphical calculator by itself had not strengthened his understanding of functions.

Only Jake regarded the graphical calculator as a tool that promoted individual work. In his response to the question: "have you felt that use of the graphical calculator has encouraged group discussions (paired or whole class)?" he commented:

> *No, I think it makes you more individual.*

The remaining five students, Marie, Carol, Roy, Nigel and Perry, felt that using the graphical calculator had not necessarily encouraged more group discussion, and that any discussion which was initiated would probably have occurred with or without the presence of the technology:

> **Nigel:** *Discussions within small groups have occurred but they may have happened anyway.*
>
> **Roy:** *I believe that the graphical calculators have not really changed much – the class often discusses things as a group anyway.*
>
> **Carol:** *No, I don't think it encourages group discussions.*
>
> **Perry:** *Not more than we would have discussed questions/problems.*
>
> **Marie:** *I don't think it encourages group discussions.*

Thus, whilst these students did acknowledge that collaboration had taken place, they did not feel that their use of the technology had generated any more intensive discussion than usual.

In practice the students' collaborative use of the technology enabled them to overcome mathematical difficulties that they experienced, as the following episode illustrates.

Episode 1- Graphical Calculators and Group Dynamics in Action

The interactions of Julian, Jake and Kirk were video and audio recorded whilst they attempted question 4 from the main exercises, which is reproduced in figure 8.1.

Given that $f(x) = x^3$, use the TI-82 to obtain the graph of $g(x) = f(x/2)$. Sketch the two graphs and write down the equation of the new function g(x).

Now use the TI-82 to perform the transformation $g(x+2) - 3$ on g(x). Sketch the resulting curve h(x) and again write down its equation, in the form $ax^3 + bx^2 + cx + d$.

Finally use the TI-82 to perform the transformation 2(h(x)) on h(x), sketching the curve l(x) and writing down the resulting equation as before.

Figure 8.1 Question 4 from the main exercises

Each of these three students had used their graphical calculators effectively to graph the functions f(x) and g(x), and had correctly specified the symbolic form of g(x) as $x^3/8$ together. The next step was to consider h(x) and the group used the graphical calculators to produce the graphs pictured in figure 8.2.

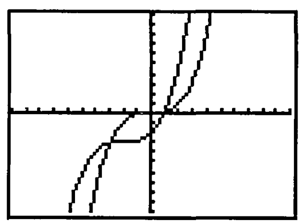


Figure 8.2 Graphs of g and h produced by the students

These graphs gave rise to the following discussion between Julian and Kirk:

1	Julian	Have you got the curve, the like bump bit at –3?	Julian showed Kirk his screen
2	Kirk	Yes.	
3	Julian	So that makes sense because it's lowered it...	Julian was describing the [0, -3] translation
4	Kirk	Yes.	
5	Julian	On the y and...	
6	Kirk	It's moved it back 2 x.	Kirk was describing the [-2, 0] translation

7	Julian	Yes.	
8	Kirk	So that actually makes sense if you think about it.	Kirk was reflecting on the links between the symbolic form of the translation and the corresponding graphical effects
9	Julian	It does.	

As was typical of the interactions between these students, comparison of the graphs produced by different members of the group using the graphical calculators provided a basis for initiating a discussion of results. In this case, having produced the graph of h(x) on his own graphical calculator, Julian initiated the dialogue with Kirk, in an attempt to discover whether his graph was the same as Kirk's. The technology fostered the interaction between these students, giving them a shared focal point for discussing their mathematics.

Kirk's acceptance of Julian's graph enabled them to begin to rationalise the relative positions of the two graphs together. In doing so, they performed a collaborative completion (lines 3-6) and it was evident that they were both thinking carefully about and reflecting on the links between the symbolic and graphical representations of the function h(x). By working together using the technology, the graphical effects of transformations were easier for these students to interpret because they were able to see the graph changing position and/or changing shape on screen. Through being able to see these transformations being applied in a dynamic way, Julian and Kirk both appeared to be making sense of their effects and accepting one another's assertions (lines 3 and 8). As the discussion continued, however, Julian began to realise that they had both transformed the wrong function:

10	Julian	Now sketch that. Ah but it's g.	Julian was realising that they had transformed f rather than g.
11	Kirk	It says sketch the....	
12	Julian	We've done f, f(x), oh no!	
13	Kirk	No that would be h(x), wouldn't it? Because that was the transformation and that will be that and you've got to write down its equation.	Kirk pointed to the new curve on the graphical calculator screen.

14	Julian	Yes. What have you done?	Julian agreed that the graphical effects of the transformation would be the same.
15	Kirk	Oh em, well I can draw that, and then I'll…	
16	Julian	Haven't we done f, haven't we f(x + 2) – 3. It's g.	
17	Kirk	No because…	
18	Julian	Y1 is f and not g.	
19	Kirk	Yes we have.	
20	Julian	So we need to clear that, go back to that, we know what that is because we've just worked it out.	At this point they both cleared their screens and applied the transformation to g.

Recognising that they had made a mistake, Julian attempted to convince Kirk that they had drawn f(x + 2) – 3 using the graphical calculator instead of g(x + 2) – 3. At first, Kirk did not accept this error and used his graphical calculator to try to convince Julian that the graph they had produced was h(x) (line 13). However, Julian realised that the transformation would have the same effect on the shape of the graphs of both f(x) and g(x) and was not swayed. Eventually Julian repaired this situation (line 18) and convinced Kirk that they needed to re-graph h(x) (line 19). In recognising this mistake it was clear that Julian was actively thinking whilst using the technology and not just accepting the results per se. He was connecting the symbolic representations with the graphical representations. The discussion also made these connections clear to Kirk.

The use of the graphical calculators enabled these students to compare individual answers quickly and easily, so that they could show each other exactly what they had done. This gave the students the same starting point from which to begin rationalising their answer together. As was typical of the pattern of interaction within the classroom, when answers differed between individual members of the group, as in this episode, these were discussed even more intensively until a shared sense of the problem emerged. It was apparent that the students' lack of familiarity with the graphical calculators and with graphical methods of solution had also prompted more discussion amongst the students than perhaps would normally have occurred, as Marty, Fay and Marvin had suggested.

Overall, the majority of these students believed that using the graphical calculators encouraged a greater degree of collaboration, particularly in small groups and pairs, through which they could support one another's use of the technology. This scaffolding between the students took the form of individual group members helping other members of the group to use the technology and to develop a shared understanding of the solution by explaining the results of their graphical investigations to them. This stimulated debate, unified the methods used by students and enabled them to overcome individual problems or problems that were encountered by the whole group.

8.1.2 Graphical Calculators and Collaboration between Visualisers and Non-Visualisers

Analysis of all of the video and audio recorded data representing whole class discussions from this phase of the research revealed that 88% of verbal contributions by students were made by students who classified themselves as visualisers. This would tend to indicate that the visualisers in this classroom were more comfortable in discussing the results of their graphical explorations with the graphical calculators than the non-visualisers. Indeed, the only non-visualisers to contribute verbally to the whole class discussions were Marie and Mick, each contributing three times. Moreover, all three of Marie's contributions and one of Mick's occurred after the class had been given time to explore their ideas further in small groups, consisting of visualisers and non-visualisers.

The transcript data from this phase provided the opportunity to study the interactions between visualisers and non-visualisers in small groups. This part of the analysis showed that non-visualisers were much more vocal during small group discussions than in the whole class setting. For example, during the audio and video recorded discussions between a non-visualiser, Kirk and two visualisers, Jake and Julian, each student made a similar number of overall contributions to these discussions; Kirk 32% (60 out of 191), Julian 38% (73 out of 191) and Jake 30% (58 out of 191). Similarly, in a conversation between Perry, a visualiser and Mick, a non-visualiser, and the teacher-researcher, Perry's comments accounted for 32% (12 out of 37) of the discourse and Mick's 27% (10 out of 37).

One possible explanation for these observations could be the way in which the non-visualisers in this classroom were able to work more closely with visualisers in the small group discussions. As was apparent during phase one, a visualiser can provide a non-visualiser with additional support with respect to their understanding of graphical approaches when using the graphical calculators together. Similarly a non-visualiser can scaffold the visualiser's symbolic approaches. This in turn gives both visualisers and non-visualisers additional confidence in sharing ideas. Episode one illustrated how Julian, a visualiser, was able to help Kirk, a non-visualiser, to make the appropriate connections between the graphs that they had produced together using the technology and their symbolic representations. This helped to repair Kirk's understanding of the problem. In a similar way, in the following episode, Mick, a non-visualiser played a vital part in alerting Perry, a visualiser, to a mistake in his algebra, which then enabled Perry to rectify his error and proceed towards a graphical solution to the problem.

<u>Episode 2 – Effective Collaboration between Visualisers and Non-Visualisers with Graphical Calculators</u>

Perry was asked to demonstrate how he solved the two simultaneous equations: $x - 3y = 16$ and $x^2 - 4y^2 = 13$ to those members of the class who had difficulty with this question. He used the overhead projector, which was connected to a graphical calculator for this purpose and this was video recorded. He began by outlining a routine symbolic approach, which involved rearranging the equations, substitution, simplification and use of the quadratic formula. His symbolic manipulation was flawless and he was able to obtain two solutions: (7, -3) and (-32.6, -16.2). He then continued to describe how to solve the problem using a graphical approach with the aid of the technology:

1	Perry	So we've got four answers there and it's very complex.	
2	SE	Yes.	
3	Perry	To do that on the TI-82. What you do is you draw the two graphs so basically Y1 equals (x – 16)/6. Oh that's not right. The other one is y over … I'll have to rearrange the formula to make y the subject.	Perry entered (x –16)/3 into the graphical calculator as Y1.
4	SE	Yes.	
5	Perry	$X^2 - 4y^2 = 13$, $4y^2 = 13 - x^2$, $Y^2 = (13 - x^2)/4$, $Y = +/- [(13 - x^2)/4]^{1/2}$ I believe. Could be wrong! Is that wrong?	Perry had made a mistake in his algebra.
6	Class	No response.	
7	Perry	Oh thanks boys! So it's the square root of $(13 - x^2)/4$. So you get two graphs.	Perry entered $[(13 - x^2)/4]^{1/2}$ as Y2.
8	SE	Right.	

This was the first time that Perry had attempted this problem using the graphical calculator and consequently he had to spend some time thinking about what form the functions would need to take in order to be entered into the graphical calculator. He also narrated every step that he was making in order to make them clear to the class, so that they could follow exactly what he was doing. Once he had completed rearranging the functions, Perry looked towards the class for acceptance (line 5). Given that there was no immediate response, Perry proceeded to graph the two rearranged functions (see figure 8.3 below).

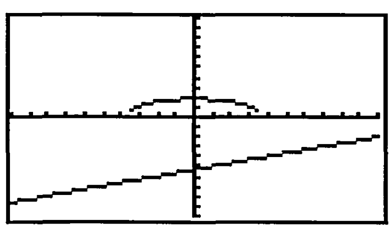


Figure 8.3 Perry's first pair of graphs

$[Y_1 = (x - 16)/3, Y_2 = [(13 - x^2)/4]^{1/2}]$

Perry was subsequently surprised that the graphs did not appear to intersect as he had anticipated and his explanation of this was that the intersection points would probably occur off screen (see line 9 below). Before he could investigate this further however, Mick was drawn into the discussion:

9	Perry	Oh? I suspect that that is extended.	Perry pointed to the curve.
10	Mick	$x^2 - 13$ though.	
11	Perry	$x^2 - 13$?	
12	Mick	Yes it is. That's what I got and I got a different graph.	Mick was confident.
13	SE	Mm.	
14	Perry	Did you?	
15	Mick	Yes that last bit.	
16	SE	Yes you have.	
17	Perry	But if you move the 4y to that side – yes you do.	
18	SE	It's going to be positive yes. Ok?	
19	Perry	Try that.	
20	SE	Try that then. Right.	Perry entered Y2 as $[(x^2-13)/4]^{1/2}$

Mick had realised that Perry had made a mistake in his rearranging of the second function. This realisation then alerted him to the fact that the graph that he produced for this example was different to the one that Perry had drawn (line 12). Consequently, Perry reconsidered his symbolic manipulations and corrected his error, producing the graphs displayed in figure 8.4.

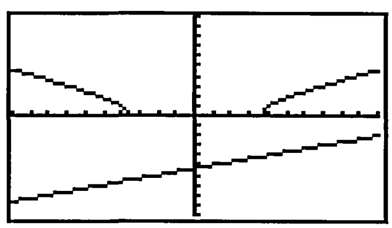


Figure 8.4 Perry's second pair of graphs
$[Y_1 = (x-16)/3, Y_2 = [(x^2-13)/4]^{1/2}]$

However, the graphs again did not appear to intersect. This puzzled Perry because if his algebraic solutions were indeed correct, then one of the intersection-points, (7, -3), should be visible on the screen. As this was not the case, Perry turned to Mick for acceptance of his graphs:

21	Perry	Is that right Mick?	
22	Mick	What, that looks right.	
23	SE	That's Ok yes.	

From this point Perry was able to find the co-ordinates of both intersection points graphically. However, in order for Perry to complete the problem, input from the teacher-researcher was also required, as can be seen in the next section.

8.1.3 The Role of the Teacher in Scaffolding Students' Understanding of Functions in a Graphical Calculator Environment

The second phase of the research emphasised the need for the teacher to scaffold the students' use of the graphical calculator to ensure that interpretations of the information produced by the technology were valid and, where appropriate, accepted by the whole class. Analysis of the transcript data from this phase shed further light on the teacher's role in this respect, which was especially important because the technology was being used to introduce functions to the students. The following episode demonstrates why and how the teacher may need to give students additional support, than that provided by the graphical calculator, when they encounter unfamiliar functions.

Episode 3 – Scaffolding Students' Understanding of the Square Root Function

In this episode, which is a continuation of episode two, Perry was contemplating the relative positions of the two graphs that he had drawn using the graphical calculator (see figure 8.4). However, he did not have the relevant prior knowledge of square root functions to be able to speculate as to why these graphs did not intersect as he had expected at this point in the discussion. Therefore, it was necessary for the teacher-researcher to repair this impasse, as the solution was not made apparent through use of the technology, and the discussion continued:

24	Perry	They don't cross.	
25	SE	So is there another thing that you could try? If you look at the square root, you've drawn the positive square root haven't you?	No response.
26	Perry	Yes.	
27	SE	What about trying the negative square root?	
28	Perry	Try the negative square root?	
29	SE	Why don't you draw them both – the next one as Y3 - to get the whole graph.	
30	Perry	This one.	
31	SE	Yes.	
32	Perry	The negative positive square root.	
33	SE	Yes, so it's exactly the same but just negative.	
34	Perry	This could be fun.	Perry typed in the negative square root and graphed the whole function.
35	SE	That was the first one, yes. Yes so you can see one crossing point there.	See fig 8.5
36	Perry	You can. I wonder if it's worth zooming out? Because that goes up there…	
37	SE	Yes, yes.	
38	Perry	Try zoom out.	See fig 8.6

Perry did not realise that his graph was actually incomplete, as he had only drawn the positive square root. When this was brought to his attention, he was able to graph the whole function (see figure 8.5). By doing this, he was able to appreciate the symmetrical graphical form of this type of square root function and could immediately see the first intersection point from having the complete curve. He was now beginning to feel more confident and suggested using the zooming facilities to view the whole graph and obtained the screen reproduced in figure 8.6.

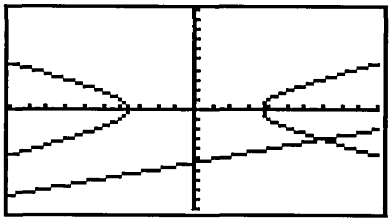


Figure 8.5

[$Y_1 = (x-16)/3$, $Y_2 = +/-[(x^2-13)/4]^{1/2}$]

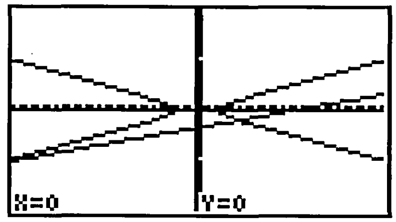


Figure 8.6

Zooming Out on Figure 8.5

Perry was then able to successfully obtain the co-ordinates of these two points of intersection by zooming in on each intersection point in turn and using the maths menu (see figures 8.7 and 8.8). The interaction that took place whilst he did this was presented in episode 1, chapter seven.

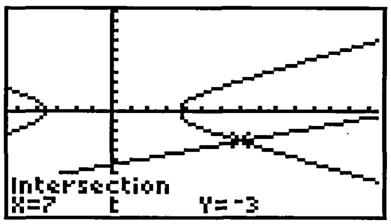


Figure 8.7

The first intersection point

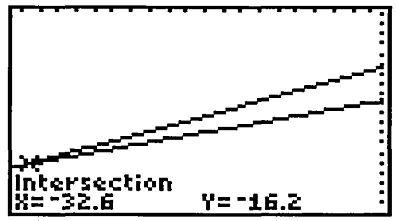


Figure 8.8

The second intersection point

This example highlights the need for the teacher to try to anticipate conceptual problems that the students may experience when using the technology and be ready to respond to these, especially when introducing new functions by means of graphical exploration. Perry was made aware that both the positive and negative components of the square root made up the graph of this particular function, which gave him an appreciation of the shapes of the graphs of square root functions. The importance of the negative part of the square root function was an aspect that was not considered when he obtained the solution to this problem symbolically.

Episode four below also illustrates the teacher's role in questioning the students' comprehension of answers obtained by way of graphical investigations using the technology. This can ensure that a proper understanding has been reached and may be particularly important when the students concerned are non-visualisers. Superficially, it might appear that students have made sense of the concepts being explored with the graphical calculators, whereas careful probing by the teacher could reveal that this is not the case. Discussion with the teacher can then bridge the gap between what the student already knows and what their exploration with the graphical calculator has told them. The additional support provided by the teacher thus provides a link that is sometimes missing between the students' use of the graphical calculator and their collective conceptual understanding.

Episode 4 – The Role of the Teacher in Prompting Cognitive Reorganisation with Graphical Calculators

Following a whole class discussion in which the students attempted to identify the graph of the function y =cos (x-90^0) (see episode 5), they were given additional time to further consider the ideas that had been posed in small groups, as no unanimous solution had been reached. Some students were then asked individually to explain how their group had determined which of the listed functions represented this graph and to justify their answer. Marie, a non-visualiser, was one such student and in this episode she was discussing the results of her work with Claire, a visualiser. Marie and Claire had identified the first graph correctly as y = cos (x – 90^0), although, as the following dialogue indicates Marie had difficulty in explaining why this was an equivalent symbolic representation of the function to y = sin x.

1	SE	So this first one here, how do you know that this is the right transformation?
2	Marie	Well it's the same as sine of x. So we looked at it and if you look at it, it looks like sin x but there wasn't a sin x.
3	SE	It is sin x, you're right. You recognised that correctly. But it's not an option so you've got to find something that's equivalent to sin x.
4	Marie	Yes.

Marie and Claire had clearly recognised that the graph could be represented symbolically as sin x and had deduced, following the argument posed by Perry in the whole class discussion, that an equivalent expression for sin x would be needed (line 2). This had prompted Marie to use an informed trial and error approach by graphing all the cosine options listed using the graphical calculator to see which one matched up with the given graph:

5	SE	So how do you know that this is correct? If you compared it to the graph of cos x?
6	Marie	Well we knew it was going to be like something like that. So then we did just to try it like x + 90 and x – 90.
7	SE	So that's trial and error isn't it?
8	Marie	Yes it was kind of trial and error but we had an idea. We knew which ones to go for. We'd go for the cos ones and not the tan ones because we knew the differences in the shapes of the graphs.
9	SE	So if you drew cos x on this picture at the same time what would cos, just cos x look like?
10	Marie	It'd be like that flipped over like going like that.

Marie insisted that she had a reasonable idea about the form of function that the given graph represented, and could immediately rule out the tangent options (line 8). However, this episode emphasises how Marie was able to obtain the correct answer by using the technology without really thinking about the significance of the result. When asked to indicate what the graph of cos x would look like in relation to the graph of cos $(x - 90^0)$ (line 9), Marie traced out the graph of cos $(x + 90^0)$ on paper rather than cos x (line 10). She thus appeared to associate the action of the transformation $f(x + 90^0)$ on cos $(x - 90^0)$ with a reflection in the x-axis ("flipped over"):

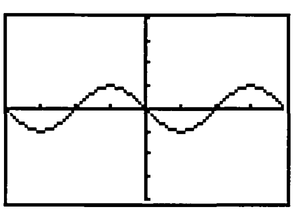


Figure 8.9

Marie's Graph y = cos $(x+90^0)$

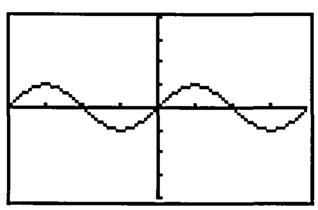


Figure 8.10

y = cos $(x - 90^0)$

In order to illuminate and repair Marie's error and to stimulate further thought, she was asked to think about what the transformation meant and the discussion continued:

11	SE	Would it? Just think about what the transformation means.
12	Marie	Oh cos $(x - 90^0)$.
13	SE	This is what you've got here on the picture.
14	Marie	Yes.
15	SE	This means it's been…
16	Marie	Has it been moved along?
17	SE	It's been moved along yes. This is cos $(x - 90^0)$ so it means that it's shifted…
18	Marie	Oh right.
19	SE	90^0 from the original cos. So where would the original cosine be on here?
20	Marie	Em it would be like that wouldn't it?

When questioned further, Marie realised that this transformation was a translation, although she appeared to lack confidence in her assertion and as such sought acceptance of her ideas (line 16). The effect of this translation was then described to Marie in an attempt to clarify her thoughts about this question and she was asked again about the relative position of the graph of cos x (line 19). However, Marie still could not visualise the effects of the transformation in reverse. For the second time she traced the graph incorrectly, giving rise to the graph of cos $(x–180^0)$ this time instead of cos x (line 20):

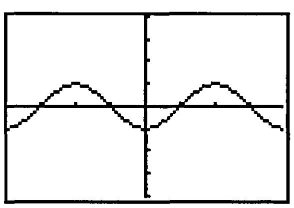


Figure 8.11
Marie's second graph

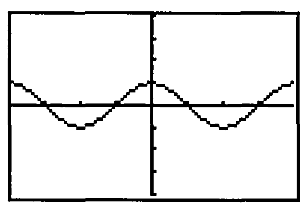


Figure 8.12
$y = \cos x$

In this case she appeared to be applying the transformation $f(x–90^0)$ to cos $(x–90^0)$, confusing the reverse transformation process with the action of the transformation. This prompted the teacher-researcher to focus her attention on the graph of cos x so that she could see that this graph was different to the one that she had just drawn.

21	SE	That's moving it 90^0 there, but what's cosine of zero, can you remember?	
22	Marie	Just normal cosine x.	
23	SE	Cosine x.	
24	Marie	Well I thought it was like that but it goes through 1.	Marie was comparing the graph of cos x to that of cos (x -90^0)
25	SE	One, that's right. So if you have a look it would look like that.	SE used the graphical calculator
26	Marie	Yes.	
27	SE	Ok. So one's crossing the other and they are separated by 90 degrees to the right.	

When Marie was prompted to consider the value of cos 0, she demonstrated that she could visualise the graph of cos x, although she was now a little unsure as this contradicted her previous graphs (line 24). Thus in order to repair any misunderstanding surrounding the shape of the graph of cos x in relation to that of cos (x – 90^0), this seemed an appropriate point to utilise the graphical calculator. Both graphs were drawn simultaneously to clearly demonstrate the relationship between them (line 25).

Marie did not initially seem to have a clear sense of the effect that the transformation f(x–90^0) would have on the graph of f(x) = cos x, even though she and Claire had used the graphical calculator to answer this question. Their trial and error approach had provided them with the correct answer, although this had not enabled Marie to develop a proper appreciation of the relationship between the graphs of translated functions. She clearly needed to think more carefully about the graphs that they had produced using technology. Marie was confused by the relationship between the graphs of cos x and cos (x –90^0). However, by the end of the discussion, she had begun to develop some appreciation of the visual effects of this translation, which was strengthened by the use of the technology because this was used to reinforce the teacher's arguments and role as a more knowledgeable person.

Another important role for the teacher lies in monitoring the language that students use when discussing the results of their explorations with the graphical calculator, especially when concepts are new to the students. As suggested by Davis (1993), students' interpretations of visual information tend to be expressed in natural language and action, and consequently, since use of the graphical calculator promotes visual approaches, teachers may need to formalise the language used. For example, in episode one, as Julian showed Kirk his screen, seeking acceptance of the graphs that he had produced, he used natural language to describe the shape of the new graph. The point of inflection was referred to as "the like bump bit". This was not the first time that Julian had used such terminology to describe mathematical concepts, shapes or ideas. When discussing the shape of the graph of the tangent function in comparison to the graphs of the sine and cosine functions with Jake, he referred to the form of the tangent curve as "daft". Presumably, he was referring to the discontinuities in the function and he may either have spontaneously used such phrases because they summarised his interpretations of the nature of the functions, or he may have deliberately chosen to speak in this informal manner to his peers through personal preference. Alternatively, he could have lacked the appropriate mathematical terms to describe what he was visualising. Jake, however, seemed to appreciate the point that Julian was making and accepted his choice of words:

> **Julian:** *Tan doesn't make a curve like that does it? It's one of those daft ones.*
> **Jake:** *Yes, it's one of those daft functions.*

Kirk also accepted the natural language being used by Julian when he attempted to describe the relationship between the sine and cosine functions:

Julian: *Sine is the same as cosine, just the humps are in a different place. Humps – that's mathematical for you, isn't it?*

Kirk: *It just humps differently.*

Julian: *That is the humps are just different, everything else is the same.*

Julian was clearly a very able student who achieved the highest overall score in the class exercises. However, as illustrated in the examples above, he often tended to use very informal language when discussing his mathematics whilst using the graphical calculator. This was also true of other students and it appeared that communicating in this way was more natural and comfortable for these students and was indeed a successful way of sharing meaning and putting their points across. However, it also indicated that the use of the graphical calculator did not imply the use of more formal mathematical language. This pointed towards a role for the teacher in making sure that students are aware of and can understand and use appropriate mathematical language, especially when using the graphical calculator.

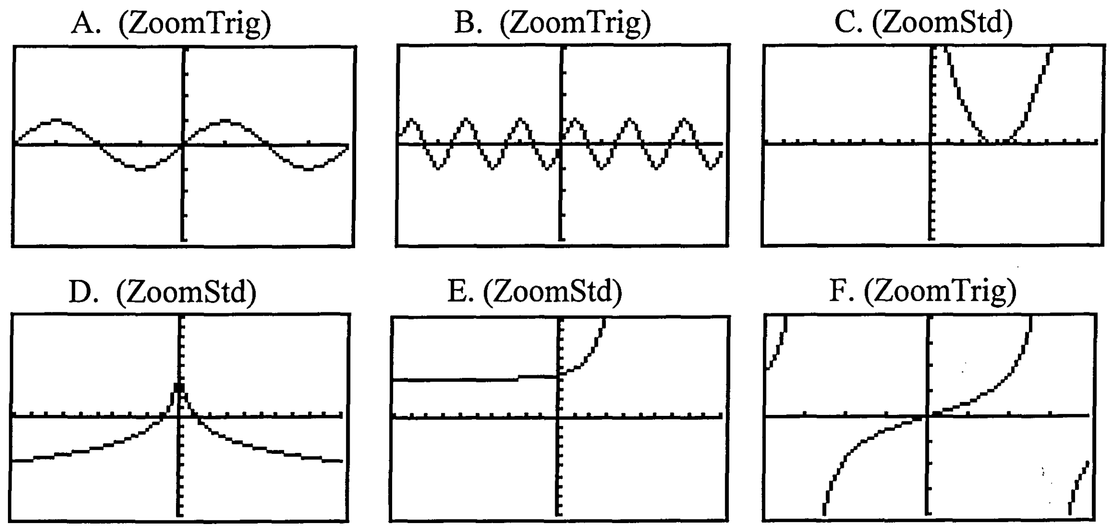
8.1.4 Graphical Calculators and Peer Tutoring

Analysis of the transcript data provided a window into the role of the graphical calculator in relation to peer tutoring. In the second phase of this study, findings suggested that the role of peer tutor could possibly be strengthened and extended through the use of technology. Consequently, further evidence was sought from this phase in order to provide additional support to these claims. The following two episodes have thus been interpreted with respect to the students' positioning within the discourse, in an attempt to draw out the role of the graphical calculator in providing a support structure for peer tutoring.

Episode 5 – Peer Tutoring with Graphical Calculators in a Whole Class Setting

As in the previous phase, the students were given the task of identifying the graphs of six functions and were invited to discuss possible solutions as part of a whole class activity (see figure 8.13).

Match up the six graphs with their corresponding functions, chosen from the list below:



1. $y = \sin(x/3)$	2. $y = \cos(x - 90^0)$	3. $y = 3\sin x$	4. $y = \cos(x + 180^0)$
5. $y = (x - 4)^2$	6. $y = \tan(x/3)$	7. $y = (4 - x)^2$	8. $y = \tan(x/6)$
9. $y = (x + 4)^2$	10. $y = \cos(x + 90^0)$	11. $y = \sin 3x$	12. $y = \ln(1/x)$
13. $y = e^{x-1} + 4$	14. $y = \ln x^2$	15. $y = e^{-(x+1)} + 4$	16. $y = 2\ln x$
17. $y = -\ln x^2$	18. $y = -e^{x+1} + 4$	19. $y = (\tan x)/3$	20. $y = (\tan x)/6$

Figure 8.13 Whole Class Activity: Identifying the Graphs of Functions

In this example, the students were trying to identify the symbolic form of the first graph:

1	SE	Any ideas for picture A?	
2	Several students	Sin x.	
3	Perry	Number 10 ($y = \cos(x + 90^0)$).	Visualiser
4	SE	You think that it might be sin x and why did you say number 10?	
5	Perry	Because we haven't got sin x, but $\cos(x + 90^0)$ is the same as sine.	Perry was thinking along the right lines.

6	SE	Cos (x + 90^0) is the same as sin x?	
7	Roy	Yes, but the period is wrong.	Visualiser

Whilst initial thoughts in the classroom focused on y=sin x as the symbolic form of the given graph (line 2), Perry was quick to realise that since y = sin x was not listed, the graph had to be of a translated cosine function which was equivalent to sin x (line 5). He did, however, incorrectly identify this function as y = cos (x + 90^0) - a function that involves a translation of 90^0 to the left rather than to the right. When Perry was questioned on this point, Roy entered the community of practice with his suggestion that the "period" was wrong (line 7). By using the term 'period' in this context he appeared to be trying to convey the idea that this graph was not obtained through a transformation of 90^0. Clearly, whilst these two students were thinking along the right lines about the symbolic form of this function, they were having difficulty in visualising the effects of particular transformations. However, despite the fact that Roy's contribution also appeared to be flawed, this did create the opportunity for further discussion. Subsequently, at this point the teacher-researcher tried to encourage the class, and these two students in particular, to use the graphical calculators to clarify their understanding (line 8). However, Perry continued to discuss the problem without the aid of the technology:

8	SE	The period is wrong? So there's some disagreement there. So I think maybe you do need to draw them out and look at the…	
9	Perry	Cos (x + 180^0) which is 4.	Perry was influenced by Roy's suggestion.
10	SE	Cos (x + 180^0) and why do you say that?	
11	Perry	Because if you move cos (x + 180^0) backwards it's the same as sin x because the intervals are the same.	
12	SE	180 – I think you might have to check…	
13	Marvin	Utters disagreement with Perry.	Visualiser
14	Perry	Every little peak, right. Its full peak is at 180^0 but the full cycle is at 360^0.	

Perry immediately accepted Roy's point that this was not a translation of 90^0 and consequently now proposed cos $(x + 180^0)$ as the symbolic form of the given graph (line 9). He then began to justify his proposal, again using the right ideas with the wrong transformation (line 11). He was clearly aware that the actual period of the sine and cosine functions was 360^0, although he was unable to picture the graph correctly in his mind – "Its full peak is at 180^0"(line 14). At this stage other students had begun to use the graphical calculators to investigate the arguments being postulated:

15	Jake	Miss it's number 2 (y = cos $(x - 90^0)$). You're wrong Perry!	Jake a visualiser was confident. The TI-82 had confirmed this.
16	Marvin	No he's right Jake.	Marvin changed his mind.
17	Perry	So it will peak and go back to the x-axis every 180^0, because if you move that back 180^0, it will peak in the gap before. It will.	
18	Jake	Because if you, oh…	Still thinking.
19	Mick	It's not 180 though.	Mick, a non-visualiser, had been using his TI-82 to graph the functions.
20	Perry	It is.	Still confident.
21	SE	I think we've got some people saying 180 and some people saying 90 degrees, so what I suggest you do, seeing that there is no consensus here is…	
22	Mick	That's the graph at 180^0 it's not the same as that one.	Mick showed Perry the graph on his graphical calculator.

Jake had been using the TI-82 to graph functions whilst listening to the discussion taking place. This had enabled him to prove to himself that Perry's suggestion was incorrect and to determine the correct symbolic form of the function, which he was then able to share confidently with the rest of the class (line 15). There was, however, still some uncertainty as to who had proposed the correct argument. Marvin, for example, disagreed with Perry at first (line 13) and then changed his mind (line 16). Perry then tried to explain the reasoning behind his suggestion for the second time, acknowledging that some students had different ideas (line 17). Whilst the discussion was continuing Mick, a non-visualiser, had also been graphing functions using the graphical calculator and was able to assert that the required translation was not of the form $f(x + 180^0)$ or $f(x - 180^0)$ (line 19). Perry, however, was not initially deterred by Mick's unsubstantiated statement (line 20). Thus, in an attempt to convince Perry that his argument was correct, he showed Perry the graphs that he had drawn.

Clearly, at the end of this class discussion there was still some disagreement as to the correct symbolic form of the function and during the last few utterances, quite a few students were beginning to hold their own small group discussions, whilst the whole class discussion was continuing. Thus, in order to enable the students to use the technology effectively to negotiate the solution, they were subsequently given time to consider the problem in small groups. During this time the whole class was busy discussing results with one another in these small groups. In particular, active participants in the discussion, Perry and Mick, and Jake and Paul were showing one another their graphical calculator screens as part of the process of sharing and exchanging ideas. In addition Marvin left his seat to consult with Mick and Perry and their discussion centred on Mick's graphical calculator screen. The debate was clearly continuing and intensifying, the outcome being that all of the students were able to correctly identify this function and justify their solution in their written work.

As the initial portion of the discussion showed, some of the students who regarded themselves as visualisers were having difficulty picturing the effects of transformations without the aid of the graphical calculator. Yet, the students who used the technology, both visualisers and non-visualisers, were able to disprove the arguments being presented and back up their own ideas. The use of the technology initiated a lively discussion and provided support to those students who needed help with their visualisations. Thus, the use of the graphical calculator in this episode enabled some of the students, especially Perry and Roy, to modify their own visual thinking, which allowed them to gain a clearer appreciation of the effects of transformations. Moreover, this occurred as a result of the students using the technology to tutor one another. Mick played an important part in convincing Perry of the validity of Jake's solution and this was achieved through the use of technology. The technology scaffolded the learning task and students' acquisition of concepts in the Vygotskian sense through facilitating the individual student's role as a peer tutor.

The students who took on the role of peer tutors in this example were given additional confidence to contribute to the discussion through their use of the technology. This was especially so in Mick's case, who had up until this point only contributed once before to the whole class discussions. By graphing functions on the graphical calculators Jake and Mick were able to verify, modify or reject their original assumptions and those of the other members of the class quickly and effectively. This gave them the impetus to share their findings with the class in an attempt to repair the ideas that were misconstrued, driving the discussion forward towards a satisfactory outcome. These students were actively using the technology to convince themselves, and others, of the validity of each-other's arguments so that eventually a consensus was reached through shared reasoning.

This example demonstrated how the students who acted as peer tutors were able to use the technology effectively to support and reshape other students' visual images of trigonometric functions and the actions of transformations on these functions. In this way, individual students were actively questioning each other and using the technology to confirm or disprove visual thinking. The use of the graphical calculator in this case stimulated further discussion amongst these students and eventually as a result of the efforts of the peer tutors, a common understanding was reached.

Episode 6 – Peer Tutoring with Graphical Calculators in Small Groups

In this episode, two visualisers and one non-visualiser: Jake, Julian and Kirk were attempting one of the questions from the main exercises together. In this question they were asked to compare the graphs of cos x, 2cos 2x, 3cos 3x, sketch them and explain why these three graphs do not cross the x-axis in exactly the same places.

1	Julian	Can I have a look at what you've done? Oh I see.	Julian viewed Jake's screen and vice versa.
2	Jake	It's got to be multiplied by different factors – the x is different.	
3	Julian	Yes.	
4	Jake	That's going to be the answer. Do we have to…	
5	Julian	It's nothing to do with the one before the cos. It's the one before the x.	
6	Jake	Yes. It makes it sort of totally different, doesn't it?	Jake was referring to the previous two questions.
7	Kirk	Right when it's 2x it halves the wavelength, when it's 3x it cuts the wavelength into three. So you get…	Kirk referred to his graphs.
8	Jake	Yes.	Emphatically.
9	Julian	Yes.	Emphatically.
10	Jake	So like they're sort of totally different, you know, totally different values.	Jake was referring to the fact that the graphs all cross the x-axis in different places.
11	Julian	It's not the coefficient of the cosine it's the coefficient of the x that moves that, yes?	Julian referred to the graphs on his screen.
12	Jake	Yes.	

As was characteristic of this groups' collaborative problem solving strategies, the initial exchange between Jake and Julian involved the students viewing one another's graphical calculator screens. This established a common starting point from which the students could begin negotiating as to the nature of the relationship between the graphs. Jake initiated this process by suggesting that it was the different x values that affected the position of the intersection points of the graphs with the x-axis (line 2). Jake's assertion was accepted immediately by Julian, and to show that he had understood Jake's argument he re-emphasised the point being made by Jake and further elaborated on Jake's utterances (line 5). However, at this point Julian and Jake were both using natural, rather than mathematical, language to explain what was happening to the graphs.

Kirk the non-visualiser was the first to introduce more formal mathematical language into the discussion in his attempt to quantify the relationship between the changing x values and the corresponding shapes of the graphs (line 7). In doing so he was building on the arguments proposed by Julian and Jake, and at the same time placing them in a more mathematical context. Moreover, as Kirk referred to his graphs when outlining his argument Jake and Julian both accepted his proposal with confidence. Kirk's use of language also prompted Julian to re-iterate the point that he made earlier, only this time using the mathematical term coefficient instead of "the one before the x" (line 11). Also by saying that "it's the coefficient of x that *moves* that", Julian appeared to be thinking about the relationship between the graphs in terms of one graph being a transformation of another. Julian sought reassurance that he was making the correct assertion and acceptance was provided by Jake (line 12).

Jake attempted to establish a context for this discussion by comparing the results of this exploration with that of the previous two questions. In these questions the students were asked to compare and comment on the main features of the graphs of cos x, 2cos x and 3cos x; and tan x, 2tan x, and 3tan x respectively. Jake emphasised that the effect of changing the values of x in this question was totally different to the effect of the changing coefficient of the cosine and tangent functions in the previous two questions (lines 6, 10). The discussion continued with each of the students trying to ensure that a common understanding had been reached:

13	Jake	They all peak on the y-axis as well.	
14	Julian	Yes, whereas sine doesn't, it crosses the origin. The coefficient of cos doesn't affect where the graph crosses. But x does, that's what you would expect.	
15	Kirk	Mm, if we've done the previous question right!	
16	Jake	As the value of x increases the wavelength increases.	
17	Kirk	Gets shorter, yes?	
~~18~~	~~Jake~~	~~Yes.~~	~~Jake recognised his error.~~
19	Jake	I've just put as the value of x increases the wavelength decreases.	
20	Kirk	Yes.	
21	Jake	Gets smaller and then it's the same as before, as the cosine increases it gets higher.	
22	Kirk	I've put that the coefficient of x moves the intersection along.	Referring to his written work.

Julian repeated his claim that the coefficient of the cosine does not affect where the graph crosses the x-axis, thereby re-enforcing his argument (line 14). By adding "but the x does, that's what you would expect" it would appear that he had been formulating ideas about what would happen to the shapes of the graphs as the coefficients are changed and that these were confirmed by his graphical exploration. In other words he was linking the changes in the symbolic forms of the function to the corresponding graphical representations. Kirk, however, seemed less certain of this relationship and proposed that the validity of their reasoning would depend on whether they had solved the previous question correctly (line 15).

To ensure that he had a shared understanding of the proposed connection between the coefficient of x and the wavelength of the function, Jake attempted to summarise this relationship (line 16). Kirk immediately queried Jake's incorrect assertion (line 17) and in doing so allowed Jake to recognise his mistake (line 18), and thus repaired any misunderstanding on Jake's part. Jake illustrated that he understood what was happening to the graphs as both of the coefficients were increased, as he emphasised that the wavelength decreases, whilst the graph is being stretched (line 21). Kirk's closing statement suggested that he, like Julian, saw that changing the symbolic form of a function by means of the transformation y = f(ax) represents a dynamic process when considered in graphical terms: "the coefficient of x *moves* the intersection *along*" (line 22).

The collaboration between the students in this example was very effective. Kirk classified himself as a non-visualiser and yet his use of the graphical calculator enabled him to talk confidently about the effect that the transformations would have on the wavelength of the function. He was able to steer Julian and Jake towards a more mathematical discussion of their findings and to repair Jake's misunderstanding of these effects.

8.1.5 Creating an Effective Classroom Environment for Learning about Functions with Graphical Calculators

The second phase of the research pointed to the importance of establishing a local community of practice within the classroom in order for students to be able to carry out and discuss the results of their investigations using the graphical calculator effectively. On building on these findings, this phase was concerned with further developing an overall approach to teaching functions with graphical calculators, which took account of how the technology could be used to introduce the idea of functions to students most productively. Transcript data was analysed for this purpose.

<u>Episode 7 – Introducing the Concept of Transformations with Graphical Calculators</u>

The students were introduced to the idea of transformations using the technology. As part of a whole class activity the students were invited to comment on the effects of different types of transformations. Each student graphed the functions being transformed individually using their graphical calculators whilst the overhead projector was used as a focal point for discussion and to bring ideas together. They were then given the opportunity to explore the effects of transformations further in small groups, before the class was brought together again to summarise the results of their investigations.

In the following episode the class had been asked to share their findings about the transformation y = *a*f(x).

1	SE	Could somebody share their results with me?	
2	Fay	They go upside down when it's negative.	
3	SE	They go upside down when it's negative. Yes that's one thing.	There were lots of students offering suggestions at once.
4	SE	OK can everybody just listen please, what did you say then?	Directed at Perry.
5	Perry	When you use the negative prefix it's reflected in the y = 0 line.	More formal mathematical explanation.
6	SE	It's reflected in the y = 0 line, yes, that's right. What were you going to say?	Directed at Marie.
7	Marie	A reflection.	
8	SE	You were going to say it was a reflection as well. What about the actual slope of the curve? What happens to it when you use another value of a?	
9	Marie	The larger *a* is the steeper.	
10	SE	The larger *a* is the steeper.	Showing acceptance.
11	Marvin	As the modulus increases the steeper.	The term 'modulus' had been introduced earlier in the same lesson.
12	SE	As the modulus increases the steeper it is. That's a good point, yes. One way of describing this is as a one way stretch parallel to the y-axis and it's factor a.	Introduces formal definition.

This example illustrates that the students were able to develop a shared understanding of the actions of this particular type of transformation through their paired/group experimentation with the technology. This was then reinforced and formalised in the whole class discussion. Each student who contributed to this discussion built on the previous students' observations. Perry, for example, attempted to provide a more mathematical description of the graphical effects of this transformation (line 5) than that which was made by Fay (line 2).

Similarly, when Marie suggested that as the value of a increased the curve became steeper (line 9), Marvin combined Perry and Marie's observations to deduce that as the modulus of a increased the curve would become steeper (line 11). Marvin used the term 'modulus' very effectively to describe the transformation effects and had clearly appropriated the mathematical meaning of this word, which had been discussed with the students at the beginning of the lesson. Following his graphical exploration of this transformation with Pierce, he was now able to share his understanding of this term with the rest of the class, with confidence. Indeed, so great was the impact of the graphical calculator on the affective domain, i.e. the motivational, confidence and interaction boosting effect, that, as was frequently observed during this phase, several students were all eager to contribute at once.

The teacher-researcher had an important function in focusing the attention of the class on particular students' responses and in encouraging the students to think about the effects of this transformation in more and more appropriate ways. There was also a further role for the teacher at the end of the discussion in introducing the students to the accepted mathematical way of describing such a transformation (line 12).

The ability of the students to explore these ideas freely with the technology in small groups, following the initial whole class introduction to the concept, allowed them to begin to develop a shared sense of the actions of transformations from a common starting point. This shared understanding was then furthered through the concluding whole class discussion, which brought the results of the independent groups of students' investigations together. In this discussion, ideas that were shared between different groups of students formed the basis for creating a new piece of mathematical knowledge that was accepted by all the participants of the classroom. The nature of this knowledge was determined by the students and formalised by the teacher. It is proposed that this approach lay at the heart of successfully introducing functions to these students with graphical calculators.

8.2 Conclusions

This chapter has further considered the third phase of the research and the ways in which the social aspects of learning with graphical calculators affected students' understanding of functions. This section summarises the findings of this phase in relation to these factors.

8.2.1 Students' Perceptions of Graphical Calculators and Group Dynamics

This study found that the introduction of the graphical calculator resulted in more intensive discussion between the students than would normally occur, especially in small groups and pairs. Moreover, the interaction between group members and the teacher-researcher was found to be essential in enabling certain students to use the technology effectively. The graphical calculator was seen to be a means by which individuals could quickly and easily demonstrate their arguments concretely to other group members and through this develop a shared understanding of the problem being considered. In this way, through using the graphical calculator, students were able to help each other overcome their individual difficulties and the difficulties experienced by the group as a whole.

8.2.2 Graphical Calculators and Collaboration between Visualisers and Non-Visualisers

This research has provided unique insight into the process of collaboration in the classroom by considering the interactions between students who classified themselves as visualisers and non-visualisers. The findings from this phase of the study suggested that the non-visualisers amongst the students were reluctant to contribute towards whole class discussions of the graphical results obtained by using the technology. Whole class discussions were consistently comprised of arguments posed by visualisers. However, this pattern of interaction was significantly different when the students worked in small groups, which paired visualisers with non-visualisers. In these groups, the quality and quantity of contributions from non-visualisers matched that of the visualisers, and it is proposed that this interaction contributed towards the comparative success of the non-visualisers in the exercises overall. This also suggested that by enabling the non-visualisers to work in small groups with visualisers the teacher of this class could promote effective collaboration and that this might give the non-visualisers more confidence to contribute towards whole class discussions.

8.2.3 The Role of the Teacher in Scaffolding Students' Understanding of Functions in a Graphical Calculator Environment

Analysis of the transcript data in this study highlighted the need for the teacher to be aware of the conceptual problems that students might experience when using technology, especially when introducing new functions to students by means of graphical exploration. Discussion with the teacher was seen to scaffold students' conceptual understanding of the results that they had obtained using the technology. Moreover, the role of the teacher in actively questioning the students' understanding of these results was found to be especially important when the student concerned was a non-visualiser. This was because the non-visualisers were less likely to contribute towards whole class discussions. It was evident that the teacher needed to monitor the type of language used by students when discussing the results obtained through use of the technology, as there was a need to initiate the students into a more appropriate mathematical discourse. In this study the use of the graphical calculator did not prompt students into talking about their mathematics in a more formalised way - this was a role for the teacher. This differs from previous research into computer environments, which found that the use of formal language was encouraged through students' interaction with the technology (Hoyles and Noss, 1992).

8.2.4 Graphical Calculators and Peer Tutoring

This phase of the research has shown that the graphical calculator can play an important part in scaffolding students' roles as peer tutors, in making these students' ideas more accessible to other students. In particular, non-visualisers may be able to tutor visualisers in visual concepts, through the shared use of technology. Moreover, students' reluctance to participate in whole class discussions may be overcome through the use of technology, giving them the confidence to challenge and tutor others by adding weight to their arguments.

8.2.5 Creating an Effective Classroom Environment for Learning about Functions with Graphical Calculators

The findings of this phase of the study build on the work of Winbourne and Watson (1998) by showing that the creation of a local community of practice played an important part in enculturating students into the meaning of new mathematical concepts with graphical calculators. Shared meaning for new function concepts was created through a combination of whole class discussions and small group activity. Following a whole class introduction to, and discussion of, each new topic the students were encouraged to explore these new concepts in small groups. Individual students were then invited to share their findings with the rest of the class. In this discussion, ideas that were held by different groups of students formed the basis for creating new pieces of mathematical knowledge. It is proposed that this type of environment formed a crucial part of the process of successfully introducing functions to these students through the use of graphical calculators.

CHAPTER 9
CONCLUSIONS, DISCUSSION AND IMPLICATIONS

9.0 Overview

This study has been undertaken to investigate the potential of graphical calculators for mediating students' learning of functions. There were three key objectives to the research. The first of these was to investigate how students acquire meaning within a graphical calculator environment. The second objective involved investigating the ways in which the imagery provided by the graphical calculator mediates students' understanding of functions. The third objective was to explore the role of the teacher in graphical calculator environments.

The study has consisted of three phases of research. The first phase encompassed the exploratory study and the second two phases comprised the main study. This chapter integrates the main findings from each of these phases and relates these to relevant theoretical positions. The first two sections of this chapter summarise the findings in terms of the various cognitive, affective and social factors in students' understanding of functions and how these are interrelated. The final section comprises a discussion of these findings and considers implications for future research.

9.1 Cognitive Factors in Students' Knowledge of Functions with Graphical Calculators

Throughout the course of the study, there were found to be a number of ways in which the students' use of the graphical calculators had a major impact on the development of their thinking about functions. Here the results of the study are presented in relation to the main themes that have influenced the students' cognitive development:

- amplification and cognitive reorganisation effects of the technology,
- graphical calculators, dependency and misinterpretation of results,

- graphical calculators and visualisation.

9.1.1 Amplification and Cognitive Reorganisation Effects in Students' Understanding of Functions

All of the students in this study appeared to be very aware of the short-term amplification effects of the graphical calculator, which refer to the speed and facility by which the learner is enabled to operate whilst using the technology. Through these effects, the students were able to access graphical images of functions quickly and easily, which enabled them to successfully attempt more difficult problems. By creating the opportunity for students to concentrate on interpretation rather than procedural tasks and through providing verification of their ideas, the amplification effects of the technology were seen to contribute towards the cognitive reorganisation effects and thus to the development of students' thinking. This study therefore provides evidence to support Berger's (1998) speculation that the amplification effects have a precursory role in inducing cognitive reorganisation.

In this study the opportunity for students to use the graphical calculator as a means of verifying or, in particular, disproving their ideas nearly always resulted in some form of cognitive reorganisation. This was seen to occur when students' initial ideas surrounding particular mathematical concepts changed significantly as a result of using the technology. For example, when Robert used the graphical calculator to explore the relationships between translations of sine and cosine functions, and a logarithmic identity in Phase 2, he was able in both cases to challenge his initial conceptions and thus move towards a deeper understanding of these concepts. His use of the graphical calculator enabled him to link the visual with the symbolic and spontaneous concepts with scientific concepts, so that he could progress from the specific to the general. Episodes 1 and 2 in chapter 6 chart this development in Robert's thinking. The fact that cognitive reorganisation had occurred more generally amongst the students was also reflected in their comments concerning how the use of the graphical calculator had furthered their understanding of functions, enabling them to use mathematical conceptions more meaningfully. The cognitive reorganisation effects of the technology therefore represented a more meaningful, holistic and general understanding of functions.

Previous research (e.g. Tall, 1991b) has mainly considered cognitive reorganisation as a product of the individual interacting with the technology. However, in this study input from the teacher and peers was also seen to be a crucial contributing factor to the cognitive reorganisation process. The role of the teacher in scaffolding the cognitive reorganisation process was particularly apparent in Phase 3. Marie, for example, clearly needed additional support other than that provided by the graphical calculator and her classroom partner, Claire, in trying to understand why cos $(x - 90^0)$ was the correct symbolic form of the pictured function (episode 4, chapter 8). In this instance my intervention proved to be a significant factor in altering Marie's perceptions through the use of the technology. Likewise, Phase 3 also provided examples of the role that peers can play in supporting the use of the graphical calculator as a cognitive reorganiser. For example, when discussing the same problem, Mick used the graphical calculator very effectively to transform Perry's thinking successfully (episode 5, chapter 8).

9.1.2 Graphical Calculators, Dependency and Misinterpretation of Results

Each phase of the study provided evidence of students regarding the solutions provided by the graphical calculators as irrefutable and thus exhibiting signs of being overly dependent on the technology. The students' over reliance on the technology led to misunderstanding, misinterpretation and misuse of some of the information provided by the graphical calculator. In this way, students from both Phase 1 and 2 completely missed and/or misinterpreted stationary points on the graphs of particular functions. Students, such as Julie (Phase 2) accepted the approximations to real numbers that were produced by the graphical calculator as a result of the zooming operations carried out, without question. Her solutions to the problems that she was set for homework contained answers that were clearly not thought out carefully, such as stationary points with co-ordinates (-2, -6.44×10^{12}). Facility with the graphical calculator, as in Julie's case, led students away from in-depth mathematical thinking and questioning of results. The limitations of the technology in terms of display resolution were also seen to give rise to misinterpretations.

Other researchers such as Guin and Trouche (1999), Smart (1995b) and Leinhardt et al (1990) have highlighted that dependency on technology is a potential danger in the classroom. This study sought to extend these findings by considering factors that contribute towards dependency. As a result, this study found that dependency was significantly influenced by three key factors: (i) students' experience of using the technology, (ii) their prior knowledge (or lack of it), and (iii) individualistic working. Regular users of graphical calculators were found to be less likely than the inexperienced graphical calculator users to be overly dependent on the technology. Dependency was also perpetuated because students did not or could not use symbolic approaches to inform the answers produced by the technology. This in turn indicated that the visual imagery produced by the graphical calculator alone might not be sufficient to strengthen students' understanding of functions. This study found that dependency occurred much less when students were working collaboratively.

9.1.3 Graphical Calculators and Visualisation

As the study developed, it became increasingly apparent that the graphical calculator played an important part in furthering students' understanding of functions through encouraging the use of visual approaches and strengthening the students' powers of visualisation. In particular, the graphical calculator provided a powerful support mechanism for those who had difficulty in visualising concepts. For example in Phase 3, Perry and Roy, who both classified themselves as 'visualisers' had obvious difficulties in visualising the effects of particular transformations for themselves, without the aid of the technology (episode 5, chapter 8). However, through the use of the graphical calculator they were each able to see the relationships between the graphs unfolding dynamically and thus to reorganise and adapt their own visual thinking. This allowed them to gain a more meaningful comprehension of the visual effects of transformations and made the connections between the symbolic and visual representations of the functions more explicit. The scaffolding role afforded by the graphical calculator in this respect was seen as a means of improving levels of student competence, especially in areas that they found difficult to visualise.

More significantly, however, the graphical calculator was found to mediate the development of students' visual capacities via more intensive student-student and student-teacher interaction. Whenever there appeared to be disagreement amongst the students as to the form that the graph of a particular function would take, this would spark a lively debate of possible solutions and the exchanging of ideas which was heightened through the use of the technology. This was particularly evident in Phase 3 (episode 5, chapter 8). The graphical calculator allowed students to communicate their ideas to one another using visual as opposed to more accepted and generally used symbolic reasoning. Prior to the use of the graphical calculators, students found that sharing ideas with others was easier using symbolic rather than visual arguments. By using the technology, however, the students were able to construct their ideas around the visual imagery being displayed and to discuss these more easily with others using the technology to back up their arguments. This led to very productive instances where students' discussions centred around the images visible on screen and enabled them to develop a visual understanding of the problems together first before applying symbolic methods of solution. This study thus highlights the importance of classroom interaction in the development of students' visualisation skills with graphical calculators.

9.2 Affective and Social Factors which Contribute Towards Students' Learning of Functions with Graphical Calculators

This study has highlighted various affective and social issues arising from the use of the graphical calculators that have had a strong impact on students' learning of functions. This section outlines these findings in terms of:

- graphical calculators, attitudes and confidence
- graphical calculators and effective collaboration
- the role of the teacher in scaffolding students' understanding of functions in a graphical calculator environment

9.2.1 Graphical Calculators, Attitudes and Confidence

The use of graphical calculator was found to be a means by which students' views surrounding the validity of visual methods in mathematics and beliefs about the nature of mathematics could be challenged. This was especially apparent during Phase 2 of the research, in which symbolic methods had been privileged over visual approaches in the existing classroom culture. For example, Robert strongly believed at the beginning of the trial that the symbolic mode of representation was of paramount importance in students' understanding of functions and was only prompted to use visual approaches as a last resort, ignoring any advice to the contrary given to him by his teachers. However, at the end of the trial whilst his overall view remained the same, he recognised the benefits afforded by using visual approaches and claimed that he would use them much more in future. It was also clear that the graphical calculator has the potential for increasing levels of motivation amongst the students by encouraging them to take more of an interest in actively creating their mathematics.

Throughout the course of the study, the graphical calculator was seen to be instrumental in improving levels of student confidence surrounding functions. This was especially so where the students concerned were initially reluctant to use visual approaches, believing the visual mode of representation to be inaccurate and/or lacking confidence in their own visual abilities. In contrast to expectations, the majority of the students who classified themselves as visualisers at the beginning of Phase 2 and 3 lacked confidence in visual representations. This was especially surprising in Phase 2 when graphical calculators were already part of the classroom culture and reflected the higher status that was given to symbolic reasoning in this classroom. However, during all three phases, use of the graphical calculator led towards improved student confidence in many different ways. In particular students became more confident in their ability to visualise functions, in their symbolic solutions, in their own graphing skills, in sharing and discussing ideas with peers and the teacher and in presenting ideas to the whole class. The ability of students to verify, modify or reject their ideas and strategies visually using the graphical calculator allowed them to not only become more confident in their chosen methods and solutions, but also to develop a sense of ownership over their mathematics. More pertinently, even when the graphical calculator was not being used all of the time by the students it still inspired confidence, which was an important affective consideration. This was because the feedback provided by the technology enabled the students to immediately develop some sense of whether their answers were right or if they were wrong, in which case they could use the technology to help them to see the solution to the problem more clearly.

9.2.2 Graphical Calculators and Effective Collaboration

The findings of this study have highlighted the great potential for use of the graphical calculator to promote and scaffold discussion and collaboration amongst the students and with the teacher. The framework for analysing collaboration developed by Teasley and Roschelle (1993) provided a useful starting point for developing a picture of how the graphical calculator functioned in terms of the collaborative activity that took place amongst the students themselves and with the teacher. Through the use of the graphical calculator, the students from each phase were able to produce shared visual representations of the problems they were exploring, which they then used to create joint problem spaces and to reinforce and extend one another's arguments. This created the opportunity for effective collaboration and prevented breakdowns in communication. This aspect was well illustrated in episode 2 of chapter 6 from Phase 2, in which Robert used the technology in an attempt to clarify his thinking when his ideas differed from those being offered by Martin and Julie. Interaction with peers and the teacher was seen to be essential for individual students to be able to use the technology effectively.

The graphical calculator represented another source of authority in the classroom, which provided rapid feedback to the students and was consequently a catalyst in the on-going process of furthering their thinking. For example in Phase 2, during Robert, Martin and Julie's discussion surrounding the graph of $y = \cos(x - \pi/2)$, the graphical calculator provided a means by which two seemingly contradictory symbolic representations for the function could be explored (episode 2, chapter 6). Robert had identified the graph as being that of the function $y = \sin x$, whilst Martin and Julie had proposed that this was $y = \cos(x - \pi/2)$. This prompted Robert to use the graphical calculator in an attempt to rectify the apparent contradiction. He only began to accept the arguments being proposed by Martin and Julie when the graphical calculator provided compelling evidence that they were correct. This allowed all the students to realise that this was an instance where the problem had two equally valid solutions and drew attention to relationships between the sine and cosine functions that they had not explored previously. Borba (1996) also refers to the role of the graphical calculator as a new 'authority' in the classroom, which he sees as a feature that contributes towards the intensification of discussion and thus reorganisation of activity. In this respect the findings to emerge from this research have parallels with those which have arisen from Borba's study.

Previous research (e.g. Smart, 1992, Doerr and Zangor, 2000) has also shown that the graphical calculator could have a beneficial effect on collaboration. However, this study has sought to exploit and extend this research. It did so by investigating collaboration between 'visualisers' and 'non-visualisers'. The results show that the pairing of visualisers and non-visualisers was extremely beneficial to each type of student in terms of the collaboration that took place. This was particularly evident in small groups in which these students each tutored one another in the areas that they were most comfortable with. The visualisers were able to provide the non-visualisers with additional support with respect to their understanding of graphical approaches. For example, in the example discussed above, Julie and Martin, who are both visualisers, were able to provide Robert, a non-visualiser, with visually based explanations as to why the graph they were investigating could be represented symbolically by $y = \cos (x - \pi/2)$. Similarly, the non-visualisers scaffolded the visualisers' symbolic approaches, as was the case in Phase 3 when Mick, a non-visualiser, pointed out the mistake in Perry's algebraic manipulations that he had entered into the graphical calculator, thus unknowingly producing the wrong graph. This in turn enabled Perry, a visualiser, to rectify his error, graph the function correctly and proceed towards the right solution using the technology (episode 2, chapter 8).

The third phase of this research also found that whole class discussions were consistently led and maintained by visualisers. The non-visualisers appeared to be reluctant to contribute towards these discussions which was in complete contrast to their behaviour whilst working in small groups with visualisers. It was found that by continuing the practice of allowing visualisers and non-visualisers to work together in small groups, the confidence of the non-visualisers improved. For example, Kirk, a non-visualiser from Phase 3, never contributed towards a whole class discussion. However, when working closely with the technology with Julian and Jake (both visualisers), he was encouraged to make a significant proportion of the contributions, which proved to be instrumental in the progress that was made by the group as a whole (episodes 1 and 6, chapter 8). In the small group setting, non-visualisers were even able to tutor visualisers in visual concepts through their shared use of the technology. For example, Mick a non-visualiser was able to use the technology effectively to convince Perry and Marvin, both visualisers, of the validity of his graphical arguments when discussing the relative forms of the sine and cosine functions and the relationships between them (episode 5, chapter 8). This was especially significant considering Mick's prior reluctance to use graphical approaches (e.g. "I don't like graphs. I try to stay clear of them as much as possible" and "I find it hard to visualise shapes and graphs"). The graphical calculator played an important part in encouraging students, such as Mick, to take on the role of peer tutors and in scaffolding this role.

9.2.3 The Role of the Teacher in Scaffolding Students' Understanding of Functions in a Graphical Calculator Environment

Throughout the study the teacher-researcher assumed a crucial role in fostering the development of student understanding through technology by providing scaffolding for student learning, which resonates with the findings of Bruner (1985). According to Bruner (ibid), through scaffolding activities, the teacher facilitates the process of internalisation, which in the Vygotskian sense allows the students to operate successfully within their zones of proximal development. Indeed, as the study progressed, the analysis pointed to the centrality of the teacher's role in initiating, maintaining and encouraging discussion between the students, especially in relation to the results produced by the graphical calculators and in providing additional verification of these results and the students' assertions. In cases where the students were unable to reach a common understanding of their findings by themselves, the teacher's input was crucial in providing clarity and explanation of the results of their exploration with the technology. This was especially evident in Phase 3, where Marie and Claire's joint investigation using the graphical calculator had not enabled Marie to make sense of the effects of the transformation $f(x - \pi/2)$ on the function $y = \cos x$ (episode 4, chapter 8). This emphasised the need for the teacher to mediate how the technology is introduced and the way it is then used by students and to monitor interactions between students, intervening when appropriate. In this way, the teacher can ensure that the technology is used effectively, results are interpreted correctly, the correct mathematical language is being used and can address any misunderstanding that may develop. Furthermore, as the use of the graphical calculator made the students' understanding or lack of it more visible, the type of teacher intervention required became clearer.

9.3 Discussion

9.3.1 Cognitive Factors

The findings of this study have illustrated the potential of the graphical calculator for helping individual students to develop their powers of visualisation and also to become more autonomous and independent learners. The graphical calculator contributed towards the depth of students' understanding of functions through the actions of the amplification and cognitive reorganisation effects of the technology. The ability of students to use the graphical calculator as an authoritative means of testing conjectures and making predictions provided scaffolding for the students' visual understanding of concepts. Through the removal of cumbersome and time-consuming procedural elements, the graphical calculator enabled the students to focus on the mathematics rather than the technical aspects. The graphical calculator allowed the students to begin to build a picture of the relationships between the graphs of related functions by quickly and easily providing lots of examples of functions to the students, especially those from Phase 3, who were learning about new function concepts throughout the trial. It also enabled students to overcome difficulties, transfer knowledge between different contexts and explore more challenging material. In this way the graphical calculator acted as a cognitive reorganiser and totally transformed the learning in the classroom.

At the same time, the research has also considered the pitfalls that can be associated with autonomous graphical calculator use. In particular it has been found that whilst there are considerable benefits to be gained in allowing students to experiment freely with technology, often the input of the teacher is required to prevent or repair the misinterpretation of results. This was found to be necessary at both the individual and group level. For example, when discussing the rationale for her solution to question 13 from the main exercises, Marie from Phase 3 was only able to make sense of the actions of the transformation $f(x - 90^0)$ on the cosine function, through discussion with myself (episode 4, chapter 8). Similarly, when Diana, Jan, Guy and Lea used the technology together inappropriately to try to find a stationary point of the function $y = 2(|x| - 1)/3$ in Phase 1, a more knowledgeable person in the Vygotskian sense was required to explain what had gone wrong. The production of an approximate value ($2.16x10^{-14}$) for the *x* co-ordinate of this non-existent minimum stationary point rather than an error message by the technology caused much confusion for the students that was only allayed through discussion with myself and the classroom teacher. As these and many other examples illustrated, the teacher was seen to play a crucial part in enabling the students to appropriate meaning from their explorations with the graphical calculator.

The individual's role in constructing meaning was clearly related to the social environment and affective factors in addition to their relative mathematical knowledge. Interaction amongst peers was seen to be an important constituent in the development of individual students' thinking. All three phases of the research provided rich examples of students using the graphical calculators collaboratively, in which understanding was developed through the ability of students to compare and contrast their visual perceptions and justifications with one another, through the use of the graphical calculators. Moreover, the cognitive reorganisation effects of the technology were sometimes manifested through one student showing another how they solved a particular problem in a different way with the technology, which challenged their initial conceptions, as well as being brought about by the teacher's input. The use of the graphical calculator was also seen to have an impact on the students' feelings, attitudes and beliefs, which led towards the creation of a more positive learning environment, where students confidently and enthusiastically explored mathematical problems.

9.3.2 Affective and Social Factors

This study has also highlighted a complex web of interrelated factors, both individual and collective, that have direct bearing on students' meaning making with graphical calculators. Affective and social considerations constituted a central part of the mathematics teaching and learning that took place, which resonates with the findings of McLeod (1992), who calls for more research on the relationship between these factors and the higher order cognitive processes of learners and teachers. The students' performances in each phase were related to their feelings and to social influences as well as to individual ability. For example, part of Marvin's inability to progress further with the second interview question in Phase 3 (episode 4, chapter 7) lay in his lack of self confidence and belief that only students of higher mathematical ability would be able to make the connections between different modes of representation.

The combined use of the graphical calculator, the creation of local communities of practice, the introduction of a new topic, new ways of working (especially visual approaches) and a new classroom teacher was seen to have an extremely positive impact on the students' attitudes, beliefs, emotions, motivation and confidence. All the students in this study made very positive comments about how the use of the graphical calculator, in particular, had been beneficial to them in one or more of these areas. Marie (Phase 3) recognised that the graphical calculator had a dual role in "making the lessons more interesting as well as helping understanding". In the local communities of practice that were established, emphasis on public recognition of competence ensured that students were given additional confidence in their strategies and solutions. In addition, students were encouraged to share their arguments freely with other group members, regardless of whether they were technically right or wrong, so that all the contributions made were valid and ideas that turned out to be incorrect were seen merely as a means of furthering the discussion. Consequently, this removed some of the anxiety associated with publicly making a mistake and created a very positive and productive atmosphere in the classroom, where students persevered with difficult problems and became excited about doing mathematics, thus stimulating the learning process. May and Sue's graphical exploration of the function $y = (x + 3)^3$ together using the graphical calculator in Phase 1 illustrated that the technology had an impact on their enthusiasm and perseverance in finding a solution, which they had failed to obtain symbolically. Episode 1, chapter 7 from the third phase also shows that the graphical calculator had a similar affect on Perry.

As proposed by Newman et al (cited in McLeod, 1992), cognitive change appeared to be as much a social as an individual process and was clearly a product of the social interactions and the social context in addition to independent thinking about problems. Furthermore, language played an important part in the development of students' understanding of functions, which fits with a Vygotskian perspective, in which thought is seen to be constituted through the internalisation of social communication (Vygotsky, 1981). Through their interaction with the teacher, peers and the technology, individual students were able to develop a more meaningful understanding of functions than they held previously. Together students were seen to overcome their individual difficulties and to create shared knowledge using the technology. For example, in Phase 3 (episode 5, chapter 8) Mick was able to help Perry to overcome the difficulty that he was having in visualising the graphs of translated functions and to convince him of the validity of his argument, through their shared use of the technology.

9.3.3 Acknowledging the Complexity of Learning about Functions with Graphical Calculators in Pedagogical Practices

The findings of this study have highlighted a number of important pedagogical considerations in relation to the use of technology in promoting students' learning of functions. Firstly, the students' tendency to neglect the symbolic form of representation whilst concentrating on the graphical in Phase 1, emphasised the need for teachers to present symbolic methods alongside visual approaches, rather than in isolation. In addition to this, there were also instances where students would avoid thinking visually for themselves when answering questions using the graphical calculators. This even occurred when the students were specifically asked to think about the possible forms certain function may take before using the graphical calculator as a means of confirming or disproving this. For example, very few students from Phase 1 actually took the time to think about what the graphs of the functions $6x^3 - 3$, $\sin x + \cos x$, e^{2x}, $\ln x$, $1/(x+5)$, $\sin (x + \pi/2)$ and $-(x-1)^2 + 3$ looked like and to sketch their ideas before using the graphical calculators as was requested. Consequently, it was found to be necessary to place emphasis on encouraging students to think visually before introducing the technology. In this way through careful structuring of activities and resources by the teacher, the situation in which students simply become proficient machine operators, no longer thinking for themselves and lacking understanding, can be avoided. This was most apparent in Phase 3, where groups of students such as Julian, Kirk and Jake were clearly seeking to explain and question the results produced by the technology (episodes 1 and 6, chapter 8).

As part of this carefully structured pedagogical approach the findings of the study suggest that it is important that the teacher takes account of:

(1) the type of technology being used and associated limitations,
(2) the timing of the introduction of the technology into the classroom,
(3) how the use of technology is combined with the use of other media in the classroom such as pencil and paper and oral communication,

(4) the role that he or she has in mediating the use of the technology.

Furthermore, this study has highlighted the benefits of using the technology to introduce the concept of functions to students. However, it also recognises that individual students have differing needs with respect to the timing of the introduction of the technology. In particular, a number of students from the final phase of the research indicated that they would have preferred to have been introduced to the technology following a period of time spent exploring the graphs of functions using pencil and paper. Overall, however, the evidence suggests that despite these particular students' feelings, use of the graphical calculators from the start, as an integral part of the teaching process of functions, might be more beneficial for the students in the long term. However, the use of pencil and paper techniques should not be discouraged. Several students in this study noted the advantages of using the graphical calculators to support their own graphing skills by hand. This occurred because they were asked to produce sketches by hand of all the graphs that they drew using the technology and occasionally to speculate over the likely shapes of graphs in small groups, producing quick sketches by hand, before they actually drew them using the graphical calculators. This encouraged them to think more about the way in which the graphs were produced and why they took on particular shapes.

This project deliberately set out to foster local communities of practice following Winbourne and Watson (1998). As a result, the creation of local communities of practice in the classroom, in which shared ways of working with the technology were developed, was found to be an important part of scaffolding the students' learning. In this type of supportive environment, the students shared ownership of their use of the technology and they and the teacher could build and maintain joint problem spaces, which led to graphical calculators being used to the greatest effect. Indeed, Teasley and Roschelle's (1993) notion of a joint problem space could be considered as a particular example of a local community of practice in action. Through the establishment of this particular type of local community of practice, the students in all phases of this study were able to establish an effective means of operating with graphical calculators, in which the knowledge generated was shared amongst the participants. For example, Julian and Kirk from Phase 3 were able to make sense of the connections between the symbolic and graphical representations of transformations through their efforts to construct and maintain a joint problem space (episode 1, chapter 8).

However, it was not essential for the students to use graphical calculators all of the time in order for learning to be successful in their community of practice. For example, where the students were able to visualise the effects of a particular transformation on the graph of a function effectively without the aid of technology, they did choose not to use the graphical calculator in that instance. However, the way in which the students operated whilst using the graphical calculators was seen to influence how they approached problems without use of the technology, as was also observed by Borba and Villarreal (1998). This was evident in episode 5, chapter 6, Phase 3 when Robert, Julie and Martin were each able to visualise the effects of the transformation f(3x) on the function y = sin x, without having to use the technology to confirm this. It was clear, however, that even though the graphical calculators were not used directly to solve this problem, the technology was having an impact on the way in which these students were thinking about the problem. Robert, in particular, was now actively looking for alternative symbolic forms for the graphed functions that he had to identify, following his discoveries about the first function through use of the graphical calculator.

A crucial part of the successful introduction of the students to the concept of functions lay in the way in which shared meaning for functions was created through a combination of whole class discussions and small group activity. Following a whole class introduction to, and discussion of, each new topic, students were encouraged to explore new concepts in small groups. Individual students were then invited to share their findings with the rest of the class. In this discussion ideas held by different groups formed the basis for creating new mathematical knowledge which were then formalised by the teacher. This approach to the teaching and learning of functions was seen to work particularly well in relation to the use of graphical calculators. Moreover, part of the success of this approach could be attributed to the use of the specially adapted graphical calculator that connected to the overhead projector as a means by which arguments could be demonstrated to the whole class. The overhead projector set-up acted as a focal point for discussion, providing an overview and a means of bringing the comments of individual students together. Another important contributing factor was the ability of students to graph these ideas for themselves whilst listening to and following the discussion because they each had been given a graphical calculator. This interactive process would be difficult to achieve using computers, unless every student could be provided with a desktop computer.

In investigating social factors, this study also highlighted the importance of language in students' success with using the graphical calculators to further their understanding of functions. In cases where students were able to collaborate effectively using the technology, they appeared to be using 'communicative speech', rather than 'egocentric speech' (Vygotsky, 1962), and were intentionally trying to convey their ideas to one another with the aim of furthering the group's collective understanding. For example, in episode 6, chapter 8 from Phase 3, Kirk, Julian and Jake each made a concerted attempt to explain their ideas to one another which played an important part in constituting the meaning that the students derived from this question as a group. Sometimes the language used by the students would lack mathematical rigour. For example Julian, Kirk and Jake had a tendency to use natural, rather than more formal mathematical language when discussing the results of their explorations with the technology. However, phrases such as "it's one of those *daft* functions", "the *humps* are in a different place" that were used by Julian were clearly laden with meaning for Kirk and Jake as they had arisen out of the group's joint exploration. Subsequently, these phrases were immediately accepted and used as part of the group's repertoire. In contrast to the findings of Smart (1995a), this study thus found that use of the graphical calculator alone did not necessarily result in students being able to talk more formally about their mathematics.

Figure 9.1 summarises the complexity of learning about functions with graphical calculators in relation to the associated implications for the teacher.

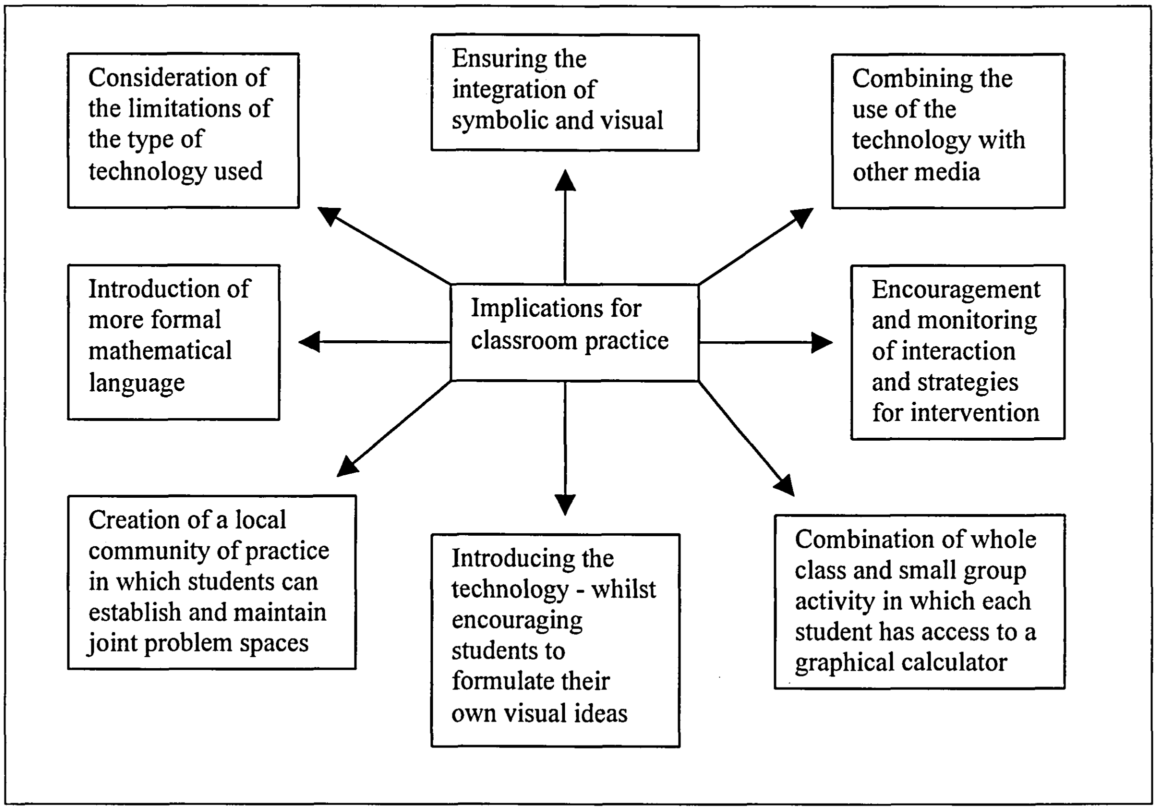


Figure 9.1 Implications of the study for the teacher

9.3.4 Implications for Further Research

This study has identified an interconnected set of factors that contribute towards students' acquisition of meaning for functions with graphical calculators in Advanced level mathematics. One of the major implications to arise is that it is important that future research takes into account the social context of the learning environment and the shared experiences and interactions between students and teacher. The graphical calculator was not seen to be a tool that had an independent existence in the classroom, rather its role was created and negotiated by the students and the teacher. As such, future work that adopts a holistic position with respect to the relationship between social communication, affect and cognition is seen to have the potential for making a substantial contribution towards understanding how students learn about different areas of mathematics with technology.

Another important implication to be drawn from this study lies in the huge potential for the graphical calculator to be used as a tool for enhancing the learning experience of students. This is especially significant with respect to the current climate in mathematics education, in which calls have been made to limit the use of technology and A level syllabuses have placed restrictions on the use of graphical calculators in examinations. This study has pointed towards the type of learning environment in which this potential can be realised, through supporting collaborative learning with technology. In developing this picture further, future work could focus more on the type of language used by teachers to enculturate students into meaning for functions and how this in turn can provide scaffolding for their cognitive development and confidence.

The analysis of the data in this study has concentrated on the way in which students come to assign meaning for functions using the graphical calculators through their shared experiences, social interactions, individual actions and mediation by the teacher. There has been no direct comparison made of how and why the images held by individual students might differ from those produced by the technology. A fruitful question for future research lies in how the external images produced by the technology correspond to and mediate the students' and teachers' internal visual representations, and how this affects the process of internalisation.

A further area for subsequent research might be to investigate the potential of the graphical calculator for mediating the learning of younger students. The theoretical framework and research methodology that has been developed in this study would provide a suitable basis for shedding light on the very early stages of the process of appropriation of meaning for function concepts using technology. This would allow contrasts to be made with the way in which meaning making was seen to occur with the older, more experienced students of this study, to provide insight into how learning develops with technology, especially in relation to the amplification and cognitive reorganisation effects.

REFERENCES

Adler, J. (1998). Resources as a Verb: Reconceptualising Resources In and For School Mathematics. *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, Stellenbosch, South Africa, 1, pp. 1-18.

Ball, S. J. (1993). Self-doubt and Soft Data: Social and Technical Trajectories in Ethnographic Fieldwork. In Hammersley, M. (Ed.) *Educational Research: Current Issues*, The Open University/Paul Chapman Publishing, pp. 32-48.

Barwise, J. and Etchemendy, J. (1991). Visual Information and Valid Reasoning. In Zimmermann, W. and Cunningham, S. (Eds.) *Visualization in Teaching and Learning Mathematics*, Providence, RI: MAA Notes Series, 19, pp. 9-24.

Bassey, M. (1995). *Creating Education through Research. A Global Perspective of Educational Research for the 21st Century.* Kirklington Moor Press / BERA.

Berger, M. (1998). Graphical Calculators: an Interpretative Framework. *For the Learning of Mathematics*, 18, 2 (June), pp. 13-20.

Bishop, A. J. (1989). Review of Research on Visualisation in Mathematics Education. *Focus on Learning Problems in Mathematics*, 11, No 1, pp. 7-15.

Bloom, L. M., Comber, G. A. and Cross, J. M. (1986). Use of the Microcomputer to Teach the Transformational Approach to Graphing Functions. *International Journal of Mathematical Education in Science and Technology*, 17, No 1, pp. 115-123.

Borba, M. C. (1996). Graphing Calculators, Functions and Reorganisation of the Classroom. *The Role of Technology in the Mathematics Classroom. Proceedings of Working Group 16 at ICME-8, the 8th International Congress on Mathematics Education,* Seville, Spain, pp. 53-60.

Borba, M. C. and Villarreal, M. E. (1998). Graphing Calculators and Reorganisation of Thinking: The Transition from Functions to Derivative. *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education,* Stellenbosch, South Africa, 2, pp. 136-143.

Breen, C. (1997). Exploring Imagery in P, M and E. *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, Lahti, Finland, 2, pp. 97-104.

Bruner, J. (1985). Vygotsky: a Historical and Conceptual Perspective. In Wertsch, J. V. (Ed.), *Culture, Communication and Cognition.* Cambridge University Press, pp. 21-34.

Carulla, C. and Gomez, P. (1996). Graphic Calculators and Precalculus. Effects on Curriculum Design. *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, Valencia, Spain, 1, p. 161.

Carulla, C. and Gomez, P. (1997). Graphics Calculators and Problem Solving. Do They Help? *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, Lahti, Finland, 1, p. 224.

Chola Nyondo, A. (1993). Mathematics Courses with a PC. *International Journal of Mathematical Education in Science and Technology*, 24, No 4, pp. 569-574.

Chronaki, A. and Kynigos, C. (1999). Teachers' Views on Pupil Collaboration in Computer Based Groupwork Settings in the Classroom. *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education*, Haifa, Israel, 2, pp. 209-216.

Cohen, L. and Manion, L. (1994). *Research Methods in Education*. Fourth Edition. Croom Helm London.

Confrey, J. (1993). The Role of Technology in Reconceptualizing Functions and Algebra. *Proceedings of the 17th Conference of the International for the Group for the Psychology of Mathematics Education*, Asilomar, California, 1, pp. 47-74.

Confrey, J. (1994). Six Approaches to Transformation of Functions Using Multi-Representational Software. *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, Lisbon, Portugal, 2, pp. 217-224.

Confrey, J. and Smith, E. (1992). Revised Accounts of the Function Concept Using Multi-Representational Software, Contextual Problems and Student Paths. *Proceedings of the 16th Conference of the International for the Group for the Psychology of Mathematics Education*, Durham, NH, USA, 1, pp. 153-160.

Cunningham, S. (1991). The Visualization Environment for Mathematics Education. In Zimmermann, W. and Cunningham, S. (Eds.) *Visualization in Teaching and Learning Mathematics*, Providence, RI: MAA Notes Series, 19, pp. 67 -76.

Cunningham, S. (1994). Some Strategies for Using Visualisation in Mathematics Teaching. *Zentralblatt fur Didaktik der Mathematik*, ZDM, 94/3, pp. 83-85.

Cuoco, A. A. (1991). Computational Media to Support the Learning and Use of Functions. In DiSessa, A. A., Hoyles, C. and Noss, R. (Eds.) *Computers and Exploratory Learning*, NATO ASI Series, pp. 79-107.

Davis, P. J. (1993). Visual Theorems. *Educational Studies in Mathematics*, 24, pp. 333-344.

Denzin, N. K. (1988). Triangulation. In Keeves, J. P. (Ed.) *Educational Research, Methodology and Measurement: An International Handbook*, pp. 511-513.

Diaz, R. M., Neal, C. J. and Amaya-Williams, M. (1990). The Social Origins of Self-Regulation. In Moll, L. C. (Ed.) *Vygotsky and Education: Instructional Implications and Applications of Sociohistorical Psychology*. Cambridge University Press, pp. 127-154.

Doerr, H. M. and Zangor, R. (1999). Creating a Tool: An Analysis of the Role of the Graphing Calculator in a Pre-Calculus Classroom. *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education*, Haifa, Israel, 2, pp. 265-272.

Doerr, H. M. and Zangor, R. (2000). Creating Meaning for and with the Graphing Calculator. *Educational Studies in Mathematics*, 41, pp. 143-163.

Dunham, P. H. and Dick, T. P. (1994). Research on Graphing Calculators. *The Mathematics Teacher*, 87, No. 6, September, pp. 440-445.

Eisenberg, T. and Dreyfus, T. (1986). On Visual Versus Analytical Thinking in Mathematics. *Proceedings of the 10th Conference of the International Group for the Psychology of Mathematics Education*, London, pp. 153-158.

Eisenberg, T. and Dreyfus, T. (1987). On The Deep Structure of Functions. *Proceedings of the 11th Conference of the International Group for the Psychology of Mathematics Education*, Montreal, Canada, 1, pp. 190-196.

Eisenberg, T. and Dreyfus, T. (1990). On Difficulties with Diagrams: Theoretical Issues. *Proceedings of the 14th Conference of the International Group for the Psychology of Mathematics Education*, Mexico, 1, pp. 27-34.

Eisenberg, T. and Dreyfus, T. (1991). On the Reluctance to Visualize in Mathematics. In Zimmermann, W. and Cunningham, S. (Eds.) *Visualization in Teaching and Learning Mathematics*, Providence, RI: MAA Notes Series, 19, pp. 25-37.

Eisenhart, M. A. (1988). The Ethnographic Research Tradition and Mathematics Education Research. *Journal for Research in Mathematics Education*, 19, No. 2, pp. 99-114.

Ernest, P. (1994). What is Social Constructivism in the Psychology of Mathematics Education? *Proceedings of the 18th Conference of the International for the Group for the Psychology of Mathematics Education*, Lisbon, Portugal, 2, pp. 304-311.

Fey, J. T. (1989). Technology and Mathematics Education: A Survey of Recent Developments and Important Problems. *Educational Studies in Mathematics*, 20, No. 3, August, pp. 237-272.

Furinghetti, F. and Bottino, R. M. (1996). Teachers' Behaviour in Teaching with Computers. *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, Valencia, Spain, 2, pp. 129-136.

Gage, J. (1999). Shifts in Confidence. The Graphical Calculator as a Space in which to do Mathematics. *Micromath*. Summer, 1999, pp. 13-17.

Goetz, J. P. and LeCompte, M. D. (1984). *Ethnography and Qualitative Design in Educational Research*. Academic Press.

Goforth, D. (1992). Visualisation and Computer Assisted Learning: The Role of Videodiscs. *British Journal of Educational Technology*, 23, No1, pp. 21-27.

Goldenberg, E. P. (1987). Believing is Seeing: How Preconceptions Influence the Perception of Graphs. *Proceedings of the 11th Conference of the International Group for the Psychology of Mathematics Education*, Montreal, Canada, 1, pp. 197-203.

Goldenberg, E. P. (1991). The Difference between Graphing Software and Educational Graphing Software. In Zimmermann, W. and Cunningham, S. (Eds.) *Visualization in Teaching and Learning Mathematics*, Providence, RI: MAA Notes Series, 19, pp. 77- 86.

Gomez, P. and Fernandez, F. (1997). Graphics Calculators Use in Precalculus and Achievement in Calculus. *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, Lahti, Finland, 3, pp. 1-8.

Guin, D. and Trouche, L. (1999). The Complex Process of Converting Tools into Mathematical Instruments: The Case of Calculators. *International Journal of Computers for Mathematical Learning*, 3, pp. 195-227.

Gutierrez, A. (1996). Visualization in 3-Dimensional Geometry: In Search of a Framework. *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, Valencia, Spain, 1, pp. 3-19.

Guttenberger, E. (1992). The Learning of Trigonometric Functions in a Graphical Computer Environment. *Proceedings of the 16th Conference of the International Group for the Psychology of Mathematics Education*, Durham, NH, USA, 3, pp. 106-113.

Hammersley, M. and Atkinson, P. (1983). *Ethnography Principles in Practice*. Cambridge University Press.

Hershkowitz, R. (1999). Where in Shared Knowledge is the Individual Knowledge Hidden? Plenary Address. *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education*, Haifa, Israel, 1, pp. 1-9.

Hershkowitz, R. and Schwarz, B. B. (1997). Unifying Cognitive and Socio-Cultural Aspects in Research on Learning the Function Concept. *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, Lahti, Finland, 1, pp. 148-164.

Hewitt, D. (1992). Train Spotters' Paradise. *Mathematics Teaching*, 140, September, pp.6-8.

Hitt Espinosa, F. (1997). Researching a Problem of Convergence with Mathematica: History and Visualisation of a Mathematical Idea. *International Journal of Mathematical Education in Science and Technology*, 28, No5, pp. 697-706.

Hollar, J. C. and Norwood, K. (1999). The Effects of a Graphing-Approach Intermediate Algebra Curriculum on Students' Understanding of Function. *Journal for Research in Mathematics Education*, 30, No. 2, pp. 220-226.

Hoyles, C. and Healy, L. (1996). Visualisation, Computers and Learning. *Final Report to the Economic and Social Research Council.* Institute of Education, University of London.

Hoyles, C., Healy, L. and Sunderland, R. (1991). Patterns of Discussion between Pupils Pairs in Computer and Non-Computer Environments. *Journal of Computer Assisted Learning*, 7, pp.210-228.

Hoyles, C. and Noss, R. (1992). A Pedagogy for Mathematical Microworlds, *Educational Studies in Mathematics*, 23, pp. 31-57.

Hudson, B. (1996). World of Number: A Critique, an Example of Classroom Use and Some Suggestions for Future Development. *Teaching Mathematics and its Applications*, 15, No 2, pp. 56-66.

Hudson, B. (1997). Group Work with Multimedia in Mathematics: the Role of the Technology and of the Teacher. *British Journal of Educational Technology*, 28, No 4, pp. 257-270.

Hughes Hallett, D. (1991). Visualization and Calculus Reform. In Zimmermann, W. and Cunningham, S. (Eds.) *Visualization in Teaching and Learning Mathematics, Providence*, RI: MAA Notes Series, 19, pp. 121-126.

Jones, A. and Mercer, N. (1993). Theories of Learning and Information Technology. In Scrimshaw, P. (Ed.) *Language, Classrooms and Computers*. Routledge, pp. 11- 26.

Kaput, J. J. (1992). Technology and Mathematics Education. In Grouws, D. A. (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, pp. 515-556.

Knuth, E. J. (2000). Understanding Connections between Equations and Graphs. *The Mathematics Teacher*, 93, No. 1, pp. 48-53.

Kynigos, C. and Argyris, M. (1999). Two Teachers' Beliefs and Practices with Computer Based Exploratory Mathematics in the Classroom. *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education*, Haifa, Israel, 3, pp.177-184.

Laridon, P. E. (1996). Knowledge of Transformations of Functions: Point-Plotting as Opposed to Use of the Graphics Calculator. *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, Valencia, Spain, 1, p 182.

Lean, G. and Clements, K. (1981). Spatial Ability, Visual Imagery, and Mathematical Performance. *Educational Studies in Mathematics*, 12, pp. 267-299.

Leinhardt, G., Zaslavsky, O. and Stein, M. K. (1990). Functions, Graphs, and Graphing: Tasks, Learning and Teaching. *Review of Educational Research*, 60, No. 1, Spring, pp. 1-64.

Lerman, S. (1989). Constructivism, Mathematics and Mathematics Education. *Educational Studies in Mathematics*, 20, pp. 211-223.

Lerman, S. (1992). The Function of Language in Radical Constructivism: A Vygotskian Perspective. *Proceedings of the 16th Conference of the International Group for the Psychology of Mathematics Education*, Durham, NH, USA, 2, pp. 40-47.

Lerman, S. (1994). Metaphors for Mind and Metaphors for Teaching and Learning Mathematics. *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, Lisbon, Portugal, 3, pp. 144-151.

Lerman, S. (1996). Intersubjectivity in Mathematics Learning: a Challenge to the Radical Constructivist Paradigm? *Journal for Research in Mathematics Education*, 27, 2, pp. 133-150.

Lerman, S. (1998). A Moment in the Zoom Lens: Towards a Discursive Psychology of Mathematics Teaching and Learning. *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, Stellenbosch, South Africa, 1, pp. 66-81.

Mannall, G. and Kenwood, M (1994). *Pure Mathematics 1.* Heinemann Modular Mathematics for London AS and A Level, pp. 35-50.

Mariotti, M. A. and Pesci. A. (1994). Visualization in Teaching - Learning Situations. *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, Lisbon, Portugal, 1, p 22.

Martin, L., Pirie, S. E. B. and Kieren, T. E. (1994). Mathematical Images for Fractions: Help or Hindrance? *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, Lisbon, Portugal, 3, pp. 247-254.

Mason, J. (1992). Towards a Research Programme for Mental Imagery. *Proceedings of the November Conference of the British Society for Research into Learning Mathematics*, pp. 24-29.

Mason, J. (1995a). What, Why and How in Mathematics. *Micromath*, 11, 1, pp. 10-14.

Mason, J. (1995b). Less may be more on a screen. In Burton, L. and Jaworski, B. (Eds.) *Technology in Mathematics Teaching: A Bridge between Teaching and Learning*, pp. 119-134.

McLeod, D. B. (1992). Research on Affect in Mathematics Education: A Reconceptualization. In Grouws, D. A. (Ed.) *Handbook of Research on Mathematics Teaching and Learning*, pp. 575-596.

Meissner, H. and Mueller-Philipp, S. (1993). Teaching Functions. *Proceedings of the 17th Conference of the International Group for the Psychology of Mathematics Education*, Ibaraki, Japan, 2, pp. 89-96.

Mercer, N. and Scrimshaw, P. (1993). Researching the Electronic Classroom. In Scrimshaw, P. (Ed.) *Language, Classrooms and Computers*. Routledge, pp. 184-191.

Mesa, V-M. (1997). The Use of The Graphing Calculator in Solving Problems on Functions. *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, Lahti, Finland, 3, pp. 240-247.

Moll, L. C. (1990). Introduction. In Moll, L. C. (Ed.) *Vygotsky and Education: Instructional Implications and Applications of Sociohistorical Psychology*. Cambridge University Press, pp. 1-27.

Nemirovsky, R and Noble, T. (1997). On Mathematical Visualisation and the Place Where We Live. *Educational Studies in Mathematics*, 33, pp. 99-131.

Noss, R., Healy, L. and Hoyles, C. (1997). The Construction of Mathematical Meanings: Connecting the Visual with the Symbolic. *Educational Studies in Mathematics*, 33, No. 2, pp. 203-233.

Noss, R. and Hoyles, C. (1996). *Windows on Mathematical Meaning: Learning Cultures and Computers*. Mathematics Education Library. Kluwer Academic Press.

Olive, J. (1992). Technology and School Mathematics. *International Journal of Educational Research*, 17, No 5, pp. 503-516.

O'Reilly, D., Pratt, D. and Winbourne, P. (1997). Constructive and Instructive Representation. *Journal of Information Technology for Teacher Education*, 6, No 1, pp 73-92.

Orhun, E. (1991). Design of Computer-Based Cognitive Tools. In DiSessa, A. A., Hoyles, C. and Noss, R. (Eds.) *Computers and Exploratory Learning*, NATO ASI Series, pp. 305-319.

Panofsky, C. P., John-Steiner, V. and Blackwell, P. J. (1990). The Development of Scientific Concepts and Discourse. In Moll, L. C. (Ed.) *Vygotsky and Education: Instructional Implications and Applications of Sociohistorical Psychology*. Cambridge University Press, pp. 251-267.

Pea, R. D. (1987). Cognitive Technologies for Mathematics Education. In Schoenfeld, A. H. (Ed.) *Cognitive Science and Mathematics Education*. Lawrence Erlbaum Associates, pp. 89-122.

Penglase, M. and Arnold, S. (1996). The Graphics Calculator in Mathematics Education: A Critical Review of Recent Research. *Mathematics Education Research Journal*, 8, No 1, pp. 58 -90.

Povey, H. (1995). Working for Change in Teacher Education. Some First Steps for Monday Morning. In Burton, L. and Jaworski, B. (Eds.) *Technology in Mathematics Teaching: A Bridge between Teaching and Learning*, pp. 135-151.

Povey, H. and Ransom, M. (2000). Some Undergraduate Students' Perceptions of Using Technology for Mathematics: Tales of Resistance. *International Journal of Computers for Mathematical Learning*, 5, pp. 47-63.

Pozzi, S., Healy, L. and Hoyles, C. (1993). Learning and Interaction in Groups with Computers: When do Ability and Gender Matter? *Social Development*, 2, 3, pp. 222-241.

Presmeg, N, C. (1985). *The Role of Visually Mediated Processes in High School Mathematics: A Classroom Investigation.* Ph.D. Thesis, University of Cambridge.

Presmeg, N. C. (1986). Visualisation and Mathematical Giftedness. *Educational Studies in Mathematics*, 17, pp. 297-311.

Presmeg, N. C. (1992). Prototypes, Metaphors, Metonymies and Imaginative Rationality in High School Mathematics. *Educational Studies in Mathematics*, 23, pp. 595-610.

Presmeg, N. C. (1995). Preference for Visual Methods: An International Study. *Proceedings of the 19th Conference of the International Group for the Psychology of Mathematics Education*, Recife, Brazil, 3, pp. 58-65.

Ruthven, K. (1990). The Influence of Graphic Calculator Use on Translation from Graphic to Symbolic Forms. *Educational Studies in Mathematics*, 21, pp. 431-450.

Ruthven, K. (1996). Calculators in the Mathematics Curriculum: The Scope of Personal Computational Technology. In Bishop, A. J., Clements, K., Keitel, C., Kilpatrick, J. and Labarde, C. (Eds.) *International Handbook of Mathematics Education*, pp. 435-468.

Selinger, M. and Pratt, D. (1997). Mediation of Mathematical Meaning Through the Graphical Calculator. *Journal of Information Technology for Teacher Education*, 6, No. 1, pp. 37-48.

Shoaf-Grubbs, M. M. (1995). Research Results on the Effect of the Graphical Calculator on Female Students' Cognitive Levels and Visual Thinking. In Burton, L. and Jaworski, B. (Eds.) *Technology in Mathematics Teaching: A Bridge between Teaching and Learning*, pp. 213-230.

Shoaf, M. M. (1996). The Long-term Effect of the Graphical Calculator on Female Students' Spatial Visualisation Skills and Level of Understanding in Elementary Graphing and Algebra Concepts. *The Role of Technology in the Mathematics Classroom. Proceedings of Working Group 16 at ICME-8, the 8th International Congress on Mathematics Education,* Seville, Spain, pp. 162-169.

Shumway, R. (1990). Super Calculators and the Curriculum. *For the Learning of Mathematics*, 10, 2 (June), pp. 2-9.

Sivasubramaniam, P. (2000). Distributed Cognition, Computers and the Interpretation of Graphs. In Rowland, T. and Morgan, C. (Eds.) *Research in Mathematics Education Volume 2, Papers of the British Society for Research into Learning Mathematics*, pp. 169-190.

Smart, T. (1992). Personal Technology in the Mathematics Classroom: Graphic Calculators and Equal Opportunities. *Proceedings of the November Conference of the British Society for Research into Learning Mathematics*, 1992, pp. 14-16.

Smart, T. (1995a). Visualising Quadratic Functions: A Study of Thirteen Year Old Girls Learning Mathematics with Graphic Calculators. *Proceedings of the 19th Conference of the International Group for the Psychology of Mathematics Education*, Recife, Brazil, 2, pp. 272-279.

Smart, T. (1995b). Visualisation, Confidence and Magic: The Role of Graphic Calculators. In Burton, L. and Jaworski, B. (Eds.) *Technology in Mathematics Teaching: A Bridge between Teaching and Learning*, pp. 195-212.

Solano, A. and Presmeg, N. C. (1995). Visualization as a Relation of Images. *Proceedings of the 19th Conference of the International Group for the Psychology of Mathematics Education*, Recife, Brazil, 3, pp. 66-73.

Souza, T. A. and Borba, M. C. (1995). Quadratic Functions and Graphing Calculator. *Proceedings of the 19th Conference of the International Group for the Psychology of Mathematics Education*, Recife, Brazil, 1, p 253.

Sutherland, R. (1991). Some Unanswered Research Questions on the Teaching and Learning of Algebra. *For the Learning of Mathematics*, 11, No. 3, November, pp. 40-46.

Sutherland, R. (1993). Connecting Theory and Practice: Results from the Teaching of Logo. *Educational Studies in Mathematics*, 24, pp. 95-113.

Sutherland, R. (1995). Algebraic Thinking: The Role of the Computer. In Burton, L. and Jaworski, B. (Eds.) *Technology in Mathematics Teaching: A Bridge between Teaching and Learning*, pp. 275-287.

Tall, D. (1986). *Building and Testing a Cognitive Approach to the Calculus Using Interactive Computer Graphics.* Ph.D. Thesis, Warwick University.

Tall, D. (1987). Whither Calculus? *Mathematics Teaching*, 118, March, pp. 50-54.

Tall, D. (1989). Concept Images, Generic Organizers, Computers, and Curriculum Change. *For the Learning of Mathematics, 9, No 3, November*, pp. 37-42.

Tall, D. (1991a). Visualizing Differentials in Integration to Picture the Fundamental Theorem of Calculus. *Mathematics Teaching*, 137, pp. 29-33.

Tall, D. (1991b). Intuition and Rigour: The Role of Visualization in the Calculus. In Zimmermann, W. and Cunningham, S. (Eds.) *Visualization in Teaching and Learning Mathematics*, Providence, RI: MAA Notes Series, 19, pp. 105-119.

Teasley, S. D. and Roschelle, J. (1993). Constructing a Joint Problem Space: The Computer as a Tool for Sharing Knowledge. In Lajoie, S. P. and Derry, S. J. (Eds.) *Computers as Cognitive Tools*. Lawrence Erlbaum, pp. 229-257.

Ubuz, B. (1994). Students' Understanding of Differentiation and Integration. *Proceedings of the November Conference of the British Society for Research into Learning Mathematics*, pp. 56-57.

Valero, P and Gomez, C. (1996). Precalculus and Graphic Calculators: The Influence on Teachers' Beliefs. *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, Valencia, Spain, 4, pp. 363-370.

Vygotsky, L. S. (1962). *Thought and Language*. MIT Press.

Vygotsky, L. S. (1978). *Mind in Society. The development of Higher Psychological Processes.* London: Harvard University Press.

Vygotsky, L. S. (1981). The Genesis of Higher Mental Functions. In J. V. Wertsch (Ed.) *The Concept of Activity in Soviet Psychology,* Armonk, NY: M. E. Sharpe, pp. 144-188.

Wazir, I. (1993). Functions: From Algebra to Calculus with a Graphing Utility. In Jaworski, B. (Ed.) *Technology in Mathematics Teaching A Bridge between Technology and Learning*. Conference Proceedings, University of Birmingham, 17-20 Sept, pp.475-490.

Wertsch, J. V. (1990). The Voice of Rationality in a Sociocultural Approach to Mind. In Moll, L. C. (Ed.) *Vygotsky and Education: Instructional Implications and Applications of Sociohistorical Psychology*. Cambridge University Press, pp. 111-126.

Wertsch, J. V. and Stone, C. A. (1985). The Concept of Internalisation in Vygotsky's Account of the Genesis of Higher Mental Functions. In Wertsch, J. V. (Ed.) *Culture, Communication and Cognition: Vygotskian Perspectives*. Cambridge University Press, pp. 162-179.

Wheatley, G. H. and Brown, D. (1994). The Construction and Re-presentation of Images in Mathematical Activity. *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, Lisbon, Portugal, 1, p 81.

Winbourne, P. and Watson, A. (1998). Participating in Learning Mathematics through Shared Local Practices. *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, Stellenbosch, South Africa, 4, pp. 177-184.

Woods, P. (1996). *Researching the Art of Teaching. Ethnography for Educational Use*. Routledge.

Zazkis, R., Dubinsky, E. and Dautermann, J. (1996). Coordinating Visual and Analytic Stategies: A Study of Students' Understanding of the Group D_4. *Journal for Research in Mathematics Education*, 27, No. 4, pp. 435-457.

Zimmermann, W. (1991). Visual Thinking in Calculus. In Zimmermann, W. and Cunningham, S. (Eds.) *Visualisation in Teaching and Learning Mathematics*, Providence, RI: MAA Notes Series, 19, pp.127-137.

Zimmermann, W. and Cunningham, S. (1991). What is Mathematical Visualisation? In Zimmermann, W. and Cunningham, S. (Eds.) *Visualisation in Teaching and Learning Mathematics*, Providence, RI: MAA Notes Series, 19, pp.1-8.

APPENDIX A

Phase 1 Data Collection

Contents

The Introductory Exercises

Use the TI-92 to draw the graphs of:

1. $y = 4x^2 - 4x + 1$ 2. $y = (x + 3)^3$ 3. $y = 2(|x| - 1) / 3$ 4. $y = \cos(x/2)$

In each case sketch the graph on paper and use the TI-92 to find the value(s) of x when y = 0. [Check these values by substituting y = 0 and solving in each case].

Also for questions 1 and 2 use the TI-92 to determine the nature of any stationary points and to ascertain their co-ordinates. [Check these values by differentiation].

Homework Questions

For each of the following functions find:

i) any values of x for which y = 0

ii) the nature and co-ordinates of any stationary points

5. $y = x - 1 + \dfrac{1}{x + 1}$ 6. $y = x^3(1 - x)$ 7. $y = \dfrac{x + 1}{(x + 2)^2}$

8. $y = \dfrac{1 + x^2}{1 + x + x^2}$ 9. $y = x^2 - x^3$ 10. $y = 1 - e^x$

11. $y = \dfrac{1}{1 - e^x}$ 12. $y = e^x \sin x$

The Main Trial Exercises

Graphing Functions Using the TI-92

1. Compare the graphs of cos x, 2cos x and 3cos x using the TI-92. Sketch the graphs and comment on the main features.

2. Repeat question 1 with tan x, 2tan x and 3tan x.

3. Compare the graphs of cos x, 2cos 2x, 3cos 3x and sketch them. Explain why these three graphs do not cross the x-axis in exactly the same places.

4. Use the TI-92 to perform the following sequence of transformations on the graph of $f(x) = x^3$: i). $f(x/2)$ ii). $f((x + 2) / 2)$ iii). $f((x+2) / 2) - 3$ iv). $2(f((x+2) / 2) - 3)$
In each case sketch the resulting graphs and write down the equation of the function, in its simplest form, i.e. $ax^3 + bx^2 + cx + d$.

5. Use the TI-92 to graph the following functions, taking a few moments before hand to try to picture what the graph will look like, and to sketch your ideas:

a). $y = 6x^3 - 3$ b). $y = \sin x + \cos x$ c). $y = e^{2x}$ d). $y = \ln (x/2)$

e). $y = \dfrac{1}{x + 5}$ f). $y = \sin (x + \pi/2)$ g). $y = -(x - 1)^2 + 3$

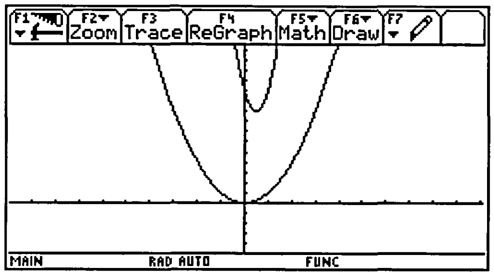
Sketch the graphs drawn by the TI-92 next to your initial thoughts of what the function might look like. If there are any discrepancies, can you explain these?

Explain how the shape of the graph of the function a) can be determined from the shape of the graph of $y = x^3$.

Answer the same question for c) - g), with e^x, ln x, 1/x, sin x and x^2.

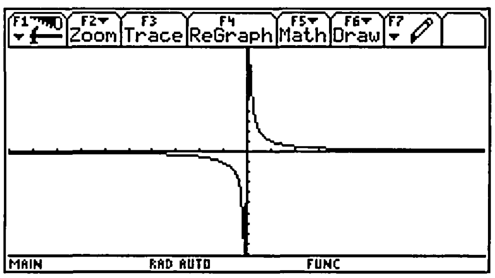
6. Investigate the relationship between the functions $y = 4x^2 + 5$ and $y = \dfrac{(x - 5)^{1/2}}{2}$ using the TI-92.

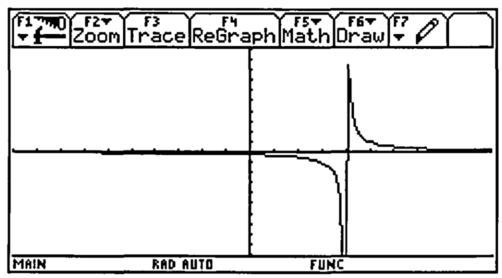
7. Use the TI-92 to graph the 4 separate transformations which when applied to the graph of $y = x^2$ form the second pictured graph, $y = 2(2x - 1)^2 + 9$. Sketch and specify each of these transformations.



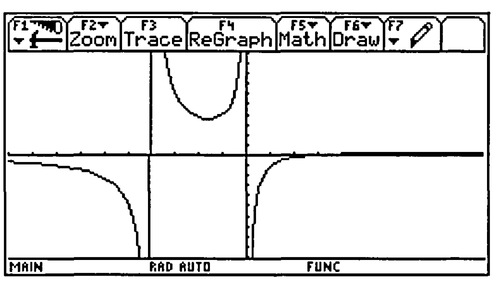
8. What transformation when applied to the given graphs below form the second pictured graphs and what is the symbolic form of each of the new functions?

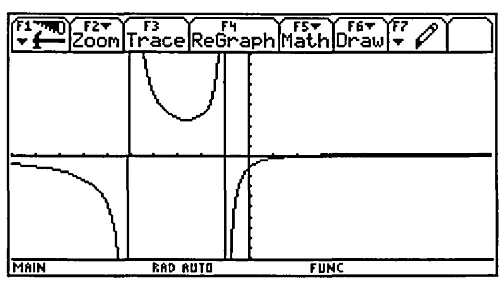
a) $y = 1/x$ (ZoomStd)



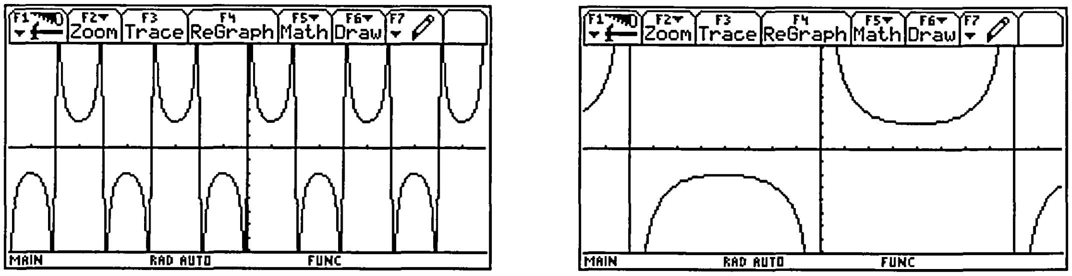


b) $y = \dfrac{3x - 9}{x^2 + 4x}$ (ZoomStd)

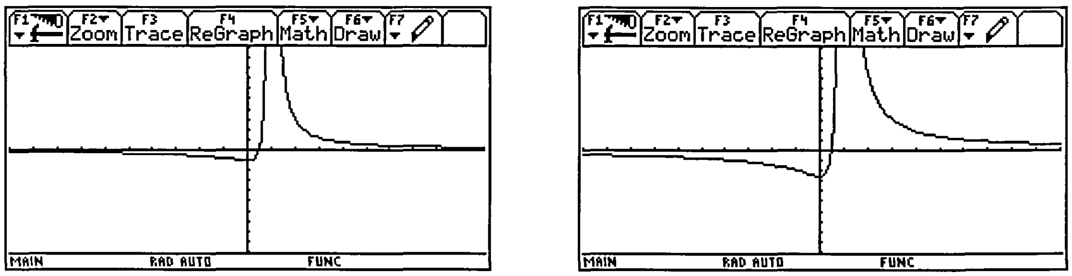




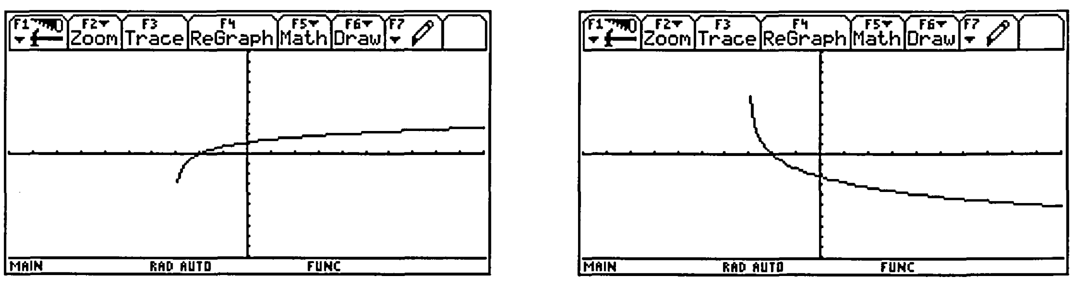
c) $y = \text{cosec } x$ (ZoomTrig)



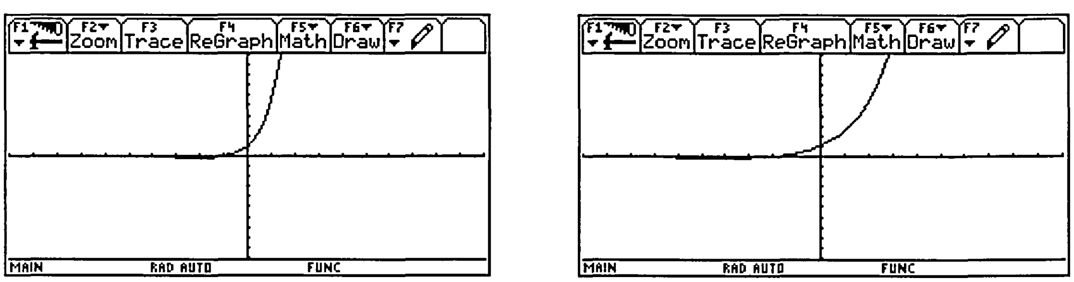
d) $y = \dfrac{2x - 1}{(x - 1)^2}$ (ZoomStd)



e) $y = \ln(x + 3)$ (ZoomStd)

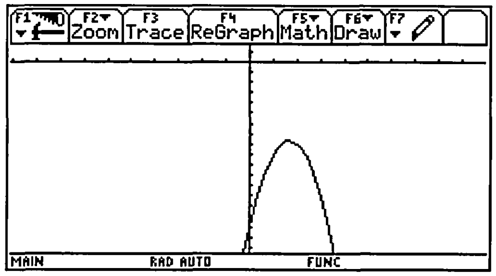


f) $y = (1+x)e^{x}$ (ZoomStd)



9. Use the TI-92 to show that a). $\ln x^a = a\ln x$ b). $\sin^2 x + \cos^2 x = 1$

10. Use the TI-92 to try to determine by a process of informed trial and error the symbolic form of the function which is graphed below using ZoomStd.



What additional information would you require to solve this problem algebraically?

11. Solve the following equations numerically, graphically and algebraically

a). $x^3 + 8x^2 + 4 = (x - 2)^2$ b). $\ln((2x+1)/(x-1)) = 2$ c). $2^{2x+1} + 2 = 5(2^x)$

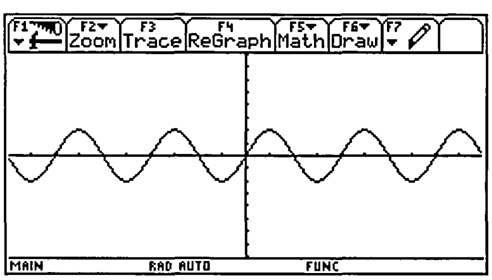
d). $\ln(x+1) + \ln(x-1) = 3$

12. Solve the following equations graphically

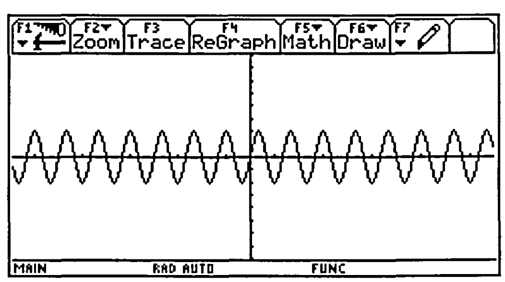
a). $e^{5x} - 3 = x^2 - 4x + 1$ b). $x^4 - 2x^3 - 2 = \dfrac{3x}{(x+1)(x-1)}$ c). $2^x = x^2$

13. Match up the six graphs with their corresponding functions, chosen from the list below:

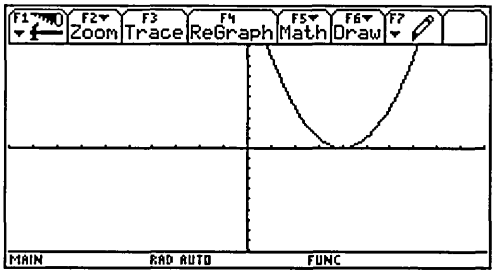
A. (ZoomTrig)



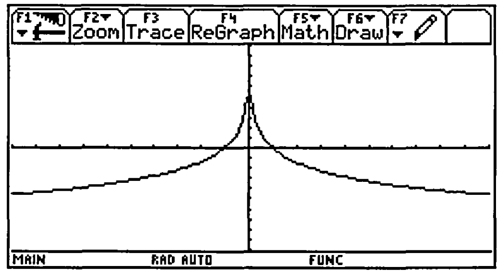
B. (ZoomTrig)



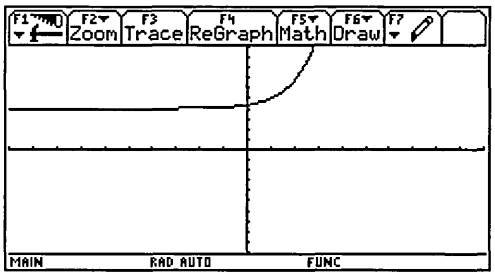
C. (ZoomStd)



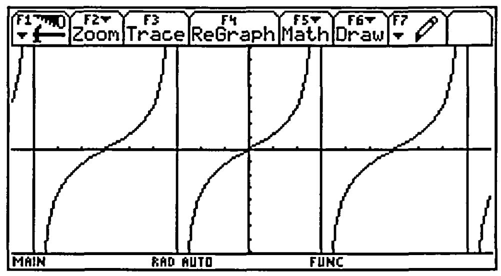
D. (ZoomStd)



E. (ZoomStd)



F. (ZoomTrig)



1. $y = \sin(x/3)$ 2. $y = \cos(x - \pi/2)$ 3. $y = 3\sin x$ 4. $y = \cos(x + \pi)$

5. $y = (x - 4)^2$ 6. $y = \tan(x/3)$ 7. $y = (4 - x)^2$ 8. $y = \tan(x/6)$

9. $y = (x + 4)^2$ 10. $y = \cos(x + \pi/2)$ 11. $y = \sin 3x$ 12. $y = \ln(1/x)$

13. $y = e^{x-1} + 4$ 14. $y = \ln x^2$ 15. $y = e^{-(x+1)} + 4$ 16. $y = 2\ln x$

17. $y = -\ln x^2$ 18. $y = -e^{x+1} + 4$ 19. $y = (\tan x)/3$ 20. $y = (\tan x)/6$

Post Trial Student Questionnaire on the Role of Technology

Analysing the Effects of the TI-92 in the A level Mathematics Classroom

Name ____

Q1. How important, in your opinion, is technology in the A level mathematics classroom?

____
____
____
____
____

Q2. Do you feel that you have benefited from the opportunity to use the TI-92?

____
____
____
____
____

Q3. Has using the TI-92 enabled you to picture functions more clearly?

____
____
____
____
____

Q4. Do you believe that using the TI-92 has strengthened your understanding of functions?

____
____
____
____
____

Q5. What do you consider to be the main advantages of using the TI-92?

____
____
____
____
____

Q6. What disadvantages do you perceive?

Q7. Would you welcome further use of the TI-92?

Q8. Would you consider yourself to be a person who forms and makes use of mental images when solving mathematical problems?

Q9. Do you have any preference for using a graphic calculator such as the TI-92 in your A level mathematics lessons rather than a computer, or vice versa?

Q10. How helpful have you found the materials and exercises designed for use with the TI-92?

I would be grateful for any additional comments or suggestions:

Post Trial Staff Questionnaire on the Role of Technology

Analysing the effects of the TI-92 in the A level Mathematics Classroom

Name ______________________________

Q1. How important, in your opinion, is technology in the A level mathematics classroom?

____
____
____
____
____

Q2. How often do you use technology in A level mathematics lessons?

____
____
____
____
____

Q3. If possible would you use technology more frequently with your A level students?

____
____
____
____
____

Q4. What do you hope to gain by using technology in A level mathematics?

____
____
____
____
____

Q5. Is it important, in your opinion, for students to be able to visualise mathematically at this level?

____
____
____
____
____

Q6. Do you feel that the TI-92 has had any affect on students' abilities to visualise the graphs of functions?

Q7. What do you consider to be the main advantages of using the TI-92?

Q8. What disadvantages do you perceive?

Q9. Do you see any potential for using the TI-92 in your classroom?

Q10. Are there any ways in which the materials used in this project with the TI-92, aimed at enhancing students' visual capabilities, could be improved?

I would be grateful for any additional comments or suggestions:

Student Performances in the Trial Exercises

The student's solutions to each of the questions from the trial exercises, were compared, evaluated and graded (see tables) with reference to the following criteria:

5 - Correct solution
4 - One omission/error
3 - Two omissions/errors
2 - Poor solution, several errors and omissions
1 - No understanding shown
0 - No solution offered

Table 1 Individual Student Performances in the Introductory Exercises

	Carl	Bet	Sal	Di	Em	May	Don	Kurt	Pat	Guy	Jan	Lea	Sue
Q1	3	5	5	4	3	1	4	5	5	5	0	0	1
Q2	3	5	5	4	3	5	4	5	5	5	0	0	5
Q3	4	5	5	0	3	4	4	5	5	4	0	0	4
Q4	4	4	1	0	3	3	4	4	4	3	0	0	3
Q5	4	3	1	1	2	5	5	5	3	1	0	0	4
Q6	1	5	5	1	2	1	1	4	4	1	0	0	1
Q7	1	5	1	1	2	2	1	2	1	2	0	0	1
Q8	5	5	4	4	5	3	5	4	5	2	0	0	2
Q9	1	1	0	1	1	1	3	5	1	1	0	0	0
Q10	5	5	0	5	5	5	5	5	5	5	0	0	0
Q11	5	5	0	5	5	5	5	5	5	3	0	0	0
Q12	3	4	0	4	3	1	5	4	4	0	0	0	0

Table 2 Percentages of Students Obtaining Each Grade in the Introductory Exercises

	Percentage Obtaining Grade 5	Percentage Obtaining Grade 4	Percentage Obtaining Grade 3	Percentage Obtaining Grade 2	Percentage Obtaining Grade 1	Percentage Obtaining Grade 0
Q1	38	15	15	0	15	15
Q2	54	15	15	0	0	15
Q3	31	38	8	0	0	23
Q4	0	38	31	0	8	23
Q5	23	15	15	8	23	15
Q6	15	15	0	8	46	15
Q7	8	0	0	31	46	15
Q8	38	23	8	15	0	15
Q9	8	0	8	0	54	31
Q10	69	0	0	0	0	31
Q11	61	0	8	0	0	31
Q12	8	31	15	0	8	38

Table 3 Mean Scores in the Introductory Exercises

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12
3.73	4.45	4.3	3.3	3.09	2.36	1.73	4	1.67	5	4.78	3.5

Table 4 Questions from the Main Exercises Involving More Than One Aspect

	First Grade	Second Grade
Q4	Accuracy of graphs	Accuracy of symbolic forms
Q5	Accuracy of graphs	Accuracy of explanations concerning the actions of transformations
Q8	Identification of the type of transformation	Deduction of the symbolic form of the transformation
Q11	Graphical solution	Algebraic solution

Table 5 Student Performances in the Main Exercises

	Carl		Bet		Sal		Di		Em		May		Don		Kurt		Pat		Guy		Jan		Lea		Sue	
Q1	4		5		5		5		5		5		4		4		4		4		4		5		4	
Q2	4		5		5		5		4		4		4		3		3		4		3		5		3	
Q3	4		5		5		4		4		4		4		4		4		4		4		5		3	
Q4I	4	4	5	5	5	4	5	4	5	1	5	4	5	5	4	0	4	4	4	4	5	0	4	5	4	0
Q4ii	4	4	5	5	5	4	5	4	5	5	5	4	5	3	4	0	4	4	4	4	5	0	0	3	4	0
Q4iii	4	4	5	5	5	4	5	4	5	4	3	4	5	3	4	0	4	4	4	4	4	0	0	3	4	0
Q4iv	4	4	5	4	5	4	5	4	5	4	4	3	5	3	4	0	4	4	0	0	3	0	0	3	4	0
Q5a	4	5	5	5	5	5	5	5	5	5	5	0	0	0	5	0	5	0	4	0	4	0	4	5	5	0
Q5b	3		3		3		3		3		4		0		4		3		3		3		3		3	
Q5c	4	5	4	5	5	5	4	5	5	5	5	0	0	0	5	0	5	0	4	0	4	0	4	5	4	0
Q5d	4	5	4	5	5	5	4	5	4	5	4	0	0	0	5	0	5	0	4	0	4	0	4	0	4	0
Q5e	4	5	5	5	4	5	4	5	4	5	5	0	0	0	4	0	4	0	4	0	1	0	5	5	4	0
Q5f	3	5	5	5	3	5	4	5	4	5	0	0	0	0	5	0	3	0	1	0	3	0	1	5	3	0
Q5g	3	5	4	5	3	5	3	5	3	0	1	5	0	0	4	0	4	0	3	0	2	0	5	4	0	0
Q6	1		3		5		3		5		0		0		5		0		4		0		4		0	
Q7	4		4		5		5		3		1		0		5		5		0		4		0		0	
Q8a	4	5	0	5	5	0	5	5	0	5	0	5	0	0	0	0	5	1	0	5	0	5	5	5	0	0
Q8b	4	1	0	1	5	0	0	0	5	0	0	1	0	0	0	0	5	1	5	0	0	0	5	0	0	0
Q8c	4	4	0	5	5	5	5	5	5	0	0	0	0	0	0	0	5	0	5	0	0	5	5	0	0	0
Q8d	3	3	0	4	4	4	0	4	3	0	0	4	0	0	0	0	1	0	1	0	0	3	1	0	0	0
Q8e	4	4	0	5	5	5	0	5	5	0	0	5	0	0	0	0	1	0	5	0	0	5	5	0	0	0
Q8f	4	1	0	4	5	5	0	1	0	0	0	4	0	0	0	0	0	3	1	0	0	1	1	0	0	0
Q9a	3		3		3		3		3		3		0		4		2		2		0		3		0	
Q9b	4		4		4		4		4		2		0		4		0		0		0		4		0	
Q10	1		4		1		0		4		2		0		1		2		0		0		0		0	
Q11a	5	5	4	5	4	3	0	5	4	5	3	4	0	0	4	5	0	5	0	0	0	0	3	5	2	2
Q11b	5	5	4	5	4	0	4	0	4	5	4	0	0	0	1	0	0	0	0	0	0	0	4	3	3	0
Q11c	0	0	4	5	3	0	3	0	4	5	4	0	0	0	3	0	3	0	0	0	0	0	3	0	0	0
Q11d	0	0	4	5	4	0	4	0	3	4	4	0	0	0	4	0	4	0	0	0	0	0	4	0	4	0
Q12a	0		4		4		4		4		1		0		0		0		0		0		4		1	
Q12b	0		4		3		3		4		4		0		0		0		0		0		4		4	
Q12c	0		0		3		3		4		4		0		0		0		0		0		4		4	
13	5		4		5		5		5		5		5		5		5		0		5		5		5	

Table 6 Percentages of Students Obtaining Each Grade in the Main Exercises

Percentages of Students Achieving Grades 0-5												
	% 5		% 4		% 3		% 2		% 1		% 0	
Q1	46		54		0		0		0		0	
Q2	31		38		31		0		0		0	
Q3	23		69		8		0		0		0	
Q4I	54	23	46	46	0	0	0	0	0	8	0	23
Q4ii	54	15	39	46	0	15	0	0	0	0	7	23
Q4iii	38	8	46	54	8	15	0	0	0	0	8	23
Q4iv	39	0	39	46	8	23	0	0	0	0	15	31
Q5a	61	46	31	0	0	0	0	0	0	0	8	54
Q5b	0		15		77		0		0		8	
Q5c	38	46	54	0	0	0	0	0	0	0	8	54
Q5d	23	38	69	0	0	0	0	0	0	0	8	62
Q5e	23	46	61	0	0	0	0	0	8	0	8	54
Q5f	15	46	15	0	38	0	0	0	15	0	15	54
Q5g	8	38	23	8	38	0	8	0	8	0	15	54
Q6	23		15		15		0		8		38	
Q7	31		23		8		0		8		31	
Q8a	38	61	8	0	0	0	0	0	0	8	54	31
Q8b	38	0	8	0	0	0	0	0	0	31	54	69
Q8c	46	31	8	8	0	0	0	0	0	0	46	61
Q8d	0	0	8	31	15	15	0	0	23	0	54	54
Q8e	31	38	8	8	0	0	0	0	8	0	54	54
Q8f	8	8	8	15	0	8	0	0	15	23	69	46
Q9a	0		8		54		15		0		23	
Q9b	0		54		0		8		0		38	
Q10	0		15		0		15		23		46	
Q11a	8	54	31	8	15	8	8	8	0	0	38	23
Q11b	8	23	46	0	8	8	0	0	8	0	31	69
Q11c	0	15	23	0	39	0	0	0	0	0	38	85
Q11d	0	8	61	8	8	0	0	0	0	0	31	85
Q12a	0		38		0		0		15		46	
Q12b	0		38		15		0		0		46	
Q12c	0		31		15		0		0		54	
Q13	85		8		0		0		0		8	

Table 7 Mean Scores in the Main Exercises

Question	Mean score		Question	Mean score		Question	Mean score	
Q1	4.46		**Q5f**	3.18	5	**Q10**	2.14	
Q2	4		**Q5g**	3.18	4.83	**Q11a**	3.63	4.4
Q3	4.15		**Q6**	3.75		**Q11b**	3.67	4.5
Q4I	4.54	4	**Q7**	4		**Q11c**	3.38	5
Q4ii	4.58	4	**Q8a**	4.8	4.56	**Q11d**	3.89	4.5
Q4iii	4.33	3.9	**Q8b**	4.83	1	**Q12a**	3.14	
Q4iv	4.36	3.67	**Q8c**	4.86	4.8	**Q12b**	3.71	
Q5a	4.67	5	**Q8d**	2.17	3.67	**Q12c**	3.67	
Q5b	3.17		**Q8e**	4.17	4.83	**Q13**	4.92	
Q5c	4.42	5	**Q8f**	2.75	2.71			
Q5d	4.25	5	**Q9a**	2.9				
Q5e	4	5	**Q9b**	3.75				

Table 8 Individual Students' Overall Performances Compared

Position	Student	Use of Imagery	Overall Score	Relation to Mean
1	Betty	Non-visualiser	256	> mean (168.4)
2	Sally	Visualiser	235	> mean
3	Emma	Visualiser	230	> mean
4	Carl	Non-Visualiser	217	> mean
5	Diana	Visualiser	213	> mean
6	May	Non-Visualiser	170	> mean
7	Pat	Visualiser	167	< mean
8	Lea	Visualiser	160	< mean
9	Kurt	Visualiser	153	< mean
10	Guy	Non-Visualiser	119	< mean
11	Don	Unknown	97	< mean
12	Sue	Visualiser	95	< mean
13	Jan	Visualiser	77	< mean

Table 9 Comparative Mean Scores and Standard Deviation for the Visualisers and Non-Visualisers

Mean overall score for visualisers	Mean overall score for non-visualisers
166.25	190.5
Standard deviation	Standard deviation
55	51.3

Transcripts from Phase 1

Discussions Surrounding the Introductory Exercises

Question 2 – May, Sue and the additional researcher, James Green

May: It's +3, isn't it what we worked out already? The same value there.

Sue: I think so.

May: That's a turning point isn't it, it changes.

Sue: Yes.

May: Yes.

Sue: Let's do question 2.

May: Right ok.

Sue: So how do you get rid of it then?

May: You go back into Y editor and you just get rid of the function don't you?

Sue: Right.

May: Just delete it.

Sue: Ok you can just get rid of the tick, can't you?

May: Yes.

Sue: Instead of just writing graph it. Don't you have to put cubed in the bracket? Or will it be alright?

May: Em see what it looks like when we've done that.

Sue: Yes enter.

May: Yes, it's fine. When $x = 3x^3$, what did you get for the intersection of the x-axis for this?

Sue: I have 0.58, I think.

May: I didn't get that, not at all.

Sue: What did you get?

May: 5. It probably doesn't work.

Sue: Let's go into graph and draw it.

May: Oh it's one of those ones.

Sue: Oh yes.

May: Oh cool.

Sue: Hang on a minute what oh? It's going to have lots of turning points as well, isn't it?

May: Graph it. Graph it.

Sue: Do you reckon we should em see if…

May: Is there only one [turning point] when y = 0?

Sue: How do you know if there's more than one though?

May: I don't know I suppose you could zoom out. Are we in standard?

Sue: Yes we are, aren't we?

May: Yes.

Sue: Em no I'm not I don't think. Oh no.

May: What have you got?

Sue: Do you think we should zoom out to see a bit more?

May: Yes but we'll have to zoom out to get ourselves to get the same.

Sue: What do you mean the same centre as before.

May: Yes.

Sue: What do you reckon?

May: I think it would have shown.

JG: You can just go to point on the graph – move it across and see what the co-ordinates are there. So you can say move it below the x-axis to about there, do you see?

Sue: Yes.

JG: That's quite a useful thing to do.

Sue: But do you have to go into the maths bit to work it?

JG: But if you want to do it, you know, yes go to maths. What are we on y = zero?

Sue: We're on y = 0.

JG: On the point of inflection?

Sue: Yes.

JG: So go down to inflection, press enter. Now then there could be – this is a nice simple curve with a single point of inflection.

Sue: Yes.

JG: You could have a wiggly curve with all sorts of points of inflections, minimums and maximums, and things.

Sue: Yes.

JG: So you've got to tell it that you're interested in the point between here and here.

Sue: Ok

JG: So if you say –5 to - it doesn't actually matter as long as you cover this because there's only the one. But in general you'd have to estimate a point here and a point here, say –5.

Sue: Do you use the cursor?

May: The cursor at all?

JG: No I think you just enter or go down.

Sue: I used that cursor thing.

May: Yes. Enter down.

JG: You're making a box round it aren't you?

Sue: Yes.

JG: So…

Sue: There's an inflection at –6.

JG: Oh yes you've just over type –5.

Sue: Ah and then enter.

JG: And then enter. Upper bound is 2 say enter.

May: There's an inflection at –3.

Sue: That's right yes. So that's a stationary point as well isn't it?

May: Yes.

Sue: Y = 0. [Working].

Question 3 – May, Sue, the teacher researcher (SE), and James Green

Sue: Modulus y =…

May: Are you setting the gradient equal to nothing? Don't you?

Sue: Yes.

SE: Without using the calculator.

May: What did she say?

Sue: Do you want us to work the point on the calculator?

SE: On paper, on your own.

Sue: Oh right ok. Em right then, y = that. Let's get rid of 2.

May: Yes. y = 2 bracket. Do we have to put all this in a bracket then?

Sue: Ah hang on.

May: Do you have to put all that in a bracket and type it in?

Sue: You type in that 2 times by bracket… How do you put modulus in?

May: You do that and press 2nd 5 or something.

Sue: Oh. Oh it's playing tricks there it is.

May: Times bracket abs. Where's the abs?

Sue: I don't know. Oh it's gone.

May: There is no abs.

Sue: Oh were's abs?

SE: Modulus.

Sue: We're trying to find the modulus. Don't you just put abs in?

May: It says press second 5 for the maths menu, which is what I'm on, and select y.

Sue: Oh

May: It tells us what to select and it's like 2nd abs 1 and 2.

Sue: -1. I've pressed abs 5 and that one, two. Oh x, bracket –1enter. Oh yes that's worked. Good. Oh no it hasn't.

May: Have you not put a bracket round?

Sue: Not put the brackets round?

May: Is that a minus or a...

Sue: It's a minus but a different thing there.

May: So which is it?

Sue: I'm not sure. Em is that a divide by or a minus?

SE: Yes. It's a mistake I'm afraid. It's supposed to be divide.

Sue: Oh right.

SE: But if you've done minus it doesn't matter.

Sue: Ok.

SE: It was only just an example to show you how to use the calculator. So it's not crucial.

May: I've done the same thing as – well I've done –5. [Working]

JG: That's -3×10^{-38} which is 0.0… It's not nought, it's not zero. It's 0 point then 38 nothings and then a 3. Alright, it's because the pixels, the accuracy of the pixel is mixed up with the window you've got.

Sue: So if I set a smaller window.

JG: If you set a – well not necessarily smaller – different window.

Sue: Yes.

JG: Where the pixels were sort of working in harmony with the window that you've set yes.

Question 2 - Kurt and Pat

Kurt: Em what do you do with those two?

Pat: It's 6x. It is isn't it? [Pause].

Kurt: Em 27.

Pat: Oh different. Oh it's a cube isn't it? Stupid! So it equals $x^3 + 9x^2 + 27x + 27$, yes?

Kurt: Yes.

Pat: So one over x would be $3x^2$.

Kurt: $3x^2$.

Pat: +18x.

Kurt: +18x.

Pat: +27.

Kurt: +27. Well that's a quadratic equation. So we can take - can we take out 3 of that – we can can't we?

Pat: Yes.

Kurt: $3x^2$, that's 18.

Pat: $x^2 + 6x + 9$. It's got to be 3 and 3. It's got to be x+3.

Kurt: x+3.

Pat: So x is equal to –3.

Kurt: Yes. It's wonderful. But we had the calculator to do it for us.

Pat: But how come you did no working?

Kurt: It's gone off, it's gone off.

Pat: How come you did no actual working.

Kurt: It's gone off. Oh thanks goodness.

Pat: How come you did no…

Kurt: Because I just copied you, because I thought – I just thought it was x + 3 squared.

Pat: That's the way I always do it, because you work out the two first.

Kurt: You work out the two and then times it.

Pat: Yes.

Kurt: Do you do them all at once?

Pat: We've done it all, let's do number 4.

Question 2 – Betty, Emma, James Green and Sally Elliott

Betty: Yes. It should exercise it for me shouldn't it?

JG: Yes. Try and do it on solve try cubed solve equal 0. $6x^2$ + that.

Betty: Right go back to the home then right sort of type it in.

JG: Yes 2.

Betty: Put all this in, I need absolute don't I. It's the wrong one it's in memory. Number 2 x bracket 3 equals 0, will that do it?

JG: No you've missed out the brackets there, where you need to end the bracket.

Betty: Oh need to end the brackets. Go back end that bracket and then equals 0.

JG: =0 comma x.

Betty: Comma, where's comma? So that's saying that I want equals 0 for x's.

JG: No it's saying that you want to solve it for x.

Betty: Right.

JG: Because you could have an equation with two variables in it and you might want to solve it for one of the variables and that is express it in terms of the other one.

Betty: Yes.

JG: If you had the x^2.

Betty: And then solve it if I had x…

JG: Say you had $y = mx + c$, you could say solve it in terms of c.

Betty: Mm

JG: So you could express the whole thing.

Betty: Right.

JG: And it would do it. I think you need another bracket there, you see, where you solve.

Betty: Does it? Oh that solve. Do I need a bracket there?

JG: You need a bracket right at the end.

Betty: Shall I take that one out and put one right at the end?

JG: No I think you need one there as well.

Betty: Ok. No it doesn't want it.

JG: Too few arguments there em.

Betty: Em so I need to put another bracket in one in there. I wonder what it will think of that em. That's what I had last time.

JG: Yes. You definitely want rid of that bracket, I think.

Betty: Oh because that's where I've put my expression in brackets, isn't it. So do I want to get rid of that one round the zero and put one round the x? That where it said there was a problem wasn't it?

JG: There's one there, there's a problem there.

Betty: Put an end one there, put one round there.

JG: Yes try it.

Betty: Brackets… Can you have a look at this for me? We're trying to make it solve this, but it won't there's a problem with the brackets. It keeps saying that we haven't got enough arguments in it.

SE: I don't think you need that bracket, try that yes.

Betty: Oh right.

SE: You put an extra bracket round the x. It doesn't need one.

Betty: Right ok.

JG: Have you done it?

Betty: Yes. That's great

JG: Right so.

Betty: Right back at the beginning.

JG: Oh right it would be nice pulling that down, so you wouldn't have to do all that.

Betty: Yes, saving that.

JG: You can pull things down from this. It's quite nice really. One of the nice things about this, the image, is that if you do 23 x 4 enter, then you get the answer and then you do enter and then so on. And then if you want to go back, it goes back through the history of all the things that you've done. And if you press enter there it pulls that down into that line again but that only works on the home screen, you see. So you can get back to that equation there and pull it down.

Betty: But you can't get from one screen to the other.

JG: But you can't get from one application to another.

Betty: Right ok.

JG: I don't know. I'm not saying you can't.

Betty: Ok.

Question 3 – Diana, Jan, Guy, Lea, Sally Elliott and classroom teacher Mr Doors

Diana: What do you get for your minimum for question 3?

Jan: I get something really horrible, -1.56.

Diana: I get +2.16 to the –14.

Jan: You get what?

Diana: 2.16 x 10^{-14}.

Jan: I get that. I get that. They're the same line though.

Diana: Yes that's really odd. That's really odd for the same function.

Jan: Because it says if you copy that it's –3 and that says divide by 3.

Diana: Have you done number three? For question 3, we've got different answers. We've both got the same equation. But when we got to the middle it's given us different ones…

Guy: Yes.

Diana: It's given the same y value.

Guy: But it's given you what?

Lea: Are you trying to do the minimum. You can't do the minimum for number 3.

Diana: Why?

Lea: You just can't. No you can't.

Guy: Why not?

Lea: I've asked and she said you can't.

Jan: It's given us all different numbers. Is it something…?

Lea: Exactly, because I had about 20 different numbers.

Jan: So your not supposed to do it. Oh right.

SE: The function actually goes to a point so it's not smooth like the others.

Jan: Right.

MrD: You can't differentiate it because the gradient is not zero, it does get to a lowest point, but it's not a turning point.

SE: In these questions I've asked you to differentiate the other ones but I left that one out because you can't differentiate it.

Jan: Right, we don't read instructions.

APPENDIX B

Phase 2 Data Collection

Contents

Pre Trial Student Visualisation Questionnaire

The Role of Mental Imagery/Visualisation in A level Mathematics

Name ____

Q1. How frequently do you form and make use of mental images when solving mathematical problems? Please circle the appropriate response.

Always Fairly Frequently Sometimes Quite Rarely Never

Q2. If you do construct mental images, how do you feel these images assist you in problem solving?

Q3. At what stage during problem solving do you usually find it necessary to formulate mental images?

Q4. Does the type of problem or topic area effect your use of mental imagery? (For example, are there certain areas of mathematics in which you use mental imagery more often?)

Q5. Are there any particular areas of mathematics that you find difficult to visualise?

Q6. Are visual methods of solution encouraged in your A level mathematics lessons?

Q7. Do you have a preference for working either symbolically or visually? Please explain your response.

Q8. In general, how often do you combine different approaches (such as visual and symbolic) when solving individual mathematical problems? Please circle as appropriate and please explain your response.

Always Fairly Frequently Sometimes Quite Rarely Never

Q9. In order to become a successful mathematician which do you regard as most important, the ability to perform symbolic manipulations or the ability to visualise mathematically?

Q10. How would you classify yourself essentially? Please indicate where you would place yourself on the following hypothetical continuum by marking the line with a cross.

Visualiser |--| Non-visualiser

Q11. How would you rate your visualisation powers overall? Please circle the appropriate response.

Very good Good Fair Poor

Q12. If you do make use of visual processes is this always obvious in your solutions, or do you tend to use a different argument for written purposes?

Q13. Please describe your understanding of the difference between a visual solution and a symbolic solution in mathematics. You may like to give an example.

I would be grateful for any additional comments that you might like to make.

Pre Trial Staff Visualisation Questionnaire

The Role of Mental Imagery/Visualisation in A level Mathematics

Name ____

Q1. Is it important, in your opinion, for students to be able to visualise mathematically at this level?

Q2. Do you encourage the use of visual solutions in your A level mathematics lessons generally?

Q3. When teaching functions to lower sixth students, do you tend to devote fairly equal amounts of time exploring the graphical, symbolic and numerical aspects, or does one particular approach dominate?

Q4. Do you personally have a preference for working either symbolically or visually? Please explain your response.

Q5. In order to become a successful mathematician which do you regard as the most important, the ability to perform symbolic manipulations or the ability to visualise mathematically?

Q6. How would you classify yourself essentially? Please indicate where you would place yourself on the following hypothetical continuum by marking the line with a cross.

Visualiser |--| Non-visualiser

Q7. How would you describe the overall visual capabilities of this particular group of A level students?

Q8. How important, in your opinion, is technology in supplementing and enriching students' visual capabilities?

Q9. How often do you use technology in A level mathematics lessons?

Q10. If possible would you use technology more frequently with your A level students?

I would be grateful for any additional comments that you might like to make.

Pre Trial Questions on Functions

1. Find the values of x for which $-2x^2-x+6>2$.

2. For which x values is $f(x) = x^3-5x^2+6x<0$.

3. Solve $\frac{x}{x+1} > 3$.

4. If $f_1(x) = |x-3|$ and $f_2(x) = 2|x+2|$, find the values of x where $f_1(x)>f_2(x)$.

5. Solve the following equations:
a) $2(x-4)^2 + 3 = 4x$ b) $\sin x = 2x^2$ c) $3e^{2x} = 4$ d) $2\ln(x+1) = 2 + \ln x$.

6. If $f(x) = x^2 + 5x$ and $g(x) = f(x-1)$, find $g(3)$ and the values of x such that $g(x) = 0$.

7. Give an example of a function which satisfies $f(2) = 3$, $f(3) = 4$ and $f(9) = 15$.

8. Which linear function passes through the points (4,-3) and (1,3)?

9. If $f(x) = 3x^3$, find the value of x for which $f^{-1}(x) = 2$.

10. Solve $3x^4 + 2x^2 + 3x = 0$.

11. Solve $2\sin x - x = 0$.

12. State whether the following functions are even, odd or neither, explaining your choice: a) $\cos^2 x$ b) $x^3 - x$ c) $\tan x$ d) $2(e^x + e^{-x})$ e) $\sin 2x$.

13. State, with justification, whether or not the following functions are periodic, and give the period of those functions which are: a) $|\sin x|$ b) $\frac{\cos x}{x}$ c) $\cos^2 x$.

14. The following transformations are applied in succession to $y = x^3$:
a) a translation of magnitude 2 units in the direction of the positive y - axis,
b) a stretch parallel to the x - axis of factor 2,
c) a stretch parallel to the y - axis of factor 3,
d) a translation of magnitude 2 units in the direction of the negative x - axis.

Give the equation of the resulting curve and the co-ordinates of three points on the curve.

15. Another curve undergoes in succession the transformations a), b), c), d) given in question 14, and the equation of the resulting curve is $y = \frac{3x^2}{4} + 3x + 9$.

Determine the original equation of this curve.

XXX

The Introductory Exercises

Use the TI-92 to draw the graphs of:

1. $y = 4x^2 - 4x + 1$ 2. $y = (x + 3)^3$ 3. $y = 2(|x| - 1) / 3$ 4. $y = \cos(x/2)$

In each case sketch the graph on paper and use the TI-92 to find the value(s) of x when $y = 0$. [Check these values by substituting $y = 0$ and solving in each case].

Also use the TI-92 to determine the nature of any stationary points and to ascertain their co-ordinates. [Check these values by differentiation].

Homework Questions

For each of the following functions find:

a) any values of x for which $y = 0$

b) the nature and co-ordinates of any stationary points

5. $y = x^2 - x^3$ 6. $y = \dfrac{x + 1}{(x + 2)^2}$ 7. $y = x - 1 + \dfrac{1}{x + 1}$ 8. $y = \dfrac{1 + x^2}{1 + x + x^2}$

The Main Trial Exercises

Graphing Functions Using the TI-92

1. Compare the graphs of cos x, 2cos x and 3cos x using the TI-92. Sketch the graphs and comment on the main features.

2. Repeat question 1 with tan x, 2tan x and 3tan x.

3. Compare the graphs of cos x, 2cos 2x, 3cos 3x and sketch them. Explain why these three graphs do not cross the x-axis in exactly the same places.

4. Given that $f(x) = x^3$, use the TI-92 to obtain the graph of $g(x) = f(x/2)$. Sketch the two graphs and write down the equation of the new function, g(x).

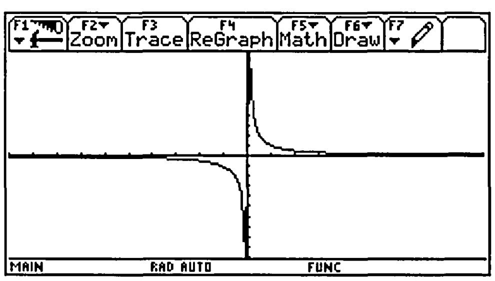
Now use the TI-92 to perform the transformation $g(x + 2) - 3$ on g(x). Sketch the resulting curve, h(x) and again write down its equation in the form $ax^3 + bx^2 + cx + d$.

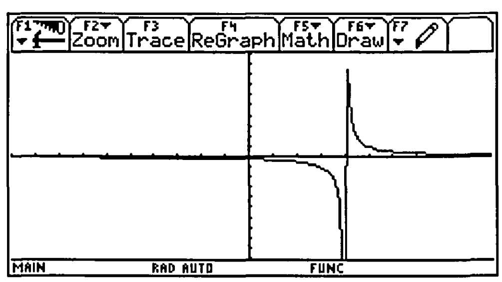
Finally use the TI-92 to perform the transformation $2(h(x))$ on h(x), sketching the curve; l(x) and writing down the resulting equation, as before.

5. Investigate the relationship between the functions $y = 4x^2 + 5$ and $y = \frac{(x - 5)^{1/2}}{2}$ using the TI-92.

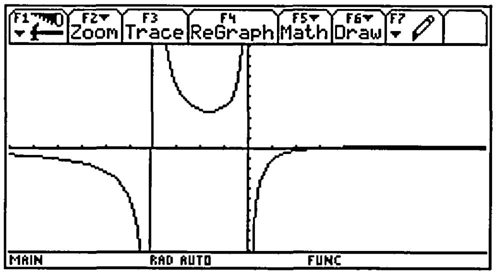
6. What transformation when applied to the given graphs below form the second pictured graphs and what is the symbolic form of each of the new functions?

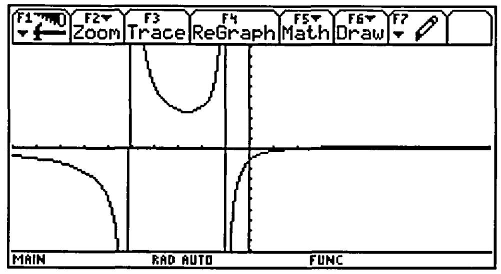
a) y = 1/x (ZoomStd)



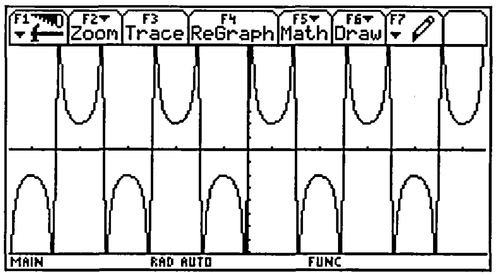


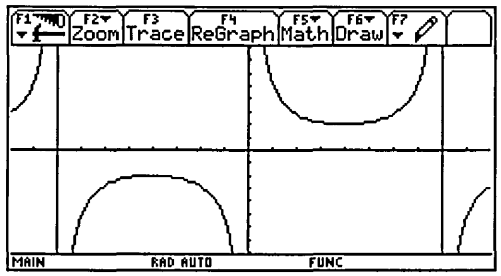
b) $y = \dfrac{3x - 9}{x^2 + 4x}$ (ZoomStd)



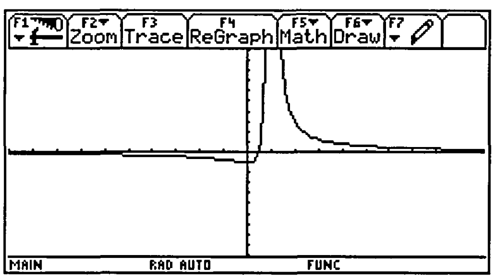


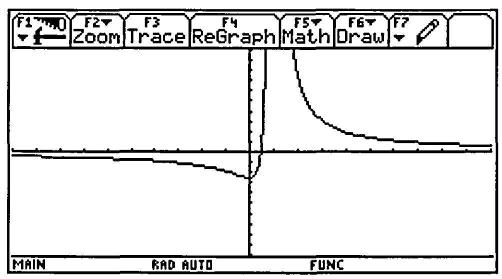
c) y = cosec x (ZoomTrig)



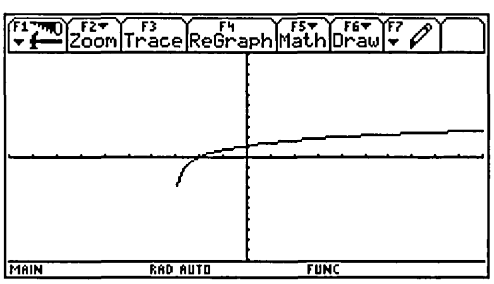


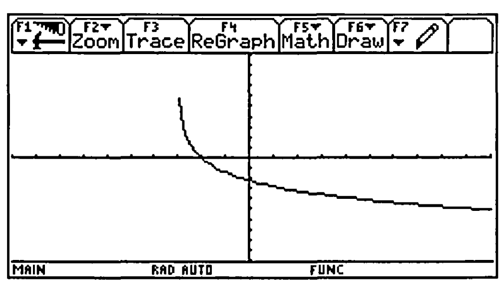
d) $y = \dfrac{2x - 1}{(x - 1)^2}$ (ZoomStd)



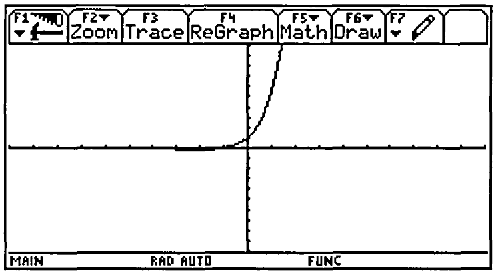


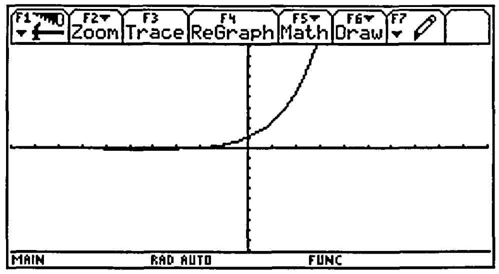
e) y = ln (x +3) (ZoomStd)





f) $y = (1+x)e^x$ (ZoomStd)





7. Use the TI-92 to show that a). $\ln x^a = a \ln x$ b). $\sin^2 x + \cos^2 x = 1$

8. Solve the following equations graphically and algebraically:

a). $x^3 + 8x^2 + 4 = (x-2)^2$ b). $\ln ((2x+1)/(x-1)) = 2$

c). $2^{2x+1} + 2 = 5(2^x)$ d). $\ln (x+1) + \ln (x-1) = 3$

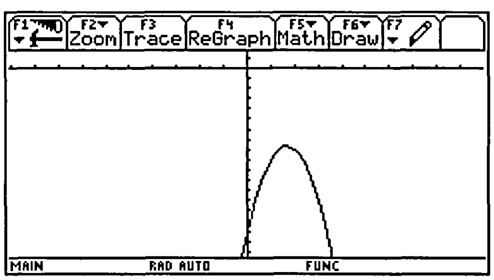
9. Without using the TI-92 sketch the following functions (each pair on the same axes): a). $y = x^3$ and $y = 6x^3 - 3$ b). $y = e^x$ and $y = e^{2x}$

c). $y = \ln x$ and $y = \ln (x/2)$ d). $y = \sin x$ and $y = \sin (x + \pi/2)$

e). $y = x^2$ and $y = 2(2x-1)^2 + 9$ f). $y = x^2$ and $y = -(x-1)^2 + 3$

Explain in words how the second graph can be obtained from the first graph in each case. Now use the TI-92 to check whether your sketches are correct.

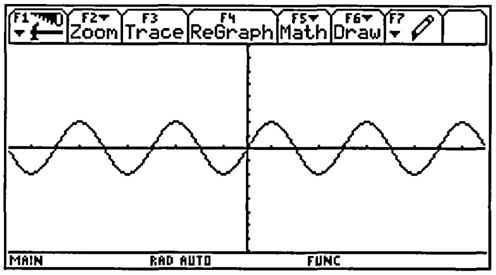
10. Use the TI-92 to try to determine by a process of informed trial and error the symbolic form of the function which is graphed below using ZoomStd. Explain the reasoning behind every step you make.



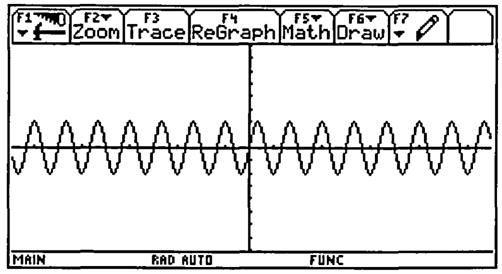
What additional information would you require to solve this problem algebraically?

11. Match up the six graphs with their corresponding functions, chosen from the list below.

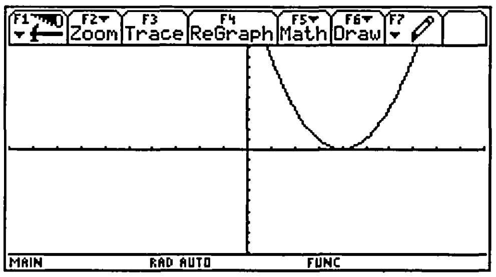
A. (ZoomTrig)



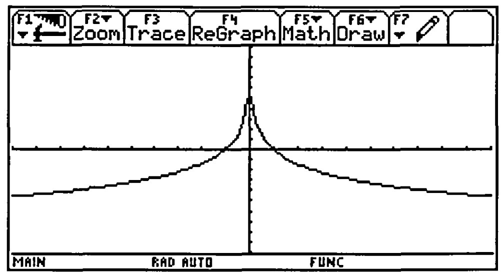
B. (ZoomTrig)



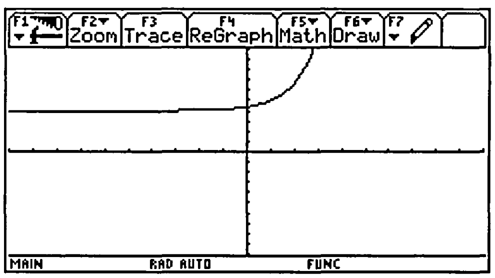
C. (ZoomStd)



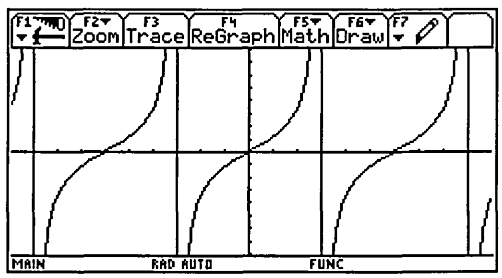
D. (ZoomStd)



E. (ZoomStd)



F. (ZoomTrig)



1. $y = \sin(x/3)$
2. $y = \cos(x - \pi/2)$
3. $y = 3\sin x$
4. $y = \cos(x + \pi)$
5. $y = (x - 4)^2$
6. $y = \tan(x/3)$
7. $y = (4 - x)^2$
8. $y = \tan(x/6)$
9. $y = (x + 4)^2$
10. $y = \cos(x + \pi/2)$
11. $y = \sin 3x$
12. $y = \ln(1/x)$
13. $y = e^{x-1} + 4$
14. $y = \ln x^2$
15. $y = e^{-(x+1)} + 4$
16. $y = 2\ln x$
17. $y = -\ln x^2$
18. $y = -e^{x+1} + 4$
19. $y = (\tan x)/3$
20. $y = (\tan x)/6$

Student Interview

Please describe how you would attempt to solve the following questions:

1. For which values of x is the graph of $y = 3x^2 + 9x - 12$ below the x-axis?

2. For which x values does the graph of $y = x^3 + 6$ intersect with the graph of $y = 2x^2 + 5x$?

3. Find the values of x for which the graph of $y = 3|x - 2|$ lies above the graph of $y = 6x^2$.

4. What effect will the transformation $3f(x + 3)$ have on the graph of the function $f(x) = 4x^5 - 3x^4 + 2x^3 - x$.

5. For a particular function $f(x)$; $f(2) = 6$, $f(3) = 14$ and $f(6) = 50$. What type of function could this be? Give an example of a function that satisfies these conditions.

6. How does the slope of the function $y = x^4 – x^2$ change from $x = -5$ to $x = 5$?

Post Trial Student Technology Questionnaire

Analysing the Effects of Technology in the A level Mathematics Classroom

Name ____

Q1. How important, in your opinion, is technology in the A level mathematics classroom?

Q2. Do you feel that you have benefited from the opportunity to use the TI-92?

Q3. Has the TI-92 enabled you to visualise functions more clearly?

Q4. Do you believe that using the TI-92 has strengthened your understanding of functions?

Q5. Will this experience influence your approach to solving problems involving functions in the future?

Q6. What do you consider to be the main advantages of using technology to teach the concept of functions to students, such as yourselves?

Q7. What disadvantages do you perceive?

Q8. Would you welcome further use of technology for exploring different areas of mathematics?

Q9. Do you have any preference for using a graphic calculator in your A level mathematics lessons rather than a computer, or vice versa?

Q10. How helpful have you found the materials and exercises designed for use with the TI-92?

I would be grateful for any additional comments or suggestions:

Student Performances in the Trial Exercises

The student's solutions to each of the questions from the trial exercises, were compared, evaluated and graded (see tables) with reference to the following criteria:

5 - Correct solution
4 - One omission/error
3 - Two omissions/errors
2 - Poor solution, several errors and omissions
1 - No understanding shown
0 - No solution offered

Table 10 Student Performances in the Pre Trial Inquiry

	Robert	Julie	Martin	Diane	Rachel
Qu1	5	5	3	5	5
Qu2	5	5	0	5	5
Qu3	1	1	0	1	1
Qu4	5	0	0	5	4
Qu5a	5	5	0	5	5
Qu5b	0	0	0	3	0
Qu5c	5	0	0	5	5
Qu5d	3	0	0	3	4
Qu6	0	0	0	5	0

Table 11 Student Performances in the Introductory Exercises

	Robert	Julie	Martin	Rachel
Qu1	4	5	1	5
Qu2	5	5	5	4
Qu3	1	5	0	3
Qu4	3	0	0	5
Qu5	4	4	4	0
Qu6	4	1	1	0
Qu7	4	3	3	0
Qu8	4	4	4	0

Table 12 Questions from the Main Exercises Involving More Than One Aspect

	First Grade	Second Grade
Q4	Accuracy of graphs	Accuracy of symbolic forms
Q6	Identification of the type of transformation	Deduction of the symbolic form of the transformation

Table 13 Student Performances in the Main Exercises

<table>
<tr><th></th><th colspan="2">Robert</th><th colspan="2">Julie</th><th colspan="2">Martin</th></tr>
<tr><td>Qu1</td><td colspan="2">4</td><td colspan="2">2</td><td colspan="2">4</td></tr>
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<tr><td>Qu3</td><td colspan="2">3</td><td colspan="2">4</td><td colspan="2">5</td></tr>
<tr><td>Qu4a</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td></tr>
<tr><td>Qu4b</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td></tr>
<tr><td>Qu4c</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td></tr>
<tr><td>Qu5</td><td colspan="2">5</td><td colspan="2">5</td><td colspan="2">5</td></tr>
<tr><td>Qu6a</td><td>5</td><td>5</td><td>5</td><td>5</td><td>0</td><td>0</td></tr>
<tr><td>Qu6b</td><td>5</td><td>5</td><td>5</td><td>5</td><td>0</td><td>0</td></tr>
<tr><td>Qu6c</td><td>5</td><td>5</td><td>5</td><td>5</td><td>0</td><td>0</td></tr>
<tr><td>Qu6d</td><td>5</td><td>4</td><td>5</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>Qu6e</td><td>5</td><td>4</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>Qu6f</td><td>5</td><td>5</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
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<tr><td>Qu8b</td><td colspan="2">5</td><td colspan="2">0</td><td colspan="2">0</td></tr>
<tr><td>Qu8c</td><td colspan="2">5</td><td colspan="2">0</td><td colspan="2">0</td></tr>
<tr><td>Qu8d</td><td colspan="2">5</td><td colspan="2">0</td><td colspan="2">0</td></tr>
</table>

Table 14 Mean Scores in the Pre Trial Inquiry

Qu1	Qu2	Qu3	Qu4	Qu5a	Qu5b	Qu5c	Qu5d	Qu6
4.6	5	1	4.67	5	3	5	3.33	5

Table 15 Mean Scores in the Introductory Exercises

Qu1	Qu2	Qu3	Qu4	Qu5	Qu6	Qu7	Qu8
3.75	4.75	3	4	4	2	3.33	4

Table 16 Mean Scores in the Main Exercises

<table>
<tr><td colspan="2">Qu1</td><td colspan="2">Qu2</td><td colspan="2">Qu3</td><td colspan="2">Qu4a</td><td colspan="2">Qu4b</td><td colspan="2">Qu4c</td><td colspan="2">Qu5</td><td colspan="2">Qu6a</td><td colspan="2">Qu6b</td><td colspan="2">Qu6c</td></tr>
<tr><td colspan="2">3.75</td><td colspan="2">4</td><td colspan="2">3.75</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td colspan="2">5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td><td>5</td></tr>
<tr><td colspan="2">Qu6d</td><td colspan="2">Qu6e</td><td colspan="2">Qu6f</td><td colspan="2">Qu7a</td><td colspan="2">Qu7b</td><td colspan="2">Qu8a</td><td colspan="2">Qu8b</td><td colspan="2">Qu8c</td><td colspan="2">Qu8d</td><td colspan="2"></td></tr>
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Table 17 Mean Scores of Individual Students

Robert	Julie	Martin	Rachel	Diane
4.47	4.32	4.06	4.31	4.11
Non-Visualiser	Visualiser	Visualiser	Visualiser	Visualiser

Transcripts from Phase 2

Class Discussions Surrounding Question 13 of the Main Trial Exercises

Graph A

1	SE	Can anybody tell me which function represents the graph drawn in the first one?	
2	Martin	Is it cos $(x + \pi/2)$?	
3	SE	And why do you say that?	
4	Robert	It's a sine graph.	Robert was confident.
5	SE	Contradiction there. Explain your choice.	Directed at Martin.
6	Martin	Er well it looks – it's got to be like sine or cos and I think that cos starts at the top and each line on the scale is 90^0 which is $\pi/2$ radians, so it's been moved …	Martin was motioning in the air, tracing the path of the graph of cos x with his finger.
7	SE	It's been moved across to the …	
8	Martin	It's got to be $-\pi/2$ rads then because it's gone the other way, so it's number 2 [cos $(x - \pi/2)$].	
9	SE	Ok so you think it's number 2. Why do you say that it might be a sine [graph]?	Directed at Robert.
10	Robert	Because sine of zero is zero and I'd say that that is in fact – because it seems that B is also a sine wave but that's more concentrated – I'd say that A is 1 [sin x/3].	
11	SE	You think that it's sin (x/3)?	
12	Robert	I wouldn't swear to it.	Robert clearly lacked confidence at this point.
13	SE	And what do you think? Have you got any ideas about this one?	Directed at Julie.
14	Julie	I think it's number 2 [cos $(x - \pi/2)$].	
15	SE	And why do you think that it's number 2?	
16	Julie	It's been moved.	
17	SE	It's been moved?	
18	Julie	Yes it's a translation.	
19	SE	And in which direction is it moved?	
20	Julie	Er $\pi/2$ in the x-axis.	
21	SE	Yes. Ok so have you tried to actually graph on the TI-92 the first one that you thought it was?	Robert had just graphed the function y = sin(x/3).
22	Robert	Yes.	
23	SE	And what did you get?	
24	Martin	Isn't that cheating drawing the graph to see which?	
25	SE	No, no he is just convincing himself.	
26	Robert	To be honest I can't remember what I typed in.	
27	SE	Well, let's think about the first one y = sin(x/3). What is the graph of that going to	

		look like?	
28	Robert	Wide, and wider than it is there.	Robert pointed to the graph to be identified.
29	SE	Yes. Ok, I'm going to say that you two are actually correct. Now it looks like a sine because it is sine of x, that is sin x.	
30	Robert	Yes.	
31	SE	But it can also be represented by $y = \cos(x - \pi/2)$ that's another…	
32	Robert	I see where that's coming from.	Robert had regained his confidence.

Graph B

1	SE	Can anybody think of a function for B?	
2	Martin	I reckon its sin 3x.	More certain than in episode 1.
3	SE	Sin 3x.	Seeking acceptance.
4	All	Yes.	Confident and firm responses.
5	SE	You seem to agree on that one. So how did you come up with that conclusion?	Question directed at the group.
6	Robert	There don't seem to be any sneaky cosine tricks.	Robert was now wary of the existence of equivalent symbolic forms.
7	SE	Not this time.	
8	Martin	It's a sine wave and it's been er…	Martin paused.
9	Robert	Three times x would condense it.	
10	Martin	It's got a stretch parallel to the x-axis of a third, because it got closer together, or so…	
11	SE	Yes, you're all right it's sin 3x.	

Graph C

1	SE	What about c?	
2	Robert	It's a quadratic so it's either 5 or 7[$y = (x-4)^2$ or $y = (4-x)^2$]. It's positive so I'd say it's – oh there are two positive ones. I'd say 5 [$y = (x-4)^2$] because it's been moved positive.	
3	SE	By four units.	
4	Robert	Yes. So you want one with –4, it's 5.	
5	SE	Do you two agree?	
6	Martin	Er I'm not sure…	
7	Julie	Yes.	
8	SE	While you're thinking about it…	
9	Martin	Yes I think so.	
10	SE	If you have a look at the formulas for 5 and 7, if you were to actually expand those, what would you notice?	

11	Robert	It would in fact still be a positive quadratic.	
12	SE	Yes.	
13	Martin	It would be $x^2 + 16$…	Martin paused to think.
14	SE	If you expanded number 5 what would you get? $x^2 - 8x +$..	
15	Robert	16.	
16	SE	16. So what would you get if you expanded number 7? $16 - 8x + x^2$.	
17	Robert	$-8x + x^2$.	Robert spoke at the same time as SE.
18	SE	So those two actually give the same expansion, so they are one in the same. Either of those formulas would have been the correct one to use, but as you recognised 5 represents the transformation more clearly, that's the one that you plumped for.	

Graph D

1	Robert	Well it's going to have a logarithm involved.	
2	SE	Log, yes.	
3	Robert	There are two of them.	Referring to the two parts of the graph.
4	Martin	It's got…	Martin paused.
5	Robert	So there's going to be a x^2 involved somewhere, 17 perhaps [$y = -\ln x^2$].	
6	SE	17?	
7	Julie	It's a reflection.	
8	SE	Yes.	
9	Robert	Although, since I've never actually seen the graph of the logarithm of x^2…	
10	Julie	Yes.	
11	Robert	No, it would work. It would always make it positive though, so yes.	
12	SE	So do you agree?	Directed at Martin.
13	Martin	I think so. I'm just working out why it's negative.	

Graph E

1	Robert	It could involve an exponential this time.	
2	SE	Yes this is an exponential.	
3	Robert	It's obviously got +4 on the end, so it's either 15, or 18 or 13 even. [$y = e^{-(x+1)} + 4$, or $y = -e^{x+1} + 4$, or $y = e^{x-1} + 4$].	Robert was able to recognise the function as exponential and thus identify the possibilities.
4	Julie	It hasn't been reflected, so it's not 18 [$y = -e^{x+1} + 4$].	Correct assertion.

5	Robert	It's probably 13 actually. [$y = e^{x-1} + 4$]	
6	SE	Why do you say that one?	
7	Robert	Because the negative sign somehow has to fit that [the graph], although I can't explain how the minus sign affects it.	At this point Robert and Julie began to conjecture incorrectly about the effects of the functions on the shape of their graphs.
8	Julie	That's some sort of reflection, isn't it?	Referring to $y = e^{x-1} + 4$.
9	Robert	15 [$y = e^{-(x+1)} + 4$] would be a reflection.	
10	Julie	Why?	
11	Robert	It would be a reflection in x, wouldn't it?	
12	Julie	I don't know.	
13	Robert	18 [$y = -e^{x+1} + 4$] would be a reflection in y. This is like ignoring the transformation of +4, which I'd say is 13 [$y = e^{x-1} + 4$].	
14	SE	Yes you are correct and if you are not sure you can always draw the graphs of them to see which is a reflection in x and which is a reflection in y.	

Graph F

1	SE	Finally F.	
2	Robert	It's a tangent.	There was a pause.
3	SE	Think about the scale the TI-92 uses.	
4	Robert	To see if it was increasing I could just draw the normal graph.	
5	SE	Ok, if it helps you can draw the, you can all draw the tan x graph and see what happens on your machine and then from there you can hopefully deduce what the function is.	
6	Robert	It's a stretch of factor 3.	
7	Martin	It's tan of x over 3.	
8	Robert	Yes.	
9	SE	Is that number 6 or number 19 [y= tan (x/3) or y =(tan x)/3], because there are two of them?	
10	Martin	Number 6 [y = tan (x/3)].	
11	SE	Number 6 and what do you think? Have you managed to get the tan?	Directed at Julie
12	Julie	Yes. That's the whole thing.	Pointing to the tanx in (tan x)/3
13	SE	That's tan of x all divided by 3.	
14	Julie	So yes number 6.	
15	SE	Number 6, yes well done you are right.	

Student Interview Transcripts

Robert

1. SE: So could you please describe to me how you would solve number one.
2. R: Em I 'd first put y = 0 ...
3. SE: Yes
4. R: And use the quadratic formula to find the values of x where y would equal zero...
5. SE: Yes
6. R: And then visualise the graph and since it would be positive I could find out precisely which values lie below the value of y = 0.
7. SE: Yes, so how do you know which way the parabola would go?
8. R: Em experience, if its positive the parabola has a bucket shape, if its negative it has the opposite.
9. SE: So when you say positive and negative what exactly are you referring to?
10. R: The (pause) the sign on the coefficient of x squared.
11. SE: Yes yes great, OK could you describe to me how you would do number two then please.
12. R: (pause) This is probably slightly more difficult. I'd put the two equations equal to each other, actually eliminating y, then get all the values on one side to equal 0 and then use the factor theorem to deduce one or more of the factors and then deduce the last one from it.
13. SE: Yes, so in this case you wouldn't find it necessary to draw a graph at all?
14. R: No I don't think it would really help in the solution.
15. SE: No OK em question 3.
16. R: Em I'd put them equal to each other again ...
17. SE: Yes.
18. R: and square each side to eliminate the modulus sign and then solve the resulting equation, although that might be quite difficult since you've got an x squared on this side (pointing to $y = 6x^2$) so you end up with an x to the power four (pause) so I'd probably end up using factor theorem on that or actually taking a factor of x if that was possible but I don't think it is in this case.
19. SE: What about a graph, do you think it would be helpful to draw a graph in this instance?

20. R: If I found that difficult I would draw a graph and actually use that as in putting three bracket x -2 equal to 6x squared on the positive side ...

21. SE: Mm.

22. R: actually choosing the positive and negative sides of the modulus graph ...

23. SE: Yes.

24. R: and seeing where they would intersect with the other graph but I would use it as a last resort.

25. SE: Yes. That would never be your first step?

26. R: No.

27. SE: OK now question 4 then please.

28. R: I would probably have to draw a graph for that one, to be honest I'm not really sure how I would do it.

29. SE: Are you not familiar with transformations?

30. R: Yes well I'm familiar with transformations, yes.

31. SE: But the effects of transformations?

32. R: Yes. I'd probably put this - I'd probably actually change the actual equation and then draw the graph and see how it changes it.

33. SE: Right, so you'd substitute in x + 3 in here (the expression) and then multiply the whole thing by three?

34. R: In fact I could actually use experience to determine what effect it has.

35. SE: Yes that was what I was thinking.

36. R: This would translate it -3 in the y-axis, this would be a stretch factor 3 parallel to the y-axis.

37. SE: Yes, are you sure about what you said first off you said it would translate it -3 in the y-axis.

38. R: I'm sorry x-axis.

39. SE: x-axis, that's what I thought, you'd just made a little error there. So in actual fact you wouldn't necessarily have to draw the graph, I mean you described a method that would work first off ...

40. R: Right.

41. SE: But you think that you could just deduce from you're experience.

42. R: Yes I would, I would prefer not to draw a graph.

43. SE: Yes yes. OK question 5 then please. This might not be a question that you are familiar with, this style of questioning.

44. R: No (pause) I can't really say I know how to do that, I mean it doesn't, it wouldn't be exponential otherwise you would have irrational numbers so I would assume that it would be quadratic.

45. SE: Right and so you've ruled out the possibility that it's linear?

46. R: Yes.

47. SE: And why have you ruled that possibility out?

48. R: Well to be honest I'm not entirely sure. I would try linear, quadratic and then possibly further, mind you in this case it would probably be easiest to draw it on a graph and see.

49. SE: Yes plot the points ...

50. R: Yes.

51. SE: and see if you could fit a straight line to them or whether its more of a curve shape.

52. R: Yes. Yes as you've probably gathered I would prefer to avoid graphs.

53. SE: Yes, so er once you've deduced what it could be, say for example, it was a quadratic what would your next step be?

54. R: Actually use a general quadratic formula, as in $ax^3 + bx^2 + cx + d$ and actually try and find what values of that would fit it.

55. SE: Mm, but you've just given me the formula for a cubic rather that a quadratic, but I know what you mean.

56. R: OK, yes.

57. SE: OK so how would these values that you have already got help you (pointing at $f(2) = 6$, $f(3) = 14$ and $f(6) = 50$)?

58. R: What do you mean by help me?

59. SE: Well you say your using a general formula ...

60. R: Yes.

61. SE: What are you going to do with that, are you going to try - use a trial and error technique try different values of x using different coefficients of a, b and c or are you going to do something else?

62. R: I would use the general formula as I said and substitute; you've got the value of x in each case and you've got the value of y in each case ...

63. SE: Yes.

64. R: So I would just substitute those in and try and do it and I can't remember the word for it.

65. SE: Simultaneous equations.

66. R: Simultaneous equations.

67. SE: Yes, that's what, that's what I thought you were meaning, but I just wanted to make sure.

68. R: I just find it hard to actually express what I try to do, I just do it instinctively.

69. SE: You just do it yes. OK and finally question 6.

70. R: Differentiation.

71. SE: You'd do some differentiation.

72. R: Yes I'd have to, I mean as soon as it says slope I know that differentiation is involved somewhere in the question, however to actually see how it would change from -5 to 5 I'd do differential - do differentiation find the differential of that (pointing to the function) find the value of the gradient at -5 and 5 and then just take them away from each other.

73. SE: Would that tell you how it actually changes in between -5 and 5?

74. R: What do you mean by change, do you mean as in the difference between the gradients at these two points?

75. SE: No, I mean how the actual slope changes throughout the graph. It might be increasing then decreasing then increasing between those values.

76. R: In that case I would differentiate again.

77. SE: You'd differentiate again?

78. R: Yes, but actually, but mind you in this instance it would be probably be easiest to draw a graph admittedly but ...

79. SE: If you were to draw a graph what would you use the graph for, would you just look at the graph and see how the slope is changing?

80. R: Yes.

81. SE: Between those values ...

82. R: Yes.

83. SE: and you think you would be able to tell what is happening to the slope by just looking at the graph?

84. R: Well I'd get a general idea but it obviously wouldn't be exact I would (pause) to be honest I m not entirely sure.

85. SE: I think that would possibly be the best solution method for this question in this case.

86. R: Actually drawing the graph?

87. SE: Mm, you might want to pick out some values between these (pointing to -5 and 5) and try those in the derivative function, but how many values would you try?

88. R: Every integer, as in -5 -4 -3 etc. up to 5.

89. SE: That could be a possibility.

90. R: But I would be very wary of that because you could get some anomaly in between that.

91. SE: Exactly.

92. R: I mean its pretty obvious that's unlikely from this function, but it would still be in the back of my mind so I would prefer to do it a calculus method, but I'm not entirely sure because I would prefer to do it for every value on that curve.

93. SE: Yes.

94. R: But I just don't have the mathematical skill to do that.

95. SE: OK well, thank you very much.

Martin

1. SE: Right could you please tell me how you would attempt to solve the first question please.
2. M: OK er I'd start off by trying to factorise it ...
3. SE: Mm.
4. M: and then if that didn't work em I'd probably use the em quadratic formula to find the answer.
5. SE: Yes.
6. M: It might end up in like a surd or whatever it would be. I'd get the answers for where it crossed so when y= 0 ...
7. SE: Yes.
8. M: and then em (pause) I think I'd have to draw a graph to see where it was - which part was greater than and which part was less than, but it's not, it's not negative so that means it'll be the right way round. It'll be em the right way round because it's an x squared curve - a quadratic so is it a parabola or something like that.
9. SE: Parabola, yes.
10. M: Yes, so it was probably going to be the least value that its greater than and the

highest value that its less than, but I'd have to check that I think.

11. SE: Yes, so you're saying that it's got a minimum turning point rather than a maximum.
12. M: Yes that's right, yes.
13. SE: OK right question 2, how would you solve that one?
14. M: Right em I'd start off by putting them equal to each other and em group it together so it would end up with em o= x^3 em it would be $-2x^2$ -5x then it would divide through by x.
15. SE: Ah you've forgotten +6.
16. M: Ah +6 yeah I didn't see it, right so it wouldn't - you wouldn't be able to do that.
17. SE: So you couldn't factorise it, you couldn't take a factor of x out rather.
18. M: Yes. Or probably a better - I don't know whether it would be better but an easier way might be to use a pair of simultaneous equations. So I'd put em - so I'd be able to get - it would be $6= y - x^3$ see if I could em swap the other line. I'm not sure I'd probably end up with the same thing.
19. SE: Yes. Would you ever consider drawing a graph?
20. M: Probably if I was really stuck. I'd do it on the calculator, but I wouldn't be sure whether I'd got the - whether I'd got the graph right unless I actually worked some values out and plotted it properly, so ...
21. SE: So you wouldn't consider drawing this graph (pointing to $y = x^3 + 6$) and then drawing this graph (pointing to $y = 2x^2 + 5x$) and seeing where they actually cross one another. Is that what you need?
22. M: Er ...
23. SE: You could do that, that would be an easier graph to draw ...
24. M: Yes, yes.
25. SE: than perhaps when you'd combined them all ...
26. M: Yes.
27. SE: to have one ...
28. M: Yes that's right.
29. SE: So that might be a possibility if working symbolically ...
30. M: Yes.
31. SE: Symbolically doesn't work.
32. M: I think.
33. SE: OK would you like to try question 3.

34. M: OK (pause) what I'd do I'd em put them equal to each to other again I think, oh it would have to be - no it would be an inequality em it would be $3|x-2| = 6x^2$ but then I'd split it up so it would be negative 3(x -2) and positive 3(x-2) and then there'd be two answers and then I'd draw a sketch of the graph to find out exactly the values like with the first one ...

35. SE: Mm.

36. M: Just to check.

37. SE: So you'd do some rearranging once you'd set up these two equations, would you?

38. M: Yes.

39. SE: And solve them and do the graph as a back up would you say?

40. M: Em ...

41. SE: Just to make sure you're answers are correct or to help you see it more clearly?

42. M: Probably to help me see it more clearly because em I'd probably get two em from the inequality I could get em two answers that would be like $x > 3$ and $x > 1/2$ or something like that and then I'd have to see which was right and which was em which was less than, because with the graph it's easier to see which values are which the right way round.

43. SE: Yes OK could you try question 4 then please.

44. M: What it would do, it would be - it would be stretching it parallel with the y axis, yes by a factor of three, so it would get taller and then it would be moving it backwards three so yeah it would be a translation of -3.

45. SE: In which axis?

46. M: The x-axis.

47. SE: Yes. So which of these happens first?

48. M: I think it would be this one (pointing to the translation).

49. SE: Yes, that's right, em ...

50. M: I'm just trying to think whether it would make any difference if it was in the other order.

51. SE: Mm.

52. M: I'd have to try it.

53. SE: So in this case you wouldn't find it necessary to actually draw a graph?

54. M: No because I know what these do, so I've got it straight in my mind what will happen to anything with those functions.

55. SE: So you know the effects that they will have, so that you can just look at the transformation and imagine the effects.

56. M: Yes.

57. SE: OK question 5 then please. (pause) Again this is a rather unusual question.

58. M: Yes. (pause) I'd probably, I don't know whether it would work because we have not done anything like it, is see if it was like a geometric or a arithmetic progression, see if it ...

59. SE: Mm.

60. M: fitted it, I don't think it's an arithmetic progression. I might see if there was a common ratio or something.

61. SE: And if there wasn't, what would you do?

62. M: I'd have to em draw it - plot it see if I could get a correlation - a quadratic or linear or whatever.

63. SE: Right so once you've decided this could be the function, the function could be of this type what would you do next? Say you've decided it looks like a quadratic what would you do then?

64. M: Em I'd probably join the rest of the curve and find the find the intercept, I think that should be it.

65. SE: Yes.

66. M: Use the values and put (them) in em $ax^2 + bx + c$.

67. SE: Using a general formula.

68. M: Using a general formula, because I've got x and y values already.

69. SE: And then you would?

70. M: What do you mean, just to get to get the function?

71. SE: Yes.

72. M: I'd rearrange it, I think.

73. SE: So you'd have three simultaneous equations ...

74. M: Yes, yes.

75. SE: and you'd just use the techniques you are already familiar with to solve those?

76. M: Yes.

77. SE: OK and finally question 6 then please.

78. M: What I'd do, because I'm not sure what the graph looks like for this one ...

79. SE: Mm.

80. M: because we've not done anything - any graph like that, so I'm not familiar with it. I'd have to differentiate it at -5 to find the gradient and differentiate it at 5 to find the gradient and then see if there was a turning point in between, to see whether it was, to see if it had changed from positive or negative to negative.
81. SE: Right, OK so you wouldn't consider actually drawing a graph at all? You wouldn't consider em substituting in values of x and then plotting the points and drawing it that way and then looking how the slope changes from the graph? You would just look at it symbolically?
82. M: Yes.
83. SE: OK right, thank you very much.

<u>Julie</u>

1. SE: OK so could you please tell me how you would solve question 1.
2. J: Em well I'd set it equal to zero to find where it crosses the x axis and then I'd say that em x was between those two values.
3. SE: And how do you know that the curve would be below the x axis between those values?
4. J: Because the x squared is positive.
5. SE: And that influences the shape of the graph, does it?
6. J: Yes.
7. SE: How does it do that?
8. J: It means it will be a bucket shape rather than a hill shape.
9. SE: OK and em question 2 then please.
10. J: (pause) Em I'd put them equal to each other and (pause) ...
11. SE: What would you do next?
12. J: (pause) I would try and get a quadratic equation em ...
13. SE: Could you do that?
14. J: I'm not sure em ...
15. SE: Because you got a cube term here, haven't you ...
16. J: Yes.
17. SE: You've also got a constant term, so you couldn't actually take a factor of x out.
18. J: No (pause) ...
19. SE: Do you think that drawing graphs might help?

20. J: Yes, yes it would.
21. SE: So if you got stuck trying to work it out symbolically, you might consider drawing a graph?
22. J: Yes if I could do it accurately to get the answer.
23. SE: So em what graphs would you draw?
24. J: Em the x^3 and the x^2 graph.
25. SE: And just have a look where they crossed one another and that would be the answer?
26. J: Yes if I drew these graphs (pointing at the two functions).
27. SE: Yes. OK, question 3.
28. J: Well I'd draw both graphs and to make sure I use - whether to use and 3(x-2) and 3(2-x) ...
29. SE: Right.
30. J: and then put them equal to each other and find where they cross.
31. SE: OK so em what would the shape of the modulus graph look like?
32. J: Em well it would be em a straight line it would be a reflection (pause) it would be reflected.
33. SE: Yes. So what kind of shape would you get?
34. J: A 'v' shape.
35. SE: Yes. OK, so in this case you would just draw the graphs and see where they intersect ...
36. J: Yes.
37. SE: and then you would be able to see where this one (pointing at $y = 3|x-2|$) lies above the other (pointing at $y = 6x^2$).
38. J: Yes.
39. SE: OK you would just consider doing it graphically then, you wouldn't use a symbolic approach?
40. J: Well I'd only sketch them.
41. SE: Right.
42. J: I wouldn't use it to find the exact values.
43. SE: So you'd be going to actually use a symbolic approach then.
44. J: Yes.
45. SE: OK, question 4 then please.

46. J: Em well it would be a translation of -3 along the x-axis and a stretch of scale factor three parallel to the y-axis.

47. SE: So you could actually picture the effects of the transformations without actually having to draw the graph?

48. J: Yes.

49. SE: Question 5 then please, little bit different.

50. J: Em well em I'd plot the points and see what kind of a graph it would give you em would it be possible to try different functions to see em how - whether they'd give you results like these?

51. SE: So you might try a linear function, a quadratic function depending how the points ...

52. J: Yes.

53. SE: Fall. So you'd use a trail and error technique then?

54. J: Yes.

55. SE: You might try $x^2 + 3x + 4$ and see how close that is and then try different values, is that what you mean?

56. J: Yes.

57. SE: Do you think that there is any other way that you could, you could do it?

58. J: I think there is, but I'm not sure, I don't know it.

59. SE: So that's the technique that you would use, you would just guess ...

60. J: Yes.

61. SE: and hopefully each guess would inform the next guess, so eventually you would hopefully get to an answer.

62. J: Yes.

63. SE: OK and finally question 6.

64. J: Em I try and draw the graph. I think that that would be the best way to do it - just use the graph and look at how the slope changes.

65. SE: Yes, between those two points.

66. J: Yes.

67. SE: So you'd be able to tell by looking at the graph what's happening to the slope?

68. J: Yes.

69. SE: OK, thank you very much.

Summary of Student Questionnaire Responses

The Role of Mental Imagery and Visualisation in A Level Mathematics

Robert

1. Robert *quite rarely* makes use of mental images in mathematical problems.
2. He constructs mental images to provide a basis for an algebraic method.
3. He usually formulates mental images at the beginning of a problem.
4. More involved mechanics problems involve greater use of mental imagery for him.
5. A few statistical applications cause him visualisation difficulties.
6. He insists that visual solutions are encouraged in class, although he "generally attempts to ignore such suggestions".
7. Robert prefers to work symbolically, since he "generally find this less prone to error".
8. Robert *fairly frequently* combines different approaches, as "a visual approach is most effective as a foundation for a symbolic solution".
9. The ability to perform symbolic manipulations is regarded as being most useful.
10. Robert classifies himself as a *non-visualiser*.
11. He considers his visualisation powers to be *good* overall.
12. He stresses that he "always tends towards a symbolic argument".
13. He describes a visual solution as one involving diagrams or a graph, however he believes "an algebraic solution to be more efficient and more accurate".

Julie

1. Julie *fairly frequently* makes use of mental images in mathematical problems.
2. She constructs mental images to further her understanding of problems - "they can tell me how the graph of an equation behaves or what might happen to a body when forces act upon it".
3. She usually formulates mental images at the beginning of a problem.
4. Certain pure and mechanics problems are considered to involve greater use of mental imagery by Julie. However, she states that "mental imagery does not seem to help in statistics".
5. Statistics questions cause Julie visualisation problems.

6. She insists that visual solutions are "definitely" encouraged in class, and that diagrams are considered to be "essential" in mechanics.
7. She prefers to work symbolically, although she "visualises things more often". In explaining this preference she states "I tend to get the right answers when I work symbolically more often than when I visualise. It can be very difficult to work visually".
8. Different approaches are *always* combined by Julie, as "usually one method alone is not the best way to tackle a problem".
9. The ability to perform symbolic manipulations is regarded as being most important, "as long as you use enough visualisation to know what you are doing".
10. Julie considers herself to be a *visualiser*.
11. She considers her visualisation powers to be *fair* overall.
12. Julie, where possible, will use a symbolic argument but believes that "examiners like to see diagrams".
13. She describes a symbolic solution as "a worked answer without diagrams". She sees visualisation as "occurring on paper or in the mind".

Martin

1. Martin *fairly frequently* makes use of mental images when solving mathematical problems
2. When mental images are constructed by Martin, they are used as tools for meaning making and verification – "I can imagine how points may fall on a graph to see the kind of result I should expect to get on paper".
3. He usually formulates mental images at the beginning of a problem.
4. Applied subjects like mechanics, "where it's modelling a real system" are considered to involve greater use of mental imagery. However, when Martin is unfamiliar with a problem, imagery may not be used.
5. When asked about areas that cause visualisation difficulties Martin re-iterated that "when I am not used to the type of problem, it is not easy to relate it to a graph or system, so I would only use algebra".
6. Martin insists that visual solutions are encouraged in class, although he would "probably try to learn an algebraic method first, until" he was "comfortable" with his "understanding of the methods", so he "may not take as much notice of learning a more visual approach".

7. Martin indicates that initially he prefers to work symbolically, stating that "when I am comfortable knowing how the methods work symbolically I would then find it easier to visualise it, as I can check that I am visualising it right".
8. Martin *fairly frequently* combines different approaches because if he "fully understand the problems" he "can visualise a solution, and then write it down with proof, algebraically".
9. The ability to successfully visualise a problem is regarded as being very important, as Martin believes that "you need a more thorough understanding of the subject".
10. Martin considers himself to be a *visualiser*.
11. He considers his visualisation powers to be *good* overall.
12. Martin "usually uses a symbolic solution, as this is a clear method to show in writing and to check".
13. He describes a visual solution as one "using a graph or model of the problem". A symbolic solution "involves algebra".

<u>Jason</u>

1. Jason *fairly frequently* makes use of mental images when solving mathematical problems.
2. When mental images are constructed by Jason, they are used as tools for meaning making and verification – "they help in the application of the mathematical equations involved and also to check the answers I get, whether they are realistic or not".
3. Jason usually formulates mental images at the beginning of a problem.
4. Pure and mechanics questions are considered to involve greater use of mental imagery. Jason rarely uses imagery in statistics.
5. There are no particular areas that cause Jason specific visualisation difficulties, although he states that "in statistics imagery is not often helpful".
6. Jason insists that visual solutions are encouraged in class, "mostly in mechanics".
7. Jason tends to combine working symbolically with working visually, as "they complement each other". Such an approach would involve "firstly finding an approximate visual answer, then applying the equations to find an exact answer".
8. Jason thus *always* combines different approaches.

9. The ability to choose one particular method, or both, where appropriate is regarded by Jason as being most important.
10. Jason considers himself to be a *visualiser*.
11. He considers his visualisation powers to be *good* overall.
12. It is often obvious that Jason has used a visual method, for it may be drawn as part of his solution.
13. Jason describes a visual solution as one involving a graph or diagram and a symbolic solution as one involving equations.

Diane

1. Diane *always* makes use of mental images when solving mathematical problems.
2. She finds mental images very helpful in problem solving.
3. She uses mental imagery for basic calculations (addition, subtraction, multiplication and division).
4. For Diane, basic calculations require the use of mental imagery.
5. The graphs of sine, cosine and tangent are difficult for Diane to visualise.
6. Diane insists that visual solutions are encouraged in class.
7. No comment is made regarding a preference for symbolic or visual methods.
8. Diane *sometimes* combines different approaches.
9. The ability to perform symbolic manipulations is regarded as being most important.
10. Diane considers herself to be a *visualiser*.
11. She considers her visualisation powers to be *fair* overall.
12. No comment is made regarding the use of visual processes in written solutions.
13. Diane describes a visual solution as one "direct from the brain without any paper work". A symbolic solution "needs paper to work it out".

Rachael

1. Rachael *sometimes* makes use of mental images when solving mathematical problems.
2. When mental images are constructed by Rachael, they are helpful if "she draws them out", and "only if" she "draws them reasonably accurately". Images are especially helpful in mechanics.

3. Mental images are usually formulated by Rachael, for example, after rearranging an equation to help find where lines cross, although not at the beginning of a problem, as she considers this to be "often too complex".
4. Mechanics problems and problems involving functions and equations are considered to involve greater use of mental imagery. Although, she feels that in statistics, imagery is very useful for the normal distribution.
5. Some pure mathematics that she finds "a bit abstract" cause her visualisation difficulties. For example, for "equations with lots of trigonometric functions, I can't really visualise the graphs". Some statistics can, also, be hard for her to visualise.
6. Rachael insists that visual solutions are encouraged in class; in pure – functions/graphs, in mechanics - diagrams, and in statistics - normal distributions.
7. Rachael finds working symbolically "easier", although recognises that it is "sometimes quicker to work things out with just numbers". She would "like to do more maths visually" but "can't apply" such an approach "to some situations, for example, the abstract".
8. Rachael *fairly frequently* combines different approaches. When doing graphs, she draws out sketches and uses them to help her with the equation. These are, also, useful for checking answers.
9. Neither the ability to perform symbolic manipulations nor the ability to visualise is regarded by Rachael as being of greater importance and she states "you have to be good at both, each will help give a balanced approach to maths".
10. Rachael considers herself to be a *visualiser*.
11. She considers her visualisation powers to be *fair* overall.
12. In mechanics it is not very obvious that Rachael has used visual processes, although she often draws diagrams. She, also, usually draws out the graphs in her solutions.
13. Rachael describes a visual solution as one involving a graph or diagram and a symbolic solution as one containing equations or numbers.

Robert

1. Robert believes that technology is "very useful in speeding up calculations", however, he would "prefer to know how to carry out calculations on paper before using alternative technology".
2. He feels that he has benefited to a "small extent" from the opportunity to use the TI-92. It has given him "greater experience of the technology applicable to mathematics".
3. Robert does not feel that the TI-92 has enabled him to visualise functions more clearly, as he has "already had reasonably extensive experience of visualising functions".
4. Similarly, he does not believe that using the TI-92 has strengthened his understanding of functions - he does not "need the aid of a graphical calculator", since he "prefers to use the ideas of transformations".
5. This experience will "possibly" influence his approach to solving problems involving functions in the future. He may "use graphical methods more when solving the more involved problems".
6. He considers the main advantages of using technology to teach the concepts of functions to be that "technology will graph very quickly and help students to recognise and visualise characteristics of many functions".
7. As a disadvantage, Robert believes that "the concept of functions is best taught in a more traditional manner, so students might gain a more profound understanding of the field".
8. He would "possibly" welcome further use of technology, although he stresses that "technology is useful as an aid for analysis, yet understanding is best developed through algebraic methods".
9. Robert felt unable to offer any preference for using either a graphical calculator or computer, as he has not, as yet, used a computer in his A level mathematics lessons.
10. The materials and exercises were considered to be "quite helpful", however, Robert prefers to "experiment on a trial and error basis, with the use of an instructional manual".

Julie

1. Julie believes that "a graphical calculator can be extremely useful" in the A level mathematics classroom.
2. She feels that she has benefited from the opportunity to use the TI-92, as "its graphical capabilities are greater" than her "calculator's own".
3. Julie does not feel that the TI-92 has enabled her to visualise functions more clearly, as "having already got a graphical calculator" means she "can do this already", as she has been "using it for a while".
4. Similarly, she does not believe that using the TI-92 has strengthened her understanding of functions - she was "taught how to do transformations before, and had to learn all the basic graphs anyway".
5. This experience will not influence her approach to solving problems involving functions in the future, as her "methods are usually visually based already".
6. She considers the main advantage of using technology to teach the concepts of functions to be that "students are free to spend time manipulating the different graphs of functions to learn how they work".
7. As disadvantages, Julie suggests that "students must be taught how to use the equipment" which "is expensive".
8. She would welcome further use of technology, as "visualisation does give a good basis for solving a problem and can make certain ideas clearer to students".
9. Julie also felt unable to offer any preference for using either a graphical calculator or computer, as she has not, as yet, used a computer in her A level mathematics lessons.
10. The materials and exercises were considered to be adequate – "the instructions for use ensure that you are able to work through the exercises without difficulty and all the instructions necessary are given".

Martin

1. Martin believes that "technology such as calculators are almost essential" in the A level mathematics classroom. In addition, he feels that "graphic calculators /computers can be helpful and speed up calculating answers" although he stresses that "if they are used all the time it would be harder to work without one".
2. He feels that he has benefited from the opportunity to use the TI-92 – "the TI-92 is useful because it can simplify equations so can help check answers. Graphs are

easy to understand when seen plotted and it has helped me to see how graphs can be manipulated".

3. Martin feels that the TI-92 has enabled him to visualise functions more clearly, although if he does not "fully understand" the mathematics "it can be confusing" when the machine "gives an unexpected result". He, also, comments that the TI-92 has "definitely" helped him to "see how functions can be related and manipulated through translations and stretches".
4. Martin, also, feels that using the TI-92 has strengthened his understanding of functions - when he "understands the theory behind the functions it helps to see how the graphs work on a screen".
5. This experience will influence his approach to solving problems involving functions in the future, as he feels that he will be "more comfortable in using a visual method such as plotting points and drawing sketches when solving problems".
6. Martin considers the main advantage of using technology to teach the concepts of functions as "the ability to quickly see how a function will change with certain things happening to it".
7. As a disadvantage, he suggests that "it is harder to take notes or revise from a lesson where lots of work was done using a calculator, as either the original functions or printouts were used".
8. Martin would welcome further use of technology. He feels that "using technology for demonstrating things in mechanics" would be a particular advantage and that "any other uses would be an advantage".
9. He would prefer to use a computer, despite their size, "because of the printer and better display". Although, he does appreciate that graphical calculators "have an advantage of being portable" they unfortunately "have limited display resolution and can't print out".
10. The worksheet and demonstrations were thought to "provide enough information to learn the functions of the calculator and they, also, helped to revise and learn methods of solving equations".

Summary of Staff Questionnaire Responses

The Role of Mental Imagery and Visualisation in A Level Mathematics

Ms. Slater – Statistics Teacher

1. Ms. Slater believes that it is "helpful" for students to be able to visualise mathematically at this level, "but not essential"- "if students can work symbolically and succeed, then the visualisation can come later".
2. She encourages the use of visual solutions in her A level mathematics lessons and seems to associate visual approaches with the use of graphical calculators – "I usually use graphical calculators very early on in the A level course. This seems to encourage some students to obtain their own calculators".
3. She has not been involved in the teaching of functions.
4. She has no preference for working either symbolically or visually. She was taught to work symbolically, but now works visually as well.
5. Neither the ability to perform symbolic manipulations nor the ability to visualise is regarded by this teacher as being of greater importance - they are seen to be of equal weight.
6. Ms. Slater considers herself to be a *visualiser*.
7. She feels unable to comment on the visual capabilities of this group of students as she teaches statistics.
8. Technology is regarded as "quite important" in supplementing and enriching students' visual capabilities, although she feels that "the benefit would be greater if all students had access to the technology at times other than lesson times".
9. Technology is used "quite often early on in the course" and "less so as we proceed to the 2nd year". "Many students do not have graphical calculators and are reluctant to use them as time goes on".
10. No comment is made regarding more frequent use of technology.

Ms. Mooney - Mechanics Teacher (with some pure mathematics)

1. Ms. Mooney believes that it is important for students to be able to visualise mathematically at this level.
2. She encourages the use of visual solutions in her A level mathematics lessons.

3. When teaching functions to year twelve students she spends "more or less equal amounts of time" on the graphical, symbolic and numerical aspects, although she does "stress graphical aspects more".
4. She prefers to work visually, since this "conveys more information" and is "easier to interpret".
5. The ability to perform symbolic manipulations and the ability to visualise are regarded by this teacher as being of equal importance – "the ability to perform symbolic manipulations depends on good logical thought processes and sequential ordering".
6. Ms. Mooney considers herself to be a *visualiser*.
7. She regards the visual capabilities of this group of students as "not particularly good on the whole". Although she emphasises that "some are obviously better than others" and that "the Chinese student (Diane) seems to have particular difficulty".
8. Technology is regarded as "very" important in supplementing and enriching students' visual capabilities, although "familiarity obviously helps" and "using technology means it is easy to become very familiar".
9. As Ms. Mooney "mostly teaches mechanics with this group", technology is not used "a great deal". She tends to use technology "much more when teaching pure".
10. If possible she would use technology more often with her A level students, stating that "having a classroom computer equipped with good software would help".

Mr. Pearson – Pure Mathematics Teacher

1. Mr. Pearson believes that it is important for students to be able to visualise mathematically at this level – "at this level most topics are a combination of symbolic and visual i.e. graphical and diagrammatic techniques".
2. He encourages the use of visual solutions in his A level mathematics lessons and also seems to associate visual approaches with the use of graphical calculators – "most students have access to a graphical calculator to graph functions and these are used in conjunction with other techniques of analysis. Diagrams, sketches, wave-forms, triangles etc give an indispensable part of most solutions".
3. When he teaches functions to Year 12 students the "symbolic aspects predominates initially then the graphical and numerical aspects gain equal weighting". He emphasises that "the functions topic requires both of these approaches be covered

in any case as the symbolic dictates understanding of the domain, range, graphical relationships etc".

4. He does not express a preference for working either symbolically or visually – "a diagram accompanies almost all" of his solutions and "in many cases is an integral part of the technique for solving a problem". He believes that "the symbolic and visual aspect of mathematics are inextricably linked".
5. He comments on the current emphasis on the ability to perform symbolic manipulations – "symbolic manipulation is the stumbling block for most would-be mathematicians, and in many cases causes the most anxiety. Even if a problem is visualised the technique for solving invariably requires algebra - and this causes most of the mistakes, errors and incorrect solutions".
6. Mr. Pearson positions himself in the middle of the continuum between *visualiser* and *non-visualiser*.
7. He regards the visual capabilities of all six of these students as "good, complemented by regular use of graphical calculators". He adds that this is a "very good group of further mathematicians".
8. Technology is regarded as of "growing importance in the teaching and enriching of mathematics in general". In particular, "visual capabilities are supplemented and learned by the use of technology".
9. Graphical calculators are used quite regularly in Mr. Pearson's lessons, since "graphical calculators are increasingly available". However, "computers have never been successfully provided as a resource for maths teaching always being distant and dominated by IT teachers and their needs".
10. "Given adequate training and time to learn the technological side so that one had confidence in using the hardware and software packages" this teacher would "welcome the more frequent use of technology with A level students". He stresses that "I have been teaching maths for 13 years and in every single year technology in the math curriculum has been an issue - but no-one has ever put the cash in to develop IT within maths".

APPENDIX C

Phase 3 Data Collection

Contents

Summary of Student Questionnaire Responses

The Role of Mental Imagery/Visualisation in A level Mathematics

In the analysis that follows the total number of students is 17 (13 male and 4 female). However, it is important to note that for some questions certain students' responses were included in more than one category of analysis and that there were occasions where no responses were given.

Question 1
How frequently do you form and make use of mental images when solving mathematical problems? Please circle the appropriate response.

Always Fairly Frequently Sometimes Quite Rarely Never

	Male	Female
Fairly Frequently	3	3
Sometimes	8	0
Quite Rarely	2	1

Question 2
If you do construct mental images, how do you feel these images assist you in problem solving?

	Male	Female
To see the problem more clearly	7	3
To verify or inform symbolic answers	1	2
To help with particular areas of mathematics	1	0
To make clear relationships	1	0
To set out and organise the information in the question in a useable format	1	0
To save time	1	0
Offers an alternative approach to problem solving	1	0

Question 3
At what stage during problem solving do you usually find it necessary to formulate mental images?

	Male	Female
At the beginning of the problem	5	3
When experiencing difficulty finding a solution	4	1
Depends on the perceived complexity of the problem	4	0
Not immediately – need to examine problem first to formulate algebraic understanding	1	1
Depends on the type of problem	1	0

Question 4

Does the type of problem or topic area effect your use of mental imagery? (For example, are there certain areas of mathematics in which you use mental imagery more often?)

	Male	Female
Graphs and functions	6	2
Geometry	5	2
Shapes	2	1
Sequences	2	0
Inequalities	1	0
Negative numbers	1	0
Statistics	1	0
More complex or new problems	1	0
No difference	1	0

Question 5

Are there any particular areas of mathematics that you find difficult to visualise?

	Male	Female
Graphs of functions	5	0
Sequences and series	3	1
Inequalities	1	0
Shapes	1	0
Mechanics	1	0
Large numbers	1	0
Negative numbers	0	1
Factorisation	0	1
No difficulties	1	0
No response	1	1

Question 6

Are visual methods of solution encouraged in your A level mathematics lessons?

	Male	Female
Encouraged for some topics	5	3
Used to explain new concepts	2	0
Encouraged for harder questions	1	0
Encouraged use at the beginning of problems	1	0
Encouraged if written down	1	0
Encouraged with substantiation	0	1
Encouraged voluntarily, not enforced	1	0
Not specifically emphasised	2	0
Not encouraged	1	0

Question 7
Do you have a preference for working either symbolically or visually? Please explain your response.

	Male	Female
Symbolically	5	1
Visually	1	1
No preference	7	2

Question 8
In general, how often do you combine different approaches (such as visual and symbolic) when solving individual mathematical problems? Please circle as appropriate.

Always Fairly Frequently Sometimes Quite Rarely Never

	Male	Female
Fairly Frequently	5	2
Sometimes	9	2

Question 9
In order to become a successful mathematician which do you regard as most important, the ability to perform symbolic manipulations or the ability to visualise mathematically?

	Male	Female
Symbolic manipulations	6	1
Ability to visualise	2	0
Neither	5	3

Question 10
How would you classify yourself essentially? Please indicate where you would place yourself on the following hypothetical continuum by marking the line with a cross.

Visualiser |---| Non-visualiser

	Male	Female
Visualiser	7	3
Non-visualiser	6	1

Question 11
How would you rate your visualisation powers overall? Please circle the appropriate response.

Very good Good Fair Poor

	Male	Female
Very good	1	0
Good	4	3
Fair	6	2
Poor	2	0

Question 12
If you do make use of visual processes is this always obvious in your solutions, or do you tend to use a different argument for written purposes?

	Male	Female
Obvious	5	2
Sometimes apparent	2	0
Not obvious / written symbolically	5	1
No response	1	1

Analysing the Effects of Technology in the A Level Mathematics Classroom

In the analysis that follows the total number of students is 16 (12 male and 4 female). However, it is again important to note that for some questions certain students' responses were included in more than one category of analysis and that there were occasions where responses were not given.

Question 1
How important, in your opinion, is technology in the A level mathematics classroom?

	Male	Female
Very important	4	1
Important	1	1
Fairly important	3	1
Not very important	4	1

Students regarded technology as important because of the following reasons:

	Male	Female
Speeds up work	4	4
Reduces errors/more accurate	1	1
Allows more work to be done	1	0
Helps when checking answers	1	0
Introduces different ways to approach problems	0	1
Helps with difficult work	1	0

Question 2
Do you feel that you have benefited from the opportunity to use the TI-82?

	Male	Female
Beneficial for learning about functions and graphs	5	3
Beneficial to learn how to use technology	3	1
Made working quicker/easier	1	3
Beneficial for learning new methods of solving problems	1	0
Given confidence in using graphical calculators	0	1
Allowed access to harder equations	0	1
Beneficial only if graphical calculators are made available in future lessons	1	0
Beneficial although it will be more difficult to solve functions without using a graphical approach in the future/some understanding has been taken away from the algebra	2	0
Have own graphical calculator already	2	0

Question 3
Has the TI-82 enabled you to visualise functions more clearly?

	Male	Female
Yes, by enabling you to draw several functions and thus to recognise the graphs	5	2
Yes, the graphic calculator sketches are clear, accurate and easy to produce which helps with visualisation	3	0
Yes, the graphic calculator has enabled me to see the graphs of functions more clearly	2	1
Yes, by helping me to imagine what graphs look like just by looking at the formula of the function	0	1
Yes, especially the more complex functions	1	0
Yes, without the graphical calculator I think that I would have struggled, I find it hard to visualise graphs and shapes	1	0

Question 4
Do you believe that using the TI-82 has strengthened your understanding of functions?

	Male	Female
Overall understanding of functions has improved	3	1
Has helped with the understanding of particular aspects of the work with functions, e.g. inverses, transformations, recognising graphs of families of functions, uses	3	3
Being able to visualise functions quickly has allowed more exploration of the subject, and has thus helped understanding	2	0
For complicated functions, any mistakes are highlighted and you are able to see what you are working with	1	0

Graphical skills have been strengthened but symbolic skills have not	3	0
Functions can be pictured clearly, but less thought about the functions is required	1	0
The technology only strengthens understanding once a certain level of initial understanding has been reached	1	0

Question 5
Will this experience influence your approach to solving problems involving functions in the future?

	Male	Female
Yes, a more graphical/visual approach will be adopted	3	1
Only if given access to a graphical calculator	2	1
Possibly depending on the situation	1	1
Would use a graphical calculator in future if possible	1	1
Yes, as a means of verifying symbolic methods	1	0
Have own graphical calculator and so am used to using a graphical approach already	2	0
No, this experience will not change the way I approach problems	1	0
No response	1	0

Question 6
What do you consider to be the main advantages of using technology to teach the concept of functions to students, such as yourselves?

	Male	Female
Problem solving becomes quicker/ more work can be done	3	4
Clear accurate solutions are provided which minimises confusion	2	0
Enables you to try many different examples	2	0
Helps with understanding the translations of functions	2	0
It is easy to sketch graphs on the graphical calculator and removes the need to draw graphs by hand	1	1
Offers an alternative approach to algebra	1	0
Beneficial for those who find visualisation difficult	1	0
Gives everyone the opportunity to obtain the right answer from which understanding can be developed	1	0
Using technology is enjoyable and motivational	1	0
The technology enables the teacher to quickly show examples to the class, which otherwise might not be possible due to time constraints – a situation which could contribute to student confusion	1	0
Good to have a better understanding of technology for the future	1	0

Question 7
What disadvantages do you perceive?

	Male	Female
Becoming too reliant on the technology and experiencing difficulty in working things out without it	8	1
Being drawn away from algebra and theory	4	0
Will do less working out and less mental/manual problem solving	1	1
Having to know how to use them properly	0	2
Expense	0	1
Easily misused or played with in lesson	1	0

Question 8
Would you welcome further use of technology for exploring different areas of mathematics?

	Male	Female
Further use of technology would be welcomed if beneficial	7	1
Use of graphical calculators is enjoyable and interesting, and would be welcomed again	1	2
As technology helps understanding future use would be beneficial	1	1
Any technological skills are important and useful	0	1
Future use of technology would be welcomed as a whole class activity	1	0
Future use would be encouraged but only in specific areas, to avoid taking the skill out of mathematics	1	0
Technology would be welcomed but not too much as students may become over reliant on the machines	0	1
Future use of technology would not be welcomed as too little thought is required	1	0

Question 9
Do you have any preference for using a graphic calculator in your A level mathematics lessons rather than a computer, or vice versa?

	Male	Female
Prefer graphical calculator – more portable and compact	6	1
Prefer graphical calculator – more accessible	3	0
Prefer graphical calculator – easier to operate	3	0
Prefer graphical calculator – specifically set up with mathematical functions, despite more advanced software and greater memory associated with computers	1	0
Prefer graphical calculator – but a computer may show clearer images/have more functions	1	1
Prefer computer – it covers more areas in mathematics	1	0

No preference for either	1	2
Not familiar with computers	1	1

Question 10
How helpful have you found the materials and exercises designed for use with the TI-82?

	Male	Female
Very helpful/assisted understanding	2	2
Helpful in exploring functions and applications of the TI-82	3	0
Quite helpful in looking at a new way of exploring functions	1	0
Covered a wide range of knowledge	0	2
Great use of overhead projector	1	0
Useful for finding functions on my own graphical calculator	1	0
Very good but repetitive towards the end	1	0
Fairly helpful but not enough detail is given and sections were covered too quickly	1	0
Good but would have liked answers to check work	1	0
Useful but the wording is sometimes confusing	1	0
Prefer teacher led explanations rather than working with a sheet of examples	0	1

Question 11
If you had permanent access to a graphical calculator do you think this would have affected the way in which you answered the student interview questions?

	Male	Female
The graphical calculator would have been used to answer more questions as this experience has resulted in increased confidence in using the technology	1	2
The graphical calculator would have been used to speed up the calculations	1	1
The graphical calculator would have made the questions easier to complete	2	0
The graphical calculator would certainly have been used more	1	1
The graphical calculator would help with working out, checking answers and experimentation	2	0
Graphical solutions would be considered before algebraic approaches as visual understanding of the problem would improve	1	0
Some questions would have been solved on the graphical calculator – others would still be easier/preferable to do symbolically	2	0
The graphical calculator would make no difference/already have own graphical calculator	3	0
No response	1	0

Question 12

Have you felt that use of the graphical calculator has encouraged group discussions (paired or whole class)?

	Male	Female
The graphical calculator has encouraged paired discussions	4	1
The graphical calculator has encouraged small group discussions	6	2
The graphical calculator has encouraged whole class discussions	1	0
The graphical calculator has encouraged discussions in general	1	0
The graphical calculator has had no particular effect on discussions – neither encouraging or discouraging	2	0
The graphical calculator has not encouraged group discussions	0	2
The graphical calculator promotes individual work	1	0

Table 18 Comparison between phases of the students' views surrounding visualisation

	Phase 1	Phase 2	Phase 3
Images are used to derive meaning from the question	N/A	67%	65%
Images are generally formed at the beginning of problem solving	N/A	83%	47%
Images are generally formed when experiencing difficulty finding a solution	N/A	17%	29%
Images are formed depending on the complexity of the problem	N/A	17%	24%
Images are not formed immediately. Symbolic understanding is sought first.	N/A	17%	12%
Students prefer working symbolically	N/A	50%	35%
Students prefer working visually	N/A	0%	12%
Students have no particular preference for working either visually or symbolically	N/A	50%	53%
Students believe that symbolic skills are more important in becoming a successful mathematician	N/A	50%	41%
Students believe that visual abilities are more important in becoming a successful mathematician	N/A	17%	12%
Students feel that visual abilities and symbolic skills are equally important	N/A	33%	47%
Students classify themselves as visualisers	62%	83%	59%
Students classify themselves as non-visualisers	31%	17%	41%
Visual methods of solution are apparent in written solutions	N/A	33%	53%

Table 19 Comparison between phases of how students viewed the use of technology

	Phase 1	Phase 2	Phase 3
Technology is viewed as a valuable addition to the A level mathematics classroom	77%	100%	81%
Using the graphical calculator was seen to be beneficial to the students' learning about functions	92%	100%	81%
Using the graphical calculator promoted the students' abilities to visualise the graphs of functions	92%	33%	100%
Using the graphical calculator strengthened the students' understanding of functions	92%	33%	75%
Amplification effects were considered to be key advantages of using the graphical calculator	77%	100%	56%
Cognitive factors were considered to be key advantages of using the graphical calculator	23%	33%	56%
The role of the graphical calculator in providing a means of verification was considered to be a key advantage of using this technology	31%	0%	6%
Students feared over reliance on technology	62%	66%	81%
Students would welcome further use of technology	92%	100%	94%
The students' experiences in this trial would influence future work	N/A	66%	75%

Student Interview

Please describe how you would attempt to solve the following questions:

1. For which values of x is the graph of $y = 3x^2 + 9x - 12$ below the x-axis?

2. For which x values does the graph of $y = 3x + 6$ intersect with the graph of $y = 2x^2 + 5x$?

3. Find the values of x for which the graph of $y = x^2 - x + 4$ lies above the graph of $y = 4x - 2$.

4. Describe how the graphs of $y = x^2 + 3x - 2$ and $y = x^2 + 3x + 2$ are related. Similarly, what is the connection between the graphs of $y = x^2 + 2$ and $y = x^2 + 2x + 3$?

5. How does the slope of the function $y = 2x^2 - 3$ change from $x = -3$ to $x = 3$?

6. When throwing a single biased dice, the probability of getting a 2 is 0.1, a 3 is 0.12 and a 6 is 0.3. All the probabilities can be worked out using a particular quadratic formula. Explain how the probabilities of getting 1, 4 and 5 can be deduced and how the quadratic formula could be obtained.

7. A car is started from rest and accelerates uniformly to a speed of v m/s in 12 seconds. The speed is maintained for a further 55 seconds and then the brakes are applied and the car decelerates uniformly to rest. The deceleration is three times greater than the initial acceleration. Sketch the velocity-time graph and calculate the deceleration time.

   The total distance travelled is 945m. Hence, calculate the value of v and the initial acceleration.

The Main Trial Exercises

Graphing Functions using the TI-82

1. Compare the graphs of cos x, 2cos x and 3cos x using the TI-82. Sketch the graphs and comment on the main features.

2. Repeat question 1 with tan x, 2tan x and 3tan x.

3. Compare the graphs of cos x, 2cos 2x, 3cos 3x and sketch them. Explain why these three graphs do not cross the x-axis in exactly the same places.

4. Given that $f(x) = x^3$, use the TI-82 to obtain the graph of $g(x) = f(x/2)$. Sketch the two graphs and write down the equation of the new function, g(x).

Now use the TI-82 to perform the transformation $g(x + 2) - 3$ on g(x). Sketch the resulting curve, h(x) and again write down its equation, in the form $ax^3 + bx^2 + cx + d$.

Finally use the TI-82 to perform the transformation 2(h (x)) on h(x), sketching the curve; l(x) and writing down the resulting equation, as before.

5. Investigate the relationship between the functions $y = 4x^2 + 5$ and $y = \frac{(x - 5)^{1/2}}{2}$ using the TI-82.

6. What transformation when applied to the given graphs below form the second pictured graphs and what is the symbolic form of each of the new functions?

a) $y = 1/x$ (ZoomStd)

b) $y = \frac{3x - 9}{x^2 + 4x}$ (ZoomStd)

c) $y = \operatorname{cosec} x$ (ZoomTrig - ZoomOut Factor 2.5)

d) $y = \dfrac{2x - 1}{(x - 1)^2}$ (ZoomStd)

e) $y = \ln (x + 3)$ (ZoomStd)

f) $y = (1+x)e^x$ (ZoomStd)

7. Use the TI-82 to show that $\sin^2 x + \cos^2 x = 1$

8. Solve the following equations graphically and algebraically:

a). $x^2 + 2x - 8 = 0$ b). $8x^2 + 4 = (x - 2)^2$

9. Solve the following inequalities graphically and algebraically:

a). $x^2 - 5x - 9 > 0$ b). $(x - 3)^2 < 6x$ c). $6x - 3x^2 < -15$

10. Solve these simultaneous equations graphically and algebraically:

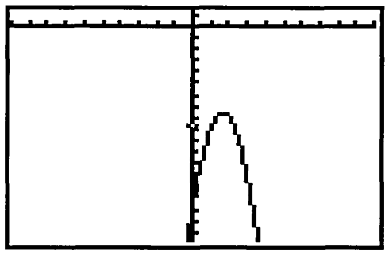
$x - 3y = 16,\ x^2 - 4y^2 = 13$

11. Without using the TI-82 sketch the following functions (each pair on the same axes): a). $y = x^3$ and $y = 6x^3 - 3$ b). $y = \sin x$ and $y = \sin (x + 90^0)$

c). $y = x^2$ and $y = 2(2x - 1)^2 + 9$ d). $y = x^2$ and $y = -(x - 1)^2 + 3$

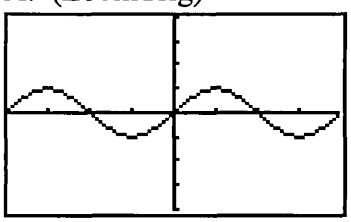
Explain in words how the second graph can be obtained from the first graph in each case. Now use the TI-82 to check whether your sketches are correct.

12. Use the TI-82 to try to determine by a process of informed trial and error the formula of the function which is graphed below using ZoomStd. Explain the reasoning behind every step you make.

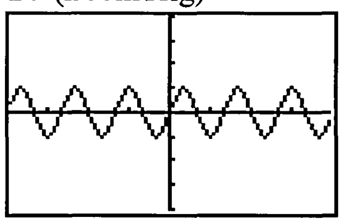


13. Match up the six graphs with their corresponding functions, chosen from the list below.

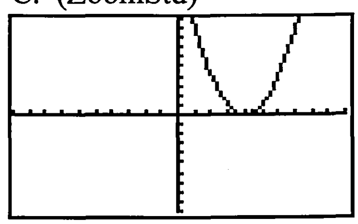
A. (ZoomTrig)



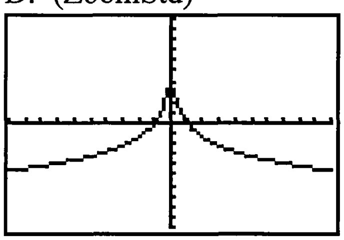
B. (ZoomTrig)



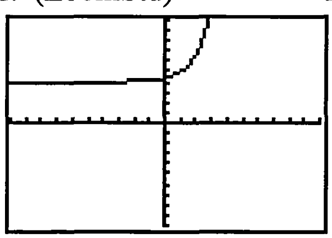
C. (ZoomStd)



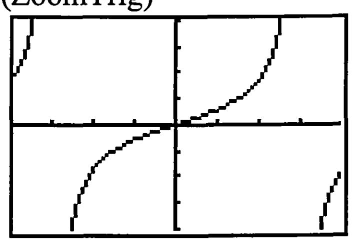
D. (ZoomStd)



E. (ZoomStd)



F. (ZoomTrig)



1. $y = \sin (x/3)$
2. $y = \cos (x - 90^0)$
3. $y = 3\sin x$
4. $y = \cos (x + 180^0)$
5. $y = (x - 4)^2$
6. $y = \tan (x/3)$
7. $y = (4 - x)^2$
8. $y = \tan (x/6)$
9. $y = (x + 4)^2$
10. $y = \cos (x + 90^0)$
11. $y = \sin 3x$
12. $y = \ln (1/x)$
13. $y = e^{x-1} + 4$
14. $y = \ln x^2$
15. $y = e^{-(x+1)} + 4$
16. $y = 2\ln x$
17. $y = -\ln x^2$
18. $y = -e^{x+1} + 4$
19. $y = (\tan x)/3$
20. $y = (\tan x)/6$

Post Trial Staff Technology Questionnaire*

Analysing the effects of technology in the A level Mathematics Classroom

Name ____

Q1. What would you hope to gain most by using technology in A level mathematics?

Q2. Do you feel that the TI-82 has had any affect on students' abilities to visualise the graphs of functions?

Q3. What do you consider to be the main advantages of using technology to teach the concept of functions to students?

Q4. What disadvantages do you perceive?

Q5. Do you see any future potential for using the TI-82 in your classroom?

Q6. Are there any ways in which the materials used in this project designed for use with the TI-82, aimed at enhancing students' visual capabilities, could be improved?

I would be grateful for any additional comments or suggestions:

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* The pre trial staff questionnaire on visualisation was identical to that used in Phase 2.

Student Performances in the Trial Exercises

The student's solutions to the questions from the main trial exercises, were compared, evaluated and graded (see tables) with reference to the following criteria:

5 - Correct solution	2 - Poor solution, several errors and omissions
4 - One omission/error	1 - No understanding shown
3 - Two omissions/errors	0 - No solution offered

Table 20 Questions from the Main Exercises involving more than one Aspect

	First Grade	**Second Grade**
Q4	Accuracy of graphs	Accuracy of symbolic forms
Q6	Identification of the type of transformation	Deduction of the symbolic form of the transformation
Q8	Graphical solution	Algebraic solution
Q9	Graphical solution	Algebraic solution
Q10	Graphical solution	Algebraic solution
Q11	Accuracy of graphs	Accuracy of explanations concerning the actions of transformations

Table 21 Student Performances in the Main Exercises

	Q1	Q2	Q3	Q4a		Q4b		Q4c		Q5	Q6a		Q6b		Q6c		Q6d		Q6e		Q6f	
Carol	5	5	4	1	4	0	1	0	0	3	5	5	5	5	5	5	5	3	0	1	0	5
Claire	5	5	5	5	5	1	1	1	1	3	5	5	5	5	5	5	5	3	5	1	5	5
Fay	5	5	4	1	4	0	0	0	0	3	5	5	5	4	5	5	5	3	5	1	0	5
Jake	4	4	4	5	5	4	5	4	5	4	5	5	4	5	4	5	0	5	0	5	4	5
Jim	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Julian	4	4	4	4	5	3	4	0	0	4	5	5	4	5	5	5	5	3	5	5	4	5
Justin	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Kirk	4	4	4	4	5	4	5	0	0	4	5	5	5	5	5	5	0	4	0	5	0	5
Marie	3	3	3	5	5	0	1	0	1	4	5	5	5	5	5	5	5	5	5	4	1	1
Marty	4	4	4	5	5	4	4	4	4	4	5	0	5	0	5	0	1	0	0	0	5	0
Mick	5	4	4	2	1	0	0	0	0	5	0	5	0	5	0	5	0	5	0	5	5	5
Marvin	3	3	3	5	5	0	0	0	0	2	5	0	5	0	5	0	5	0	5	0	5	0
Nigel	4	4	2	3	1	0	0	0	0	4	0	4	0	5	0	5	0	0	1	0	2	0
Paul	4	4	4	4	5	4	4	0	0	3	5	5	5	0	5	1	5	0	5	5	1	1
Perry	4	4	4	4	1	0	0	0	0	4	5	5	5	5	5	5	0	5	0	1	5	5
Pierce	3	3	3	5	5	0	0	0	0	2	5	0	1	0	5	0	5	0	5	0	4	0
Roy	4	4	4	5	5	1	4	4	4	5	0	5	0	5	0	5	0	1	0	5	0	5

Table 22 Student Performances in the Main Exercises (continued)

	Q7	Q8a		Q8b		Q9a		Q9b		Q9c		Q10		Q11a		Q11b		Q11c		Q11d		Q12
Carol	5	5	5	4	4	2	1	0	3	0	2	0	2	0	0	0	0	0	0	0	0	0
Claire	5	5	5	1	3	5	5	5	5	3	1	0	2	4	0	2	0	1	0	0	0	0
Fay	5	1	5	2	3	3	3	4	5	3	1	0	2	4	0	2	0	4	0	4	0	0
Jake	5	0	4	0	5	0	2	0	2	0	2	0	0	4	0	4	0	1	1	1	1	5
Jim	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Julian	5	0	5	0	4	5	5	5	5	5	5	4	5	5	5	3	5	1	5	3	3	5
Justin	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Kirk	5	0	5	0	5	0	5	0	4	0	4	0	1	3	0	3	0	3	0	3	0	0
Marie	5	5	5	3	4	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Marty	5	5	5	5	5	5	4	4	3	4	3	0	3	5	5	1	5	1	5	1	5	0
Mick	1	0	5	0	5	0	2	0	4	0	3	0	4	5	0	5	0	5	5	5	5	5
Marv	5	3	5	0	5	4	4	4	4	0	2	0	0	0	0	0	0	0	0	0	0	0
Nigel	0	2	5	2	5	0	2	0	0	0	0	0	0	4	4	4	4	1	1	4	2	0
Paul	5	5	5	1	4	5	5	5	5	3	2	0	5	5	5	4	4	5	5	5	0	5
Perry	1	0	5	0	4	0	1	0	3	0	2	0	5	5	5	4	5	5	5	5	5	5
Pierce	5	4	5	4	5	5	4	5	4	5	4	0	0	0	0	0	0	0	0	0	0	0
Roy	0	5	5	4	5	0	5	0	5	0	2	0	4	5	0	4	0	4	0	4	0	0

The students' work from their textbook exercises was also assessed. The grades that were awarded still ranged from 0-5, but also included half point credits to account for the fact that the students were being given one overall grade for each exercise that consisted of several questions.

Table 23 Student Performances in their Textbook Exercises and Question 13 from the Main Trial Exercises

	Exercise 3A	Exercise 3B	Exercise 3C	Exercise 3D	Identifying functions (Q13)
Carol	4	2.5	3	3	3.5
Claire	4	5	4.5	4	3.5
Fay	4	2.5	3	3	3.5
Jake	3	3.5	3.5	2.5	0
Jim	0	3.5	0	0	0
Julian	5	4	3.5	4	3.5
Justin	4	4.5	3	0	0
Kirk	3	3.5	3.5	3.5	0
Marie	3	4.5	3.5	5	3
Marty	3.5	4	4.5	4	4.5
Mick	4	4	3.5	5	4.5
Marvin	4.5	4.5	4.5	4.5	4
Nigel	3	4.5	3	2.5	2
Paul	3	4	3.5	4	4
Perry	4	3	3.5	3.5	4
Pierce	5	4	3.5	4	3.5
Roy	4.5	5	5	4.5	5

Transcripts from Phase 3
Classroom Discussions Surrounding New Function Concepts

The class had just been introduced to the modulus functions and were asked to use the graphical calculators to graph the function y = |2x –3|:

1	SE	Ok so before you actually graph it, I know some people might have done already, can anybody tell me where this is going to make the v shape? At what point on the x-axis will this make the v?	
2	Marvin	-3, when x is 0 – What?	Seemed confused.
3	SE	Any other ideas?	
4	Perry	+ 1.5.	
5	SE	+ 1.5 and how did you come up with that?	
6	Perry	Well, if you say 0 = 2x –3 and rearrange it you get 3 = 2x so x = 1.5.	
7	SE	Right so does everybody understand that? Let's just get the graph up. So you've entered it, and then graph.	
8	Marvin	Wee. Oh right.	Marvin imitated the graph being drawn.
9	SE	You can see that it's actually cutting here. Now what's happening here is this is the line of 2x-3 but for all the negative values the modulus makes them positive values. So what you're doing is seeing where the line cuts the x-axis – where it is equal to zero and then these negative values become positive. So you want to get 2x – 3 equal to nought and then solve it and that would give you this point here.	SE used the LCD screen and OHP for demonstration. The line y = 2x–3 was also drawn on the OHP.
10	Marvin	It drew the wrong graph on mine. That's why I gave you the wrong answer.	
11	SE	It drew the wrong graph?	
12	Marvin	Because I didn't put the brackets in.	
13	SE	You didn't put the brackets in. Well I was afraid of that. Ok so remember use brackets when you are unsure.	

The graphical calculators were then used to introduce the students to the idea of odd and even functions:

1	SE	The question is to determine whether the following functions are odd or even. So it's probably a good idea to draw the graph and see which of these definitions it fits. So if we go back to Y= and we clear what we've got. The first one is y = x, enter that and graph it. Now looking at these definitions what kind of function is that? If it is even then it is symmetrical about the line x = 0 or the y-axis.	The class was silent.
2	Marvin	It's an odd function.	
3	SE	So is it symmetrical about the y-axis?	
4	Class	No.	
5	SE	No. Has it got half turn rotational symmetry about 0?	
6	Class	Yes mostly, some no's.	
7	SE	Yes. So if you've got this piece then it does a half turn and it's the same here, so this is an odd function. Right we'll have a look at the next one y = x^2. So clear that.	SE demonstrated using the OHP and LCD screen.
8	Someone	It's even.	
9	SE	Right. Ok what kind is that?	
10	Class	Even.	

11	SE	Even. It is symmetrical about the y-axis. Now the next one is $y = x^3$ so we're going up. We did x^1, x^2 now we're looking at x^3.	
12	Roy	Odd. It's odd.	
13	SE	Right let's just type it in – you're getting ahead of me again. Right that one is odd again because you've got this half-turn rotational symmetry about 0. The next one is x^4.	
14	Marvin	Even.	
15	SE	Right so that one is even. So what we've got is y = x was odd, $y = x^2$ was even, x^3 was odd, x^4 is even. So would anybody like to guess what x^5 might be?	
16	Perry	Odd.	
17	SE	So these particular values follow this pattern. We'll just check that, some of you might have done this already. Ok so that follows that pattern. But then you are asked to actually look at y = 1/x. So we'll go back and type that one in, graph it. What's that one?	
18	Class	Odd.	
19	SE	That one is odd, yes. And the final one you are asked to draw is $x^{1/2}$. So type in x to the power – you'll have to put the half in brackets. What do you think about that one?	
20	Someone	Neither.	Puzzled looks.
21	SE	Neither, you are right. Not all functions have to be odd or even. This one hasn't got half turn rotational symmetry so it's not odd and it's not symmetrical about the y-axis, so it's neither. So sometimes in the examination you might be asked the question what kind of function is this and you can draw it but you've got to remember that not necessarily is it either one of odd or even.	
22	Marvin	Is it called anything then?	
23	SE	No it's not called anything.	

The next whole class activity was to consider the actions of particular types of transformations, beginning with $y = af(x)$:

1	SE	Could somebody share their results with me?	
2	Fay	They go upside down when it's negative.	
3	SE	They go upside down when it's negative. Yes that's one thing.	There were lots of students offering suggestions at once.
4	SE	OK can everybody just listen please, what did you say then?	Directed at Perry.
5	Perry	When you use the negative prefix it's reflected in the y = 0 line.	More formal mathematical explanation.
6	SE	It's reflected in the y = 0 line, yes that's right. What were you going to say?	Directed at Marie.
7	Marie	A reflection.	
8	SE	You were going to say it was a reflection as well. What about the actual slope of the curve? What happens to it when you use another value of *a*?	
9	Marie	The larger *a* is the steeper.	
10	SE	The larger *a* is the steeper.	
11	Marvin	As the modulus increases the steeper.	The term 'modulus' had been introduced earlier in the same lesson.
12	SE	As the modulus increases the steeper it is. That's a good point, yes. One way of describing this is as a one way stretch parallel to the y-axis and it's factor *a*. You've all noticed the right things there. That's really good.	

The students were then asked to comment on the effects of other types of transformations as they were being drawn. They graphed the functions individually using the graphical calculators and the over head projector was again used as a focal point for discussion and to bring ideas together.

Discussion of the transformation f(x) + a

1	SE	So that gives you what y = 1/x + 3 would look like and what's happened to that?	
2	Julian	It's moved up.	
3	SE	It's moved up and how many units has it moved up?	
4	Class	3.	
5	SE	3. And what do you think for the second one, without drawing it?	
6	Class	Moves down.	
7	SE	Moves down 3 again.	

Discussion of the transformation f(x + a)

1	SE	So what's happened to that one?	
2	Fay	It's moved.	
3	SE	It's moved.	
4	Fay	3 units to the left.	
5	SE	3 units across the x to the left. So what do you think will happen for x is –4, sorry x –4?	
6	Jake	It goes the other way by 4.	
7	SE	It goes the other way by 4 units. So you can see when it's plus it goes to the left and when it's minus it goes to the right.	

Discussion of the transformation f(ax)

1	SE	So we'll finally have a look at the last transformation. Clear that we're looking at x^3 to start with. So it's 2nd draw, number 6, draw function and again we want it in this form. So what will I be typing in? What's the first step?	
2	Julian	Y1.	
3	SE	Y1 Ok. So it's 2nd vars, function Y1. What's next?	
4	Someone	Bracket.	
5	SE	Bracket. The value in the first one is 2, so it's 2x close bracket, enter. And what's happened to that?	
6	Marvin	Steeper.	
7	SE	It's got steeper. So I'll let you continue and look at what happens for the other values in that example.	There was a lot of discussion amongst individual groups of students.

Following the initial class discussion, the students were asked to use the graphical calculators to further explore the effects of the transformations. The results were then drawn together in a concluding discussion:

1	SE	Right so the first one we discussed that when we did it so we'll move on to the next one. What was – remind us what happened here for this one when you've got f (x)+a.	SE pointed to the form f(x) + a, written on the board.
2	Mick	When it's positive, when *a* is positive then it moves negative along the x-axis. And when it's negative it moves positive along the x-axis.	Here Mick was confusing f(x) + a with f(x + a)

3	SE	Right. Did everybody agree with that, did everybody find that?	
4	Class	Some disagreement.	Some students realised Mick had made a mistake.
5	SE	So what you can actually say about that is that it's a translation. It's a movement and it moves by 0 in the x-axis and *a* is the y-axis, Ok. What about this one - here? What did you discover in that case?	SE clarified the effects of f(x) + a. SE pointed to f(x + a), written on the board.
6	Roy	It moved along the x-axis in the opposite direction to the sign.	
7	SE	The sign of the *a*. Yes that's right. Did everybody else find that? Yes?	
8	Class	Agreement.	
9	SE	So that one you would call it a translation as well and this time it doesn't move in the y it moves in the x. And what about the last one, where you've got y = f(ax)?	SE pointed at f(ax) on the board.
10	Roy	When it's positive it just gets closer to the x-axis and you get (pause).	
11	SE	You get?	
12	Roy	I've forgotten what it's called. What's that number?	
13	SE	The coefficient?	
14	Roy	The coefficient.	
15	SE	The coefficient.	
16	Roy	And as that gets larger it gets closer to the y-axis.	
17	SE	Yes.	
18	Roy	And if it's a negative number then it's reflected in the x-axis.	
19	SE	Yes you're right. Did everybody else find that?	
20	Class	Agreement.	
21	SE	So what you can say about that is that it's a one way stretch parallel to the x-axis and it's got a factor, scale factor of 1/a. I'll write these out for you so you can copy them down before you go. Em the final thing I wanted to say was if you look at this here, $y = (2x)^3$. How can you simplify that? Is there any way of simplifying it?	SE introduced a more formal definition of this transformation. These definitions were written out on the board following the discussion.
22	Someone	8.	
23	SE	8, yes. You could write that one as $8x^3$, cube the 2 and cube the x. So what about this one?	Referring to $(x/2)^3$
24	Marvin	-8.	
25	SE	-8? I'm not sure how used to you are with…	
26	Marvin	Oh it's 8.	
27	Perry	It's 1/8.	
28	SE	It's 1/8. This is the same as x/2 all cubed because it's a half x all cubed.	
29	Marvin	Oh I'm looking at the wrong one sorry.	
30	SE	Oh that's Ok. So it's $x^3/8$ that's another way of writing it. And this one would be?	Referring to $(-1x)^3$
31	Class	Some suggest $1x^3$ some suggest $-1x^3$.	
32	SE	$1x^3$, or $-1x^3$. So which one is it going to be, is it $-x^3$ or x^3?	
33	Marie	Minus.	
34	SE	It's minus because if you cube –1 it becomes -1 again. It's: -1 times -1 is 1 times another -1 is -1, Ok?	
35	Marvin	Is this one -8 then?	Referring to $(-2x)^3$
36	SE	And is this one –8? Yes.	

The class was asked to identify the family of functions to which a couple of pre-drawn graphs belonged:

1	SE	Can anybody suggest to me what the form of that function would be?	Referring to the graph of $x^3 - 4$ on the OHP screen.
2	Marvin	Tan. (Pause) x^3.	
3	Class	x^3.	
4	Roy	It's $x^3 - 4$.	
5	SE	Right you said tan at first...	
6	Marvin	No I ...	
7	SE	And that's kind of the shape that tan does take but because we're in this mode you can see that we're not in trig mode.	The graph was drawn in Zoom standard mode.
8	Marvin	Yes, yes it's not in trig mode.	
9	SE	So you think it's $x^3 - 4$?	
10	Perry	It might be $2x^3$.	
11	SE	So how did you come up with that? What made you think of that?	
12	Mick	It's the shape of the x^3.	
13	Roy	It's the x^3 shape.	
14	SE	It's the x^3 shape.	
15	Roy	And it's moved four down the y-axis.	
16	SE	And it's moved four down the y-axis yes. So if you wanted to check that – you say it might be two.	
17	Perry	It might be two.	
18	SE	It might be $2x^3 - 4$. How would you check that?	
19	Perry	Draw it on here.	
20	SE	You would draw it on there and what would you compare it to?	
21	Perry	To $x^3 - 4$.	
22	SE	$x^3 - 4$, right. So you would be able to see using your graphical calculator which it was. And what about the other function. Can anybody guess the form of that?	
23	Julian	2 or $3x^2$.	
24	SE	It's an x^2, it looks like an x^2. 2 or $3x^2$. So if that's your initial thought then you would have to perhaps compare it with the ordinary x^2.	
25	Perry	It might be x^4.	
26	SE	It might be x^4. So you have to check it out on the graphical calculator, all the possibilities that you think it might be and see which one matches up.	The class was busy using the TI-82's to check their initial thoughts.

A class discussion was then initiated surrounding question 13 from the main exercises:

1	SE	Picture A, what kind of graph is that going to be?	
2	Marvin	Cosine.	
3	Marie	Sine.	
4	SE	It looks like sine or cosine, yes. It could be either one. What about B?	
5	Marvin	Sine or cosine again.	
6	SE	Sine or cosine again. So you've just got to look at the shape of the curve and think oh this could possibly be this. What about the next one, C?	
7	Class	x^2.	
8	SE	It looks like an x^2 and for some of the others you might have to draw the options to see which you can match up. So lets go back to A. Any ideas for picture A?	

9	Several students	Sin x.	
10	Perry	Number 10 ($y = \cos(x + 90^0)$).	
11	SE	You think that it might be sin x and why did you say number 10?	
12	Perry	Because we haven't got sin x, but cos ($x + 90^0$) is the same as sine.	Perry was thinking along the right lines.
13	SE	Cos ($x + 90^0$) is the same as sin x?	
14	Roy	Yes, but the period is wrong.	
15	SE	The period is wrong? So there's some disagreement there. So I think maybe you do need to draw them out and look at the…	Attempt to persuade the students to use the TI-82 to mediate their thinking.
16	Perry	Cos ($x + 180^0$) which is 4.	Perry was influenced by Roy's suggestion.
17	SE	Cos ($x + 180^0$) and why do you say that?	
18	Perry	Because if you move cos ($x + 180^0$) backwards it's the same as sin x because the intervals are the same.	
19	SE	180 – I think you might have to check…	
20	Marvin	Utters disagreement with Perry.	
21	Perry	Every little peak, right. Its full peak is at 180^0 but the full cycle is at 360^0.	
22	Jake	Miss it's number 2 ($y = \cos(x - 90^0)$). You're wrong Perry!	Jake was confident. The TI-82 had confirmed this.
23	Marvin	No he's right Jake.	Marvin changed his mind.
24	Perry	So it will peak and go back to the x-axis every 180^0, because if you move that back 180^0, it will peak in the gap before. It will.	
25	Jake	Because if you, oh…	Still thinking.
26	Mick	It's not 180 though.	Mick had been using his TI-82 to graph the functions.
27	Perry	It is.	Still confident.
28	SE	I think we've got some people saying 180 and some people saying 90 degrees, so what I suggest you do, seeing that there is no consensus here is…	
29	Mick	That's the graph at 180^0 it's not the same as that one.	Mick showed Perry the graph on his graphical calculator.

Small Group Discussions

Nigel and Marvin were discussing question 2 from the main trial exercises:

1	Nigel	Marvin?	
2	Marvin	Yes.	
3	Nigel	Did you do number 2?	
4	Marvin	Yes you change the window.	
5	Nigel	That's it?	
6	Marvin	-1 and 1.	
7	Nigel	What y?	
8	Marvin	No the x bit, you want x to be – hold on I'll tell you if it's worked. No it doesn't seem to be.	
9	JG	Is that radians or degrees?	James Green entered the conversation.
10	Marvin	Erm no zoom. Just wait until it finishes.	
11	JG	Oh right.	
12	Marvin	It won't take long. You have to change it to zoom trig then. It's worked now. You have to be in zoom trig.	
13	Nigel	I did!	
14	Marvin	You did? All right then it doesn't work if you change it?	
15	Nigel	Still didn't work!	
16	Marvin	Come here. Yes yours is right.	Marvin grabbed Nigel's Calculator.
17	Nigel	Is that right?	
18	Marvin	Yes.	
19	Nigel	How do you draw it?	
20	Marvin	Em…	

Nigel was clearly having difficulty seeing the relationship between the graphs of tan x, 2tan x and 3tan x in question two, so I intervened:

1	SE	What you might want to do is watch it draw them again and perhaps draw them individually rather than all at the same time. That might help you see the picture. So if you just de-highlight those. So you can see what that picture is like. You can draw that one and then you could try and see what the next one is. So if you just graph that.	Nigel drew the first graph separately on the graphical calculator, as was suggested. He then drew just the second graph.
2	Nigel	Yes.	
3	SE	Can you see anything? I know that you haven't got the other one to compare but with the shapes of those lines, can you see anything?	
4	Nigel	Em, I've forgotten what it looked like the first time, so em…	
5	SE	Why don't you just draw those two to start with and compare those.	
6	Nigel	Oh right yes, the…	
7	SE	What's happened to it?	
8	Nigel	The second one is steeper.	
9	SE	Yes. So the third one, what do you think will happen to that?	
10	Nigel	It will be even less of a sharp gradient. It will be – it's got to go through on the other side.	Nigel extrapolated the pattern incorrectly.
11	SE	Are you sure that you've got these the right way round, which one's the first one you drew? Can you remember?	
12	Nigel	Oh…	
13	SE	Just do it again, em zoom… That's the first one.	
14	Nigel	Ah yes. So the third one would be even steeper.	

15	SE	Yes. The next one comes in like that, so the next one follows the pattern and goes over that side. So you just try and draw those, the best you can. I know that it's quite difficult because they're very close together, aren't they?	
16	Nigel	Yes. So when you put the third one in – the second one is steeper and the third one is even steeper.	SE departed. Nigel repeated his conclusion for Justin.
17	Justin	Mm.	
18	Nigel	Let's draw it.	
19	Justin	Which one, this bit?	
20	Nigel	I don't know.	Both students sketched the graphs on paper.

Carol and Fay were attempting question three from the main trial exercises:

1	Fay	The minus just like runs everywhere.	
2	Carol	I've got that.	Carol pointed to her calculator screen.
3	SE	Do you know what these points are?	Referring to the divisions on the x-axis on the graphs in their solutions.
4	Fay	Are they 90^0, 180^0?	
5	SE	Yes. So you just need to label them. What about this point though, there and then there and then there – zero, ninety there's a point that crosses there so that's the 180^0 and that's the next one. What's the next one?	
6	Fay	270^0.	
7	SE	Yes, Ok.	

I departed and Fay correctly labelled the axes for her cosine graphs, but made a mistake when labelling the axes of the tangent graphs:

8	Carol	What's that there?	Spotting her mistake, pointing to the incorrectly labelled graph.
9	Fay	Oh why have I put that there and there? One of them is not there. Have you got a rubber?	Fay watched Carol working on the TI-82 and began writing out an explanation for her findings.
10	Fay	What do you call it? Would you call it a wavelength? Could you call it the wavelength?	
11	Carol	No it's not right, it's just like points.	
12	Fay	I know it's just wiggles but…	
13	JG	Amplitude and wavelength. Wavelength.	
14	Fay	So could you call it a wavelength?	
15	Carol	The wavelength is the…	
16	JG	There's no reason why – the wavelength is from when it does one thing to when it does exactly the same thing in the same direction. So it's from there to there.	James referred to the graph on the TI-82.
17	Fay	Yes.	In agreement.
18	Carol	Yes.	In agreement.
19	Fay	So could you call it a wavelength?	
20	JG	Yes, yes.	

Julian and Kirk discussed the first question from the main trial exercises:

1	Julian	We've done something similar before. Put them all on one graph. Oh it's playing up again.	Julian was using zoom standard instead of zoom trig.
2	Kirk	You have to use the zoom.	
3	Julian	Oh yes I see what you mean. Don't know what's going to happen though. When the coefficient increases, will it get steeper?	
4	Kirk	Nods in agreement.	
5	Julian	It goes through the same points.	Julian had drawn the three graphs.

They then moved on to question two and Jake joined in the discussion:

1	Jake	Repeat with tan.	
2	Julian	Tan doesn't make a curve like that does it – it's one of those daft ones.	Julian referred to the graphs of sine and cosine.
3	Jake	It's one of those daft functions. What did you get for tan?	
4	Julian	Haven't done it.	
5	Jake	The line gets steeper.	
6	Julian	Have you put all B on at once?	Wondering if Kirk had sketched the three graphs in question 2 simultaneously on the TI-82.
7	Kirk	Yes.	Kirk showed Julian his screen.
8	Julian	See what you mean Jake it is the same as before.	Referring to question 1.
9	Kirk	Yes it looks better.	The graphs were more easily distinguishable than in the previous question.
10	Jake	The same – the same explanation isn't it.	
11	Kirk	They all go through the same points.	
12	Julian	Yes as it gets bigger it goes steeper but it goes through the same points.	Julian elaborated on the pattern.
13	Jake	Yes.	
14	Julian	The same outline.	
15	Jake	It's quite hard to sort of get them all the same making the graphs look halfway decent.	
16	Kirk	I know look at mine. Yours looks good.	Referring to Julian's sketches.
17	Julian	As long as you can understand it doesn't really matter does it? It's only a sketch. You do know that they're like different formula graphs anyway, don't you from before. So you kind of know the shape that they should be so you kind of guess because sine is the same as cosine just the humps are in a different place. Humps - that's mathematical for you isn't it.	Julian now referred to the cosine graphs from the first question.
18	Kirk	Yes but Julian you always just guess. It just humps differently.	
19	Julian	That's it, the humps are just different everything else is the same.	

The groups' discussion was now focused on question 3:

1	Jake	Are you having a go at three yet Joe?	
2	Julian	I've just got there.	
3	Jake	Can I have your…	
4	Julian	Can I have a look at what you've done? Oh I see.	Julian views Jake's TI-82 screen.
5	Jake	It's got to be multiplied by different factors -	

		the x is different.	
6	Julian	Yes.	
7	Jake	That's going to be the answer. Do we have to…	
8	Julian	It's nothing to do with the one before the cos. It's the one before the x.	
9	Jake	Yes. It makes it sort of totally different, doesn't it?	Referring to the previous two questions.
10	Kirk	Yes. Have you drawn it on your graph, yet?	
11	Jake	No.	
12	Kirk	Right when it's 2x it halves the wavelength, when it's 3x it cuts the wavelength into three. So you get…	
13	Jake	Yes.	Emphatically.
14	Julian	Yes.	Emphatically.
15	Jake	So like they're sort of totally different, you know, totally different values.	Referring to the fact that the graphs all cross the x-axis in different places.
16	Julian	It's not the coefficient of the cosine, it's the coefficient of the x that moves that, yes?	
17	Jake	Yes.	
18	Jake	I'm trying to draw them all but I'm finding it right hard.	
19	Kirk	It just looks like a big scribble.	
20	Jake	I don't know. It does a bit.	
21	Kirk	Those graphs look really neat and mine doesn't.	Referring to the graphs produced by the TI-82.
22	Jake	They all peak on the y-axis as well.	
23	Julian	Yes whereas sine doesn't, it crosses the origin. The coefficient of cos doesn't affect where the graph crosses. [Pause] but x does that's what you would expect.	
24	Kirk	Mm, If we've done the previous question right.	

As the discussion continued, Julian and Kirk attempted to make a start on question 4, however Jake was still contemplating question 3:

1	Julian	Right question 4. It's getting more difficult now.	
2	Kirk	The answer is given to you. You know the equation of the x.	
3	Jake	You know what I mean as the value of x increases the wavelength increases.	Jake was still referring to question 3. He makes the wrong assertion.
4	Kirk	Gets shorter, yes?	Kirk queried this.
5	Jake	Yes. In three I've just put as the value of x increases the wavelength decreases.	Jake realised his error.
6	Kirk	Yes.	
7	Jake	Gets smaller and then it's the same as before as the cosine increases it gets higher.	
8	Kirk	I've put that the coefficient of x moves the intersection along.	Referring to his written work.

With Jake's questions answered the group were now free to begin tackling question 4:

1	Jake	Are you on four yet Julian?	
2	Julian	Yes. What did you do Jake?	
3	Jake	I think you know the g(x)…	
4	Julian	Yes.	
5	Jake	And the f(x) combined.	
6	Julian	Combined?	
7	Jake	Combined, it's x over 2 all cubed, I think.	

8	Kirk	Well it would be x^3 over 8.	
9	Julian	Yes x^3 over 8.	
10	Kirk	Yes.	
11	Jake	Yes.	
12	Julian	Yes it would be. Miss…	
13	SE	Yes.	
14	Jake	You know on this one, would it be $x^3/8$?	Jake sought reassurance.
15	SE	Yes.	
16	Julian	Because that's $x^3/8$, if that's what you got using the draw function, the graph.	He compared the graph on his calculator with Kirk's.
17	SE	Have you got onto question 4?	Directed at Julian.
18	Julian	Yes that's …	
19	SE	x^3.	
20	Julian	Yes x^3.	
21	Jake	You've got to sketch the graphs first. So I've got the equation.	
22	Julian	And there's x there and you get that there which is $x^3/8$.	Julian pointed to his written work.
23	SE	$x^3/8$, yes.	
24	Julian	Because you cube the x's.	
25	SE	That's right.	
26	Julian	So we've got the equation of the next function.	
27	Jake	Have we got – we've got to draw the other two haven't we?	
28	Kirk	Yes.	
29	Jake	We have haven't we - I mean the two.	
30	Kirk	Have you done the second bit of 4?	
31	Julian	No. Do they want you to draw a graph or not?	
32	Kirk	Yes.	
33	Jake	Oh have I got - I bet I haven't got it in degrees. How did you get it into degrees again because I've got that? Is that what you got?	Jake showed his calculator image to Kirk. Kirk then considered the image more carefully.
34	Kirk	No.	
35	Jake	You didn't get that?	
36	Kirk	No I didn't.	
37	Julian	I'm on B now Kirk.	
38	Jake	Did you type in x^3 and x^3 over 8?	
39	Kirk	No. Press Y=	
40	Jake	What did you put in the Y=	Jake worked on the TI-82 whilst Kirk watched.
41	Kirk	You want x over 2 that's why.	
42	Jake	Oh right I want x^3.	
43	Kirk	Yes.	
44	Jake	And then what?	
45	Kirk	x^3 over 8.	
46	Jake	Is that it?	
47	Kirk	Yes.	
48	Julian	When you got onto stage B did you go to draw again?	
49	Kirk	That's what I've done. You do...	
50	Julian	It's Y1 again isn't it? Is it Y1plus 2 in brackets minus 3?	Using the draw function command.
51	Kirk	It's Y1 and presumably you can change it.	
52	Julian	Yes but not all of it, what you put in.	
53	Kirk	Yes. Is it Y1 plus 2 in brackets and then minus 3.	Kirk agreed with Julian's suggestion.
54	Julian	I put Y1and then plus 2 in brackets and then minus 3.	

After some individual work the discussion continued:

1	Jake	What number are you on?	
2	Kirk	The second bit.	
3	Julian	Mm, have you got the curve, the like bump bit at –3?	
4	Kirk	Yes.	
5	Julian	So that makes sense because it's lowered it…	Describing the [0, -3] transformation
6	Kirk	Yes.	
7	Julian	on the y and…	
8	Kirk	It's moved it back 2 x.	Describing the [-2, 0] transformation
9	Julian	Yes.	
10	Kirk	So that actually makes sense if you think about it.	Reflecting on their knowledge of the graphical effects of transformations
11	Julian	It does. Now sketch that. Ah but it's g.	Julian was confusing the function g for h
12	Kirk	It says sketch the….	
13	Julian	We've done f, f(x), oh no.	Julian was still confused
14	Kirk	No that would be h(x), wouldn't it? Because that was the transformation and that will be that and you've got to right down its equation.	Kirk pointed to the new curve on the TI-82 screen
15	Julian	Yes. What have you done?	
16	Kirk	Oh em, well I can draw that, and then I'll…	
17	Julian	Haven't we done f, haven't we f(x + 2) – 3. It's g.	
18	Kirk	No because…	
19	Julian	Y1 is f and not g.	
20	Kirk	Yes we have.	
21	Julian	So we need to clear that, go back to that we know what that is because we've just worked it out.	Referring to the function g.
22	Kirk	What is that anyway?	

At this point Jake rejoined the groups' discussion:

1	Jake	Is it substituting it in again?	
2	Julian	Pardon?	
3	Jake	I'm on the second part, is it substituting it in again?	Jake had fallen behind the other two.
4	Julian	Yes. Yes using the draw function on your calculator.	
5	Jake	Oh right yes. What on Y1 or Y2?	
6	Kirk	Y2.	
7	Julian	Have you got a Y2? I didn't have one.	
8	Kirk	I've got a Y2 and yes it's Y2 and it gives you a similar graph but it's still not on the em…	
9	Jake	It goes through the middle.	Referring to the point of inflection passing through the origin.
10	Julian	Oh yes that's better that's what it is. Before we did it with f.	
11	Jake	What is it second function, draw then what?	
12	Julian	6. If you press the number you don't have to do anything to it.	
13	Kirk	Now we need a formula for that.	
14	Jake	Have you got that Kirk?	
15	Kirk	I got that. So we're putting it into the $x^3/8$, aren't we? We do…	
16	Julian	What?	
17	Kirk	We do it we substitute in, that into $x^3/8$, don't we?	

18	Jake	Yes.	
19	Kirk	So what's this? It's going to be this all cubed isn't it?	
20	Julian	$x^3/8$ as x. So we've got…	
21	Jake	So is that, is the resulting curve called h(x)?	
22	Kirk	Yes. I've got that.	
23	Julian	Let's have a look.	
24	Kirk	Now we got to substitute it in, don't we?	
25	Julian	We've got x^3 so we're going to have $x^3/8$ so we're going to have $x^3 + 8$ minus…	
26	Kirk	Why plus 8?	
27	Julian	Because we're substituting that in. That's going to be $x^3 + 8$, isn't it?	
28	Kirk	G is $x^3/8$. Oh it's going to be $x^3 + 2$ brackets – 3 over 8, isn't it? If you put g into that, yes?	
29	Jake	You write h(x) as an equation.	
30	Julian	Yes that's what we're doing now.	
31	Julian	You make that the x, don't you?	
32	Kirk	So that's going to be x^3…	
33	Julian	That's going to be $x^3/8$.	
34	Kirk	Oh yes it is, isn't it.	
35	Julian	That's going to be x + 2.	
36	Kirk	x + 2 yes.	
37	Julian	So I was right before. Yes.	
38	Kirk	Then minus 3.	
39	Julian	Ah, how would you simplify that?	
40	Kirk	Wait a min.	
41	Julian	Do you do $1/8\ x^3$ over…	
42	Kirk	So it's going to be –1/8.	
43	Jake	Have you got it wrong?	
44	Kirk	Yes.	
45	Jake	What have you got?	
46	Julian	Do they want it in the form of a normal cubic?	

Fay, Carol and Claire were asked about question 6 from the main trial exercises:

1	SE	Right so what question have you got onto then?	
2	Fay	Pardon? Question 6e.	
3	SE	Question 6e. Right ok so how have you been having a go at these questions? Have you been aware of what transformation it would be and then just trying to use the graphical calculator to find it or have you just been using a trial and error approach?	
4	Fay	I've been guessing.	
5	SE	You've been guessing?	
6	Carol	In some cases like the first couple they look like relatively easy but then the others were just a bit baffling so sort of trial and error. Guessed a couple. Claire did them already.	
7	SE	Oh you've finished these have you?	
8	Claire	I have finished them but I did them but the first one I just looked and thought how many like units it had been moved and then I found it. But these ones, they were harder.	
9	SE	They are more difficult yes, but do you think by doing the question that you are getting more of an idea of what transformations do to the graphs?	
10	Claire	Yes.	
11	SE	Yes or no?	Directed at Carol and Fay.
12	Carol	Yes.	
13	SE	Yes. Well that was the object of the questions anyway.	

Jim was also asked about the transformations in question six:

1	Jim	I think –4 because it's moved 4 across that way.	Jim indicated a movement to the right.
2	SE	Yes.	
3	Jim	So I knew it was a minus but I wasn't sure how you got 4, because that's what I got.	
4	SE	You count four, the values on the x.	SE demonstrated on the graphical calculator.
5	Jim	Yes. And then I got that as x + 1. I got that to draw first time round as well, no problem.	Jim used the graphical calculator to confirm his assertion.
6	SE	Oh that was good. So this one has moved across by 1.	
7	Jim	A plus number.	
8	SE	So it's plus. Yes, you've got those right. So what about this one here how are you going to try doing that one?	
9	Jim	Oh em let me think.	
10	SE	Have you got any idea?	
11	Jim	It's enlarged that one.	
12	SE	It's enlarged, yes. So it's a stretch.	
13	Jim	A stretch of about two I reckon.	
14	SE	So you've got to think about what… of about 2. So you're going to do 2 times the function.	
15	Jim	Yes.	
16	SE	Ok I think that's a good starting point. Do you agree?	
17	Jim	It could possibly be.	

Nigel was having difficulty in obtaining the symbolic form of the new function in question 6b:

1	Nigel	I can't work out how you would get that one. I've tried putting plus one after it but that won't do it but it gives me exactly the same graph.	The transformation f(x+1) gave Nigel the second graph, using the graphical calculator. However, he was then trying to graph y = f(x) + 1.
2	SE	Right ok so you've just tried to type it in here, into the function that you already have. Right so maybe it would be a good idea to just look at the pictures and see what's happened to them.	I was trying to help Nigel appreciate the difference between y = f(x+1) and y = f(x) + 1.
3	Nigel	Yes it's gone one along the x-axis.	Nigel recognised this movement.
4	SE	Right so what kind of transformation makes that happen?	
5	Nigel	Forgotten what it is. Is it em…	
6	SE	Well let's have a look in the book. If you've got a transformation like that, what does that one do?	
7	Nigel	Don't know.	Nigel was looking at the symbolic forms of the transformations.
8	SE	It stretches it.	
9	Nigel	Right.	
10	SE	So if you have a look at these pictures.	The pictures helped him.
11	Nigel	Yes, oh right.	
12	SE	And this one here will move it up or down. A transformation like this…	
13	Nigel	Moves it across.	
14	SE	Moves it across.	

15	Nigel	Yes.	
16	SE	So we've got one like that.	
17	Nigel	That's what I was trying to put in plus one after it.	He was still confused by the symbolic representation.
18	SE	So yes right you had plus one. But the trouble is here every x has to become x+1 now, not just x+1 at the end. So it's 3(x+1) because it's three times x+1.	
19	Nigel	I need to put everything in brackets then.	
20	SE	Yes. It might be a better idea if you do it on paper first and then type it in here.	
21	Nigel	Yes.	
22	SE	Because it's going to have a lot of brackets and be very complicated. So that needs to be all in brackets because it's three times x and x is now x+1. Yes so you need a bracket there. It's three times x+1.	
23	Nigel	3 (x+1). That's right weird that. Then brackets. Is that right?	Nigel was still puzzled.
24	SE	Yes.	
25	Nigel	Then divide that by…	
26	SE	So it's in brackets x +1 all squared plus four.	
27	Nigel	Plus four x + 1.	
28	SE	4(x + 1) yes. If you going to type that into the calculator you just need to have brackets round…	
29	Nigel	Put brackets round that bit.	
30	SE	That's right. You've got that yes. And it should graph it now. So that should give you this picture. If it doesn't just call me back.	
31	Nigel	Yes.	Nigel was able to obtain the correct picture.

Marvin was asked about how he was going to approach question 8 from the main exercises:

1	Marvin	I'm on 8a now. Em I've just done 7.	
2	SE	So how are you going to tackle this problem then?	
3	Marvin	Well first of all I'm going to put the formula into the Y1.	
4	SE	Yes.	
5	Marvin	And work out where it crosses using the table.	
6	SE	Right.	
7	Marvin	And then do it, putting it in brackets, actually solving it algebraically.	
8	SE	So if you were doing this ordinarily and didn't have the graphical calculator would you still do the same thing, do a quick sketch and then…	
9	Marvin	No. I wouldn't do the sketch. I'd do it…	
10	SE	You wouldn't do the sketch, you'd do the algebra.	
11	Marvin	Because yes. But that kind of …	
12	SE	But if it was an inequality would that change?	
13	Marvin	If it was an inequality then it would change yes because em where the zero is greater than the formula of less than determines whether it's a one inequality answer or two separate ones.	
14	SE	Yes.	
15	Marvin	So in that way I'd draw it.	
16	SE	You'd draw it.	
17	Marvin	Yes. Or imagine it would be easier.	

Kirk, Jake and Julian were also asked about question 8:

1	SE	Right so which question are we on at the moment?	
2	Kirk	8b.	
3	Jake	Yes.	
4	SE	8b. So how did you get on with 8a?	
5	Julian	Quite simple algebraically we just factorised it.	
6	Jake	Put it into two brackets: (x+4) and (x-2).	
7	SE	Right so did you draw it on the calculator as well?	
8	Jake	Yes and it was right as well.	
9	SE	And it was right.	
10	Jake	We just compared the co-ordinates in I mean the places on the x-axis, 2 and –4.	
11	SE	Yes and how are you going to tackle this b? You've started doing it with algebra again.	
12	Jake	The same we always do.	
13	Julian	Yes.	
14	SE	Yes.	
15	Julian	Because the co-efficient of x^2 is large it's going to make it quite difficult.	
16	SE	Yes.	
17	Julian	So we're going to now put it into the calculator and see if that helps. So we're going to do that now.	
18	SE	There is one thing that you could do with that though. You've got $7x^2+4x$ if you've done it right. What's common to those terms?	
19	Julian	x, ah you can take an x out.	
20	SE	Yes you can take an x out. So I think that would make things easier for you.	
21	Julian	Yes. x equals…	Julian completed the factorisation.
22	SE	Yes so you've got two solutions there. Ok so now you can check them on the graphical calculator, to see if you've got the right ones.	
23	Jake	It is right. It is because when I typed it in it was really small values because it was small values so I thought…	
24	SE	If you didn't have the graphical calculator would you actually draw these…	
25	Jake	No.	
26	SE	Sketch these out or would you just rely on your algebra?	
27	Jake	Rely on algebra probably.	
28	Julian	Which ones do we sketch because in some we sketch what was that…	
29	Kirk	Inequalities, isn't it?	
30	Jake	Inequalities we sketch yes.	
31	SE	Inequalities.	
32	Julian	We sketch inequalities.	
33	SE	Do you know which area you're looking at?	
34	Julian	Yes, but the rest we tend to just battle on with the algebra.	

Perry and Mick also discussed question 8:

1	SE	Right I've just come to talk to you about what you're doing at the moment.
2	Perry	Oh right.
3	SE	Which question are you on?
4	Perry	I'm on question 8b.
5	Mick	8a.
6	SE	8a and 8b. So how did you do 8a?
7	Perry	Well I did it the silly way em I ended up using the quadratic formula when actually you could have factorised it, but it's Friday afternoon and I'm tired.

8	SE	Well that's OK, I mean as long as you've got that right.
9	Perry	Yes. You know calculating you get em when y = 0, x = 2 or –4, and then you can just check it graphically, just put $y = x^2 + 2x - 8$ into the calculator, push graph and it graphs it and you can calculate the points...
10	SE	So is that what you're doing?
11	Mick	I've done the graph first.
12	SE	Oh you've done the graphical approach first.
13	Mick	Yes.
14	Perry	So you know what you're working out before you work it out.
15	Mick	Yes.
16	SE	Yes, oh right well that's fine.
17	Mick	I think I find that way easier than well using the calculator it would be but if we didn't have the calculators then we would have to do it algebraically we wouldn't probably consider drawing it.
18	SE	Yes. What about inequalities the boys down there said that they would draw graphs for inequalities?
19	Mick	Em inequalities, yes possibly just to work out the regions which we probably wouldn't know. Yes that's probably the only time.
20	Perry	Yes. I always sketch a graph anyway. I always like to do it algebraically as well.
21	Mick	I don't like graphs. I try to stay clear of them as much as possible.
22	SE	Oh dear. Well how do you find the graphical calculator then?
23	Mick	It's been useful and I'm thinking about buying one myself, but I think I would be lost if I didn't have it for doing the functions work now, because I don't know until I try some without it.
24	SE	Because you've got used to using it.
25	Perry	I just don't know how I also could do it, how to sketch the graph of $y = (x^3 + 2)/(x - 1)$ or something. To draw that would be really hard.
26	SE	When you get onto the graph sketching section later on in the upper sixth I'm sure you will be much more confident than you are.
27	Perry	Yes but we're not at the moment, are we?
28	SE	Do you think that that's damaging then using the graphical calculator instead?
29	Mick	Em, if you haven't got one afterwards then yes but it is in a way but it's not in other ways because it helps you. It would have taken me a lot longer to get to grips with these functions if I didn't have the calculator but I'd probably understood it a bit more and in a bit more depth if I hadn't have had it.
30	Perry	Yes for demonstrating new graphs, for new functions and stuff they're really, really useful.
31	SE	Yes.
32	Perry	But as I say you can loose all the algebra just by relying totally on the calculator. You can make big mistakes or something else.
33	SE	That's why I making you do both in these questions.
34	Perry	Yes that's how we're doing it.
35	Mick	Yes.
36	SE	Maybe I ought to take the graphical calculator away for the final session.
37	Perry	No, no, no, no. (jokingly)

Perry volunteered to demonstrate how to solve question 10 from the main trial exercises using the blackboard and OHP.

1	Perry	Right question 10, all you've got to do is solve two simultaneous equations to solve x and y. The two equations: $x - 3y = 16$ and $x^2 - 4y^2 = 13$. First of all I make x the subject in the simple form: $x = 16 + 3y$ then substitute that into this formula ($x^2 - 4y^2 = 13$). So I get $(16 + 3y)^2 - 4y^2 = 13$. Multiply out the brackets: $256 + 9y^2 + 48y + 48y - 4y^2 = 13$ and you get that. So you have to simplify that and you get: $5y^2 + 96y + 256 = 13$. So we can say from that $0 = 5y^2 + 96y + 243$.	

		You can try to factorise that but you won't get very far so you have to use the big formula which is... And what you get is: $\frac{-96+/-(4356)^{1/2}}{10}$ So we can get two answers from this, so we get: y = -3 or y = -16.2. So when y = -3, x = 16 + (3.-3) = 7 or when y = -16.2, x = 16 + (3.-16.2) = -32.6. So we've got four answers there and it's very complex.	Perry wrote out the quadratic formula correctly.
2	SE	Yes.	
3	Perry	To do that on the TI-82. What you do is you draw the two graphs so basically Y1 equals (x – 16)/6. Oh that's not right. The other one is y over ... I'll have to rearrange the formula to make y the subject.	Perry entered (x –16)/3 into the graphical calculator as Y1.
4	SE	Yes.	
5	Perry	$X^2 – 4y^2 = 13$, $4y^2 = 13 – x^2$, $Y^2 = (13 – x^2)/4$, $Y = +/- [(13 – x^2)/4]^{1/2}$ I believe. Could be wrong! Is that wrong?	Perry had made a mistake in his algebra.
6	Class	No response.	
7	Perry	Oh thanks boys! So it's the square root of $(13 – x^2)/4$. So you get two graphs.	Perry entered $[(13 – x^2)/4]^{1/2}$ as Y2.
8	SE	Right.	
9	Perry	Oh? I suspect that that is extended.	Perry pointed to the curve.
10	Mick	$x^2 – 13$ though.	
11	Perry	$x^2 – 13$?	
12	Mick	Yes it is. That's what I got and I got a different graph.	Mick was confident.
13	SE	Mm.	
14	Perry	Did you?	
15	Mick	Yes that last bit.	
16	SE	Yes you have.	
17	Perry	But if you move the 4y to that side – yes you do.	
18	SE	It's going to be positive yes. Ok?	
19	Perry	Try that.	
20	SE	Try that then. Right.	Perry entered Y2 as $[(x^2–13)/4]^{1/2}$
21	Perry	Is that right Mick?	
22	Mick	What, that looks right.	
23	SE	That's Ok yes.	
24	Perry	They don't cross.	
25	SE	So is there another thing that you could try? If you look at the square root, you've drawn the positive square root haven't you?	No response.
26	Perry	Yes.	
27	SE	What about trying the negative square root?	
28	Perry	Try the negative square root?	
29	SE	Why don't you draw them both – the next one as Y3 - to get the whole graph.	
30	Perry	This one.	
31	SE	Yes.	
32	Perry	The negative positive square root.	
33	SE	Yes, so it's exactly the same but just negative.	
34	Perry	This could be fun.	Perry typed in the negative square root and graphed the whole function.
35	SE	That was the first one, yes. Yes so you can see one	

		crossing point there.	
36	Perry	You can. I wonder if it's worth zooming out? Because that goes up there…	
37	SE	Yes, yes.	
38	Perry	Try zoom out.	
39	SE	You can see the two crossing points there can't you?	Perry zoomed out again.
40	Perry	Take them one at a time. Zoom in.	Perry zoomed in on the intersection point on the right hand side.
41	SE	Right so if you trace that you can see what the co-ordinates are can't you?	
42	Perry	Work out the intersections.	
43	SE	Or you can calculate yes sure.	
45	Perry	Hurray!	Perry was jubilant.
46	SE	Yes and that's what we've got isn't it, yes.	
47	Perry	We did get that.	
48	Mick	I got that as well.	Mick added reassurance.
49	SE	And the other one is on the other side of the screen that you haven't got on at the moment, but you'd have to zoom around that wouldn't you.	Perry zoomed in to the left and calculated the intersection point.
50	Perry	You could do a dance whilst waiting for it! Which curve am I on now? The intersection point is –2.6, -16.2.	
51	SE	Yes. Ok that's what we got yes.	
52	Perry	I got that, success!	Perry was really pleased.
53	SE	That's great. So that shows that you got the correct answers with your algebra.	
54	Perry	Yes, I did yes. It's very nice to know that I worked it out right for a change.	Perry was given confidence in his algebra and in the validity of the graphical approach.
55	SE	But you would have chosen to do the symbolic rather than the graphical approach if you didn't have a graphical calculator?	
56	Perry	Well I would have done both.	
57	SE	You would have done both.	
58	Perry	Yes because I would have done this bit because I'm pretty confident with my algebra.	Perry pointed to the algebra on the whiteboard.
59	SE	Mm.	
60	Perry	And then I would have checked it using the graphical calculator.	
61	SE	Yes.	
62	Perry	If I had one – although I wouldn't mind but I prefer to check.	
63	SE	Ok right thanks a lot for your demonstration.	

Carol was asked to describe how she had tackled question 13 from the main exercises:

1	SE	Did you actually just draw these out and then match them up? Or did you think that this might be this? Did you have some reasoning before you attempted the questions?	
2	Carol	I knew roughly what each was. You know that that's going to be some sort of trigonometric function.	Carol pointed to graph A.
3	SE	Yes.	
4	Carol	Because it's a – one of those ones.	Carol was hesitant.
5	SE	And it's in zoom trig so that's a big clue.	
6	Carol	And it's in zoom trig yes. So you know that those three are going to be them and you know that a tan looks like that.	Carol referred to graphs A, B and F and then pointed again to graph F.
7	SE	Yes.	

8	Carol	And the other ones – you know that's an x^2 and that's an e^x.	Carol pointed to graphs C and E.
9	SE	Yes. That shape?	SE pointed to graph D.
10	Carol	Yes and you know that's an inverse of something.	
11	SE	That's actually a logarithm but you haven't actually met logarithms yet…	
12	Carol	Natural log?	
13	SE	Yes natural log …Because I wanted to introduce you to some of the functions that you would meet later on and what they would look like.	
14	Carol	We have done natural log, haven't we?	
15	SE	You have?	
16	Carol	Yes. It's just that it always looks like it's an inverse. It looks like that is the inverse of the other one.	Carol is not yet familiar with the shapes of logarithmic functions.
17	SE	So how could you show that this was the correct transformation?	
18	Carol	Em I don't know. It's just it went the right way round on my calculator, rather than the other way round. I actually did them by trial and error using the cos function.	
19	SE	So this one here with this transformation, if you drew the picture of cos as well, what would that be like?	SE pointed to graph A.
20	Carol	It would be the other way round. That would be there wouldn't it?	Carol traced the position of cos x.
21	SE	Yes because it's shifted 90^0…	
22	Carol	To the right.	Referring to the action of $f(x - 90^0)$.
23	SE	To the right.	
24	Carol	Right.	
25	SE	So yes that's right well done. And this one why is this sin 3x?	SE pointed to graph B.
26	Carol	Because it's 3 times smaller.	
27	SE	Yes that's right, it's been squashed.	
28	Carol	It's been like compressed three times smaller than sine.	
29	SE	And this one here?	SE pointed to graph F.
30	Carol	That's three times wider.	
31	SE	Wider yes.	

Marie was also asked about question 13:

1	SE	How did you do these questions, did you work through them drawing every one on the graphical calculator or did you have an idea of which it would be before you started?	
2	Marie	Yes I had an idea.	
3	SE	So this first one here how do you know that this is the right transformation?	
4	Marie	Well it's the same as sine of x. So we looked at it and if you look at it, it looks like sin x, but there wasn't a sin x.	
5	SE	It is sin x you're right. You recognised that correctly. But it's not an option so you've got to find something that's equivalent to sin x.	
6	Marie	Yes.	
7	SE	So em how do you know that this is correct? If you compared it to the graph of cos x?	
8	Marie	Well we knew it was going to be like something like that. So then we did just to try it like x + 90 and x – 90.	
9	SE	So that's trial and error isn't it?	
10	Marie	Yes it was kind of trial and error but we had an idea. We knew which ones to go for. We'd go for the cos ones and not the tan ones because we knew the differences in the shapes of the graphs.	

11	SE	So if you drew cos x on this picture at the same time what would cos, just cos x look like?	
12	Marie	It'd be like that flipped over like going like that.	Marie traced the graph of y = cos (x+90) onto the paper.
13	SE	Would it? Just think about what the transformation means.	
14	Marie	Oh cos (x – 90).	
15	SE	This is what you've got here on the picture.	
16	Marie	Yes.	
17	SE	This means it's been…	
18	Marie	Has it been moved along?	
19	SE	It's been moved along yes. This is cos (x – 90) so it means that it's shifted…	
20	Marie	Oh right.	
21	SE	90^0 from the original cos. So where would the original cosine be on here?	
22	Marie	Em it would be like that wouldn't it?	Marie traced the graph of cos (x-180).
23	SE	That's moving it 90^0 there, but what's cos of zero, can you remember?	
24	Marie	Just normal cosine x.	
25	SE	Cosine x.	
26	Marie	Well I thought it was like that but it goes through 1.	Marie was referring to the graph of cos(x –90)
27	SE	1 that's right. So if you have a look it would look like that.	SE drew the graphs of cosx and cos (x –90) on the graphical calculator.
28	Marie	Yes.	
29	SE	Ok. So it's like crossing with the other one and they're separated by 90^0 in the right direction, to the right, which corresponds if you remember to a form of the transformations, which we did.	
30	SE	What about this one, how do you know that that's the right form?	
31	Marie	Because to what you'd normally expect there are three more kind of peaks.	
32	SE	That's right, yes it's all compacted, and this one here?	
33	Marie	Em well we knew that it was a tangent but we didn't know which one, so we just tried them. So, just trial and error…	
34	SE	What does this transformation do to the tangent? What would it do to it?	
35	Marie	Spreads it out.	
36	SE	Spreads it out yes, and what about this picture here, C?	
37	Marie	We knew it was like a x^2 graph and it would be like – and you have to move it along that line.	
38	SE	It's been moved along by 4 units...	
39	Marie	Yes.	
40	SE	And it's in the positive direction. So that tells you that it's…	
41	Marie	It'll be negative.	
42	SE	-4 there.	
43	Marie	Yes.	

Marvin

1. SE: Right so can you just tell me how you were having problems with that question 3, then?
2. M: Em well with question 3 what I did is I started by trying to do it algebraically and I haven't, actually brought question 3 with me but I did try doing it by doing it where $x^2 - x + 4$ is equal to $4x - 2$.
3. SE: Yes.
4. M: Where the values are - the values of x^2-x+4 are greater than - I can't remember how I did it I worked it out as an inequality em but then I was really getting confused with that because it seemed like it was the wrong way to do it. So what I did was used the calculator and I drew them both and you get one straight line and one u shaped one …
5. SE: Yes.
6. M: …and it looked like all the values of the u shaped one were above but when you zoom in, some of them are on the line and some are just slightly below because I did that trace thing with it.
7. SE: Yes.
8. M: Em and that confused me I don't know if I - could I have done it straight away.
9. SE: You could have done it that way …
10. M: Right.
11. SE: … and you could have done it symbolically as well. You were on the right track.
12. M: I did the table, you know where you draw the table and you get x down one and then you get two sets of y because you've got two graphs, right?
13. SE: Yes.
14. M: Now what I was trying to do is work out the relationship between the two - where one graph was higher than the other graph. And there was a - it worked yes, some, a lot of the values were higher than the straight line graph, but the points where I think - there were two points where they were 6 and 10 both of the y values. I didn't know whether they would be classed as em above the graph of 4x-2. So if it had of been it would have been - it would have either been all values minus 6 and 10 between 6 and 10 or the lot, every one. So, but I did get confused with that and after that I was on a down-hill tread the whole way em I did try these two next (pointing to questions six and seven).
15. SE: Yes.
16. M: And em well.
17. SE: I did understand that perhaps em some of you hadn't met questions like this before, I was just throwing them in, you know just to …
18. M: Well we actually did an investigation at GCSE, now when it says, assuming it says throwing a single biased dice, you *see* a dice, it's the first thing. I don't know if that is what you are looking for, but I do definitely.
19. SE: Yes.
20. M: And then that's one aspect of it, but then saying that the probability of getting two is 0.1 then that's when the algebra comes in and em I mean I don't know if it says the number of faces on the dice em ...

21. SE: Yes, it going to be six, actually.
22. M: Yes, because you've got six there and you've got ...
23. SE: 1, 2, 3 ...
24. M: Yes.
25. SE: 4, 5, 6 and it doesn't make that clear, no. Maybe the question could be ...
26. M: Em well no, no - it's kin of obvious isn't it. I mean em how the quadratic formula could be obtained, we've never done that before so em but I did try my best, I really did, I had a good go.
27. SE: Yes I understand that. But thinking about this (question six) you've got a value of the probability when it's 2, when its 3 and when its 6.
28. M: Right.
29. SE: And it's on a quadratic curve, if you're thinking in graphical terms.
30. M: Yes, the curve will ...
31. SE: Will go through those points.
32. M: Right, yes.
33. SE: Join them up.
34. M: Yes.
35. SE: So does that suggest anything to you, how you could have ...
36. M: Em (pause) a quadratic curve starts at zero and it accelerates up almost.
37. SE: Yes.
38. M: So the probability of 1 would be below 0.1 more towards zero, between 0 and 0.1 I would say and then - is that what you mean?
39. SE: Well it's on the right lines, but do all quadratics actually go through 0?
40. M: No I don't mean - I mean between 0 and 0.1 on the y because that's at 0.1.
41. SE: Mm.
42. M: So it would be somewhere between - see what I mean?
43. SE: I see what you mean, I think, yes.
44. M: I mean if the graph is like x^2+3 it would go 3 up the y axis, so it will be between 3 and whatever the next value of x is because that's 0.1 and you don't know the first value, it's going to be between. Do you see what I mean? It might be below the x.
45. SE: It could be possible with a quadratic though that if you had x is 1 and x is 2...
46. M: Right.
47. SE: And you've got a reflection in the curve, don't you, it goes like that (demonstrating the shape of the curve in the air) so sometimes two x values can give the same y value.
48. M: Right.
49. SE: So it could be that when x is one this could be 0.1 as well...
50. M: Right.
51. SE: And at 2 it could be 0.1 and it could have a minimum point in between, I mean there are different possibilities.
52. M: Right I see what you mean, right yes, yes. Am I getting the right idea then that you're saying that the quadratic formula would be in the form of $ax^2 + bx + c$.
53. SE: Yes.
54. M: Ah right, not just $ax^2 + c$.
55. SE: No.
56. M: Right, I see.
57. SE: Ok and...
58. M: Because they are the u-shape aren't they?
59. SE: Yes.
60. M: Right.

61. SE: There are some em algebraic techniques that you can apply to actually work out these values.
62. M: Right.
63. SE: What I was trying to suggest to you is if you did plot these points…
64. M: Right.
65. SE: They would lie on a curve, so you could look at what 1 is and approximate the y value…
66. M: Right. I see, yes, right.
67. SE: As part of the curve and then for three…
68. M: Right.
69. SE: Is it three or is three given. Oh it's given, then for four and for five.
70. M: Right, now what I was em confused with - if that's the case can I actually do that. I mean with the graphic calculator you write Y= and you write in the formula and then you draw the curve and then you trace it.
71. SE: Yes, you wouldn't be able to do this at the moment on the calculator.
72. M: Right.
73. SE: But there is a statistics plot…
74. M: Right.
75. SE: Where you can put in particular values of x with the corresponding y's and it will just plot them as points.
76. M: Right.
77. SE: So if I had shown you how to do that you could use the graphical calculator.
78. M: Right.
79. SE: But you would have had to have done this on paper at the moment.
80. M: Ok.
81. SE: So I'm just going to have a look at what you've written down here for (question) one.
82. M: Em am I alright to take this back and have a go at it tonight?
83. SE: Yes that's fine.
84. M: I mean I will do.
85. SE: Yes, Ok. "Algebraically. We were taught to roughly imagine the graph in your mind" (reading out Marvin's written solution).
86. M: When we were doing inequalities oh quadratic inequalities is where the formula is less than or equal to zero or greater than. If it's greater than zero it's em oh it's two areas, two separate areas, they're not joined together. So it would be x is greater than or something like that and x is less than that instead of being em it's like that (pointing to his work) that's two separate areas isn't it?
87. SE: Yes.
88. M: If it was the other way round, if it was below the x axis, it would be em for example –4 is less than x but one is greater than x as the other one. Do you understand?
89. SE: I know what you mean, yes. So in this one - you didn't use any images in question two?
90. M: No, no, not at all. Em…
91. SE: Would you check - did you check it using the graphical calculator?
92. M: I did - I did check it, yes.
93. SE: Yes.
94. M: Em I mean the reason why I didn't use any imagery with that, I mean em I myself, I mean, I can imagine roughly what the graph would look like but I wouldn't be able to link the two together, you see what I mean. Some people might be able to…

95. SE: Yes.
96. M: Some mathematical genius's in the world might be able to, but I couldn't so em that way I tried - I tried, I don't know whether you can, but I tried - I went for the easy option and used the formula. I'm quite proud of that actually. I learned that off by heart.
97. SE: Some people made a mistake in their test in that, didn't they?
98. M: Yes that's right.
99. SE: Yes, they got it wrong.
100. M: Yes. Is that - I mean are those answers right?
101. SE: I have got the answer sheet here so I'll look for you. I mean I've got all the solutions written out.
102. M: Right.
103. SE: I've only done them symbolically on here…
104. M: Right. I see.
105. SE: But I mean I'll photocopy these and distribute them.
106. M: Right, that's nice of you.
107. SE: Right, yes (It's correct).
108. M: Oh that's alright.
109. SE: Em so do you feel that the graphical calculator helped you with these questions? Or only on certain questions? Or not at all?
110. M: Not all - not all the questions and I wouldn't say not at all either em it is - it is a help definitely. I mean it's spurred me on to go and get one.
111. SE: Yes.
112. M: I'm getting one tonight actually.
113. SE: Oh are you?
114. M: Yes I am. I'm having a Casio one em as I say it's spurred me on to get one. I do find them very useful and I mean I said that I checked it there and just to check it, it's an advantage.
115. SE: Yes.
116. M: It's so good. I mean I thought it was great when you had that LCD thing on the board and used it.
117. SE: Yes it's quite impressive, isn't it?
118. M: Yes but, yes…
119. SE: So useful for demonstrating to the group.
120. M: Yes I mean it's very easy for you isn't it, you don't have to draw the graph or anything.
121. SE: Oh no, it's much better.
122. M: But I mean em I would use it, I have used it, you know I even did my physics homework on it. I don't know if that's all right or…
123. SE: That's fine.
124. M: Em I drew a graph on that.
125. SE: Ok.
126. M: Yes.
127. SE: Right thanks a lot for your time, I really appreciate it.

Nigel

1. SE: Ok so maybe if you just start describing to me how you'd have a go at these questions.
2. N: Right. That one (pointing at number one) I'd factorise that bit first (pointing at the formula of the function).

3. SE: Yes.
4. N: So that I would get two values for x.
5. SE: Mm.
6. N: And then I wasn't sure - I think I would… I would know how to do it if it was an inequality, otherwise apart from that I don't know. I think is it just the two values of x so say, I don't know what it is, but say it's just in the two brackets it was x-3 and say x+4.
7. SE: Yes.
8. N: It would be x is 3 or –4.
9. SE: That would be the values when then actual graph cuts the x-axis.
10. N: Yes.
11. SE: But this is asking for whereabouts it's actually below the x-axis. So you've got some points on the x-axis where it's actually equal to 0.
12. N: Yes.
13. SE: So how could you determine where it was below?
14. N: You'd have to - would you use an inequality?
15. SE: Yes, well you could express it as an inequality, because you're looking at below the x-axis so it would be less than or equal to 0.
16. N: Yes.
17. SE: So does that clarify things for you?
18. N: I think so em I'll write it out to work it all out now. I can't just do it.
19. SE: Would you have to plot the graph, do you think, or would it not be necessary to plot the graph?
20. N: Em I think I would have to plot the graph just so I could just like check the values. It would be just easier to plot the graph, so you could like picture it rather than just picture it in your head.
21. SE: Yes. And question two?
22. N: I use two simultaneous equations to work out x, so x is (pause).
23. SE: And that would give you?
24. N: That would give me two values of x.
25. SE: And would that be the answer in this case?
26. N: Yes.
27. SE: Yes, that would be the answer because you want to know where the graphs are equal to one another, don't you?
28. N: Yes.
29. SE: So would you not find it necessary at all to use any pictures in your head or on the calculator or to draw them yourself for that one?
30. N: No I'd just work it out in algebra.
31. SE: Ok and what about question three?
32. N: (Pause) I wasn't really sure about this one. What I would have to do is I'd have to plot it onto a pair of axes and then I could actually picture what it would look like and it would help me to understand what I've actually got to find out.
33. SE: So you would have the two pictures of the graphs and they would intersect at some points.
34. N: Yes, mm.
35. SE: And then you would be able to see whereabouts one lies above the other.
36. N: Yes.
37. SE: Do you think that there is an algebraic way of doing it? Or would you just do it graphically?
38. N: There probably is but I don't know what it is.

39. SE: That's a fair comment, but would you try for it? If you had the knowledge would you do the graphical approach or would you…?
40. N: If I knew how to do it algebraically, I would always do it algebraically.
41. SE: You would prefer to do it algebraically. And what about question four?
42. N: (Pause) em right with that one - the first one I can just picture it in my head because they are fairly similar to each other. So one would cut yes one would be just 2 above the x-axis and one would be just 2 below it so that would be really easy.
43. SE: Yes.
44. N: That one (the second pair) I would have to draw them onto a graph.
45. SE: Ok and this one here (the first pair) would they have exactly the same shape?
46. N: Yes.
47. SE: Yes Ok, so they're just - one's shifted up.
48. N: Above yes.
49. SE: Ok and so this one (the second pair) you'd have to draw them out and see how they were related. Right and what about question five?
50. N: (pause) em if x is there, it just em it's just a reflection (pause). Yes it's hard to explain actually. All right yes on one side of the graph say I don't know what the gradient would be say it was 0.5, it would be 0.5, on the other side it would be minus 0.5, just change the sign round.
51. SE: Yes that's right, yes your right it's a reflection.
52. N: Yes.
53. SE: Good and what about question six. I can see that you're a little unsure about those aren't you?
54. N: Yes.
55. SE: And seven. Em if we think about a graphical approach these values are part of a quadratic formula, so you could plot them as points on a graph. So when x is 2 you could put y is 0.1, when x is 3 it will be 0.12 and put 6 on as well and a quadratic curve would join up the points. Would that help you at all in trying to work out what the probabilities of these x values would be?
56. N: Yes you could just, yes, find one just work up to the line.
57. SE: That's right and you can have an approximate value for the y value.
58. N: Right, yes.
59. SE: Yes. That's one way of doing it, but it might not give you exact values and there is a symbolic approach you could try to work those out.
60. N: Yes.
61. SE: But obviously you've not come across that, so that's not important at the moment.
62. N: Yes.
63. SE: So the final question I wanted to ask you was, have you found the graphical calculator to be useful?
64. N: Em no because I don't know enough about the calculator to use it.
65. SE: You're not that happy with it at the moment?
66. N: No it's the first time I had seen one on Tuesday, so…
67. SE: So how are you feeling with it now are you still a little uncomfortable?
68. N: I was fine with the stuff we did in that lesson.
69. SE: That was ok.
70. N: But that was fairly simple, wasn't it?
71. SE: Yes, yes I mean this part here - to actually plot these points on the calculator you would have to use the statistics mode.
72. N: Yes.

73. SE: Which I haven't shown anybody how to use yet so unless you've had prior experience you would have had to have done this one on paper.
74. N: Yes. If you did have the graphical calculator you could just put it in and you would be able to work out those exactly.
75. SE: Yes that's right, well thanks a lot for your time.

Fay

1. SE: Ok so if we just have a look at the questions, you've got a sheet there, em so could you talk me through how you would have a go at these questions then please.
2. F: Em first of all I didn't think about them as if I had a graphical calculator because I don't actually have one myself so I wouldn't be using one normally.
3. SE: Right.
4. F: So I'd first of all sketch the graph and then from looking at the graph I could see that the values must be less than 0 when substituted in the formula. So then I would be able to see from the graph that they would be below the x axis, so then I just did like an equation to work out whereabouts - what sort of size they would have to be, but I only really got a rough estimate.
5. SE: Right so em you would definitely draw the graph first?
6. F: Yes.
7. SE: And how would that help you?
8. F: Because I'd be able to see from drawing the graph - I'd be able to see whereabouts on the axis they would be to equal, well to be below the x axis.
9. SE: But you would find the points where it was on the x-axis first...
10. F: Yes.
11. SE: So then you would know whereabouts it was.
12. F: Yes.
13. SE: Ok what about question two?
14. F: Well for that one I would either draw the graphs or use the graphical calculator to find the intercepts, just by looking at it and using the little trace.
15. SE: Yes. That would give you an exact value.
16. F: Yes.
17. SE: Would you not apply a symbolic approach at all?
18. F: Em I would draw the graphs if I was going to do it without the graphical calculator but apart from that I would just type it in.
19. SE: Ok so what about question three?
20. F: I would have done pretty much the same by drawing the graphs having a look at them and seeing what was there.
21. SE: Can you think of a way of doing that symbolically? Or would you prefer not to do it symbolically?
22. F: I'd prefer not to I think. I think I would just use the calculator and put them in and draw two lines. To have to plot values for two graphs, it would take too long.
23. SE: And, actually if you took this one and you set it equal to that one (pointing to the question sheet) and you re-arranged it what would that give you? If you said that was equal to that, put them equal to one-another and then re-arranged it.
24. F: You would get that $-2+4 = 2$. Would you do that would you?
25. SE: Yes.
26. F: So $4x+2 = x^2-x$ so you get $x^2=3x+2$ Would that be...?

27. SE: You could write it like that, but if you wanted to solve it using the quadratic formula, then you would have to have it all…
28. F: x^2- (3x +2).
29. SE: All on one side.
30. F: To put it into your quadratic formula.
31. SE: And that would give you the co-ordinates of when these graphs were actually intersecting.
32. F: Crossing.
33. SE: That's right crossing, yes so that would give you another way of working this problem out.
34. F: Yes.
35. SE: To give you those exact values. But if you did use the graphical calculator it would…
36. F: Save a lot of time.
37. SE: and give you those as well.
38. F: I'm not very good at re-arranging.
39. SE: Em is that why you prefer using a graphical approach?
40. F: Yes because I find it harder to swap - swapping sides and balancing equations and things end up taking me longer like to balance the equation than just to solve the problem using the graphic calculator. So that's on my Christmas list.
41. SE: And what if you didn't have one, though, like you don't at the moment?
42. F: Em.
43. SE: Do you still find it easier to sketch?
44. F: I still find it easier to draw them rather than - because if you draw them you get a rough idea and then you can… I just find it easier than putting things into formulas.
45. SE: Ok and what about question four?
46. F: Em well by looking at the equations you can see that they both have the same gradient and direction so they're parallel and then from looking at the equations I can see that one intersects the y axis at 2 and one at –2 and roughly what they look like, just from imagining the graphs. Then the connection between the other two graphs is by looking at it they are the same but whereabouts they are on the x-axis – it's further along.
47. SE: Yes, that's right. Did you work that out by drawing those out?
48. F: No I think we'd done some work on this before previously and I just remembered that looking at this where that that would be –2 on the y axis and that would be 2 and that these would be –2 and –3 or something similar, but they would be placed - whereabouts they would be placed on the axis.
49. SE: Mm so on this one (the first part of the question) have they got exactly the same shape?
50. F: Yes they're just one's above and one's below.
51. SE: That's right and are these the same shape as well?
52. F: Yes they're just moved across.
53. SE: Yes well done and what about question five?
54. F: Em for that one I would also look at the graphs pretty much the same as in this one, because you can see they're both $2x^2$ and it's just where they intercept that's different. So one at 3 and one at –3. And that was just from previous knowledge and I wouldn't have used the calculator for that if I didn't have it.
55. SE: So how do you get this one, where does this one come into it, the +3. Oh I see you've substituted that (the –3) for +3...
56. F: Yes.

57. SE: Instead. These are actually x values.
58. F: So I've done the question wrong.
59. SE: It's no problem. So it's just that the –3 would have been substituted in for x.
60. F: Yes.
61. SE: And then here if you wanted to sketch this graph…
62. F: You would have got the graphs the same on top of each other, wouldn't you. They wouldn't have changed at all.
63. SE: It is the same graph.
64. F: The same graph, yes.
65. SE: But you're just looking at different points on it.
66. F: Yes. I just didn't look at the x equals.
67. SE: It's ok that's fine.
68. F: Yes.
69. SE: So on this one you're looking at the slope, so you're looking at the gradient…
70. F: Yes.
71. SE: As you go along the curve.
72. F: Mm.
73. SE: Have you ever considered the gradients before? You've probably looked at straight line gradients before.
74. F: Yes we've done gradients on curves.
75. SE: Yes.
76. F: Using the area of the curve or something but that was last year. We haven't done that since.
77. SE: Right.
78. F: And it doesn't really come to mind.
79. SE: Ok and have you left it at question five then?
80. F: Well I tried question six but I got nowhere.
81. SE: It's probably nothing like what you've ever met before.
82. F: No it was - I just had no idea what to do and for this one I started it but then I couldn't get anywhere because I worked out that your speed went from 0 metres per second - 0 to 3 metres per second in 12 seconds and then stayed at that for 55 seconds…
83. SE: Yes.
84. F: And then went back down to 0 in x seconds.
85. SE: So you picked out that information.
86. F: I picked out that information but I couldn't get anywhere from it.
87. SE: Yes.
88. F: I couldn't see how to progress from there.
89. SE: Well you obviously gave them a good try anyway.
90. F: Yes.
91. SE: Ok thank you for your time.

Perry

1. SE: Ok Perry so could you actually show me how you've gone through these questions, starting at number one.
2. P: Well first of all for number one em there are two ways of doing it. You can either do it algebraically by saying y=0, solving it by factorising and working out the values of x, alternatively you could draw it on the graphical calculator and use the table function to see if its an integer. The problem is if its not an

integer you have to do it algebraically, because I haven't worked out how to use Y= yet.

3. SE: Right.
4. P: I found that it was actually 1 and –4 the values that are below. Em number two, well this one was easier by using the graphical calculator because you can draw both the graphs and you can actually calculate the intersect on it …
5. SE: Yes.
6. P: By using the calculate function and it was really useful because it was just a case of drawing it and typing in things. I got the correct values I hope.
7. SE: Yes looks like it to me. So going back to question one, which of these approaches would you prefer to use - do you have a preference?
8. P: Well I prefer to do it on the calculator, just by drawing the graph, because it would save doing work. But as I say if it's not an integer, it can't actually recall the values, so you have to do it algebraically. So either would be all right, but I much prefer doing it just drawing it because it saves a lot of fuss.
9. SE: Mm, but you could have done this one (pointing to question two) algebraically as well couldn't you?
10. P: Em you could, but it would be a lot more complex, I think. I haven't quite got it into my head how to do it, so I thought it would be a lot easier to draw it, to get an idea what the lines look like. Then just play around with it to see if there is another way of calculating it, which there was.
11. SE: Yes. What about question three?
12. P: Em well again I thought that that would be hard to do algebraically. It would mean a lot of playing around with perhaps trial and error because I haven't quite sorted out how to work it out. So again by drawing the line it gives you a clear picture of what it looks like and so you can go on from there.
13. SE: So if you became more confident, after more experience with you're A level, with using the algebra.
14. P: Yes.
15. SE: Do you think you would still draw graphs and use that approach?
16. P: Em yes I would to make sure that my algebra was right, because it's easy to do a page of calculations in an exam and you don't know whether its right or wrong, you've just got to hope. But if you can actually draw that, you know and have an idea of what the lines look like and you get 2.34 something, and it looks like its between 2.2 and 2.5 then you know that you've got it reasonably right. So it's a good use.
17. SE: Ok and question four?
18. P: Em well I know what the x^2 graph would look like, you know the parabola.
19. SE: Yes the general shape.
20. P: And I know by being –2 and 2 that they would be 4 apart, because I know that through drawing graphs and experience.
21. SE: Yes.
22. P: But if I had no idea then just drawing it you can see the different values.
23. SE: Yes, so have those graphs got the exact same shape?
24. P: Yes the same shape, except one is 4 lower down the y-axis.
25. SE: Yes. What about the other two?
26. P: The other two is that the $y= x^2 +2x+3$ is one above - no it isn't. Well again you can draw the graphs on the calculator.
27. SE: Yes.
28. P: And compare the shapes and just compare it by visually doing it I suppose.

29. SE: And if you didn't have the graphical calculator you would just do this by hand?
30. P: Yes I would have to draw that by hand because I don't know how to compare them algebraically really, because I have some idea of what they look like but its not much use really em when you're looking at numbers and stuff.
31. SE: Maybe it will become more clear to you when you've had more experience of functions?
32. P: Yes it will yes. But while you're getting the grasp of new ideas it's good to have a visual aid.
33. SE: That's right. Ok question five.
34. P: Em well this one, I know - I would know personally that $y = 2x^2 - 3$ would be symmetrical through the x=0 line, but again by drawing that, it just helps you so that you can see that the gradient changes round, as it were, from a negative to a positive.
35. SE: Yes.
36. P: I couldn't do six, but I did seven. Now seven, I would find very hard if I didn't draw it.
37. SE: Yes.
38. P: And I couldn't actually draw it on the graphical calculator.
39. SE: No.
40. P: So, a quick sketch really. I can draw it and write down on the graph what I know - a basic velocity time graph and by working out the area of a trapezium and stuff em you can actually work it out algebraically by visual aid.
41. SE: Yes.
42. P: But without actually drawing a graph you'll struggle on that one.
43. SE: I think your right. Going back to question six, this biased dice. All these values are part of a quadratic formula.
44. P: Yes.
45. SE: So if you were to plot them individually on a graph say, using a graphical approach. When x is 2, plot y is 0.1, when x is 3 it will be 0.12, and then you have x is 6 and you would go up to 0.3. Then you could join those points with a curve because you know they lie on a curve because it says so – it's a quadratic formula.
46. P: Yes.
47. SE: So does that give you any idea of how you might be able to estimate the values?
48. P: Well if you have a good idea of what the curve would look like then you can draw it, but it's hard to get the graph so it looks right and then you can estimate the values of 1, 4 and 5.
49. SE: Yes that's one way you could have done it, using a graphical approach, but it would be approximate perhaps depending on how accurate your graph was. So there is an algebraic approach that you'll see that when I hand out the solution sheets.
50. P: Oh right.
51. SE: Ok. But you probably haven't come across that yet, so I wouldn't really expect you to come up with that. So have you found the graphical calculator useful?
52. P: Yes it's been very useful em I suppose using physics and stuff as well. If I get a graph and I need to draw it quick, I've got some idea what it looks like or I might have to draw a graph in the exam on a piece of graph paper, which

happened at GCSE. You know what the graph looks like before you draw it so it's an advantage really, you know what you are doing.

53. SE: You don't think that if you concentrate on the graphical calculators you skills in drawing by hand will suffer?
54. P: Well I think you have got to be careful. I think if you do just concentrate totally on the calculator, completely leaving the paper behind and you make a silly mistake - you push add instead of minus or something, or times instead of minus, em then you're just going to go completely wrong and you're going to get bad scores. So you need to be able to do the stuff on paper to understand it then you can do it on the calculator. I think that to be able to work things out algebraically first shows understanding rather than just typing a few numbers into the calculator.
55. SE: Yes. Ok well thanks very much.

Roy

1. SE: Ok so can you explain to me how you would attempt to solve these questions.
2. R: For the first one I'd try the equation, either by using the formula - the quadratic formula or by just factorising it.
3. SE: Yes.
4. R: And I'd have to see whether, which way it - and then I'd sketch the graph and find the critical values. I'd be able to sketch the graph so that I could see that em the two critical values - in between the two of them, I'd be able to see that that's below zero so I know that my answer.
5. SE: Yes so you would sketch the graph...
6. R: Yes.
7. SE: To accompany the algebra - with the graph?
8. R: Just to make sure that I don't make a mistake.
9. SE: Ok and what about question two?
10. R: Em I'd probably rearrange the first formula, the equation and substitute it for x into the second one. So I could see, so that I could work out my two equations my two values of x and from that I can use that back in the first equation to work out the two values for y. Then I've got the two points.
11. SE: Right em and would you draw a sketch in this case?
12. R: Em I find that the calculators tend to be a bit inaccurate, so I don't tend to draw them on there and sometimes if you are doing it with graphs it can take a bit of time because you're plotting all the values, you've got to work them out and then you have got to calculate which values they actually cross at. So sometimes it's a dodgy decimal…
13. SE: Yes.
14. R: Then it can be a bit difficult, so I find it much easier to just use the equations.
15. SE: Right Ok, what about question three?
16. R: Em that one I would use my calculator to plot them em and see if I can em - so that way I would be able to see the actual graphs. So I could have an idea of how to start out by solving it. I shall probably use the same way as the first one but it's a bit more complicated then the first one, using the two ways for the first two questions.
17. SE: Yes but in this case you prefer to do it graphically, would you?
18. R: I'd start off by doing it graphically and then see if I could work in say algebra.
19. SE: And then check it with the algebra that way?

20. R: Yes.
21. SE: Ok and what about question four?
22. R: The way I'd use my calculator because I find it much easier just to put it into the calculator and work it out from there. So I can see that how the graphs are related, then if there is, if it is slightly confused I would sit down and plot the graph myself.
23. SE: Yes.
24. R: And see if I could see the relationship that way.
25. SE: So by looking at these two formulas, can you see a relationship without drawing it?
26. R: I could actually - I believe that one crosses the y intercept at –2 and one crosses at 2.
27. SE: Yes.
28. R: But I'd check that because I'm not entirely happy.
29. SE: What about the shapes of these graphs?
30. R: Well I'd know that they were both parabolas, the same - they've got the same shape, the y's are parallel. I'd know that they were both the same shape because they're both x^2+3x.
31. SE: Yes Ok. So how long have you had your graphical calculator?
32. R: I've had it about a year now since Christmas.
33. SE: Do you feel like you're ordinary plotting skills yourself have suffered at all because you've been using a graphical calculator or because you don't always rely on it and do sketches yourself do you think that it hasn't made any difference or has it helped?
34. R: I find that it's helped because I can plot the graph on my calculator and by hand and then I can check that I am getting it right, because if there's a problem I know that there is a problem.
35. SE: Mm.
36. R: I don't have to wait for it to be marked. I can see the problem immediately and try it again so that way I can make sure that I get it right. It's helped in that way.
37. SE: Ok em, question five?
38. R: Em I'd also plot it on my calculator again, but I can see - I can see from the question that em as you go down the curve you can see that as it passes the y axis it changes its gradient but I'd be able to plot that using the calculator because I can plot tangent…
39. SE: Oh, yes.
40. R: At various points and the equation of that.
41. SE: So when you say you could see it from the question, what do you mean by that?
42. R: Well em certain questions I am able to see in my head and so that I could - that one I know that at a certain point the gradient on the curve the gradient would be –3 coming down and on the other side it would be 3 because it's a symmetrical curve. So it would just change the shape of the gradient.
43. SE: Ok and what about question six?
44. R: Em I wasn't entirely sure because we haven't exactly done this yet.
45. SE: No I was thinking that probably you hadn't done that…
46. R: Yes.
47. SE: Even before I set the question but I thought I would just throw it in.
48. R: I had a look to see if there was any relationship on a graph to start with, because the way that the values are going it looks as though it might be a

straight line graph. I'd plot this out and see if there was a formula to it and then em once I'd got all the values I'd probably need to have a look at a formula.

49. SE: So using your graph can you actually plot statistical points?
50. R: Yes.
51. SE: So you would do it like that on your graphical calculator because you can on these but I haven't shown anybody yet how to do that.
52. R: Yes. And if I was unsure how to do that I would do it all by hand.
53. SE: Yes.
54. R: But I would still plot the graph because it would help.
55. SE: Yes that would be one way of doing it yes and what about question 7?
56. R: Em from physics I know that a velocity-time graph's a trapezium. The area underneath equals the total distance travelled. So em I know that and then by using the - by knowing what's on the axes I can use the formula for the area of a trapezium to calculate it - to actually even work out the value of v.
57. SE: Yes. You wouldn't draw a graph, because you know what the shape is like?
58. R: I'd probably not, no but if it was a more complicated question i.e. something that I had not done before then I would probably sit down and draw the graph. It just depends on what I've done and what I haven't done.
59. SE: So you tend to use the graph if you are unfamiliar with the situation?
60. R: Yes.
61. SE: Ok right, thanks very much.

Carol

1. SE: Right so could you please describe to me how you'd have attempted these questions.
2. C: Right ok well for question one seeing as I'd got the calculator I used it and like used the Y= and typed in the graph and then went to the table to get the values.
3. SE: Yes.
4. C: If I hadn't had had it though, usually I would have just like drawn the graph and put in my own values and worked it out that way. So em that's how I would have done that one.
5. SE: Would you not have used any algebra at all – any symbolic techniques?
6. C: Em no I don't think I would. I would have just done the graph for that.
7. SE: Why do you think that is - because you wouldn't be happy with the techniques using algebra or is it because you just feel it's not necessary in this case or?
8. C: It's just I'm used to doing it with graphs, you know, I find it easier.
9. SE: Ok what about question two?
10. C: Right em I'd also draw the graphs for this one so I could see where em (pause) they meet. So that's how I would have done that one as well. A lot of these I would have done like that actually.
11. SE: So even if you hadn't got the graphical calculator you would have still used this approach, like you say plotting the values?
12. C: Yes, yes.
13. SE: But do you find that having the graphical calculator makes it easier?
14. C: Oh definitely yes. It definitely does.
15. SE: Do you think that if you used it a lot your skills using your own diagrams, drawing them yourself would suffer at all or do you think it would help improve those?

16. C: I don't think that it would change really either way just … I do find them a lot easier because it gives you all the values as well so you don't have to work that out …
17. SE: Yes.
18. C: But perhaps that might be worse, because if sort of you didn't know how to make your own values up, it was just given to you, you have to know how to do it but because I already know how to do it then it's fine, you know.
19. SE: Yes, and what about question three?
20. C: Question three em (pause) I typed in the two equations so that I could plot the graph on the calculator and then em I could look on the table to find the values of x em of where the graph was above the other. I used the calculator but I haven't got one myself but, so I would have done it drawing the graphs em I find them a lot better though, the calculators, they're good.
21. SE: Mm. So would you have known how to do this algebraically? Would you have had an idea of how to do it?
22. C: Algebraically? How do you mean?
23. SE: Using the formulas and rearranging them in some way to find out what x is.
24. C: Em I don't think I could just say now what it would be, but I might be able to get my head round it if I sort of got stuck into it. But that would take longer really so …
25. SE: Is that why you prefer to do it graphically?
26. C: Yes, it's a lot quicker.
27. SE: What about question four?
28. C: Question four (pause) em I did this on the calculator and typed the graphs into the calculator and em I found that they were the same graph except its moved up by four.
29. SE: For the first one.
30. C: Yes but I did actually know that already em I didn't really need to use the calculator but I did seeing as it was there, because I quite like it because it's fun. But I know about the, you know, the number on the end to work it out so that's what I did for that one. For question five (pause) em I used the calculator and looked at the table and so that I could work out the graph went across the x axis twice and that's how I used that. If I didn't have had it, again I would have used the graph and for these type of questions I'm very into the graph drawing.
31. SE: So it crosses the x- axis twice.
32. C: Yes.
33. SE: What happens to the slope, did you consider that - the gradient of the graph?
34. C: It would decrease and then increase again.
35. SE: Decrease and then increase?
36. C: I didn't do that actually.
37. SE: You didn't do that.
38. C: No.
39. SE: It doesn't matter.
40. C: And I didn't do the last two questions as well because they just went over my head.
41. SE: Yes, I thought that maybe you would not have met anything like this before.
42. C: No.
43. SE: With being fairly new to the A level. Em, so on this one, the slope changing you're right about this part its increasing (pointing to the right of the curve).
44. C: That's a minus gradient (pointing to the left part).

45. SE: That's a minus gradient, so what's happening to it? It's quite steep and then its getting more shallow.
46. C: Yes.
47. SE: So it's actually increasing the whole way through.
48. C: Yes, yes because it's … yes.
49. SE: Can you see that now. Right Ok then thanks a lot for your help. Just before you go, do you think that the graphical calculators have been beneficial?
50. C: Yes, for this type of work I used them a lot it's been good.
51. SE: Yes, you've enjoyed it?
52. C: Yes I have.
53. SE: Thank you.

Julian

1. SE: Could you please tell me how you answered these questions.
2. J: Right for question one I just typed it into the graphic calculator. Do you want me to do it now or?
3. SE: No it's no problem just describe what you did.
4. J: And em when it came up, I used the trace facility on the calculator to em highlight the first value that was below the x axis and then wrote them out in an inequality.
5. SE: Mm and did you think at all about using algebra – a symbolic technique to solve the problem?
6. J: Em well I was going to but then I thought this is a new toy so I thought I'd try that.
7. SE: So having the graphical calculator lead in the path of using a graphical approach for this question?
8. J: Yes, definitely and I've got a graphical calculator of my own.
9. SE: Yes.
10. J: So I'm a bit familiar with it so I was fine.
11. SE: So em would you have considered, without this graphical calculator, solving this problem symbolically? Is the graphical calculator the thing that has made you do it graphically? If you didn't have it what would you have done?
12. J: Oh you mean if I didn't have a calculator yeah I would probably have attempted it algebraically but as I have my calculator it was easier to do it that way, so I chose that method.
13. SE: Ok, what about question two?
14. J: What I did here was I typed in both of the functions onto the calculator, so it would graph both at the same time, put that on there (pointing to a button on the calculator) and used the trace facility again to see where it intersected and em wrote my answer down again.
15. SE: And what about question three?
16. J: Same again - typed them both in to the calculator and used trace to show where it first did it and then wrote an inequality for my answer.
17. SE: So in these two questions, as well…
18. J: Yes.
19. SE: You used this graphical approach seeing as this graphical calculator facilitates that.
20. J: Yes.
21. SE: Em what kind of approach would you have used without it? Would it still have been graphical? Or would it have been symbolic?

22. J: Well I would have tried to have done it with algebra first, but if I couldn't struggle that way I would tend to draw a graph for myself plotting some values and then do it that way.
23. SE: Em how long have you had your own graphical calculator?
24. J: Em I had it that last year of GCSE and it helped me then when we were doing functions and it was quite handy in the exam actually. You have to wipe the memory so any other use is minimal.
25. SE: Right, what about question four?
26. J: Em this is another one. I'd put them into the calculator again and they - as it shows them both at the same time it helps you to see how they are connected.
27. SE: Em by looking at the formulas of these (pointing to the first two equations)…
28. J: Yes.
29. SE: Without drawing it could you have seen, for these two, what the relationship was?
30. J: Em they intersect the y at different heights.
31. SE: Yes. Are they the same shape?
32. J: Yes other than that they will be the same shape. The –2 and the 2 show you that one's just higher up than the other.
33. SE: So do you actually need to draw the graph?
34. J: No.
35. SE: or would you have drawn it anyway to just check that you were right?
36. J: Yes, I think. Say I had twenty questions like that I would have drawn the graph on the first one just to check my method was sound and then I would have just not bothered and would have just carried on like that.
37. SE: Em this ones a little bit different (pointing to the second pair of equations)
38. J: Yes.
39. SE: Would you have needed to have graphed those?
40. J: Yes I would have graphed that one definitely.
41. SE: Yes and what about question five?
42. J: (pause) I would have graphed that one too but I presume that it gets steeper.
43. SE: It gets steeper?
44. J: As it veers towards 3.
45. SE: Right.
46. J: Because I know the shape of x^2 is a parabola.
47. SE: Em and what about –3?
48. J: Oh it's going to be a negative one isn't it?
49. SE: Mm.
50. J: It's going to go from negative to - it's going to stay negative, pretty steep then it will level off and it will get steep again and that will be the same as that (indicating the gradient at 3 and –3) but one will be minus.
51. SE: Right ok yes that's fine em so you can see that in your head if you need to?
52. J: I didn't need the calculator no, but I just it's a way of checking because there's no problem of doing it. It's not against any exam law so you might as well make sure you are right instead of just hoping.
53. SE: Yes and what about six and seven, because some people had problems with these because obviously they're very different from what you might have met before. Did you manage to get anything out of them?
54. J: Em I found them very hard. I can't remember exactly how I did them, so I'll just scan them. (Pause) No I just couldn't do these. To do something like this, because I don't understand it straight away I'd either get help or tackle it algebraically as far as I could go to try and get a function I could type into the

calculator and then hopefully makes sense from it from there. But at first I think I'd definitely need to get some instruction.

55. SE: Yes I think it would be quite difficult because you don't know the value of v and you don't really know…
56. J: All these you knew what to put into the calculator straight away.
57. SE: Yes.
58. J: Here it's less clear.
59: SE: Here it's totally different, yes.
60. J: So I'd try to work out what to put in at first and then go for it from there.
61. SE: So have you found the graphical calculator a real help then.
62. J: Yes it's been really good. It was good at GCSE and been more useful now at A level em even if you don't want to use it to do the work, it's definitely a good back up.
63. SE: Mm.
64. J: Because it gives you confidence. You know you've got it right because you've seen it happen.
65. SE: Yes. Ok well thanks a lot.
66. J: Thank you.

Marie

1. M: Do you want to know how I did it?
2. SE: Yes, can you please just explain how you would have done these questions.
3. M: Right ok em I'd draw that graph.
4. SE: For the first one?
5. M: For the first one on the calculator and then probably just read off the graph to find the values of x.
6. SE: Right, is that because you had the graphical calculator or do you particularly like working with a graphical approach?
7. M: I think it would just be easier for this one, yes.
8 SE: Would you know how to do it symbolically – well algebraically?
9 M: Em yes probably. Yes I'll just do that and put it's less than 0 and just work it out that way probably. Em Ok number two (pause) probably again I'd do a graph em and read off the graph. If I couldn't - if it wasn't that clear then I would do it algebraically.
10. SE: So you would rather do the graphical approach first?
11. M: Yes to see if it's an obvious - if it's an integer or not.
12. SE: So is that because you've got the graphical calculator or would you do that even without the graphical calculator?
13. M: What draw a graph by hand? No because I wouldn't trust myself at being able to draw it well enough.
14. SE: So the graphical calculator is having an influence in the way you might solve these questions?
15. M: Yes.
16. SE: What about question three?
17. M: Em (pause) again I would draw a graph on the calculator but not by hand (pause).
18. SE: What about question four?
19. M: Question four I'd do algebraically.
20. SE: Algebraically?

21. M: Well I'd probably do a bit of both. I'd look at the graphs and then prove it, but it's saying to describe it, so I'd try and prove it algebraically as well.
22. SE: So how do you think you could do that algebraically?
23. M: Well hang on I might have got the question wrong (re-reads the question). Well when it says the connection there I'd put $x^2+2 = x^2+2x+3$ and simplify it. I'm not sure that that's right.
24. SE: That's one way you could approach the problem.
25. M: And then I'd draw that graph as well to show how they are related, because it might be that they are reflected or something.
26. SE: Yes. That's not the approach I was looking for in the question actually setting them equal to one another, but if you did that it might tell you something interesting. It is something that you could try because you don't know the relationship that I've got in mind.
27. M: Yes, that's why I would draw the graph first to get a rough idea.
28. SE: Ok and what about question five?
29. M: Em I'd draw the graphs, yes. I'm not quite sure what that means though.
30. SE: The slope means the gradient of the function.
31. M: Oh right yes then I would find the gradients at each of those points, where it crosses it.
32. SE: Did you actually manage to do questions six and seven?
33. M: No I probably didn't get chance to do them no.
34. SE: Because a lot of people had trouble with those because they're not like any questions that you have met before.
35. M: No I didn't get chance to do those.
36. SE: That's Ok that's fine. So how have you felt about the graphical calculator?
37. M: Em once I can use it, it is good to use, it's just getting to know it first of all that's more difficult.
38. SE: So have you found it useful?
39. M: Yes definitely.
40. SE: Do you think it's helped your understanding of functions?
41. M: Yes.
42. SE: Yes. Ok right thanks a lot.

APPENDIX D

Associated Publications

Contents

Associated Publications Listing

Elliott, S. (1998). Visualisation and Using Technology in A Level Mathematics, *Proceedings of the British Society for Research into Learning Mathematics Conference*, University of Birmingham, June 1998, pp. 45-50.

Elliott, S. (1999). How Does the Way in Which Individual Students Behave Affect the Shared Construction of Meaning? *Proceedings of the British Society for Research into Learning Mathematics Conference*, University of Warwick, Nov 1999, pp. 13-18.

Elliott, S., Hudson, B. and O'Reilly, D. (2000). Visualisation and the Influence of Technology in 'A' Level Mathematics: A Classroom Investigation. In T. Rowland and C. Morgan (Eds.), *Research in Mathematics Education Volume 2. Papers of the British Society for Research into Learning Mathematics,* pp. 151-168.

Elliott, S., Hudson, B. and O'Reilly, D. (In press). How Does the Way in Which Individual Students Behave Affect the Shared Construction of Meaning? In K. Jones and C. Morgan (Eds.), *Research in Mathematics Education Volume 3. Papers of the British Society for Research into Learning Mathematics*.

Poster presentation

Elliott, S., Hudson, B. and O'Reilly, D. (1999). Visualisation and the Influence of Graphical Calculators, *Proceedings of* the *23rd Annual Conference of the International Group for the Psychology of Mathematics Education,* Haifa, Israel, 1, p. 347.

VISUALISATION AND USING TECHNOLOGY IN A LEVEL MATHEMATICS

This project seeks to identify and evaluate the ways in which existing technology can be utilised to promote and develop students' powers of visualisation and to encourage the usage of these skills in A level mathematics lessons. At present, a small scale pilot study has been carried out and the resulting data has been analysed. After briefly summarising current research in the area of visualisation and technology in this paper, I will report on the findings of this initial pilot study, in which materials developed for use with the TI-92, aimed at promoting students' abilities to visualise the graphs of functions, were trialled with a class of thirteen year twelve students. The subsequent consequences for future directions of the research will, also, be discussed.

Introduction

Visualisation is increasingly being accepted as an important aspect of mathematical reasoning. Studies have revealed that 'activities encouraging the construction of images can greatly enhance mathematics learning' (Wheatley and Brown, 1994). Indeed, potentially, technology could assume a very powerful and influential role in stimulating and shaping students' powers of visualisation, and as such may prove to contribute significantly to the depth of students' understanding.

Zimmerman and Cunningham (1991) insist that mathematical visualisation is not merely 'math appreciation through pictures' - a superficial substitute for understanding. Rather they maintain that visualisation supplies depth and meaning to understanding, serving as a reliable guide to problem solving, and inspiring creative discoveries. In order to achieve this understanding, however, they propose that visualisation cannot be isolated from the rest of mathematics, implying that symbolical, numerical and visual representations of ideas must be formulated and connected. This project is conceptualised on the basis that visual thinking and graphical representation must be linked to other modes of mathematical thinking and other forms of representation (Tall, 1989).

Issues Surrounding Visualisation

Within the current literature there exist many differing notions of the key terms associated with the area of visualisation in the learning of mathematics, each developed with respect to a specific research purpose/focus, and each drawing on and expanding previous ideas.

Mariottii and Pesci (1994) acknowledge *visualisation* occurring when 'thinking is spontaneously accompanied and supported by images'. Mason (1992) regards *visualising* as 'making the unseen visible' and *imagery* as 'the power to imagine the possible and the impossible'. Solano and Presmeg (1995) see *visualisation* as 'the relationship between images' - 'in order to visualise there is a need to create many images to construct relationships that will facilitate visualisation and reasoning'. Hitt Espinosa (1997) suggests that *visualisation* of mathematical concepts is 'not a trivial cognitive activity: to visualise is not the same as to see'. To *visualise* is the 'ability to create rich, mental images which the individual can manipulate in his mind, rehearse different representations of the concept and, if necessary, use paper or a computer screen to express the idea in question'.

Unfortunately, despite the current views of researchers surrounding the importance of visualisation, there is still a tendency for visualisation to be undervalued in mathematics classrooms and consequently some students, whilst able to visualise mathematically, often opt for non-visual, more 'conventional' approaches to problem solving (Presmeg, 1995). Traditionally, a greater emphasis has been placed on algebraic or analytic proof, despite the proposed legitimacy of visual theorems. Presmeg's findings (1986) indicate that an ability to apply and interchange both visual and non-visual methods in problem solving is particularly advantageous for students, especially where one mode is more appropriate. However, the teaching of school mathematics is predominately non-visual and 'visualisers are seriously under-represented amongst high mathematical achievers' (ibid).

Although, images presented to students by teachers will influence the students' understanding and individual construction of such images, the students' conception of these images will not necessarily correspond to that of the teachers' (Mason, 1992). Indeed, 'visual ideas often considered intuitive by an experienced mathematician are not necessarily intuitive to an inexperienced student' (Tall, 1991). Students should be encouraged to create and explore their own images (Cunningham 1994) - a visual understanding of a given situation is more robust and is thus more likely to be remembered by the student in the longer term than a purely algebraic proof. Yet, Presmeg (1986) outlines four particular difficulties involving imagery; images/diagrams viewed inappropriately, inflexible thinking when dealing with a non-standard diagram, rigid uncontrollable images and vague imagery. She (ibid), also, suggests that 'less imagery is used with greater experience or learning'.

Visualisation skills may be employed by students privately to clarify, interpret and make sense of the given problem intuitively, as tools for 'meaning-making' (Wheatley and Brown, 1994) although, such processes are unlikely to be explicit in written arguments (Presmeg, 1995). Furthermore, the usage of visual techniques is comparatively time intensive suggesting that tests and examinations will tend to favour the non-visual thinker (Presmeg, 1986). In addition, visual thinking requires non-sequential, parallel processing of information, and as such poses a greater cognitive challenge to students than step by step sequential algorithmic reasoning (Eisenberg and Dreyfus, 1991).

<u>The Perceived Role of Technology</u>

In light of the recent advancements in technology, a whole range of computer programmes and scientific instruments are currently available with the potential to assist students in the formation of visual mathematical images. One of the main objectives of this research is to evaluate and develop materials and strategies which aim, as far as possible, to maximise this potential, with particular emphasis on the graphical calculator. However, the role of the computer in this respect is, also, regarded by the researcher as extremely important and influential and is thus explored in this review of current literature. Overall, the findings of studies involving graphical calculators appear to be very similar to those which utilised computer technology, although the similarities and differences between these two types of technologies should not be overlooked.

Many researchers realise the potential of utilising technology to promote and encourage visualisation skills (Souza and Borba, 1995; Smart, 1995). In particular, computer based visual approaches in teaching mathematics can i) increase motivation and ii) provide an opportunity to pursue an alternative and yet complimentary mode of thought to the traditional symbolic approach (Cunningham, 1994). Technology can be utilised to enable students to develop a deeper insight into the relationship between functions and their graphs (Carulla and Gomez, 1997). Furthermore, technology can be particularly useful in exploratory learning, where students are able to formulate concepts for themselves and benefit from visualisation in the process (Tall, 1991).

However, despite the advantages, students may still misunderstand, misinterpret and therefore misuse information provided by graphic calculators (Carulla and Gomez, 1997). Furthermore, such technology may encourage students to focus primarily on graphical representation whilst neglecting other modes (ibid). In contrast, other researchers report that graphical calculators can be utilised to foster the transitions between and exploration of different modes of representation (Ruthven, 1990). Multi-representational software, however, could contribute towards misunderstanding and confusion amongst students; any difficulty experienced with one particular representation could be intensified by the presence of other forms of representations (O'Reilly et al,1997). There is a danger that students could become 'saturated by images' (Mason, 1992). Alternatively, students may become too dependent on technology, regarding the solutions generated as irrefutable (Smart, 1995). Zimmerman and Cunningham (1991) believe that certain fundamental visualisation skills are prerequisite for meaningful computer based visualisation.

Many researchers maintain that the use of technology can promote collaborative learning and equal opportunities (Smart, 1995). In particular, female students, have benefited from the private nature of the graphics calculator (ibid). Ruthven (1990), also, found that a reduction in student uncertainty and anxiety accompanied regular use of the graphics calculator, and hence stimulated improvement in the 'confidence, competence and performance' of all students, especially that of the females. This study will investigate how graphical calculators affect the visualisation capabilities of the female students in comparison with the males, with the aim of determining whether female students benefit, in this manner, to a greater or lesser extent.

<u>The First Pilot Study</u>

Initial classroom trials were carried out at a school in Sheffield for a period of six hours during February 1998, with a group of five male and eight female year twelve students. The fieldwork involved participant observation and a post-trial questionnaire. Each individual student was given a TI-92, although they generally worked together in pairs, sharing ideas. The exercises featured graphing functions, and involved exploring and identifying the effects of transformations, finding inverse functions, solving equations - graphically and algebraically, and investigating trigonometric and logarithmic identities. The main aim of this pilot study was to enable the researcher to assess the suitability of early materials and techniques, to elicit preliminary reactions to the use of technology and to provide a framework for further data collection.

Student Questionnaire Responses

The questionnaire responses indicated that, generally, this particular group of students viewed technology as an important addition to the A level mathematics classroom - a quick and accurate means of strengthening their understanding and visualisations of functions. However, some feared over-dependency and accompanying laziness, but would nevertheless welcome further use of technology in the future. Thus, these students appeared to appreciate the opportunity to use the TI-92 and seem to have benefited mathematically from the experience.

The Students' Work

A preliminary examination of the students' work revealed that a high proportion of students often assumed that the TI-92 was displaying the whole graph without using the zoom in and out facilities. The function $x^2 - x^3$ caused particular problems. The students were asked to sketch the graph of the function and to determine the nature and co-ordinates of any turning points. The first graph (fig 1) is drawn using ZoomStd (where the x and y axes vary from -10 to 10, in divisions of 1 unit), and the second graph (fig 2) results from zooming in on the first to a degree of factor six, centred on the origin. The second graph provides a much better picture of the actual shape of the graph. Yet, all of the students who attempted this question failed to use the zoom facilities and thus drew a sketch of the function which resembled fig 1. Consequently, they mistook the point (0,0) as a point of inflection (clearly a local minimum in fig 2) and were unaware that a local maximum existed and as no-one checked their results by differentiation these errors were undetected. Clearly, these turning points were missed because they were not initially visible on screen.

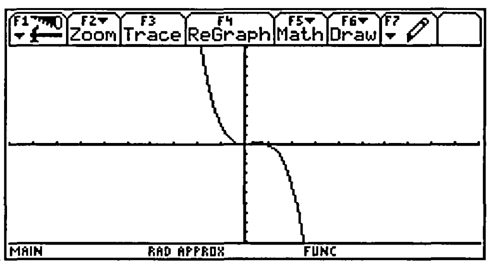
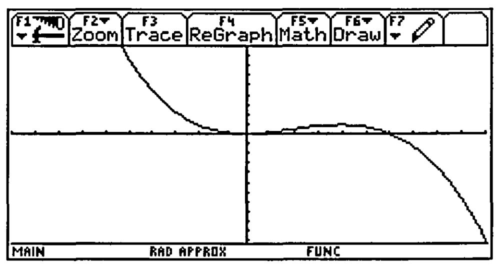
fig 1 

fig 2 

In contrast, some students believed that the graph of $y = (x+1)/(x+2)^2$ had a minimum turning point at $x = -2$. These students failed to realise that the function is undefined at this x value as they completely misinterpreted the graphs displayed by the TI-92 and neglected to inspect the equation.

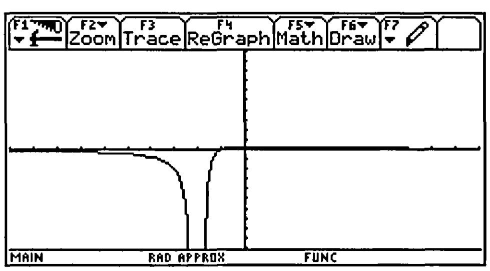
fig 3 

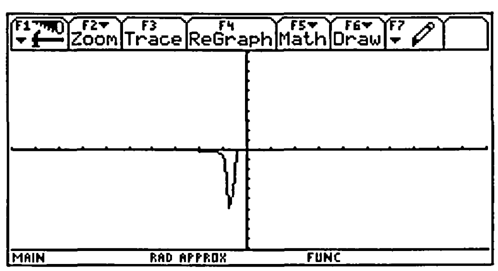
fig 4 

Fig 3 shows the graph of $y = (x+1)/(x+2)^2$, using ZoomStd again, whereas fig 4 is obtained by zooming out on the original graph, centred on the origin, by a factor of three. Since only part of the graph appeared to be visible on the screen in ZoomStd, some students choose to zoom out, producing graphs resembling fig 4, which seemed to have a minimum stationary point, and so these particular students (who were using the zoom facilities) were fooled. As before, these students did not spend time thinking logically about the function or picturing what the function might look like for themselves - they were confident that the technology provided them with the correct answer. Smart's (1995) research emphasises this problem, referred to as the 'magic' element of technology.

In addition, few students successfully completed the algebraic components of certain questions, and fewer still actually specified the symbolic form of the graphs resulting from a series of successive transformations, even though this was requested. Thus, these students tended to concentrate on graphical representation in questions involving both graphical and algebraic aspects. However, the remaining questions were completed satisfactorily. In particular, eleven students were able to identify the graphs of all six functions in the final exercise, which was an encouraging outcome.

Implications for Future Data Collection

The data collected in this pilot study has enabled the first evaluation of the classroom materials and approaches devised by the researcher, aimed at promoting the development of student's powers of visualisation using technology to be undertaken, thereby permitting some initial progress in terms of achieving the second objective of the research. However, there was insufficient data to provide notable insight into the third and fourth objectives; to investigate the ways in which the technology acts as a tool in mediating the development of students' powers of visualisation and to investigate how powers of visualisation might be evoked and be developed by the use of mathematical software. To what extent did the materials encourage, if at all, visual thinking?

Thus, preliminary results suggest that it would be useful to establish a means which would indicate how students initially approach problems involving functions. In other words, before the research takes place, do they adopt a predominately visual, algebraic or numeric approach? If their approach tends to be visual, how successful are they? If their approach is not visual, how do they perform when asked to work visually? Moreover, following the introduction of technology does their preferred mode of operation change? Do the visualisation skills of all students (not only those who prefer to work visually) improve? Do all students necessarily have a visual approach? Teachers will be interviewed to determine the extent to which they have used visual methods in their teaching of functions, and in lessons generally. It is, also, recognised that to try to distinguish between visualisers and non-visualisers is problematic; there is a continuum between students who can be regarded as almost entirely visual thinkers and those who are virtually exclusively non-visual and furthermore there are few students at the extremes - for different types of problems individual students may use different methods of solution. In the future pre-trial exercises, questionnaires and interviews will be utilised in an attempt to ascertain answers to these central research questions.

References

Borba, M. (1996) Graphing Calculators, Functions and Reorganisation of the Classroom. *Proceedings of Working Group16, ICME-8*, pp. 53-60.

Carulla, C. & Gomez, P. (1997) Graphics Calculators and Problem Solving. Do They Help? *Proceedings of PME 21*, 1, p 224.

Cunningham, S (1994) Some Strategies for Using Visualisation in Mathematics Teaching. *Zentralblatt fur Didaktik der Mathematik, ZDM,* 94/3, pp. 83-85.

Eisenberg, T. & Dreyfus, T. (1991) On the Reluctance to Visualize in Mathematics. In Zimmerman, W. & Cunningham, S. (Eds.), *Visualization in Teaching & Learning Mathematics*, pp 25-37.

Hitt Espinosa, F. (1997) Researching a Problem of Convergence with Mathematica: History and Visualisation of a Mathematical Idea. *International Journal of Mathematical Education in Science and Technology,* 28, No5, pp. 697-706.

Mariotti, M. A. and Pesci. A. (1994) Visualization in Teaching - Learning Situations. *Proceedings of PME 18,* 1, p 22.

Mason, J. (1992) Towards a Research Programme for Mental Imagery. *Proceedings of the November Conference of BSRLM,* pp. 24-29.

O'Reilly, D. et al (1997) Constructive and Instructive Representation. *Journal of Information Technology for Teacher Education,* 6, No 1, pp 73-92.

Presmeg, N. (1995) Preference for Visual Methods: An International Study. *Proceedings of PME 19,* 3, pp.58-65.

Presmeg, N. (1986) Visualisation and Mathematical Giftedness. *Educational Studies in Mathematics,* 17, pp. 297-311.

Ruthven, K. (1990) The Influence of Graphic Calculator Use on Translation from Graphic to Symbolic Forms. *Educational Studies in Mathematics,* 21, pp. 431-450.

Smart, T. (1995) Visualisation, Confidence and Magic: The Role of Graphic Calculators. In Burton, L. & Jaworski, B. (Eds), *Technology in Mathematics Teaching,* pp 195-212.

Solano, A. & Presmeg, N. C. (1995) Visualisation as a Relation of Images. *Proceedings of PME 19,* 3, pp. 66-73.

Souza, T. & Borba, M. C. (1995) Quadratic Functions and Graphing Calculator. *Proceedings of PME 19,* 1, p 253.

Tall, D. (1991) Intuition and Rigour: The Role of Visualization in the Calculus. In Zimmerman, W. & Cunningham, S. (Eds.), *Visualization in Teaching & Learning Mathematics*, pp 105-119.

Tall, D. (1989) Concept Images, Generic Organizers, Computers, and Curriculum Change. *For the Learning of Mathematics,* 9, No 3, November, pp. 37-42.

Wheatley, G. & Brown, D. (1994) The Construction and Re-presentation of Images in Mathematical Activity. *Proceedings of PME 18,* 1, p 81.

Zimmerman, W. & Cunningham, S. (1991) What is Mathematical Visualisation? In Zimmerman, W. & Cunningham, S. (Eds), *Visualisation in Teaching and Learning Mathematics,* pp. 1-9.

HOW DOES THE WAY IN WHICH INDIVIDUAL STUDENTS BEHAVE AFFECT THE SHARED CONSTRUCTION OF MEANING?

Audio taped discussions between three students have been examined to shed light on the way in which the behaviour of individual students may affect the shared construction of meaning. These discussions revealed a complex pattern of interaction between the students. Each student was responsible for defining his or her own role within the discourse and these roles appeared to change as the discussion progressed. With reference to the framework offered by Winbourne and Watson (1998), it is proposed that local communities of practice have been established and that the individual student's positioning within the community of practice determines their success as a learner and contributes towards the creation of shared knowledge.

Introduction

This paper seeks to investigate whether three GCE Advanced level further mathematics students were able to develop a joint conception of the problems that they worked on together as part of a class discussion. Of particular interest was the part that each individual student played in creating shared meaning. The theoretical position adopted in this study is based on the Vygotskian idea that all learning is essentially social and that meaning is derived through interactions between students and with the teacher, and is mediated by tools. Each participant occupies a different role in the construction and negotiation of meaning and these roles are developed through participation in local communities of practice. These ideas which form the basis for this study are elaborated below.

Social Construction of Meaning

Lerman (1994) regards meaning as socio-cultural in nature, a product of discourse and discourse positions and he argues that individuals are thus acculturated into meanings. The individual student's input into meaning making changes and is changed by the discourse. In this way the student derives meaning from their *positioning* in social practices (Lerman, 1994). Meaning is seen to be *appropriated* by individual students, whereby each student forms his or her own something, from that which already belongs to others (ibid). Appropriation occurs through communication and tool use. Hershkowitz (1999) identifies a need for focusing on the individual student's development as he or she participates in the collective construction of shared cognition in small groups or in the whole class community. She claims (ibid) that socio-cultural studies focus mostly on the interaction or the interactional event itself and that the individual student is generally an anonymous participant in classroom episodes. This paper thus attempts to draw out the individual student's role in creating, maintaining and deriving meaning from the discourse.

Local Communities of Practice

Winbourne and Watson (1998) identify six key features of *local communities of practice*:

- Pupils see themselves as functioning mathematically within the lesson;
- There is a public recognition of competence;

- There are shared ways of behaving, language, habits, values and tool-use;
- The shape of the lesson is dependent upon the active participation of the students;
- Learners and teachers see themselves as engaged in the same activity;
- Learners see themselves as working together towards achieving a common understanding.

They propose that any classroom can be regarded as an intersection of a multiplicity of these practices and trajectories. They also argue, as does Lerman, that the individual student's *positioning* within a community of practice will determine their learning success. Ultimately, the students can come to operate masterfully, within the constraints of the social setting. The students fulfil their ultimate positions within the community of practice through smaller-scale "becomings" in which they join the practice and begin to assume their eventual position. The student's experiences at school are mediated by the images of themselves as learners that they bring with them.

The Role of the Teacher

Both the teacher and the students play a mutual and active part in creating the social environment. The teacher is seen as a mediator of student learning and assumes an active and necessary role in the learning process (Lerman, 1994). An important objective for the teacher is to apprentice students into the discourse of the mathematics classroom (Lerman, 1994). The teacher assists the students in "appropriating the culture of the community of mathematicians as a further social practice", so that the students will be able to operate masterfully in this setting. To establish *local communities of practice* the teacher must constrain the foci for attention, and recognise and work with pre-dispositions, rather than ignore them (Winbourne and Watson, 1998).

The Role of Technology

Borba (1996) proposes that the use of graphical calculators can enhance mathematical discussions and "reorganise" the way that knowledge is constructed. The graphical calculator is seen as a mediator of both the teacher-student relationships and the interactions between students. Pea (1987) argues that "social environments that establish an interactive social context for discussing, reflecting upon, and collaborating in the mathematical thinking necessary to solve a problem also motivate mathematical thinking" (p.104). He emphasises that technology can play a fundamental mediational role in promoting dialogue and collaboration in mathematical problem solving.

The Class Discussions

Robert, Martin and Julie were asked to identify the symbolic forms of six graphed functions from a list of twenty possibilities and discuss their ideas. The discussions surrounding three of the graphs are presented below. Terminology developed by Teasley and Rochelle (1993) was used to analyse the interaction. This involved identifying student 'initiation' of the discourse, student 'acceptance' of arguments and cases of students 'repairing' misunderstandings. There were also instances that appeared to involve 'collaborative completions' between students, where one partner's turn would begin a sentence or idea and the other partner would use their turn to complete it.

Discussion of Graph B [y = sin 3x]

1. SE: Can anybody think of a function for B?
2. M: I reckon its sin3x.
3. SE: Sin3x.
4. All: Yes.
5. SE: You seem to agree on that one. So how did you come up with that conclusion?
6. M: It's a sine wave and it's been er...
7. R: Three times x would condense it.
8. M: It's got a stretch parallel to the x-axis of a third, because it got closer together.
9. SE: Yes, you're all right it's sin3x.

Martin initiated the discussion by asserting that this was the graph of sin3x. The other two students immediately accepted that this was the correct form of the function. When asked to give reasons why, Martin and Robert took turns to give an explanation, each building and elaborating on the previous utterances (lines 6, 7, 8), thereby producing a collaborative completion. When Martin paused to think (line 6), Robert anticipated what he may have intended to say and completed his statement. Together they provide a convincing argument for their choice of function. Although, Julie did not participate verbally in this part of the discussion, she did make gestures that indicated her agreement with the arguments being put forward. The knowledge constructed by the students in this example appears to be shared between the students, especially Martin and Robert. Instead of concentrating on developing their own arguments separately in the discussion (which had occurred during the discussion of graph A), they produced a joint explanation of why sin3x was the correct function. In this case each of the students appeared to be able to clearly picture the effects of the transformation, without using the technology.

Discussion of Graph E [$y = e^{x-1} + 4$]

1. R: It could involve an exponential this time.
2. SE: Yes this is an exponential.
3. R: It's got +4 on the end, so it's either $y = e^{-(x+1)} + 4$, $y = -e^{x+1} + 4$, or $y = e^{x-1} + 4$.
4. J: It hasn't been reflected, so it's not $y = -e^{x+1} + 4$.
5. R: It's probably $y = e^{x-1} + 4$ actually.
6. SE: Why do you say that one?
7. R: Because the negative sign somehow has to fit that [the graph], although I can't explain how
   the minus sign affects it.
8. J: That's some sort of reflection, isn't it? [referring to $y = e^{x-1} + 4$].
9. R: $y = e^{-(x+1)} + 4$ would be a reflection.
10. J: Why?
11. R: It would be a reflection in x, wouldn't it?
12. J: I don't know.
13. R: $y = -e^{x+1} + 4$ would be a reflection in y. This is like ignoring the transformation of +4, which
    I'd say is $y = e^{x-1} + 4$.
14. SE: Yes you are correct. If you two are not sure you can always draw their graphs.

The fifth graph to be considered was of a type unfamiliar to the students and resulted in Robert assuming the role of peer tutor. This discussion also provided Julie with an opportunity to share her thoughts with Robert, and was the first example of her engaging fully in the discourse. Robert was the first to comment on the possible forms of the graphed function. Julie then voluntarily presented her argument to eliminate $y = -e^{x+1} + 4$ (line 4). Following this Robert guessed the correct function. At this point Robert and Julie began to conjecture incorrectly, about the effects of the functions on the shape of their graphs. They were unsure about their ideas, turned to each other for help and steered the conversation accordingly. Julie proposed that one of the functions was a reflection and invited acceptance or repair from Robert (line 8). Robert responded by suggesting that another of the functions would be a reflection, thus dismissing Julie's proposal (line 9). Julie could not see why this would be a reflection and sought an explanation from Robert (line 10). In response Robert merely restated that this would be a reflection, adding that it would be in the x-axis, inviting acceptance or repair from Julie (line 11). Julie was still unsure and Robert's utterances did not make things clearer (line 12). Robert finished by proposing that another of the functions would be a reflection in the y-axis and re-emphasising his choice of function (line 13).

Whilst Robert and Julie turned to one another for support, they were unable to answer each other's questions satisfactorily. Julie was confused about which of the functions are reflections (line 8) and Robert was confusing a reflection in the x-axis with a reflection in the y-axis and vice versa (lines 11, 13). In this way they were able to develop a shared, albeit flawed understanding of the problem. Martin on the other hand does not offer any comments, although he appeared to be considering the arguments posed by Julie and Robert. The evidence suggests that these students would need additional support to enable them to visualise the effects of certain transformations on exponential functions correctly. This is an occasion where technology and the teacher could be particularly effective in mediating the students' visualisation powers. The students needed to test their conjectures and investigate the visual connections between the various exponential functions.

Discussion of Graph F [y = tan(x/3)]

1. R: It's a tangent.
2. SE: Think about the scale the TI-92 uses.
3. R: To see if it was increasing, I could just draw the normal graph.
4. SE: Ok, if it helps you can draw the - you can all draw the tanx graph and see what happens on your machine and then from there you can hopefully deduce what the function is.
5. R: It's a stretch of factor 3.
6. M: It's tan of x over 3.
7. R: Yes.
8. SE: Is that y = tan(x/3) or y = tanx/3 because there are two of them?
9. M: y = tan(x/3).
10. SE: y = tan(x/3) and what do you think? Have you managed to get the tan?
11. J: Yes. That's the whole thing. [Julie pointed to the tanx in tanx/3].
12. SE: That's tan of x all divided by 3.
13. J: So yes y = tan(x/3).
14. SE: y = tan(x/3), yes well done you are right.

Robert was the first to state that this graph belonged to the tangent family of functions. There was however some uncertainty amongst the students as to what the graph of y = tanx would look like in relation to graph F. Recognising this problem, I asked the students to think about the scale that the graphical calculator uses to draw trigonometric functions (line 2). Robert then suggested that he could draw the graph of tanx using the TI-92 and compare this with graph F to deduce the relationship (line 4). I then advised all three students to try this approach. Robert compared the graphs and deduced that graph F was obtained using a stretch of factor three. To complete Robert's statement, Martin added that the correct function was 'tan of x over three', again producing a collaborative completion and Robert immediately agreed. As there were two functions which could be verbalised as 'tan of x over three', I sought confirmation that Martin had identified the function correctly and was quickly satisfied that he had. Up until this point Julie had not contributed to the discussion and I drew her into the conversation again to see if she was following the arguments being presented. Julie accepted the choice of function offered by Martin and provided some evidence that she had understood why this was the correct function (line 12). The students had thus been able to develop some shared understanding of the transformations used in this example.

Reflections

During these discussions *local communities of practice* appear to have been established. The students each showed willingness to explore and explain, and they began actively working together towards achieving a common sense of each problem through the sharing of ideas and by questioning one another. Each student created their own role in the practice, which varied accordingly and they shared behavioural traits, language, and technology use. I tried to ensure that they received public recognition of their competence and we saw ourselves as being involved in the same activity. Finally, the students considered themselves to be functioning mathematically within the lesson, as they were each offering suggestions as to which functions represented the given graphs, based on mathematical reasoning, which enabled them to obtain the correct form of the function in each case.

The patterns of interactions between the students changed as each new graph was considered. Throughout the discussions the individual students appeared to occupy different positions within the discourse, modifying their roles depending on their needs. Martin initiated the discussion around the first two graphs, and Robert took over this role for the discussions concerning the remaining four graphs. Robert began to act as a peer tutor (Graphs E, F). He continually made verbal contributions to the discussions and at times took control of the discussion, whilst the Julie and Martin spent some time actively listening and thinking rather than speaking. Robert, in particular, adopted the role of steering the discussions, whilst reacting to the arguments presented by the other students. So as the discussion developed, Robert's positioning within the discourse evolved and he proceeded to occupy a central role. Martin was initially quite instrumental in moving the group towards the correct solutions (eg Graph B). However, as Robert took over initiation and steering of the discourse, Martin seemed to fade into the background. Martin indicated that he was unsure about the symbolic forms of some of the graphs and he made fewer contributions when discussing these (eg graph E) and appeared to be listening to the arguments being presented by the others and thinking about their validity. He needed time to take into full consideration the arguments offered, to enable him to form his own ideas and to convince himself of their meaning. In this way Martin was attempting to derive his 'own something' from that which already belonged to Robert and Julie.

Julie operated as an active listener during the majority of the discussion, offering her suggestions in the main when specifically asked to do so. She had to be drawn into the discourse. However, during the discussion of graph E this pattern changed and her contributions were more spontaneous. She seemed to be particularly unsure about this question and appeared keen to further her understanding. She actively questioned Robert about his arguments, whilst offering her own for acceptance or repair. In this instance Julie's needs appeared to change. Her difficulties with this graph encouraged her to share her ideas more freely, in an attempt to derive meaning from the interaction.

None of the students appeared to be self-reliant. They learned from each other, my comments, using the technology, by participating in the community of practice and through their discourse. Robert was an eager and very active participant. His position developed into that of a peer tutor and he began to initiate and steer the discussion. His thought processes were more apparent as he more often verbalised his ideas. Martin's role in steering and initiating the discussions was superseded by Robert, and during the questions that he found difficult he became more of an active listener. Julie's position as a reluctant participant, was changed into that of active contributor when the need arose. The positions that the students occupied within the discourse were their ways of appropriating meaning and contributed towards their success as learners. Each made their own contribution to the construction of shared meaning, yet in cases when students were acting more as active listeners it is difficult to determine whether they have developed a joint conception of the problem's solution.

References

Borba, M. C. (1996) Graphing Calculators, Functions and Reorganisation of the Classroom. *Proceedings of Working Group 16, ICME-8*, Spain, pp. 53-60.

Hershkowitz, R (1999) Where in Shared Knowledge is the Individual Knowledge Hidden? Plenary Address, *Proceedings of PME23*, Haifa, Israel, 1, pp. 9-24.

Lerman, S. (1994) Metaphors for Mind and Metaphors for Teaching and Learning Mathematics. *Proceedings of PME18*, Lisbon, Portugal, pp. 144-151.

Pea, R. D. (1987) Cognitive Technologies for Mathematics Education. In Schoenfeld, A. H. *Cognitive Science and Mathematics Education,* pp. 89-122.

Teasley, S. D. and Rochelle. J. (1993) Constructing a Joint Problem Space. In Lajoie, S. P. and Derry, S. J., (Eds) *Computers as Cognitive Tools.*

Winbourne, P. and Watson, A. (1998) Participating in Learning Mathematics Through Shared Local Practices. *Proceedings of PME22*, South Africa, 4, pp. 177-184.

VISUALISATION AND THE INFLUENCE OF TECHNOLOGY IN A LEVEL MATHEMATICS: A CLASSROOM INVESTIGATION

The study reported here is part of a wider study, which aims to investigate the potential of the graphical calculator for mediating the development of students' abilities to visualise the graphs of functions at GCE Advanced level. This paper focuses on how the graphical calculator influenced six particular students' work with functions. Initial results have illuminated ways in which the technology can have a positive impact on students' visualisation capabilities. It is proposed that visual thinking forms a significant part of many students' mathematical reasoning, enabling students to derive richer meaning from given problems. It is suggested further that use of the technology mediates the development of students' visual capacities, by helping to highlight the links between complementary modes of representation.

Background

Visualisation is increasingly being recognised as a fundamental aspect of mathematical reasoning. Potentially, technology could assume a very powerful and influential role in stimulating and shaping students' powers of visualisation, and as such may contribute significantly to the depth of students' learning. The study reported here explored how one form of technology, the graphical calculator, influenced six particular students' approaches to solving problems involving functions.

Visualisation

Many researchers have stressed the importance of mental imagery in the construction of meaningful mathematics (Tall, 1991; Mason, 1992; Wheatley and Brown, 1994). Studies have revealed that "activities which encourage the construction of images can greatly enhance mathematics learning" (Wheatley and Brown, 1994, p. 81). Indeed, Breen (1997) insists that there is "enormous potential for using images as a powerful starting point" for providing "rich learning situations" (p. 97). Cunningham (1994) proposes that "some students learn more effectively from visually based discussions and experiences than from symbolic and analytic work", suggesting that adding images to words is instrumental in providing students with a "richer set of ways to communicate their mathematics" (p. 84). Dubinsky et al. (1996) ultimately argue that virtually all thinking is based on visualisation.

Cunningham (1991) describes the key benefits of visualisation as:

> the ability to focus on specific components and details of very complex problems, to show the dynamics of systems and processes, and to increase the intuition and understanding of mathematical problems and processes (p. 70).

In addition, he claims that the inclusion of visualisation in mathematics education, permits a broader coverage of mathematical topics and, most importantly, allows students access to new ways to approach their own mathematics. In particular, visual arguments can prove to be extremely useful in helping students to conceptualise particular mathematical ideas. In "real visual thinking, the students' visual understanding becomes the primary vehicle for delivering and developing concepts", which depends on "interactive student experiences" (Cunningham, 1994, p. 84).

Barwise and Etchemendy (1991, p. 16) outline three ways in which visual reasoning can be considered as valid reasoning:

1. visual information is part of the given information from which we reason,
2. visual information can be integral to the reasoning itself,
3. visual representations can play a role in the conclusion of a piece of reasoning.

The visual component of mathematical reasoning needs to be presented alongside the symbolic to enable students to develop more than merely a mechanical understanding of mathematical concepts, ideas and processes. Mason (1992) suggests that "imagery often forms a key for a rich network of connections and associations, and so has a crystallising effect" (p. 27).

It has been argued that visual reasoning is not mathematically adequate. As Tall (1991) proposes:

> intuition involves parallel processing quite distinct from the step by step sequential processing required by rigorous deduction. An intuition arrives whole in the mind and it may be difficult to separate its components into a logical deductive order (p. 107).

He therefore stresses that since visual information is processed simultaneously, an intuitive approach may be unsuitable in satisfying the logic of mathematics. Conversely, though, a "purely logical view" is similarly "cognitively unsuitable for students" (p. 108). Thus, ideally, both types of processing should be integrated, through an approach that "appeals to the intuition and yet can be given a rigorous formulation" (p. 108). Tall (1989) believes that although traditional mathematics has emphasised the "symbolic and sequential", algebraic symbolism, at the expense of the "integrative and holistic", visual symbolism, both are necessary requirements in the study of mathematics (p. 42). Whilst the proof of mathematical ideas involves algebraic symbolism, the construction of such ideas requires some form of visual symbolism. Thus, Tall (ibid) stresses the importance of the many facets of a student's concept image.

Similarly, Zimmerman and Cunningham (1991) emphasise the need for a multi-faceted approach. They argue that mathematical visualisation is not merely "math appreciation through pictures" - a superficial substitute for understanding (p. 4). Rather they maintain that visualisation supplies depth and meaning to understanding, serving as a reliable guide to problem solving, and inspiring creative discoveries. In order to achieve this level of understanding, however, they propose, like Tall, that visualisation cannot be isolated from the rest of mathematics, implying that symbolic, numerical and visual representations of ideas must be formulated and connected. This study is conceptualised on the basis that "visual thinking and graphical representation must be linked to other modes of mathematical thinking and other forms of representation" (p. 4).

Connections between different modes of representation need to be made by students, the significance of particular links must be recognised and most importantly an appropriate balance of approaches should be introduced (Hughes Hallett, 1991). Indeed, many researchers support the view that whilst visualisation stimulates and reinforces conceptual understanding, this particular mode of representation is no more important than other modes. What is required is a multi-representational approach to mathematics, incorporating the symbolic, visual and numerical modes of representation, where each mode complements and strengthens the understanding the student acquires when operating in an alternative mode.

Studies of problem solving (e.g. Presmeg, 1986) suggest that the ability for students to apply and interchange both visual and non-visual methods in problem solving is particularly advantageous, especially where one mode appears to be more appropriate. Students who are able to work in this way are likely to develop a deeper, more holistic understanding of mathematics. Dubinsky et al. (1996) propose that visualisation and analysis are "mutually dependent in mathematical problem solving" and reject the conventional notion of an analyser/visualiser dichotomy or continuum (p. 435). They argue that for "most people both visual and analytic thinking may need to be present and integrated" in order for them to be able to "construct rich understandings of mathematics concepts" (p. 438). Most visualisations contain some form of analysis (Presmeg 1986; Dubinsky et al., 1996) and conversely most analyses involves some use of visualisation (Dubinsky et al., 1996). Moreover, Dubinsky et al. claim that visual approaches benefit from analytical thinking and that analytical approaches are enriched by visualisation. These studies highlight the necessity of empowering students with multiple interchangeable approaches to problem solving.

The Role of Technology

Whilst appreciation of the importance of visualisation in mathematics is growing, technology is rapidly being developed which may revolutionise the mathematics curriculum. Following recent advances in technology, a whole range of computer-programmes and scientific instruments are currently available with the potential to assist students in the formation of visual mathematical images. One of the main objectives of the study reported here is to develop and evaluate materials and strategies that aim as far as possible to maximise this potential, with particular emphasis on the graphical calculator.

Many researchers have realised the potential of utilising technology to promote and encourage visualisation skills (Souza and Borba, 1995; Smart, 1995, Goforth, 1992). In particular, Cunningham (1994) recognises two essential features that contribute to the success of computer based visual approaches in teaching mathematics: the motivational aspect and the opportunity to pursue an alternative, yet complementary mode of thought to the traditional symbolic approach. He further acknowledges that:

> one of the most remarkable things about visualisation is the amount of mathematics students will learn and the amount of work students will do in order to create images describing a mathematical concept, especially when the computer is used as part of the process (p. 83).

The development of new technological tools means that ideas about mathematical reasoning, meanings and learning need to be examined. In particular, the variety of forms of representation and its effects need to be studied. Barwise and Etchemendy (1991) contend that "much, if not most, reasoning makes use of some form of visual representation" and that

> as the computer gives us ever richer tools for representing information, we must begin to study the logical aspects of reasoning that uses non-linguistic forms of representation (p. 22).

Confrey (1994) argues for a "epistemology of multiple representations", in which the contrast between representations is recognised as significant in establishing meaning through the convergence of understanding from each mode of representation (p. 218). By identifying multiple representations, we can encourage students to find multiple ways to make sense of their results and to develop their sense of flexibility and elegance. Multiple approaches support the "diversity in students' preferences and provide alternative approaches to use when faced with cognitive obstacles" (p. 218). She, also, advises that "in a multi-representational tool, no representation should dominate others, and, in every representation there is both a loss and a gain" (Confrey, 1993, p. 66).

Kaput (1992) identifies two key purposes of multiple linked notations: firstly, "to expose different aspects of a complex idea" and secondly, "to illuminate the meanings of actions in one notation by exhibiting their consequences in another notation" (p. 542). He contends that since "all aspects of a complex idea cannot be adequately represented within a single notation system", multiple systems are required for their full expression (p. 530).

Multi-representational software, however, could also contribute towards misunderstanding and confusion amongst students. According to O'Reilly, Pratt and Winbourne (1997):

> multiple representation software runs the risk that the difficulties of reading a representation are simply multiplied up by the number of modalities represented on the screen simultaneously. The child has to make sense of each modality in turn and the links between them (p. 88).

The relationship between functions and graphs is one area in which many researchers have addressed the use of technology. For example, Carulla and Gomez (1997) claim that technology can be utilised to enable students to develop a deeper insight into the relationship. When students are given the opportunity to approach functions in a visual manner, they are more likely to develop an intuitive understanding of translations (Confrey, 1994). Technology enables the teacher to demonstrate numerous functions and their graphs in an effective manner that could not possibly be achieved using relatively unsophisticated resources such as a black/white board. Consequently students gain a deeper insight into the relationship between functions and their graphs (Chola Nyondo, 1993). The ability for a student to recognise a given graph as belonging to or resembling some member of a family of functions is a fundamental stage in the development of a solution (Ruthven, 1990). For, only when a student has been able to successfully identify the family of functions to which the graph belongs can the correct symbolisation be constructed (referred to by Ruthven as the process of refinement).

Students must be visually aware of the effects of particular transformations and of the corresponding symbolic modifications. Goldenberg (1991) suggests that graphical exploration “provides valuable scaffolding for the required symbolic manipulations” (p. 85). Bloom et al. (1986) found that students who were taught to recognise the graphs of functions as compositions of certain transformations of standard functions developed a "greater understanding" of functions and their graphs and "in less time" than those taught in the traditional manner (p. 123). Indeed, students who are given the opportunity to develop graphical and numerical algorithms for understanding functions, and are able to use these effectively could legitimately question the need for symbolic skills (O'Reilly et al., 1997).

Carulla and Gomez (1997) appreciate that whilst the use of graphic calculators can "enhance the learning of functions and graphing concepts" (p. 224), there may be associated problems. Their findings indicate that these instruments can also encourage students to concentrate primarily on graphical representation systems at the expense of verbal and symbolic representations. In contrast, Penglase and Arnold (1996) reported that graphic calculators could promote the transition between symbolic manipulation and graphical investigation and exploration of the different modes of representation associated with particular concepts. Ruthven's study (1990), also, supported the view that regular use of graphic calculators will probably “strengthen” and “rehearse” relationships between certain symbolic and graphic forms (p. 447).

THE STUDY

The aims of the study reported here were:

1. to develop a picture of the students' preferences for visual or non-visual methods;
2. to gain some insight into their perceptions of imagery;
3. to explore how and why they used images;
4. to investigate the inter-relationship between visual and symbolic methods.

This paper reports on the findings of the second phase of data collection which followed an initial pilot phase (see Elliott, 1998). Data was collected during June 1998 and involved the teacher-researcher working individually with a small group of Year 12 students, all aged 17, for a period of six hours (two three-hour sessions). In the first session the group consisted of six students: Diane, Martin, Jason, Julie, Rachael and Robert. The group for the second session consisted of three of these students: Martin, Julie and Robert. These students, all of whom were studying GCE Advanced level Further Mathematics, were described by their teachers as being very capable mathematicians. Each student had purchased his or her own graphical calculator and was familiar with using this type of technology. In addition, the staff, Mr. Pearson, Ms. Slater and Ms. Mooney, each of whom taught one of the components of the course (pure mathematics, statistics and mechanics, respectively) positively encouraged the use of the graphical calculators in lessons. For the purposes of this study, each student was given a TI-92. Although they had individual calculators, they generally worked together as a group, sharing ideas.

Methods of enquiry

The notion of 'study of singularities' (Bassey, 1995) is considered to be of direct relevance to this research. Bassey proposes that "the term study of a singularity embraces virtually every kind of empirical study" and is preferable because the phrase 'case study' is often associated with generalisation (p. 112). The findings of singularity studies are merely related to populations outside the boundary in space and time. This study does not seek to generalise but rather to illuminate how one particular group of students solved problems involving functions with the use of the TI-92.

This study consisted of four distinct stages of data collection, which are summarised in figure 1.

Two questionnaires were administered to the students, one regarding the role of visualisation in A level mathematics and the other concerning their reactions to the technology. These questionnaires were devised to illuminate why, when and how the students used imagery and whether the students viewed technology as a resource that provides support for visual learning. The questionnaire involving the students' views on visualisation was answered during the first lesson. The technology questionnaire was distributed at the end of the second and final session. The three members of staff who each taught this group of students were also given a questionnaire on visualisation. The staff questionnaire was intended to provide a background to the study, in clarifying their views on visualisation and the extent to which visual methods are encouraged in the classroom.

Data Collection Activity	Timing of Activity	Students Involved	Staff Involved
Student and staff questionnaires concerning visualisation	Beginning of 1st session, 30mins	Diane, Jason, Julie, Martin, Rachael and Robert	Mr. Pearson, Ms. Slater and Ms. Mooney
Student interviews explaining how they would attempt to solve problems involving functions	Beginning of 2nd session, 50mins	Julie, Martin and Robert	N/A
Students' work on pre-prepared exercises	End of 1st session and the majority of 2nd session, 3hrs 30mins	Julie, Martin and Robert (Julie also submitted her work from the 1st session)	N/A
Student questionnaires concerning the role of technology	End of 2nd session, 25mins	Martin, Julie and Robert	N/A

Figure 1: Data Collection Activities

The student interview questions were devised in order to illuminate how these six students would initially attempt to solve questions which involved finding intersection points, solving inequalities, identifying the effects of transformations, gradients and identifying functions. The main purpose of this was to see whether the activities that the students engaged in during this study would have any effect on the way in which these students would solve such problems. One of the subsidiary aims of the study was to highlight the validity of visual approaches and to encourage these students to combine visual and symbolic methods as much as possible.

Each question was selected to provide a range of different problems for the students to consider. In addition, styles of questions that would possibly be unfamiliar to the students were chosen, especially question 5 (see figure 4). The intention of this was to see if the students would tend to use more imagery with unfamiliar, non-typical problems.

The questions were phrased in a manner that would not necessarily promote a symbolic approach. For example, the first question was worded: for which values of x is the graph of $y = 3x^2 + 9x - 12$ below the x-axis? (See figure 4). The same question could have been written: solve $3x^2 + 9x - 12 < 0$ or for which x values is the expression $3x^2 + 9x - 12$ less than zero? This type of wording could be considered as a prerequisite/precursor to a symbolic approach, and certainly as indicative of such a method.

The class activities included an introduction to the TI-92 and its applications, questions intended to familiarise the students with the various functions of the calculator and a sequence of questions designed to draw out the students' visual abilities and develop key skills in understanding the concept of function. These questions featured graphing functions, exploring and identifying the effects of transformations, finding inverse functions, solving equations - graphically and algebraically, and investigating trigonometric and logarithmic identities.

The TI-92 was chosen for this study because of its facility to perform combined transformations. However, a further study has since involved the use of the TI-82. The actual type of graphical calculator used is not seen as significant in the aims of this study.

DATA ANALYSIS

Student and Staff Questionnaire Responses

The student questionnaire concerning the role of mental imagery and visualisation in A level mathematics was administered during the first session in which all six students were present.

When asked to classify how often they used mental images when solving mathematical problems in general, the students responded as shown in figure 2.

	Female	Male
Always	1	0
Fairly Frequently	1	2
Sometimes	1	0
Quite Rarely	0	1
Never	0	0

Figure 2: Frequency of Use of Mental Images

Five out of the six students would generally formulate mental images at the beginning of problem solving, all recognising some benefits in using images:

> **Jason** They help me in the application of the mathematical equations involved and also to check the answers I get, whether they are realistic or not
>
> **Robert** They give a basis on which to use an algebraic method

Martin I can imagine how points may fall on a graph to see the kind of result I should expect to get on paper

Martin found visualisation particularly difficult when unfamiliar with the type of problem, declaring that "when I am not used to the type of problem, it is not easy to relate it to a graph or system, so I would use algebra". Robert expressed a reluctance to explore visual solutions, insisting that he "generally attempts to ignore such suggestions".

Julie and Robert preferred working symbolically, claiming that this approach involved fewer errors for them and "less thought" for Robert. However, despite her preference, Julie "visualises things more often", although, she stressed that "it can be very difficult to visualise" sometimes. Jason and Rachael had no such preference. Jason believed that the two approaches "complement each other". Rachael, though, did find working symbolically easier, but expressed a desire to "do more maths visually". Martin, did not state a clear preference, although he seemed more cautious about a visual approach and stated that "when I am comfortable knowing how the methods work symbolically I would then find it easier to visualise it, as I can check that I am visualising it right".

When asked to categorise how often different approaches are combined, the students responded as shown in figure 3.

	Female	Male
Always	1	1
Fairly Frequently	1	2
Sometimes	1	0

Figure 3: Frequency of Combined Approaches

The value of combining different solution techniques was clearly recognised by these students:

Jason I tend to combine the two techniques, as they complement each other. First finding an approximate visual answer then applying equations to find an exact answer.

Julie Usually one method alone is not the best way to tackle a question.

Rachael You need to be good at both. Each will help give a balanced approach to work.

Julie, Robert and Diane regarded the ability to perform symbolic manipulations as being of greater importance than the ability to work visually, in order to become a successful mathematician. However Julie added the stipulation "as long as you use enough visualisation to know what you are doing". In contrast, Martin felt that visual capabilities were of greater importance, to enable a more thorough understanding to be reached. Rachael and Jason expressed the opinion that neither of the two was more significant - that the ability to combine or alternate approaches where appropriate, resulting in a balanced approach, was of paramount importance.

Five of the students regarded themselves as being more visually orientated, whilst Robert, in contrast, considered himself to be almost entirely a non-visualiser. The three male students considered their visualisation powers to be Good overall, whereas the three females rated them as Fair.

The staff questionnaire responses indicated that the graphical calculator is recognised as a tool that helps students to explore complementary modes of representation and highlights the links between them. The students' teachers all encouraged combined approaches to problems solving and Mr Pearson commented that "the symbolic and visual aspects of mathematics are inextricably linked". The teachers' attitudes appeared to have been influential in the development of their students' problem solving strategies, as was access to technology. Views expressed by these teachers concerning the use and validity of visual approaches were echoed in the students' responses.

The second student questionnaire provided Martin, Julie and Robert the opportunity to evaluate the role of the technology they had been using in promoting visualisation. These students regarded technology as an extremely useful addition to the A level mathematics classroom, providing a means of "speeding up calculations", enabling student's to "see how the graphs of functions can be related and manipulated" and to "recognise and visualise characteristics of many functions". However, reservations regarding potential over-dependency and possible replacement of pencil and paper techniques were expressed.

Following the investigation, Robert indicated that he may "use graphical methods more", in the future, "when solving the more involved problems". He had recognised the additional benefits of using graphical approaches and was more confident in using them. Martin's confidence had also improved and he commented, "I think that I will be more comfortable in using a visual method such as plotting points and drawing sketches when solving problems".

1. For which values of x is the graph of $y = 3x^2 + 9x - 12$ below the x-axis?
2. For which x values does the graph of $y = x^3 + 6$ intersect the graph of $y = 2x^2 + 5x$?
3. Find the values of x where the graph of $y = 3|x - 2|$ lies above the graph of $y = 6x^2$
4. What effect will the transformation $3f(x + 3)$ have on the graph of the function $f(x) = 4x^5 - 3x^4 + 2x^3 - x$.
5. For a particular function f(x); f(2) = 6, f(3) = 14 and f(6) = 50. What type of function could this be? Give an example of a function that satisfies these conditions.
6. How does the slope of the function $y = x^4 - x^2$ change from x = -5 to x = 5?

Figure 4 - Student Interview Questions

Student Interviews

Martin, Julie and Robert were asked to describe how they would attempt to solve the questions reproduced in Figure 4. For the different types of questions the students adopted different approaches, as is illustrated in figure 5.

The Class Activities

The questions from the exercises completed by the students were answered extremely well. Each student used the TI-92 effectively to graph and translate functions and to check their own visualisations, which provided a basis for discussion amongst themselves. The discussions between students' generally involved attempts to derive shared meaning from the images portrayed on screen. The students were actively pursuing visual solutions to problems.

	Wholly Symbolic	Primarily Symbolic	Symbolic & Graphical Combined	Primarily Graphical	Wholly Graphical
Q1			3		
Q2	1	2			
Q3		1	2		
Q4					3
Q5			2		1
Q6	1		1		1

Figure 5 - A Summary of Student Approaches

In addition, the students' attempts at questions that involved symbolic and graphical representations and required solutions incorporating both of these aspects were very encouraging. The students each provided valid solutions to these problems in which symbolic and graphical approaches were effectively combined. In this case these students did not appear to concentrate more on the graphical mode of representation as a result of using the technology, which was a feature of the pilot study (see Elliott, 1998). On the contrary, use of the technology encouraged the students to use a combination of symbolic and graphical techniques and to explore the links between these two modes of representation. Possible explanations for these findings could be the students' familiarity with graphical calculators and their everyday use in class, which has led to the transparency of this resource in this particular classroom (Adler, 1998).

An Individual Student's Work

Robert was the only student in the group who classified himself as non-visualiser. Despite the fact that visual solutions were usually encouraged in class, Robert 'generally attempts to ignore such suggestions'. Furthermore, the only reason that he claimed to use visual images was to 'provide a basis for an algebraic method' where needed, often only using visualisation as 'a last resort'. Robert's reluctance to use visual methods in problem solving made his use of the technology particularly interesting, as is illustrated below in the following example.

Robert's use of the TI-92 was video taped. He was showing that $\ln x^a = a\ln x$. Initially, Robert used the TI-92 to draw the graphs of $\ln x^2$ and $2\ln x$ simultaneously. However, at this time he was unaware that this approach was unlikely to yield any discoveries, as one of the graphs would mask any differences between the two. When it was suggested that he draw them separately, Robert obtained the two graphs shown in figures 6 and 7 and the dialogue below was initiated:

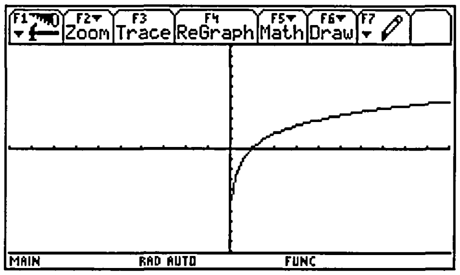


Figure 6: $y = 2\ln x$

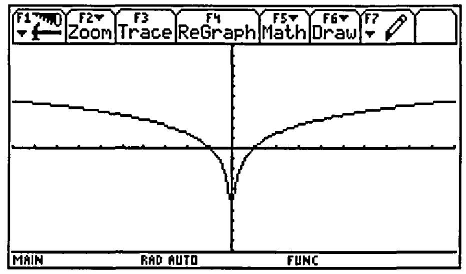


Figure 7: $y = \ln x^2$

1 SE: Does the graph of $2\ln x$ surprise you?

2 Robert: Not really, you can't have logarithms of negative numbers.

3 SE: Exactly. So would you say that the two expressions were the same or not?

4 Robert: I'm hesitant to say. I would say that algebraically they were the same.

Robert recalled being taught during his previous experience of logarithms that these two expressions are equivalent (line 4). However, the graphs produced by the TI-92 here seemed to contradict this assumption, although he had established why the two graphs are not identical for negative x values (line 2).

5 SE: What will happen, do you think, for x^3: $\ln x^3$ and $3\ln x$?

6 Robert: I wouldn't have thought that there would have been any difference.

7 SE: Think about the graph of $\ln x^3$. Why would that be different to $\ln x^2$?

Robert drew the graph of $\ln x^3$ on the TI-92 (figure 8).

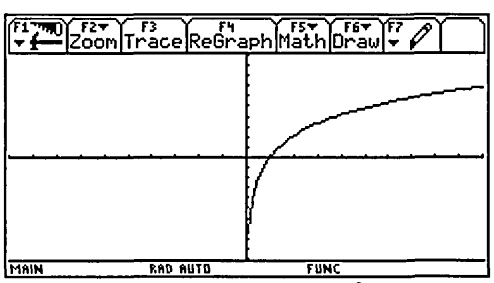


Figure 8: $y = \ln x^3$

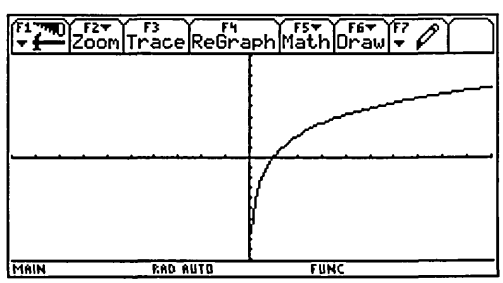


Figure 9: $y = 3\ln x$

8 SE: Why isn't there a part of that graph for negative x values?

9 Robert: For the same reason that there isn't a negative part for 2lnx - you can't have a logarithm of a negative x value.

10 SE: So is 3lnx going to be the same as $\ln x^3$ then?

Robert plotted *3*lnx and agreed that they are the same (figure 9).

When confronted by the question involving x^3, Robert's initial reaction was to assume that the same thing would happen (line 6). By employing the technology, he was able to begin to formulate ideas as to why this is not the case. Robert was now developing additional insight into the problem and was able to predict what would happen in the case of x^4:

11 SE: What do you think will happen with x^4?

12 Robert: The same thing as with x^2, but it would become steeper, as in stretched.

Robert established that the two graphs would only be identical for odd values of a and the reason why this is the case. His experimentation with the TI-92 enabling him to make sense of this standard logarithmic rule and graphical aspects that he would not have considered before, using a purely symbolic approach.

Discussion of Results

The study that has been reported here is part of a larger research project. It does not attempt to generalise these findings. Rather its aim was to provide data on how a small group of students used visualisation in problem solving, and in what ways the graphical calculator facilitated and encouraged this.

Initial findings suggest that visual thinking formed a significant part of these students' mathematical reasoning. Imagery tended to be used to enable the students to make sense of problems and to clarify their ideas, which supports the findings of Wheatley and Brown (1994). Consequently, mental images were usually formulated by these students at the beginning of the problem, as was also proposed by Dubinsky et al. (1996), which provided a basis for mathematical meaning making. The graphical calculator enabled the students to access graphical images of functions quickly and easily, when perhaps they would have had difficulty otherwise. This in turn allowed them to see the problem more clearly and proceed towards a solution.

The students generally found working symbolically easier than working visually, and appeared more comfortable when performing symbolic manipulations, reflecting the findings of Eisenberg and Dreyfus (1991). According to the students, symbolic arguments require less thought and are less prone to error than corresponding visual approaches, due to the sequential and logical ordering of each step in the problem. The students felt that they achieved greater success when using symbolic arguments and believed the symbolic approach to be more efficient. Yet, surprisingly perhaps, all of these students except Robert regarded themselves as visualisers.

As these students are highly successful students studying Further Mathematics, one might assume that they would be highly efficient symbolisers, who spend little time visualising, considering the current emphasis on symbolism, especially in examinations. However, with the exception of Robert, this was clearly not the case. Yet, there was initially a definite lack of confidence surrounding the accuracy and validity of visual solutions. Martin illustrated this point when describing how he would normally approach new topic areas:

> When I'm comfortable with knowing how the methods work symbolically I would then find it easier to visualise it, as I can check that I am visualising it right. Visual methods are encouraged but I would probably try to learn an algebraic method first until I am comfortable with my understanding of the methods, so I may not take as much notice of learning a more visual approach.

This lack of confidence was less noticeable towards the end of the study and suggested that as a future focus of the research, technology could possibly be utilised to encourage students to have greater confidence in visualisation and to help them overcome their initial difficulties. The graphical calculator appeared to influence the students' perceptions in a positive way towards the validity of visual methods in mathematics. This was especially so in Robert's case.

These students seldom used imagery in isolation from other techniques, as has also been found by Dubinsky et al. (1996), who propose that visualisation and analysis are mutually dependent. For different types of questions the students adopted different approaches. Initial findings suggest that certain areas of mathematics and particular types of questions encourage students to make more use of imagery, and thus, the use of technology in these areas is especially rewarding. The students indicated that questions involving functions and their graphs, equations, inequalities and certain applied topics would require more visual thinking than in other areas. Observations also suggest that questions with which students are unfamiliar may provoke greater use of imagery, as might questions phrased in such a way as to appeal to the use of imagery. In addition, materials have been developed which are aimed at encouraging students to visualise when otherwise different approaches may predominate.

When the students did choose graphical approaches, these tended to be used to help them view the problem more clearly and as a means of verifying the results of symbolic approaches. Question 2 may have been answered in a symbolic way because solving the equation would give the exact values of the intersection point, whereas when an inequality is involved, the range of x values concerned is not always immediately apparent. This may explain why the students tended to combine symbolic and graphical approaches for the problems involving inequalities - questions 1 and 3. The students were all fairly confident with the actions of transformations and were thus able to describe the effects of the particular transformation in question 4, without needing to picture the graph of the function in any way. This supports the representational-development hypothesis, which suggests that "less imagery is used with greater experience or learning" (Presmeg, 1985, p. 279).

Question 5 was rather different from the type of questions these students were used to and they initially concentrated on the graphical representation of the function. Robert and Martin were then able to suggest a symbolic approach to find the exact form of the function, while Julie indicated that she would have used a trial and error technique based on her graph. Dubinsky et al. (1996) have proposed that initial thoughts in problem solving involve some form of visualisation. They further suggest that weaker students are less likely to progress beyond the visualisation stage. The novelty of question five may have prompted a visual approach. On the other hand, the phrasing in question six was also new to these students and in this case the student's approaches diversified. This was an example of how the students adapted their approaches to compensate for their own individual uncertainties about the question.

Overall, the students tended to combine symbolic and graphical approaches much more often than concentrating on one of these methods alone, as was also found by Dubinsky et al. (1996). Willingness and ability to combine approaches whenever appropriate was a theme that the teacher-researcher was keen to promote and the fact that these students already used a variety of solution techniques was extremely encouraging. Further, their choice of solution techniques benefited from access to technology. The graphical calculator enabled the students to make connections between visual and symbolic modes of representation more easily. In this way the students were able to move confidently between representations and were given the flexibility to be able to make sense of, and to formulate solutions to unfamiliar and challenging problems. In turn they were able to achieve a level of understanding which would not have been available to them if they concentrated on one mode of representation alone. This strongly supports the findings of researchers such as Zimmerman and Cunningham (1991), Tall (1989) and Presmeg (1986) who argue for multi-representational approaches to mathematics.

References

Adler, J.: 1998, 'Resources as a Verb: Reconceptualising Resources In and For School Mathematics', *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, 1, 1-18.

Barwise, J. and Etchemendy, J.: 1991, 'Visual Information and Valid Reasoning.' In W. Zimmerman and S. Cunningham (eds.), *Visualization in Teaching and Learning Mathematics* (pp. 9-24). Providence, RI: MAA Notes Series, 19.

Bassey, M.: 1995, *Creating Education Through Research. A Global Perspective of Educational Research for the 21st Century.* Kirklington Moor Press in conjunction with the British Educational Research Association.

Bloom, L. M. et al.: 1986, 'Use of the Microcomputer to Teach the Transformational Approach to Graphing Functions'. *International Journal of Mathematical Education in Science and Technology*, 17(1), 115-123.

Breen, C.: 1997, 'Exploring Imagery in P, M and E'. *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, 2, 97-104.

Carulla, C. and Gomez, P.: 1997, 'Graphics Calculators and Problem Solving. Do They Help?' *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, 1, p224.

Chola Nyondo, A.: 1993, 'Mathematics Courses with a PC'. *International Journal of Mathematical Education in Science and Technology*, 24(4), 569-574.

Confrey, J.: 1993, 'The Role of Technology in Reconceptualizing Functions and Algebra'. *An Invited Address for the Group for the Psychology of Mathematics Education*, 1, 47-74.

Confrey, J.: 1994, 'Six Approaches to Transformation of Functions Using Multi-Representational Software', *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, 2, 217-224.

Cunningham, S.: 1991, 'The Visualization Environment for Mathematics Education'. In W. Zimmerman and S. Cunningham (eds.), *Visualization in Teaching and Learning Mathematics* (pp. 67-76). Providence, RI: MAA Notes Series, 19.

Cunningham, S.: 1994, 'Some Strategies for Using Visualisation in Mathematics Teaching', *Zentralblatt fur Didaktik der Mathematik, ZDM*, 94(3), 83-85.

Dubinsky, E. et al.: 1996, 'Coordinating Visual and Analytic Stategies: A Study of Students' Understanding of the Group D4, *Journal for Research in Mathematics Education*, 27(4), 435-457.

Eisenberg, T. and Dreyfus, T.: 1991, 'On the Reluctance to Visualize in Mathematics'. In W. Zimmerman and S. Cunningham (eds.), *Visualization in Teaching and Learning Mathematics* (pp. 25-37). Providence, RI: MAA Notes Series 19.

Elliott, S.: 1998, 'Visualisation and Using Technology in A Level Mathematics', *Proceedings of the Birmingham Conference of the British Society for Research into Learning Mathematics*, 45-50.

Goforth, D.: 1992, 'Visualisation and Computer Assisted Learning: The Role of Videodiscs', *British Journal of Educational Technology*, 23(1), 21-27.

Goldenberg, E. P.: 1991, 'The Difference Between Graphing Software and Educational Graphing Software'. In W. Zimmerman and S. Cunningham (eds.), *Visualization in Teaching and Learning Mathematics* (pp. 78-86). Providence, RI: MAA Notes Series, 19.

Hughes Hallett, D.: 1991, 'Visualisation and Calculus Reform'. In W. Zimmerman and S. Cunningham (eds.), *Visualization in Teaching and Learning Mathematics* (pp. 121-126). Providence, RI: MAA Notes Series 19.

Hammersley, M. and Atkinson, P.: 1983, *Ethnography Principles in Practice*. Tavistock Publications, London.

Kaput, J. J.: 1992, 'Technology and Mathematics Education'. In Grouws, D. A. (ed.), *Handbook of Research on Mathematics Teaching and Learning*, (pp. 515-556).

Mason, J.: 1992, 'Towards a Research Programme for Mental Imagery'. *Proceedings of the November Conference of the British Society for Research into Learning Mathematics*, 24-29.

O'Reilly, D., Pratt, D. and Winbourne, P.: 1997, 'Constructive and Instructive Representation', *Journal of Information Technology for Teacher Education*, 6(1), 73-92.

Penglase, M. and Arnold, S.: 1996, 'The Graphics Calculator in Mathematics Education: A Critical Review of Recent Research', *Mathematics Education Research Journal*, 8(1), 58 -90.

Presmeg, N, C.: 1985, *The Role of Visually Mediated Processes in High School Mathematics: A Classroom Investigation*, Ph.D. Thesis, University of Cambridge.

Presmeg, N. C.: 1986, 'Visualisation and Mathematical Giftedness', *Educational Studies in Mathematics*, 17, 297-311.

Ruthven, K.: 1990, 'The Influence of Graphic Calculator Use on Translation from Graphic to Symbolic Forms', *Educational Studies in Mathematics*, 21, 431-450.

Smart, T.: 1995, 'Visualising Quadratic Functions: A Study of Thirteen Year Old Girls Learning Mathematics with Graphic Calculators', *Proceedings of the 19th Conference of the International Group for the Psychology of Mathematics Education*, 2, 272-279.

Souza, T. A. and Borba, M. C.: 1995, 'Quadratic Functions and Graphing Calculator', *Proceedings of the 19th Conference of the International Group for the Psychology of Mathematics Education*, 1, p253.

Tall, D.: 1989, 'Concept Images, Generic Organizers, Computers, and Curriculum Change', *For the Learning of Mathematics*, 9(3), November, 37-42.

Tall, D.: 1991, 'Intuition and Rigour: The Role of Visualization in the Calculus'. In W. Zimmerman and S. Cunningham (Eds.), *Visualization in Teaching and Learning Mathematics* (pp. 105-119). Providence, RI: MAA Notes Series, 19.

Wheatley, G. H. and Brown, D.: 1994, 'The Construction and Re-presentation of Images in Mathematical Activity', *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, 1, p81.

Zimmerman, W. and Cunningham, S.: 1991, 'What is Mathematical Visualisation?' In W. Zimmerman and S. Cunningham (Eds.), *Visualisation in Teaching and Learning Mathematics* (pp. 1-9). Providence, RI: MAA Notes Series, 19.

HOW DOES THE WAY IN WHICH INDIVIDUAL STUDENTS BEHAVE AFFECT THE SHARED CONSTRUCTION OF MEANING?

Audio taped discussions between three students have been examined to shed light on the way in which the behaviour of individual students may affect the shared construction of meaning with graphical calculators. These discussions revealed a complex pattern of interaction between the students. Each student was responsible for defining his or her own role within the discourse and these roles appeared to change as the discussion progressed. With reference to the framework offered by Winbourne and Watson (1998), it is proposed that local communities of practice have been established and that the individual student's positioning within the community of practice determines their success as a learner and contributes towards the creation of shared knowledge.

Background

The study reported in this paper seeks to investigate whether three GCE (General Certificate of Education) Advanced level Further Mathematics students were able to develop a joint conception of the problems that they worked on together. They discussed as a group, with graphical calculators available. Of particular interest was the part that each individual student played in creating shared meaning in the context of the technology. The theoretical position adopted in this study is based on the Vygotskian idea that all learning is essentially social and is mediated by tools. Meaning is derived through interactions between students and with the teacher. Each participant occupies a different role in the construction and negotiation of meaning and these roles are developed through participation in local communities of practice. These ideas, which form the basis for this study, are elaborated below and discussions between students and teacher working on a mathematical task are then analysed using these theoretical constructs.

Socio-Cultural Learning

Vygotsky proposed that all individual mental processes are based on social interactions. Interactions experienced within the social context are internalised by the individual and learning proceeds from the interpsychological to the intrapsychological. Furthermore, the learning process is mediated by the use of tools, such as speech, symbols, writing and technology. Within a Vygotskian perspective, tools are seen to fundamentally shape and define activity. They are used firstly as a means of communicating with others, to "mediate contact with our social worlds", and eventually "these artifacts come to mediate our interactions with self; to help us think, we internalise their use" (Moll, 1990, p. 11-12). In particular, Vygotsky regarded language as the means through which thought is developed: "thought is not merely expressed in words; it comes to exist through them" (p. 125).

The site in which learning takes place is the *zone of proximal development (ZPD)*. Vygotsky defined this as "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). Individuals learn from interaction with other more knowledgeable persons in the *ZPD*. Consequently peer tutoring, peer collaboration and teacher intervention play an important part in constituting the *ZPD*.

Social Construction of Meaning

In developing a Vygotskian perspective, Lerman (1994) regards meaning as socio-cultural in nature - a product of discourse and discourse positions. Individuals are acculturated into meanings and thus the intersubjective becomes the intrasubjective. The individual student's input into meaning making changes and is changed by the discourse. In this way the student derives meaning from their *positioning* in social practices. Meaning is *appropriated* by individual students, whereby each student forms his or her own something, from that which already belongs to other people. This appropriation occurs through communication and tool use.

As concepts derive their meaning from being used, the acquisition of a concept or understanding can be interpreted as the result of an individual coming to share in that meaning through negotiation and discussion (Lerman 1996). Mathematical concepts are social acts and tools; as these concepts are socially determined, they are socially acquired. Jones & Mercer (1993) propose that successful learning occurs when two or more people manage to share their knowledge and understanding, so that a new cultural resource is created which is greater than the knowledge and understanding any of the individuals hitherto possessed. They stress that much learning, not least in relation to information technology, consists of sharing knowledge.

Local Communities of Practice

The study was concerned with creating a classroom environment that would facilitate and support the negotiation of meaning between students and teacher, thereby giving rise to successful learning opportunities. As such the teacher-researcher deliberately set out to establish *local communities of practice* in order to achieve this aim. Winbourne and Watson (1998) identify six key features necessary for initiating *local communities of practice (LCP)*:

1. Pupils see themselves as functioning mathematically within the lesson;
2. There is a public recognition of competence;
3. Learners see themselves as working together towards the achievement of a common understanding;
4. There are shared ways of behaving, language, habits, values and tool-use;
5. The shape of the lesson is dependent upon the active participation of the students;
6. Learners and teachers see themselves as engaged in the same activity.

They propose that any classroom can be regarded as an intersection of a multiplicity of practices and trajectories. They further argue that the individual student's *positioning* within the community of practice will determine their success as a learner. Ultimately, the students can come to operate masterfully, within the constraints of the social setting. The process by which the individual achieves his or her position within a community of practice is explained by the notion of *telos*. This notion presupposes a common direction of learning and Winbourne and Watson broadly describe telos as "an unfulfilled potential to move or change in many different ways" (p. 182). They contend that "telos could be conceptualised as a set of constraints in some sense inherent in situations and in the individual's pre-dispositions to respond to situations as she does" (p. 182). In this sense the individual student's learning is both determinant of the common direction of learning and in part determined by the complex paths that the students have taken to be where they are. The students fulfil their ultimate positions within the community of practice through smaller-scale "becomings" in which they join the practice and begin to assume their eventual position. For example participation in the practice of asking questions can enable students to generate mathematical questions themselves. Similarly, participation in the practice of using graphical calculators can allow students to become "masters" in the use of these tools. The student's experiences at school are mediated by the images of themselves as learners that they bring with them.

The Role of Technology

In exploring how the use of technology mediates students' learning of mathematics, Pea distinguished between the *amplification* and the *cognitive reorganisation* effects of technology (cited in Berger, 1998). The *amplification* effects refer to the speed and ease by which the student is able to operate whilst using the technology. In this study, the graphical calculator is seen to *amplify* the zone of proximal development by creating a situation where the student is able to complete more conceptually demanding tasks effectively and easily. The benefits of *amplification* are regarded as short-term phenomenon, providing the student with immediate assistance during problem solving.

Use of the graphical calculator may also enrich or change student's conceptions in some way and thus may function as a tool which helps the student's thinking to develop. This is referred to as the *cognitive reorganisation* effect of the technology, which is defined by Berger as "a systematic change in the consciousness of the learner, occurring as a result of interaction with a new and alternate semiotic system" (1998, p. 16). Long-term changes in the quality of learning arise through *cognitive re-organisation*. Berger argues that the learner needs to engage thoughtfully with the technology if internalisation is to occur. It is not sufficient for a student merely to be introduced to the technology. Berger further suggests that in order for the learner to "interact in such a mindful way" he or she needs to "use the technology actively and consciously in a socially or educationally significant way" (1998, p. 19).

In a study conducted by Borba (1996), the use of the graphical calculator was seen to "enhance mathematical discussions" and this in turn "reorganised" the way in which knowledge was constructed in the classroom. In such a social context, the graphical calculator is seen as a mediator of both the teacher-student relationships and the interactions between individual students. Pea (1987) argues that "social environments that establish an interactive social context for discussing, reflecting upon, and collaborating in the mathematical thinking necessary to solve a problem also motivate mathematical thinking" (p.104). Furthermore, he also emphasises that technology can play a "fundamental mediational" role in promoting dialogue and collaboration in mathematical problem solving.

The Role of the Teacher

In a Vygotskian framework both the teacher and the students play a mutual and active part in creating the social environment (Moll, 1990). The function of the teacher is seen as an integral part of any learning situation. To discuss teaching and learning separately would thus make no sense from a Vygotskian viewpoint. The teacher is seen as a mediator of student learning and assumes an active and necessary role in the learning process (Lerman, 1994). An important objective for the teacher is to apprentice students into the discourse of the mathematics classroom. The teacher assists the students in "appropriating the culture of the community of mathematicians as a further social practice" (Lerman, 1996, p. 146). Consequently the students will be able to operate masterfully in this setting. Likewise Moll (1990) argues that a major role for the teacher is in creating social contexts for mastery of and conscious awareness in the use of cultural tools. By "constraining the foci for attention, and by recognising and working with pre-dispositions, rather than ignoring them, a teacher is more likely to be able to initiate *local communities of practice* which enable learners to see themselves as members of a mathematical community" (Winbourne and Watson, 1998, p. 183).

The theoretical framework discussed above illustrates that interaction between peers, with the teacher, and with technology, within a supportive learning environment are key elements in students' meaning making in the mathematics classroom. It also raises some important questions - particularly concerning the nature of shared knowledge. For example, when can knowledge be taken as shared? What role does each individual play in constructing such knowledge and how is this then "appropriated" (Lerman, 1994)? How does the teacher or use of the technology facilitate the appropriation process? How does the individual student make further use of shared knowledge? This study has sought to address these issues and in doing so has attempted to elaborate further on the complex process of knowledge acquisition in the mathematics classroom.

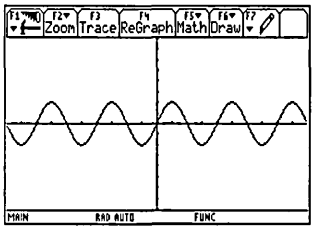
Methodology and Data Collection

The work reported in this paper forms part of a broader study of the way in which the graphical calculator mediates students' learning of functions, and the data was collected during the second phase of this research (see Elliott (1998) for details on the first phase). The methodological approach adopted in this study is both qualitative and ethnographic and is based on the underlying assumption that "all human activity is fundamentally a social and meaning making experience" (Eisenhart, 1988, p. 102). The principles of ethnography have thus governed the whole approach to carrying out the classroom-based research to date, from the choice of methods of enquiry, to the way in which each episode has been interpreted within the world of the participants. For a more detailed overview of this phase of the study and the way in which data was collected see Elliott, Hudson and O'Reilly (2000).

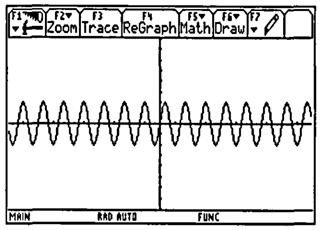
The data examined in this paper pertains to a lesson where three GCE Advanced level Further Mathematics students, Robert, Martin and Julie were asked to identify the symbolic forms of six graphed functions from a list of twenty possibilities and to discuss their ideas. These students were all experienced graphical calculator users and were each provided with a Texas Instrument TI-92 to assist them in their task. The task presented to the students is reproduced in figure 1. The discussions surrounding three of the graphs are presented below.

Match up the six graphs with their corresponding functions, chosen from the list below:

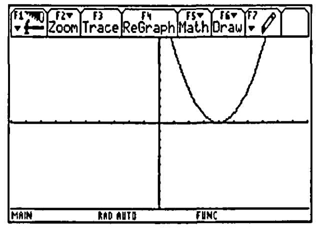
A. (ZoomTrig)



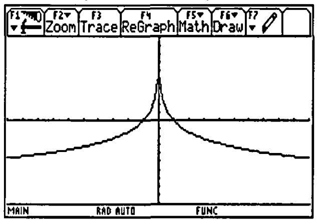
B. (ZoomTrig)



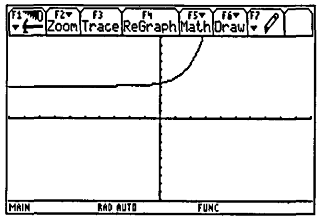
C. (ZoomStd)



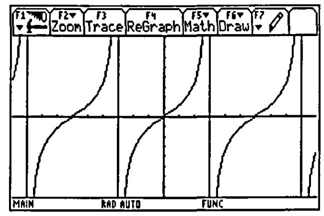
D. (ZoomStd)



E. (ZoomStd)



F. (ZoomTrig)



1. $y = \sin(x/3)$	2. $y = \cos(x - \pi/2)$	3. $y = 3\sin x$	4. $y = \cos(x + \pi)$
5. $y = (x - 4)^2$	6. $y = \tan(x/3)$	7. $y = (4 - x)^2$	8. $y = \tan(x/6)$
9. $y = (x + 4)^2$	10. $y = \cos(x + \pi/2)$	11. $y = \sin 3x$	12. $y = \ln(1/x)$
13. $y = e^{x-1} + 4$	14. $y = \ln x^2$	15. $y = e^{-(x+1)} + 4$	16. $y = 2\ln x$
17. $y = -\ln x^2$	18. $y = -e^{x+1} + 4$	19. $y = \tan x/3$	20. $y = \tan x/6$

Figure 1 Class Activity: Identifying the Graphs of Functions

Data Analysis

Notions developed by Teasley and Roschelle (1993) were used to analyse the interaction. Teasley and Roschelle propose that social interactions in the context of problem solving activity occur in relation to a *Joint Problem Space (JPS)*. They maintain that the *JPS* is a shared knowledge structure that supports problem solving activity by integrating (a) goals, (b) descriptions of the current problem state, (c) awareness of available problem solving actions, and (d) associations that relate goals, features of the current problem state and available actions.

In Teasley and Roschelle's model, collaborative problem solving consists of two concurrent activities: solving the problem together and building a *JPS.*

> Conversation in the context of problem solving activity is the process by which collaborators construct and maintain a *JPS.* Simultaneously, the *JPS* is the structure that enables meaningful conversation about problem solving to occur. Students can use the structure of conversation to continually build, monitor and repair a *JPS.* (Teasley and Roschelle, 1993, p. 236)

The analysis of the data thus involved finding evidence for the construction of a joint problem space as well as identifying student 'initiation' of the discourse, student 'acceptance' of arguments and cases of students 'repairing' misunderstandings. Evidence was also sought for instances that involved 'collaborative completions' between students, in which one student's turn would begin a sentence and the other student would use their turn to complete it. We shall present three examples of discussions to illustrate how the behaviour of individual students contributes towards the development of shared meanings.

Discussion of Graph A [$\cos(x - \pi/2)$]

The students' discussion of the possible symbolic representations for the first graph highlighted the way in which Robert's use of the graphical calculator provided a means through which he could become part of the *JPS* that was being created by Julie, Martin and the teacher-researcher. The discussion also signified the importance of the teacher's role in promoting collaboration and meaning making in a graphical calculator environment.

1	SE:	Can anybody tell me which function represents the graph in the first one?
2	Martin:	Is it $\cos(x + \pi/2)$?
3	SE:	And why do you say that?
4	Robert:	It's a sine graph.
5	SE:	Contradiction there. Explain your choice. [Directed at Martin].
6	Martin:	Er well it looks - it's got to be like sine or cos and I think that cos starts at the top and each line on the scale is 90^0 which is $\pi/2$ radians, so it's been moved …
7	SE:	It's been moved across to the …
8	Martin:	It's got to be $-\pi/2$ rads then because it's gone the other way, so it's $\cos(x - \pi/2)$.
9	SE:	Ok so you think it's $\cos(x - \pi/2)$. Why do you say that it might be a sine [graph]?
10	Robert:	Because sine of zero is zero and I'd say that that is in fact – because it seems that B is also a sine wave but that's more concentrated – I'd say that A is sin x/3.
11	SE:	You think that it's sin (x/3)?
12	Robert:	I wouldn't swear to it.
13	SE:	And what do you think? Have you got any ideas about this one?
14	Julie:	I think it's $\cos(x - \pi/2)$.
15	SE:	And why do you think that it's $\cos(x - \pi/2)$?
16	Julie:	It's been moved.
17	SE:	It's been moved?
18	Julie:	Yes it's a translation.
19	SE:	And in which direction is it moved?
20	Julie:	Er $\pi/2$ in the x-axis.
21	SE:	Yes. Ok so have you tried to actually graph on the TI-92 the first one that you thought it was?

22	Robert:	Yes.
23	SE:	And what did you get?
24	Martin:	Isn't that cheating drawing the graph to see which?
25	SE:	No, no he is just convincing himself.
26	Robert:	To be honest I can't remember what I typed in.
27	SE:	Well, let's think about the first one y = sin(x/3). What is the graph of that going to look like?
28	Robert:	Wide, and wider than it is there. [Robert pointed to graph A].
29	SE:	Yes. Ok, I'm going to say that you two are actually correct. Now it looks like a sine because it is sine of x, that is sinx.
30	Robert:	Yes.
31	SE:	But it can also be represented by y = cos(x - π/2) that's another…
32	Robert:	I see where that's coming from.

When asked to identify graph A, Martin was the first to offer a suggestion: 'Is it cos (x + π/2)?' (line 2) in which he invited acceptance or repair. Robert on the other hand seemed more confident and asserted that this *is* a sine graph (line 4). Robert recognised the distinctive shape of the graph as being of the form y=sinx and as such did not initially think of the graph in terms of a translation of the cosine function, as Martin had suggested. When asked to explain his initial response Martin pictured the graph of cosx in his mind and then considered the effect that the transformation cos (x + π/2) would have. This allowed him to perform a self-repair, by recognising his original error and realising that the correct form of the given graph was actually cos (x - π/2) (line 8).

In contrast, Robert who had immediately recognised the graph as that of sinx, was somewhat confused by the fact that this was not one of the listed options. His initial image of this function as a sine graph was strong and instead of considering the graph as a translation of cosx, he began to consider the other sine functions listed, focusing on sin(x/3) (line 10). Yet, he was still uncertain that this was the correct function (line 12). At this point Julie who had remained silent throughout was drawn into the conversation (line 13). The teacher-researcher's questions encouraged Julie to elaborate on her initial explanation of why she had accepted Martin's argument and showed she had made sense of the problem. However, Robert seemed unaffected by the arguments proposed by Martin and Julie and in an attempt to clarify his thoughts he began using the graphical calculator.

When asked to consider what the graph of sin(x/3) would look like in relation to graph A (line 27), Robert was able to recognise that the graph of sin(x/3) would be wider than graph A. Use of the technology and the teacher-researcher's question aimed at making him think about the relationship between the two graphs had helped Robert to perform a self-repair. He now realised that sin(x/3) was not the correct form of this function, and he started to question his initial thoughts and to eliminate the other sine functions listed. When it was explained that the graph could be represented symbolically by either y = cos (x - π/2) or y = sinx, Robert remarked 'I see where that's coming from'. This suggested that he could visualise the action of the transformation f(x-π/2) on the graph of f(x)=cosx and how this would produce the graph of sinx. He appeared to have internalised the argument that the teacher-researcher was presenting.

Thus, the use of the technology and the discussion in this example appeared to have resulted in some form of *cognitive reorganisation* for Robert. His thinking during the course of the episode had changed and by the end of this part of the discussion he was able to transfer his prior knowledge of trigonometric functions to this context. The concept of transformations became more meaningful to him, adding greater depth to his overall understanding of functions. He seemed to have begun to make the important visual connections between sine and cosine graphs and translations, which were also made explicit to Julie and Martin in this example. The graphical calculator provided an authoritative means by which Robert could investigate the ideas being discussed and modify his own visual images of the graphs accordingly. Consequently, Robert's use of the technology was an important part of the process by which he was able to enter the *JPS*. Robert began to have more confidence in the arguments being posed by his peers following his graphical exploration with the technology. He was further convinced of the validity of these arguments through SE's concluding remarks. Here the teacher-researcher assumed the role of a more knowledgeable person in the Vygotskian zone of proximal development and helped all of the students to make sense of the apparent contradictions.

The use of the graphical calculator also provided a means of furthering the discussion and preventing a breakdown in communication. When Robert was unable to move forward he turned to the graphical calculator in an attempt to clarify his thoughts, rather than merely accepting the arguments put forward by Martin and Julie without really understanding them. The teacher-researcher's repeated questioning of the students' reasoning was also a factor in maintaining and steering the discourse and in the creation of a *JPS*. This allowed the students to make discoveries for themselves whilst receiving appropriate guidance and reinforcement of their solutions.

In this example the students were working in a group to find the solution to the problem. However, close examination of their dialogue indicates that they actually seemed to be working separately, within the group situation, towards achieving this common goal. There were plenty of interactions between each student and the teacher-researcher, but there was no direct interaction between the students themselves. Each of the students made their own independent contributions to the discussion, which appeared separate from previous utterances. They did take turns in the conversation, but did not question one another's contributions or request further elaboration. Robert did not seem to consider the contributions made by Martin and Julie. Martin and Julie did not question Robert's argument. Julie did not make clear her agreement with Martin's line of reasoning until she was specifically asked to share her viewpoint. Each student appeared to take his or her turn in the conversation in a linear way. There were no interruptions from the other two students when one of them was presenting their argument.

The way in which these students were interacting made it necessary for the teacher-researcher to take an active role in maintaining and encouraging the discussion between students, in verifying students' assertions and in providing clarification and explanation of solutions where needed. There was also a need to promote the use of the technology, especially in Martin's case, as he saw this as a means of cheating (line 24) rather than as a tool that could help the students to clarify their thoughts and move towards a different level of understanding.

At the end of the episode the students had gained some shared understanding of the problem posed and of the solution through the creation of a *JPS*. They had each participated in the same activity at the same time and they had all listened to each other's contributions and those made by the teacher-researcher. The knowledge and understanding that these students had gained as a group appeared to be greater than that which they already possessed as individuals and as Jones and Mercer (1993) argue, this indicates that successful learning had taken place.

Discussion of Graph B [y = sin 3x]

The students' discussion of graph B showed that successful collaboration can occur in a graphical calculator environment without the use of this technology. However, we propose that even when the graphical calculator was not being used, as is the case here, it was still having an effect on the students' thinking.

1	SE:	Can anybody think of a function for B?
2	Martin:	I reckon its sin3x.
3	SE:	Sin3x.
4	All:	Yes.
5	SE:	You seem to agree on that one. So how did you come up with that conclusion?
6	Robert:	There doesn't seem to be any sneaky cosine tricks.
7	SE:	Not this time.
8	Martin:	It's a sine wave and it's been er...
9	Robert:	Three times x would condense it.
10	Martin:	It's got a stretch parallel to the x-axis of a third, because it got closer together.
11	SE:	Yes, you're all right it's sin3x.

Martin initiated the discussion by asserting that this was the graph of sin3x (line 2) and this time appeared to be much more confident with his suggestion. The other two students immediately accepted that this was the correct form of the function and when asked to give reasons why, both Martin and Robert took turns to construct an explanation, each building and elaborating on the previous utterances, thereby producing a collaborative completion (lines 8, 9, 10). When Martin paused to think (line 8), Robert anticipated what he may have intended to say and completed his statement. Together they provided a convincing argument for their choice of function. The students were thus all confident that they had identified the function correctly. Although, Julie did not participate verbally in this part of the discussion, she did make gestures that indicated her agreement with the arguments being put forward.

The knowledge constructed by the students in this example appears to be shared between them, especially Martin and Robert, and these two students appear to have constructed a *JPS*. However, it is uncertain whether Julie had actually developed a fully shared sense of the solution to the problem, as she did not verbalise her ideas through interaction with the other students. Martin and Robert seemed to be more supportive of each other when discussing this graph than when discussing the previous one. Rather than concentrating on developing their own arguments separately, they produced a joint explanation of why sin3x was the symbolic representation of graph B.

Martin and Robert were able to perform a collaborative completion together because it seems their visual images of the function were strong and corresponded to one another. In this case each student appeared to be able to visualise clearly the effects of the transformation, without using the technology. Yet, there was evidence that the graphical calculator was having an impact on the way in which these students were thinking about the problem. In particular, Robert was now actively looking for alternative symbolic forms for the graphed functions following his exploration of the function represented by graph A with the graphical calculator (line 6). We therefore propose that the use of the graphical calculator and its continued presence in the environment restructure the way in which students think about problems, and that this is most productive when used as part of a local community of practice. This will be discussed further below.

Discussion of Graph F [y = tan(x/3)]

The discussion of the final graph serves to illustrate how collective use of the graphical calculators enabled the students to establish a *JPS*.

1	Robert:	It's a tangent.
2	SE:	Think about the scale the TI-92 uses.
3	Robert:	To see if it was increasing, I could just draw the normal graph.
4	SE:	Ok, if it helps you can draw the – you can all draw the tanx graph and see what happens on your machine and then from there you can hopefully deduce what the function is.
5	Robert:	It's a stretch of factor 3.
6	Martin:	It's tan of x over 3.
7	Robert:	Yes.
8	SE:	Is that y = tan(x/3) or y = tanx/3 because there are two of them?
9	Martin:	y = tan(x/3).
10	SE:	y = tan(x/3) and what do you think? Have you managed to get the tan?
11	Julie:	Yes. That's the whole thing. [Julie pointed to the tanx in tanx/3].
12	SE:	That's tan of x all divided by 3.
13	Julie:	So yes y = tan(x/3).

14 SE: $y = \tan(x/3)$, yes well done you are right.

Robert was the first to state that this graph belonged to the tangent family of functions (line 1). There was, however, some uncertainty amongst the students as to what the graph of y = tanx would look like in relation to graph F. This was evident in the silence that ensued Robert's initial suggestion. Recognising this problem, the teacher-researcher asked the students to think about the scale that the graphical calculator uses to draw trigonometric functions (line 2). Robert then suggested that he could draw the graph of tanx using the TI-92 and compare this with graph F to deduce the relationship (line 3). The teacher-researcher then advised all three students to try this approach. Robert compared the graphs and deduced that graph F was obtained using a stretch of factor three. To complete Robert's statement, Martin added that the correct function was 'tan of x over three' and Robert immediately agreed. As in the discussion of graph B, Robert and Martin attempted to construct shared knowledge and again produced a collaborative completion. However, as there were two functions which could be verbalised as 'tan of x over three', the teacher-researcher sought confirmation that Martin had identified the function correctly and was quickly satisfied that he had. Up until this point Julie had not contributed to the discussion and the teacher-researcher drew her into the conversation again to see if she was following the arguments being presented. Julie accepted the choice of function offered by Martin and provided some evidence that she had understood why this was the correct function (line 11), which was then confirmed by the teacher-researcher's closing comments.

The students had thus been able to develop some shared understanding of the transformations used in this example through the creation of a *JPS*. Moreover, the use of the graphical calculator was an important part of this process. The interaction between the students was constructed in relation to the graphs produced by the graphical calculator and it was this factor that led directly to the collaborative completion between Martin and Robert. This occurred because the students were able to establish a shared visual interpretation of the function using the graphical calculator. In other words, in this episode use of the technology provided Julie, Martin and Robert with a common starting point from which they were able to think about the problem in the same visual terms. From this position they were each able to contribute towards correctly identifying the symbolic form of the function. As their discussion was structured around their shared use of the graphical calculator, this facilitated successful interaction. The decision to use the graphical calculator in this case thus proved extremely productive and resulted in the collaborative completion.

Local Communities of Practice in Action

During these discussions a *local community of practice* appears to have been established by the students and the teacher-researcher. Figure 2 summarises the kinds of interactions used by each participant throughout the discussion of all six graphs.

Firstly, as shown by figure 2, the students each showed willingness to explore and explain ideas to one another. They clearly saw themselves as functioning mathematically within the lesson, as they were each offering suggestions as to which functions represented the given graphs, based on some mathematical reasoning, which enabled them to obtain the correct form of the function in each case.

Secondly, the teacher-researcher ensured that the students received public recognition of their competence. This was achieved through acceptance of the students' ideas ('yes, well done you are right', 'yes, you are all right, it's sin3x').

	Robert	Julie	Martin	SE
Presenting ideas	4	2	1	1
Explaining ideas	3	3	0	3
Making assertions	7	2	3	1
Making statements	6	1	2	5
Showing acceptance	4	4	3	6
Repairing ideas	2	0	0	3
Self repairing ideas	1	0	1	0
Questioning	1	2	2	23
Performing collaborative completions	4	0	2	2
Number of interactions involving natural language	18	9	8	25
Number of interactions involving scientific language	14	5	6	19
Total number of interactions	**32**	**14**	**14**	**44**

Figure 2 Types of interaction used by the participants

Thirdly and most significantly, as the discussions progressed, the students began actively working together towards achieving a common understanding of each problem, through the sharing of ideas and questioning of one another. This led to the creation of *joint problem spaces* and successful collaboration in the form of collaborative completions between the students themselves and with the teacher-researcher.

Fourthly, the students each shared behavioural traits, such as presenting and justifying their own arguments and listening to, accepting and questioning the arguments of others. The language used by the students was both scientific and natural, and the students appeared to have shared conceptions of the scientific language that was used. The students also used the graphical calculator together as a group in their attempts to identify the fifth graph (not presented here).

The shape of the lesson depended on the active participation of the students. Each student created his or her own role in the practice, which varied accordingly. During the discussions the patterns of interactions between the students were continually changing as each new graph was considered. In each case the individual students appeared to occupy different positions within the discussion, modifying their roles depending on their needs. Martin initiated the discussion around the first two graphs, and Robert took over this role for the discussions concerning the remaining four graphs. Robert also began to act as a more capable peer in the Vygotskian *zone of proximal development* (graphs C-F). He continually made verbal contributions to the discussions and at times took control of the discussion, while Julie and Martin spent more time actively listening and thinking rather than speaking. In most cases Julie did not contribute voluntarily to the discussions and needed to be drawn into the discourse.

Martin was initially quite instrumental in moving the group towards the correct solutions (graphs A and B). However, as Robert took over initiation and steering of the discourse, Martin seemed to fade into the background. He was unsure about the symbolic forms of some of the graphs and he made fewer verbal contributions when discussing these, especially graph E (see Elliott, (1999)). He needed time to take into full consideration the arguments offered, to enable him to form his own ideas and to convince himself of their meaning. In this way Martin was attempting to derive his own meaning from that which already belonged to Robert and Julie.

Julie operated as an active listener during the majority of the discussion, only offering her suggestions when specifically asked to do so. It was only during the discussion of graph E that this pattern changed and her contributions became more spontaneous. Her difficulties with this graph encouraged her to share her ideas more freely, in an attempt to derive meaning from the interaction. However, at all other times she was a participant with a relatively voiceless role in the *local community of practice.*

Finally, both the students and the teacher-researcher regarded themselves as being involved in the same activity. In each case the teacher-researcher attempted to initiate each student into the discourse with the aim of encouraging the construction of shared knowledge. Through these actions the teacher-researcher attempted to maintain and repair the *JPS* that was being created. Figure 2 highlights that questioning formed an extremely important part of the teacher-researcher's strategy for encouraging participation and the construction and maintenance of a *JPS* amongst the students and herself. It also illustrates the quality and frequency of interactions made by Robert in comparison to Martin and Julie. Robert performed more successfully overall in the class activities in this trial, taking into account the scores that he obtained in the entire series of exercises, too numerous to present here, which comprised this investigation as a whole. This may have been the result of his additional willingness to share ideas and difficulties with the other students and the teacher-researcher and to use the technology without being prompted to do so.

This type of environment, where *local communities of practice* are established in which the students and teacher are able to construct and maintain *joint problem spaces*, is seen to be conducive to successful collaborative work involving graphical calculators. Within this supportive environment students are able to establish an effective means of operating with graphical calculators, in which the knowledge generated is shared amongst the participants. As seen in the discussion of graph B, this may not necessarily involve working with graphical calculators all of the time. For example, if the students are able to visualise the effects of a particular transformation on the graph of a function effectively without the aid of technology, then they may choose not to use the graphical calculator in that instant. However, the way in which they approach the problem is likely to reflect their prior use of the technology.

Reflections

The use of the graphical calculator in this study enabled the students to perform collaborative completions together and thereby created the opportunity for effective collaboration. This was achieved because the students were able to produce a shared visual representation of the problem using the graphical calculator and thereby create *joint problem spaces*. Successful collaboration was also promoted by the ability for students to reinforce their arguments through use of the graphical calculator. This discouraged breakdowns in communication between the students. The ability of students to use the graphical calculator as a means of verifying or, in particular, disproving their ideas also led to *cognitive reorganisation*, especially in Robert. This enabled Robert to gain a better understanding of the actions of transformations and the corresponding relationships between the graphs of sine and cosine functions. Input from the teacher-researcher was also seen to be a contributing factor in this cognitive reorganisation process.

There is an important role for the teacher in initiating the students into the *local community of practice* through initiating discussion of the results obtained by the graphical calculator. This could be on a one-to-one basis, in small groups or as a whole class. Through the interaction with the teacher, peers and the technology the individual student is able to develop a more meaningful understanding than was hitherto possessed. Analysis of the episodes has pointed to the centrality of the teacher's role in maintaining and encouraging discussion between the students, especially in relation to the results produced by the graphical calculators, and in providing additional verification of these results and the students' assertions. A further important function of the teacher lay in providing clarity and explanation of the results of the students' exploration with the technology, especially when the students were unable to reach a common understanding of their findings by themselves. The teacher needs to scaffold the students' learning with the graphical calculators to ensure that the technology is used effectively and results are interpreted correctly by the students, so that any misunderstandings are not perpetuated.

The establishment of *local communities of practices* in the classroom was seen to be conducive to successful collaborative work involving graphical calculators. In this type of supportive environment the students shared ownership of their use of the technology and they and the teacher-researcher could build and maintain *joint problem spaces*, which can lead to graphical calculators being used to greatest effect. However, it was not essential for the students to use graphical calculators all of the time in order for learning to be successful in their community of practice. Yet, the way in which the students operated whilst using the graphical calculators was seen to influence the way in which they approached problems without use of the technology.

The ways in which students define their eventual roles in a *local community of practice* is a complex process. The patterns of interactions between the students changed as each new graph was considered. Throughout the discussions the individual students appeared to occupy different positions within the discourse, modifying their roles depending on their needs. Robert, in particular, adopted the role of initiating and steering the discussions, whilst reacting to the arguments presented by the other students. So as the discussion developed, Robert's positioning within the discourse evolved and he proceeded to occupy a central role. He was an eager and very active participant and his position developed into that of a more capable peer in the *zone of proximal development.*

None of the students appeared to be self-reliant. They learned from each other, from the teacher-researcher's comments, from using the technology and by participating in the community of practice, through their discourse. The positions that the students occupied within the discourse were their ways of appropriating meaning and contributed towards their success as learners. Moreover, the successful learning that took place in these discussions is attributed to the creation of *joint problem spaces*, which can be thought of as particular examples of *local communities of practice* in action and both the technology and the teacher-researcher played an important part in contributing towards their development.

References

Berger, M.: 1998, 'Graphical calculators: an interpretative framework.' *For the Learning of Mathematics*, 18(2), 13-20.

Borba, M. C.: 1996, 'Graphing calculators, functions and reorganisation of the classroom.' *The Role of Technology in the Mathematics Classroom. Proceedings of Working Group 16 at ICME-8, the 8th International Congress on Mathematics Education* (pp. 53-60). Seville: Spain.

Eisenhart, M. A.: 1988, 'The Ethnographic Research Tradition and Mathematics Education Research.' *Journal for Research in Mathematics Education*, 19(2), 99-114.

Elliott, S.: 1998, 'Visualisation and using technology in A level mathematics.' In E. Bills (ed.), *Proceedings of the BSRLM Day Conference at the University of Birmingham* (pp. 45-50). Coventry: University of Warwick.

Elliott, S.: 1999, 'How does the way in which individual students behave affect the shared construction of meaning?' In E. Bills (ed.), *Proceedings of the BSRLM Day Conference at the University of Warwick* (pp. 13-18). Coventry: University of Warwick.

Elliott, S., Hudson, B. and O'Reilly, D.: 2000, 'Visualisation and the influence of technology in 'A' level mathematics: a classroom investigation.' In T. Rowland and C. Morgan (eds.), *Research in Mathematics Education Volume 2. Papers of the British Society for Research into Learning Mathematics* (pp. 151-168). London: British Society for Research into the Learning of Mathematics

Jones, A. and Mercer, N.: 1993, 'Theories of learning and information technology.' In P. Scrimshaw (ed.), *Language, Classrooms and Computers* (pp. 11- 26). London: Routledge.

Lerman, S.: 1994, 'Metaphors for mind and metaphors for teaching and learning mathematics.' In J. Ponte and J. Matos (eds.), *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education,* Volume 3 (pp. 144-151). Lisbon, Portugal.

Lerman, S.: 1996, 'Intersubjectivity in mathematics learning: a challenge to the radical constructivist paradigm?' *Journal for Research in Mathematics Education,* 27(2), 133-150.

Moll, L. C.: 1990, 'Introduction.' In L. C. Moll (ed.), *Vygotsky and Education: Instructional Implications and Applications of Sociohistorical Psychology* (pp. 1-27). Cambridge: Cambridge University Press.

Pea, R. D.: 1987, 'Cognitive technologies for mathematics education.' In A. H. Schoenfeld (ed.), *Cognitive Science and Mathematics Education* (pp. 89-122). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Teasley, S. D. and Roschelle. J.: 1993, 'Constructing a joint problem space: the computer as a tool for sharing knowledge.' In S. P. Lajoie and S. J. Derry (eds.), *Computers as Cognitive Tools* (pp.229-257). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Winbourne, P. and Watson, A.: 1998, 'Participating in learning mathematics through shared local practices.' In A. Olivier and K. Newstead (eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education,* Volume 4 (pp. 177-184). Stellenbosch, South Africa.

Vygotsky, L. S.: 1978, *Mind in Society. The development of higher psychological processes.* Cambridge, Massachusetts: Harvard University Press.

VISUALISATION AND THE INFLUENCE OF GRAPHICAL CALCULATORS

Visualisation is increasingly being accepted as a fundamental aspect of mathematical reasoning. Indeed many researchers stress the importance of mental imagery in the construction of meaningful mathematics (Presmeg, 1995; Wheatley and Brown, 1994). Zimmerman and Cunningham (1991) further argue that visual thinking needs to be linked to other modes of representation in order for students to learn optimally.

The potential of utilising graphical calculators to promote and encourage visualisation skills has been recognised in numerous studies. In particular, graphical calculators can be used to enable students to develop a deeper insight into functions and their graphs (Carulla and Gomez, 1997, Ruthven, 1990). Borba (1996) suggests that the use of graphical calculators mediates both teacher-student relationships and interactions between students.

This study investigated ways in which the graphical calculator mediated students' powers of visualising functions. The findings indicate that this occurred in three distinct ways. Firstly, it appeared that the graphical calculator enabled the students to access graphical images of functions quickly and easily, when perhaps they may have had difficulty otherwise. This in turn allowed them to see the problem more clearly and proceed towards a solution. Secondly, it seemed that the graphical calculator influenced students' perceptions in a positive way towards the validity of visual methods in mathematics. Thirdly, the observations suggested that the graphical calculator was instrumental in improving levels of student confidence surrounding functions. It did so by providing scaffolding for student-student interactions, which enabled students to make connections between visual and symbolic modes of representation more easily.

References

Borba, M. C. (1996) *Graphing Calculators, Functions and Reorganisation of the Classroom.* Proceedings of working Group 16 at ICME-8, pp. 53-60.

Carulla, C. and Gomez, P. (1997) *Graphics Calculators and Problem Solving. Do They Help?* Proceedings of PME 21, 1, p 224.

Presmeg, N. C. (1995) *Preference for Visual Methods: An International Study.* Proceedings of PME 19, 3, pp. 58-65.

Ruthven, K. (1990) *The Influence of Graphic Calculator Use on Translation from Graphic to Symbolic Forms.* Educational Studies in Mathematics, 21, pp. 431-450.

Wheatley, G. and Brown, D. (1994) *The Construction and Re-presentation of Images in Mathematical Activity.* Proceedings of PME 18, 1, p 81.

Zimmerman, W. and Cunningham, S. (1991) Visualisation in Teaching and Learning Mathematics, Providence, RI: MAA Notes Series, 19, pp. 1-9.