A GENERALISED SEMI-MARKOV RELIABILITY MODEL

by

Anthony Bendell B.Sc.(Econ), M.Sc., F.I.S., F.S.S., M.Sa.R.S.

Thesis submitted to the CNAA in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

Sponsoring establishment: Sheffield City Polytechnic.

Collaborating establishment: National Centre of Systems Reliability, UKAEA.

July, 1982

Anthony Bendell: A Generalised Semi-Markov Reliability Model

The thesis reviews the history and literature of reliability theory. The implicit assumptions of the basic reliability model are identified and their potential for generalisation investigated. A generalised model of reliability is constructed, in which components and systems can take any values in an ordered discrete or continuous state-space representing various levels of partial operation.

For the discrete state-space case, the enumeration of suitable system structure functions is discussed, and related to the problem posed by Dedekind in 1897 on the cardinality of the free distributive lattice. Some numerical enumerations are evaluated, and several recursive bounds are derived. In the special case of the usual dichotomic reliability model, a new upper bound is shown to be superior to the best explicit and non-asymptotic upper bound previously derived. The relationship of structure functions to event networks is also examined. Some specific results for the state probabilities of components with small numbers of states are derived.

Discrete and continuous examples of the generalised model of reliability are investigated, and properties of the model are derived. Various forms of independence between components are shown to be equivalent, but this equivalence does not completely generalise to the property of zero-covariance. Alternative forms of series and parallel connections are compared, together with the effects of replacement. Multiple time scales are incorporated into the formulation.

The above generalised reliability model is subsequently specialised and extended so as to study the optimal tuning of partially operating components. Simple drift and catastrophic failure mechanisms are considered. Explicit and graphical solutions are derived, together with several bounds. The optimal retuning of such units is also studied and bounds are again obtained, together with some explicit solutions.

OBJECTIVES

The main objectives of the research in this thesis are:-

(i) To investigate the nature of the basic reliability model, to identify its implicit assumptions, and to examine their realism and potential for generalisation.

(ii) To construct a generalised model of reliability incorporating states of partial operation.

(iii) To consider the associated enumeration of such systems.

(iv) To consider related optimisation problems in systems management.

ADVANCED STUDIES

The following advanced studies were undertaken in connection with the programme of research for this thesis:

(i) Participation in research seminars at Sheffield City Polytechnic, Dundee College of Technology and the Universities of Sheffield and St. Andrews.

(ii) Attendance at meetings/lectures of local groups of the Royal Statistical Society, the Dundee Mathematical Association, and the Institute of Mathematics and its Applications.

(iii) Participation in the following conferences:

IMA Conference on the Mathematics and Statistics of Reliability, London 1975,

1st National Reliability Conference, Nottingham 1977,

11th European Meeting of Statisticians, Oslo 1978,

2nd National Reliability Conference, Birmingham 1979,

3rd National Reliability Conference, Birmingham 1981,

7th Advances in Reliability Technology Symposium, Bradford 1982,

EUROCON '82, Copenhagen 1982.

(iv) Short secondments to:

The National Centre of Systems Reliability, during 1977,

The Royal Military College of Science, during 1980.

(v) Appropriate reading.

Throughout the period of registration the author has taught honours degree courses across a wide spectrum. These include courses in Applied Probability to final year B.Sc.(Hons) Applied Statistics candidates, and Reliability to final year honours engineering students.

During the registration period he has also acted as a referee for the IEEE Transactions on Reliability and the Journal of Physics Series A (Mathematical).

ACKNOWLEDGEMENTS AND INDIVIDUAL CONTRIBUTIONS TO JOINT RESEARCH

Work for this thesis has been completed over a considerable period of part-time registration whilst the author was lecturing full-time at Sheffield City Polytechnic and subsequently Dundee College of Technology. He would like to thank both of these institutions for their assistance and encouragement.

The thesis was originally conceived as concerned with the construction of a model for systems reliability at the level of greatest generality, with later chapters concerned with specific aspects of this generalisation. However, the breadth of this topic, together with the length of the period of registration and the considerable amount of work published by the author in this area during this period, meant that the first draft of the thesis was long and unwieldy. In consequence, the thesis was redrafted to concentrate on the partial operation extension to the dichotomic reliability model, which formed Chapter 3 of the original version.

Though the general framework of the model and the treatment of each of the extensions to the usual dichotomic reliability formulation was entirely the responsibility of the author, collaborative work took place with a number of colleagues on specific aspects of some of these extensions. This collaborative work was intended for direct publication and has in part already been published, with some of the remainder currently submitted to various journals. The Appendix, which contains copies of published papers, indicates the extent and nature of this joint work in the area that finally forms the thesis. Other joint work of more peripheral interest to the work of the thesis is referenced as appropriate in the text. The author's contribution to this collaborative part of the basis for the thesis

was the major part of the identification, formulation and (where appropriate) analytic solution of the problems discussed. Numerical analysis, as well as the solution of illustrative examples, was largely done by the co-authors.

The author would like to thank his collaborators on the above works, in particular Jake Ansell (now at the University of Keele) and Stephen Humble (now at the Royal Military College of Science) for their continued interest and perseverance in this collaboration. Thanks are also due to Colin Fraser (Dundee College of Technology) for his assistance with, and confirmation of, the evaluation of so many nasty integrals during the writing up of this thesis. He would also like to thank the staff of the National Centre of Systems Reliability, and in particular Mr. A.J. Bourne, for useful discussion and assistance over a number of years. He is immensely grateful to his supervisors, Dr. M. Knott, Dr. W.G. Gilchrist and Dr. D.N. Shanbhag for their help and guidance, and for not giving up on him. Finally, the author would like to thank his wife, Sulli and Mary Combe for their help in the preparation and typing of this thesis, and the CNAA itself for having sufficient regulations to prevent him from avoiding to write up the thesis for any longer.

DECLARATION

During the period of registration for M.Phil/Ph.D. the author has not been registered as a candidate for any other award of the CNAA, nor any award of a University.

CONTENTS

* Other papers by the author, published within the period of registration for Ph.D. but of more peripheral interest to the work of this thesis, appear in the reference list under his name or that of a co-author, as referenced in the text.

CHAPTER 1

INTRODUCTION

1.1 The Literature

Whilst the exact age of Reliability Theory is subject to some dispute (e.g. Barlow and Proschan (1965), Lomnicki (1973)), there is general agreement that it has not yet celebrated its thirtieth birthday. Of the main reliability journals, the IEEE Transactions on Reliability first appeared in 1951, Technometrics in 1959, Microelectronics and Reliability in 1961, and Reliability Engineering in 1980. More recent journals also aim specifically to publish material on reliability, e.g. Stochastic Processes and Their Applications (since 1972) and the Journal of Statistical Planning and Inference (since 1977). Papers on reliability and related replacement problems are also more widely dispersed across the literature, and important in this context are Operational Research Quarterly (1950-1977), Journal of the Operational Research Society (since 1978), Operations Research (since 1953), Naval Research Logistics Quarterly (since 1954), and Management Science (since 1955). Microelectronics and Reliability incorporates the World Abstracts on Reliability.

The major reliability books started appearing in the 1960s with Bazovsky (1961), Lloyd and Lipow (1962), Cox (1962), Zelen ed. (1963), Polovko (1964, English edition 1968), Roberts (1964), Barlow and Proschan (1965), Gnedenko, Bielajev and Solovier (1965, English edition 1970), and Shooman (1968). In the seventies and early eighties, many more have appeared, notably Green and Bourne (1972), Mann, Schafer and Singpurwalla (1974), and Barlow and Proschan (1975).

Today regular symposia and conferences on reliability take place in the U.K., U.S.A. and elsewhere. The relevant organisations and professional bodies in the field, as well as the industrial history of reliability are reviewed by Green (1977). See also Brewer (1977).

The important early papers which established the basic characteristics of the reliability model and the problem of systems reliability predictions were von Neuman (1956), Moore and Shannon (1956) and Birnbaum, Esary and Saunders (1961).

1.2 Component Reliability

Reliability of a component of age t ($t \geqslant 0$), denoted by R(t), is defined as the probability that the item is still operating satisfactorily at that age, and R(t) is taken to be monotonically non-increasing with $R(0) \equiv 1$, $R(\infty) \equiv 0$. Related functions to this reliability or survivor function are the distribution function of time to failure

$$F(t) = 1-R(t), \tag{1.1}$$

its derivative the probability density function of time to failure (if it exists everywhere)

$$f(t) = F'(t), \tag{1.2}$$

and the hazard function, age-specific failure rate or failure intensity

$$h(t) = f(t)/R(t). \tag{1.3}$$

Clearly,

$$\begin{aligned} R(t) &= \exp\left[-\int_0^t h(x)dx\right] \\ f(t) &= h(t)\exp\left[-\int_0^t h(x)dx\right]. \end{aligned} \tag{1.4}$$

Also of interest is the cumulative hazard

$$H(t) = \int_0^t h(x)dx, \tag{1.5}$$

and the moments of the time to failure distribution (if they

exist), in particular the mean time to failure (MTTF)

$$E(t) = \int_0^\infty tf(t)dt = \int_0^\infty R(t)dt \,, \qquad (1.6)$$

It is common to restrict attention to specific classes of life or time to failure distributions defined in terms of the above functions. Of greatest practical interest are classes of distributions which in some sense correspond to wearout or aging. Barlow, Marshall and Proschan (1963) and Barlow and Proschan (1965) consider the class of distrubutions with increasing hazard or failure rate (IFR), for which h(t) is increasing in t. Birnbaum, Esary and Marshall (1966) consider the increasing failure rate average (IFRA) class for which H(t)/t is increasing. Bryson and Siddiqui (1969) consider the class with decreasing mean residual life (DMRL) for which $\left[\int_0^\infty R(s+t)/R(s)\right] dt$ is decreasing. Marshall and Proschan (1965) consider both the new better than used (NBU) class for which $R(s+t) \leqslant R(s)R(t)$, and the new better than used in expectation (NBUE) class for which $\int_0^\infty R(s+t)dt \leqslant R(s) \int_0^\infty R(t)dt$. Haines and Singpurwalla (1974) introduce a further class with decreasing percentile residual life (DPRL), whilst Muth (1980a) defines the class with convex decreasing mean residual life, which is a proper subset of the IFR class. See also Marshall and Proschan (1972), Esary, Marshall and Proschan (1973), Proschan and Serfling (1974), Barlow and Proschan (1975), Hollander (1978) and Ross (1979). Tests of the appropriateness of the various classes are developed by Proschan and Pyke (1967), Barlow and Proschan (1969), Bickel and Doksom (1969), Bickel (1969), Hollander and Proschan (1972, 1974) and Koul (1977).

Of course, dual classes to the above can be defined by reversing the direction of monotonicity or inequality in order to describe the life distributions of items that improve with age.

Multivariate equivalents have also been considered (e.g. Harris (1970), Brindley and Thompson (1972), Marshall (1975), Buchanan and Singpurwalla (1977), Esary and Marshall (1979), Block and Savits (1980, 1981a)).

Whilst any continuous density on $[0,\infty]$ may be hypothesised for f(t), interest in the reliability literature (and especially amongst reliability engineers) has concentrated upon the one-parameter exponential density

$$\begin{aligned} f(t) &= \frac{1}{\phi}\exp(-t/\phi) \\ R(t) &= \exp(-t/\phi) \\ h(t) &= \frac{1}{\phi} \\ E(t) &= \phi,\ \mathrm{var}(t) = \phi^2 \quad ,\ \phi>0 \ , \end{aligned} \tag{1.7}$$

since its constant hazard corresponds to random failure, or the central section of the so-called <u>bath-tub</u> curve popular amongst engineers (e.g. Shooman (1968), Lomnicki (1973)).

Despite e.g. Shooman (1968)'s early warning, it is still apparently true that many reliability engineers assume a constant hazard or age-specific failure rate unless there is evidence to the contrary (see e.g. Bourne (1973), Lomnicki (1973), Cottrell (1977), Dorey (1979)), and this often causes serious error (Yasuda (1977), Moss (1978)). Indeed, in the literature reliability data is often presented implicitly based upon this assumption (e.g. Kujawski and Rypka (1978), Gibson (1979), Snaith (1979), Henley and Kumamoto (1981)), and the administration of reliability data banks often shares this approach (e.g. George (1978), Silberberg (1979), Holmberg and Markling (1980), Colombo and Jaarsma (1980)). See also Shooman (1968)'s comments on MIL-HDBK-217 and other published reliability data sources, and more recently Gaertner et al (1977), and O'Connor (1977).
The assumption of exponentiality corresponds (1 to 1) to specifying

a Markov Process, in this case a simple Poisson process, for the component (e.g. Feller (1968)). If upon failure the component is repaired and has an independent exponential repair time distribution, the alternating renewal process so generated forms a simple two-state Markov process. Some justification for the use of the exponential in systems is its arising as a limit; e.g. Feller (1971), Gnedenko, Belyayev and Solovyev (1970) . See also Gaver (1963), and Schüeller and Schwarz (1976).

The other distributional form given increasing prominence in the literature is the Weibull distribution named after Weibull (1939, 1951) but originally derived by Fisher and Tippett (1928). For this

$$
\begin{aligned}
f(t) &= \frac{\beta}{\phi}(t/\phi)^{\beta-1}\exp\left[-(t/\phi)^{\beta}\right] \\
R(t) &= \exp\left[-(t/\phi)^{\beta}\right] \\
h(t) &= \frac{\beta}{\phi}(t/\phi)^{\beta-1} \qquad (1.8) \\
E(t) &= \phi\,\Gamma(1+\beta^{-1}),\ \mathrm{var}(t) = \phi^2\left\{\Gamma(1+2\beta^{-1})-\left[\Gamma(1+\beta^{-1})\right]^2\right\}, \\
&\qquad \phi>0,\ \beta>0.
\end{aligned}
$$

The hazard function is monotonically increasing in t if $\beta>1$ (corresponding to aging, wearout or the third section of the bath-tub curve), whilst it is monotonically decreasing if $\beta<1$ (corresponding to initial or burn-in failures, or the first section of the bath-tub). If $\beta = 1$, the distribution reduces to the exponential (1.7).

The simplicity of the form of the hazard and its ability to model any section of the bath-tub curve partially explain the Weibull's popularity in reliability work, as does its relationship to extreme value theory (e.g. Mann (1968)). One of its disadvantages is that standard methods of estimation are inconvenient; maximum likelihood estimation for example requiring iterative solution (e.g. Cohen (1965), Harter and Moore (1965, 1967), Wingo (1972), Ringer and Sprinkle (1972),

Rockette, Antle and Klimko (1974), Zanakis (1979a), Archer (1980)). However, alternative explicit estimation methods are available; Mann (1968) and Mann, Schafer and Singpurwalla (1974) give extensive bibliographies. See also Hinds, Newton and Jardine (1977), Gross and Lurie (1977), Saylor (1977), Bennett (1977), Martz and Lian (1977), Kuchii, Kaio and Osaki (1979). In particular simple graphical estimation methods exist (e.g. Kao (1959, 1960), King (1971), Cran (1976), Kaio and Osaki (1980)) and the appropriate special graph papers are commercially available (e.g. Chartwell 6572-3). Another disadvantage of the Weibull, relative say to the Gamma, is the complexity of results in renewal theory to which it leads (e.g. Cox (1962), Smith and Leadbetter (1963), Lomnicki (1966), Kay (1973), Nakagawa and Yasui (1978)).

The exponential and Weibull distributions above are respectively one and two parameter distributions. The fit to data can often be improved substantially by the addition of an additional threshold parameter $\alpha>0$, so that each t in the right hand sides of f(t), R(t) and h(t) in (1.7) and (1.8) is replaced by $(t-\alpha)$. (See Bob Moss's contribution to the discussion of Lomnicki (1973)). Estimation, however, is correspondingly more complicated; e.g. Wingo (1973), Mann, Schafer and Singpurwalla (1974), Lemon (1975), Zanakis (1977, 1979a,b), Lehtinen (1979), Archer (1980), Dyer and Keating (1980).

The analytic inconvenience of the Weibull distribution has meant that a number of authors have investigated whether one can work satisfactorily with methods based on another distribution, usually the exponential, when the Weibull distribution applies. Zelen and Dannemiller (1961), although misquoted by Mann, Schafer and Singpurwalla (1974), considered the robustness of four widely used

acceptance sampling procedures based upon the one-parameter exponential when the time to failure distribution was really two-parameter Weibull with an increasing hazard rate but the same mean life. They found that procedures based upon the recommendations of Task Group Two in A.G.R.E.E. (1957) were very sensitive to departures from exponentiality, and that consequently applying them to data from a Weibull distribution with increasing hazard rate might result in substantially increasing the probability of accepting components having poor mean times to failure. Harter and Moore (1976) show by Monte Carlo that the exponential based sampling plans in MIL-STD-781B are not robust under departures from exponentiality and further give simple modifications for use when the Weibull distribution is appropriate. Posten (1973), also building on the work of Zelen and Dannemiller, investigates the robustness of exponential-based reliability (point) predictions for series systems of up to 15 identical components when the Weibull distribution is valid. Powers and Posten (1975) extend this to parallel systems. These two papers provide ranges of β in which the error in using the exponential procedure is within an acceptable limit. Generally these ranges are broader for the smaller numbers of components considered. Hager, Bain and Antle (1971) also demonstrate the lack of robustness of exponential-based reliability estimation. For a connected Bayesian problem, see Higgins and Tsokos (1977).

In the author's own joint work, Bendell, Humble and Mudhar (1979), the robustness of exponential-based interval estimators of a number of characteristics of interest were considered when the Weibull distribution applied. It was found that the confidence intervals for most characteristics of the failure distribution were relatively robust. The only exceptions being the first percentile of the failure

distribution, and the reliability for large time t.

1.3 System Reliability

The early papers on reliability by von Neuman (1956), Moore and Shannon (1956), and Birnbaum, Esary and Saunders (1961) established the mathematical basis for the evaluation of the reliability of complex systems of components from knowledge of component reliability, and for the construction of reliable systems from relatively unreliable components. A methodology for the computation of systems reliability from component reliability is necessary as in most cases data on the reliability of complete systems or subsystems is virtually non-existent (e.g. Bourne (1973), Green and Bourne (1972), Snaith (1979)), and the complexity of the system and its often high reliability precludes the estimation of systems reliability by life tests on identical systems on time and cost criteria (e.g. Lomnicki (1973)). This point is given emphasis by the steady increase in the reliability of many components and thus systems through time (e.g. Kooi (1967), Shooman (1968)).

Whilst for single components for which life testing would take prohibitively long, the solution is accelerated life testing (i.e. life testing at environments more severe than those at which the component is expected to operate), this possibility is not available for complex systems. Accelerated testing (e.g. Mann, Schafer and Singpurwalla (1974)) was originally devised to provide failed components to be analysed so as to improve design. However, there is no guarantee that the basic physical processes of failure encountered under excessively severe environments should be common with those which would be encountered under long term exposure to a normal environment. According to Cox (1972) this is likely to

happen only when there is a single predominant mode of failure, but see Kimball (1980).

For accelerated testing to provide a measure of the reliability of a component under normal usage it is necessary to have some connection between the component's reliability under a normal environment and its reliability under the excessively severe environments. Such a connection is sometimes no more than a purely graphical technique, though other times it is analytic and based on a theoretical model of the mechanisms of failure. In fact, no satisfactory simple connections exist for most components, although some generalised models with a theoretical background such as the Arrhenuis equation, are of some value. Often it is required to investigate in detail the physical structure of the particular component, model the operation of failure mechanisms upon these components, and then employ these theoretical mathematical models to obtain the connection between reliability under normal and excessively severe conditions (Jacobi (1968)). For complex systems, however, the assumption that the physical processes of failure under accelerated and normal environments are common is unlikely to be valid, as e.g. there will not be a single predominant mode of failure, and any connection between reliability under the normal and accelerated environments is likely to be prohibitively complex. (However, see Nelson (1975)). Mann, Schafer and Singpurwalla (1974) discuss additional problems of accelerated life tests.

In order to be able to evaluate systems reliability from knowledge of component reliabilities it is necessary to possess information about the structure of the system; specifically which combinations of component failures result in system failure,

or equivalently which combinations of operating components result in system operation. With the basic binary definition of reliability introduced in Section 1.2 and the usual implicit assumptions of the basic systems reliability model (given below), there are a number of equivalent representations of this aspect of system structure. These are notably the structure function or truth table, the reliability, event or switching network, and the Boolean hindrance or admission functions. See e.g. Hohn (1962),Flegg (1971), Green and Bourne (1972), Lomnicki (1973), Evans (1976).

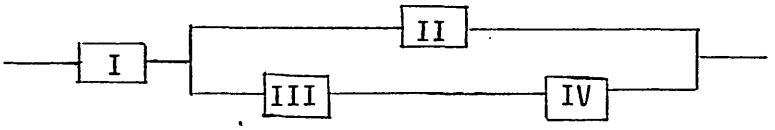
As an example of their application in the basic systems reliability model, we show in Figure 1.1 these equivalent representations for a simple system which will only work if componentI and either II or (III and IV) work. An alternative representation which is not shown but which is gaining interest in the literature is event or fault trees (e.g. Barlow and Proschan (1975), Bazovsky (1977), Dhillon and Singh (1978)).

It is apparent from the figure that the implicit assumptions which make these representations equivalent - apart from the assumptions that the system has a single function, the system's structure is static and components and system can each only take one of two states - are that component and system operation is instantaneous, the order of component failures does not affect the state of the system, and that there is one unambiguous and homogeneous failure mode, failure to operate (failure to idle is impossible). With these assumptions, simple bounds of known accuracy can be put on the systems reliability for given component reliabilities by making use of the Inclusion-Exclusion Theorem and Bonferroni's Inequality (e.g. Feller (1968), Lomnicki (1973)).

FIGURE 1.1

Alternative representations of system structure.

Reliability network



Structure function
('1' denotes operating state, '0' denotes failed state)

component				system
I	II	III	IV	
1	1	1	1	1
1	1	1	0	1
1	1	0	1	1
1	1	0	0	1
1	0	1	1	1
1	0	1	0	0
1	0	0	1	0
1	0	0	0	0
0	1	1	1	0
0	1	1	0	0
0	1	0	1	0
0	1	0	0	0
0	0	1	1	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	0

Boolean representation

Hindrance function:
(A_i denotes failure of component i, F denotes system failure)

$$F = A_I + A_{II}(A_{III} + A_{IV})$$

Admission function:
(B_i denotes operation of component i, R denotes system operation)

$$R = B_I(B_{II} + B_{III}\ B_{IV})$$

(Whilst the above Boolean representation in which Boolean addition (+) corresponds to the logical 'or' and Boolean multiplication (·) corresponds to the logical 'and', is the form usually applied in the reliability literature (e.g. Green and Bourne (1972), Lomnicki (1973)), notation in the mathematics texts varies. Following e.g. Stoll (1961), Flegg (1971), Kuratowski (1972), these symbols are respectively replaced by the set theory symbols $\cup$ (cup) and $\cap$ (cap) denoting the join or union, and the meet or intersection respectively. Other authors, e.g. MacLane and Birkhoff (1967) and Cohn (1977) use the equivalent symbols $\vee$ and $\wedge$ instead.)

To proceed further and be able to evaluate the reliability function of the system (for any t) from the reliability of the components it is necessary to have information or make assumptions about the interrelationships of component failures, or equivalently of the dependence between the states of the various components. The basic system reliability formulation assumes that component failures are independent, or equivalently the absence of common-mode or common-cause failures, so that systems reliability can be obtained from the usual rules for combining the probabilities of independent events (e.g. Birnbaum, Esary and Saunders (1961),Shooman (1968), Bourne (1973), Lomnicki (1973), Edwards and Watson (1979)). With this assumption, the reliability function for the system of Figure 1.1 is

$$R(t) = R_I(t)\left[R_{II}(t)+R_{III}(t)R_{IV}(t)-R_{II}(t)R_{III}(t)R_{IV}(t)\right] .$$

For complex systems, however, the computation of system's reliability by such a direct method can be difficult even under the assumption of independence, and methods for simplifying and computerising evaluation are of practical interest to the reliability assessor (e.g. Shooman (1968), Misra (1970), Woodcock (1971), Green and Bourne (1972), Rosenthal (1975), Aggarwal, Misra and Gupta (1975 a, b, c), Fussell (1975), Sharma (1976), Lin, Leon and Hwang (1976), Blin et al (1977), Nakazawa (1977), Satyanarayana and Prabhakar (1978), Arnborg (1978), Aggarwal and Rai (1978), Rai and Aggarwal (1978), Gupta and Sharma (1978a), Gopal, Aggarwal and Gupta (1978b), Locks (1978, 1979b), Singh (1979), Boffey and Waters (1979), Laviron, Berard and Quenee (1979), Misra (1979), Easton and Wong (1980)).

The structure function in Figure 1.1 illustrates the fact

that with the above assumptions (but not necessarily including independence of component failures) corresponding to n distinct components each of which may be in either of two states - operating (1) or failed (0) - there are 2^n states for the system corresponding to all combinations of operating and failed components. To each of these states of the system may be assigned one of the two levels 1 or 0, so that there are $_2 2^n$ possible systems of n components. Thus the number of possible systems gets large very fast; for two components there are 16 possible systems, for three components 256, and for four components 65,536. In theory, the smaller is the number of possible system structures the less information is needed for, and the easier is the identification of, the appropriate structure function and reliability function for a real physical system. Consequently, there has been considerable interest in the literature (e.g. Birnbaum, Esary and Saunders (1961), Esary and Proschan (1963), Lomnicki (1972, 1973, 1977), Barlow and Proschan (1975)) in restricting the class of possible structure functions to a sub-set which corresponds to the systems with real physical analogues.

The class of series-parallel systems discussed e.g. by MacMahon (1892), Riordan and Shannon (1942), Knödel (1950), Carlitz and Riordan (1956), Lomnicki (1972, 1977), is too restrictive to represent all such realistic systems, and does not contain all the real systems to be found in the reliability texts. In particular it excludes the so-called <u>k-out-of-n</u> (or k-out-of-n:G) systems (whereby the system operates if any k or more of its components operate), which are examples of symmetric Boolean functions (e.g. Flegg (1971)), and are of great physical interest to the engineer (Birnbaum, Esary and Saunders (1961), Phillips (1980),

Ansell and Bendell (1982a)). Birnbaum in the discussion of Lomnicki (1973) suggested a generalised class of realistic systems which contains the series-parallel and k-out-of-n systems, and which is based on replacing single modules in a simple system by k-out-of-n structures of components.

The class of 'realistic' systems which has received most attention in the reliability literature, however, is the class of so-called coherent (or monotonic) systems introduced by Birnbaum, Esary and Saunders (1961). This class contains the two-terminal systems of Moore and Shannon (1956), as well as all series-parallel and k-out-of-n systems. A coherent system is a system of components such that the system's state does not deteriorate from 1 to 0 if a failed component is replaced by an operating one, and does not improve from 0 to 1 if an operating component is replaced by a failed one, and operates if all its components operate and fails if all its components fail. Formally, we describe the state of an n-component system by the state vector

$$\underline{s} = (s_1, s_2, \ldots, s_n)$$

where s_α the state of the α^{th} component may be 1 (operating) or 0 (failed), and the resulting state of the system (1 or 0) is described by the structure function $f(\underline{s})$. Then if we define

$$\underline{1} = (1, 1, \ldots, 1)$$

$$\underline{0} = (0, 0, \ldots, 0)$$

and $\underline{x} > \underline{y}$ if $x_\alpha \geqslant y_\alpha$ for all $\alpha = 1, \ldots, n$

and $x_\alpha > y_\alpha$ for some α ,

it follows that a coherent system is defined by the requirements

$$f(\underline{x}) \geqslant f(\underline{y}) \qquad \text{for all } \underline{x} > \underline{y} \tag{1.9}$$

$$f(\underline{1}) = 1 \tag{1.10}$$

$$f(\underline{0}) = 0 .$$

The definition of coherent systems in the literature vary somewhat from the original one above due to Birnbaum, Esary and Saunders (1961), although this is also used by some other authors, e.g. Esary and Marshall (1974). Lomnicki (1973, 1977) neglects to include the conditions (1.10) in his definition of coherent systems, so that his 'coherent systems' correspond to semi-coherent systems as defined by Birnbaum, Esary and Saunders (1961). However, this is an omission rather than an alternative definition, since his enumerations correspond to the original definition. Barlow and Proschan (1965) refer to the coherent systems of Birnbaum, Esary and Saunders (1961) as 'monotonic systems' due to the obvious algebraic connotation. However, Barlow and Proschan (1975) perhaps following Kaufmann (1969) re-define monotonic systems to be those satisfying (1.9) alone; i.e. the semi-coherent systems of Birnbaum, Esary and Saunders (1961). Their definition of coherent systems is composed of the condition (1.9) together with the requirement that every component is relevant; i.e. that there is no component α for which

$$f(s_1, \ldots, s_{\alpha-1}, 1, s_{\alpha+1}, \ldots, s_n) = f(s_1, \ldots, s_{\alpha-1}, 0, s_{\alpha+1}, \ldots, s_n)$$

$$\text{for all } s_1, \ldots s_{\alpha-1}, s_{\alpha+1}, \ldots, s_n \quad . \qquad (1.11)$$

It follows from this requirement of relevancy and from (1.9) that (1.10) must also hold, but the definition is more restrictive than the original one of Birnbaum, Esary and Saunders (1961). Phillips (1977) uses monotonic in the same way as Barlow and Proschan (1965), and coherent in the same way as Barlow and Proschan (1975). In this thesis we shall follow the original definitions of coherency and semi-coherency due to Birnbaum, Esary and Saunders (1961), except where we specify otherwise.

Since coherent systems (by either the Birnbaum, Esary and Saunders (1961) or the Barlow and Proschan (1975) definitions) are generally regarded as corresponding to the set of real physical systems of practical interest, a lot of attention in the literature has been devoted to the study of the desirable properties possessed by coherent systems. See for example Birnbaum, Esary and Saunders (1961), Esary and Proschan (1963), Birnbaum, Esary and Marshall (1966), Esary and Marshall (1974), Haines and Singpurwalla (1974), and Barlow and Proschan (1975).

The reliability texts generally neglect to indicate that the set of non-coherent systems does contain some very plausible and indeed simple physical systems (see Evans (1978)'s review of Barlow and Proschan (1975)), so that whilst coherency is to date the most satisfactory criterion for systems to be of physical interest, it is not completely satisfactory. It is perhaps interesting that when John Bourne in the discussion of Lomnicki (1973) raised the question of whether there may not be other classes of structures of more physical importance than coherent systems, and ".... whether Mr. Lomnicki has established coherent systems by a personal examination or whether he has seen ways of doing so automatically", no response was received. However, Lapp and Powers (1977) do consider a non-coherent system associated with a nitric acid cooling process, and this is subject to further debate in the December 1977, April 1979 and June 1980 issues of the IEEE Transactions on Reliability. Locks (1979a) provides an interesting discussion of aspects of this system. Fussell (1975), Locks (1978) and Amendola and Contini (1980) discuss the occurrence of non-coherent systems, and Worrell produced a relevant computer program as long ago as 1961 (Bell Telephone

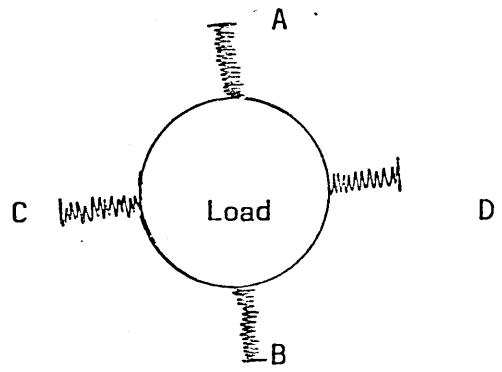
Laboratories (1961)). More recent programs and approaches that can deal with non-coherent as well as coherent systems are described by Bennetts (1975), Caldarola and Wickenhauser (1977), Kumamato and Henley (1978) and Locks (1979c). See also Ogunbiyi and Henley (1981).

As an example of a simple non-coherent system, the system represented in Figure 1.2 is composed of four springs designed to keep a load in place. As is apparent from its structure function, the system is non-coherent since failure (breakdown) of a single spring causes system failure as the load is pulled to one side, whilst failure of two opposite springs leaves the system operating (although less stable in relation to outside disturbances).

Whilst the number of series-parallel systems of n components is now known for both the cases where all components are distinct and where some are identical (MacMahon (1892), Knödel (1950), Carlitz and Riordan (1956), Lomnicki (1972)), similar results are not available for the class of coherent systems, for which the number of systems of distinct components is only known explicitly up to n = 7. The enumeration problem for coherent systems is in fact identical to the problem posed by Dedekind in 1897 on the cardinality of the free distributive lattice generated by the symbols $s_1, \ldots, s_n$. The known numbers of coherent systems of n distinct components following the original Birnbaum, Esary and Saunders (1961) definition (Δ_n) are given in column two of Table 1.1, whilst the numbers corresponding to the more restrictive definition of Barlow and Proschan (1975) (t_n) are given in column three. It is perhaps noteworthy that the plot of date of publication against n = 4, 5, 6, 7 is approximately linear (especially if one takes into consideration the fact that the publication of

FIGURE 1.2

A non-coherent system.



Structure function

component				system
A	B	C	D	
1	1	1	1	1
1	1	1	0	0
1	1	0	1	0
1	1	0	0	1
1	0	1	1	0
1	0	1	0	0
1	0	0	1	0
1	0	0	0	0
0	1	1	1	0
0	1	1	0	0
0	1	0	1	0
0	1	0	0	0
0	0	1	1	1
0	0	1	0	0
0	0	0	1	0
0	0	0	0	0

TABLE 1.1

Known numbers of coherent systems of n distinct components, with first published sources.

n	Birnbaum et al (1961) definition $\mathcal{S}_n$		Barlow and Proschan (1975) definition t_n	
1	1		1	
2	4	Dedekind (1897)	2	
3	18		9	
4	166		114	Lomnicki (1977)
5	7,579	Church (1940)	6,894	
6	7,828,352	Ward (1946)	7,785,062	
7	2,414,682,040,996	Church (1965)	2,414,627,400,434	

Church's result for n = 5 was unduly delayed) and a simple least-squares fit predicts 1990 approximately for the publication of the number of coherent systems of eight distinct components!

Since the numbers of coherent systems is in general unknown, the obtaining of sharp upperbounds (and to a lesser extent lowerbounds) for these has long been of interest; Dedekind (1897), Gilbert (1954), Korobkov (1963), Hansell (1966) (misquoted by Lomnicki (1977)), Kleitman (1969), Hanish, Hilton and Hirsch (1969), Alekseev (1973), Kleitman and Markowsky (1975), Korsunov (1977). The sharpest explicit and non-asymptotic upperbound published to date is due to Hansel (1966) who proved that

$$\mathcal{D}_n < 3^{M_n} \tag{1.12}$$

where M_n is the middle Binomial coefficient i.e.

$$M_n = \begin{cases} \dfrac{n!}{(n/2)!\,(n/2)!} & , \text{ if } n \text{ even} \\ \dfrac{n!}{([n+1]/2)!\,([n-1]/2)!} & , \text{ if } n \text{ odd} \end{cases} .$$

However, in Chapter 2 we derive improvements to this bound as bi-products of the generalisation to multistate systems.

The above discussion is for the case where all the components are distinct. Lomnicki (1977) discusses enumeration for the case where some or all of the components are identical, and tabulates the corresponding numbers of possible coherent systems (according to the Barlow and Proschan definition) of up to five components. Coherent systems with all the components identical have received some attention in the reliability literature since they are of interest in the context of the problem of optimal redundancy in the presence of opposite failure modes (e.g. Lomnicki (1977), Phillips (1980), Ansell and Bendell (1982a)). It is also true that for series-parallel systems the enumeration problem was first solved for

identical components (MacMahon (1892)). For coherent systems, making all the components identical also greatly reduces the numbers of possible systems and consequently substantially simplifies (numerical) enumeration. The available numbers of coherent systems of identical components (due to Lomnicki (1977) are presented in column two of Table 1.2

However, despite the earlier work of Phillips (1976) in a related context, Lomnicki failed to consider the fact that a further substantive reduction in the number of coherent systems could be achieved by assuming independence of component failures and considering the reliability function instead of the structure function or Boolean function. With the usual assumption of independence, systems with different Boolean or structure functions reduce to the same reliability function. The corresponding reduced numbers of distinct coherent systems up to n = 5 are shown in column three of Table 1.2. As an example of the reduction, Figure 1.3 shows two systems of four identical components with distinct Boolean representations (even after rotation of the indices 1, 2, 3, 4) or equivalently structure functions, but identical reliability functions under the assumption of independence of component failures. (The reliability of each identical component is denoted by x). It follows from Phillips (1980) that the reliability functions of coherent systems of n identical components are, under the assumption of independence, convex combinations of the reliability functions of the k-out-of-n systems. Ansell and Bendell (1982a) generalise this result to dependent components.

As a final point in this section, we note that Figure 1.3 serves to illustrate the fact that the equivalence in the dichotomic reliability

TABLE 1.2

Number of coherent systems of exactly n identical components.

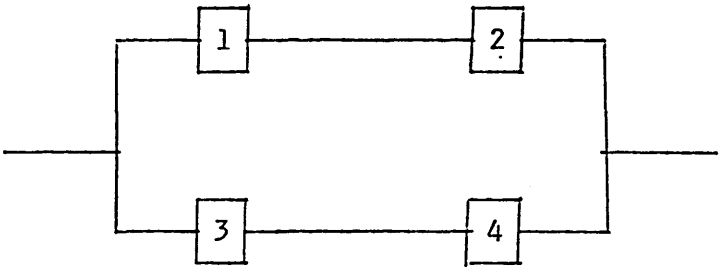
(Barlow and Proschan (1975)'s definition of coherency).

n	distinct structure functions Lomnicki (1977)	distinct reliability functions
1	1	1
2	2	2
3	5	5
4	20	17
5	180	78

FIGURE 1.3

Two systems of identical components with distinct Boolean representations, but identical reliability functions.

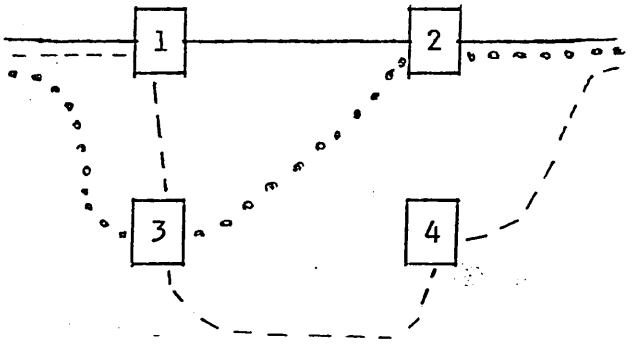
(i)



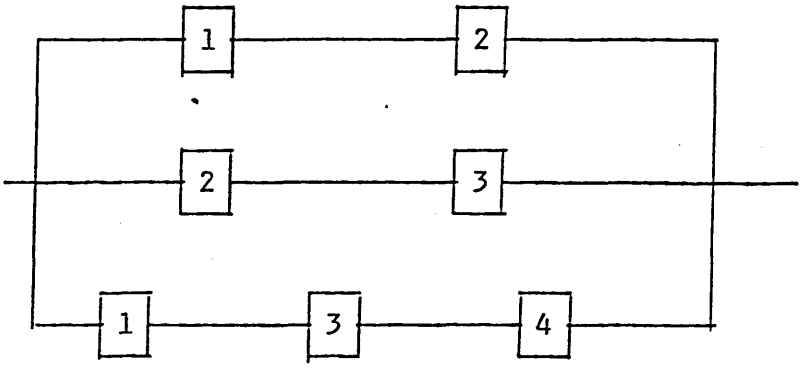
Boolean function: $B_1B_2 + B_3B_4$

Reliability function: $2x^2 - x^4$

(ii)



or equivalently



Boolean function: $B_1B_2 + B_2B_3 + B_1B_3B_4$

Reliability function: $2x^2 - x^4$

model between event networks and Boolean and structure functions (e.g. Flegg (1971)) can only be accomplished for non-series-parallel systems by multiple representation of single components (which reduces the event network to a formal statement of logical structure without intuitive physical back-up), or by the multiple route notation used in Figure 1.3 (which soon gets confusing as the numbers of components and routes rise) or by ad-hoc logic devices such as the k-out-of-n gate (e.g. Buzacott (1967,1970)) or the priority-AND gate (e.g. Fussell, Aber and Rahl (1976)) which have become standard notation in the engineering literature. The limitations of the event network as a conveyor of system structure is a subject to which we shall be returning when we consider the generalisation of the usual dichotomic systems reliability model.

1.4 Possible Generalisations

The basic systems reliability model introduced in the previous two sections is based on the following (often implicit) assumptions:

(i) Time t is continuous and perfectly ordered on $[0,\infty]$.

(ii) The system has a single function (or output) which is required to be performed continuously and does not vary in time, and which depends for satisfactory operation upon the operation of the components in the way specified in the event network or structure function or alternative representation of system structure.

(iii) The system's structure, the environment of the system, and the conditions that define component failure are stationary in time.

(iv) Components and the system can each only be in one of two homogeneous states at any point in time; 1 (operating) or 0 (failed). At time zero they are in state 1.

(v) In the absence of a repair or replacement mechanism the component state 0 is an absorbing state whilst the initial state 1 is not, so that by time ∞ components and coherent systems are in state 0, and the probability that any component, or coherent system, is in state 1 is a monotonically non-increasing function of time.

(vi) The operation of the components and system are simultaneous and instantaneous.

(vii) The order of component failure does not affect the state of the system for a given set of failed components.

(viii) The states of the components are independent.

The above assumptions, which are in increasing order of

specialisation, limit the application of the basic systems reliability model. They are expressed as generally as is consistent with the procedure for evaluating systems reliability introduced in the last section. Whilst these assumptions form the norm in reliability work (e.g. Barlow and Proschan (1965), Shooman (1968), Kaufmann (1969), Lomnicki (1973)), they are often fully or partly implicit (especially assumptions (i), (ii), (iii), (vi) and (vii)) and the situation is further confused by the fact that in parts of the theoretical development of reliability certain assumptions are unnecessary or can be treated more generally. For example, the paper by Birnbaum, Esary and Saunders (1966) avoids the element of time, and of the remaining assumptions only makes explicit the assumptions that components and systems only have two states and that the states of the components are independent. For much of their development Barlow and Proschan (1975) avoid the restrictive assumption of independence.

As one would expect from its wide-spread literature and use, the basic systems reliability model corresponding to the above assumptions fits reasonably well many real systems, or at least can be regarded as a first approximation (e.g. Green (1977), Cannon and Jones (1977), Snaith (1979)). Practical reliability engineers find the complexity of reliability theory hard enough even in this restricted form, without the complications of further mathematical models (as the discussion of papers at the First National Reliability Conference indicates; e.g. National Centre of Systems Reliability (1978)). Further, limitations in the availability of data, and in certain cases of appropriate statistical techniques, for even this basic reliability model (e.g. Evans (1971, 1974), Konarski and Evert (1975), Rosenthal (1975), Levine and Vesely (1977), Green (1977),

Shooman and Sinkar (1977), Anderson (1979), Dhillon (1979c), Haas and Batt (1980)), imply that developing more sophisticated models may not be worthwhile. Certainly in my experience of attempting to obtain appropriate data for generalisations to the reliability model I found that even with the assistance and data banks of the National Centre of Systems Reliability I was only rarely able to find appropriate data.

However, the unavailability of data etc. is a 'chicken-and-egg' argument, since it is not unreasonable to suppose that appropriate data will by and large not be collected until there is a purpose for it in terms of a method of analysis based on a generalised reliability model. Indeed, apart from the hand of chance, problems must be defined in appropriate generalised terms before appropriate data can be collected to help solve them (Venton (1977)). (Maruvada, Weise and Chamow (1978) discuss a related 'chicken-and-egg' problem whilst the comments of Evans (1977) on the gulf between the reliability researcher and the reliability practitioner are also relevant).

Whilst the above arguments of the realism and good fit of the basic system reliability model, the complexity of potential generalisations and the unavailability of appropriate data for them, may be taken to preclude the development of a generalised model of reliability, the questions remain as to what would be the effects of liberalising the assumptions, and whether a more general model of reliability could be constructed without such restrictive assumptions. Further, the reliability literature already contains many partial extensions to the basic system reliability model, which are usually in the form of generalisations of one specific aspect applied in the context of a very specific system. Such papers are common in the

IEEE Transactions on Reliability and Microelectronics and Reliability, and specific cases are discussed as appropriate below. However, some more systematic attempts at generalising the basic reliability model are to be found in Green and Bourne (1972), Murchland (1975) and Barlow and Proschan (1975). See also Papazoglou and Gyftopoulos (1977), Virtanen (1977), Singh (1978) and Garribba, Mussio and Naldi (1980). The existence of such generalisations of the basic systems model in the literature, although largely in a piecemeal fashion, does suggest that there is a general feeling amongst reliability workers that the basic model is inadequate to describe many real physical systems, and a more general model is required. (See e.g. Barlow, Fussell and Singpurwalla (1975) and the introduction by the editors to Section 2 of Apostolakis, Garribba and Volta (1980)).

For reasons of brevity, this thesis concentrates on the partial operation extension to assumption (iv), though one can easily broaden the model to generalise the other assumptions. Of course, there is much justification for the generalisation of a single assumption at a time, since it clarifies the effect of that assumption on the methodology and results, it prevents the development of a prohibitively complex model with associated data and inferential problems, and it corresponds to the situation in applications where it is often the case that only one or two standard assumptions are in question at any one time.

There has, in particular, been much recent interest in the extension of assumption (iv) to incorporate multistate systems with ordered states, in an attempt to describe partial or degraded operation. A number of plausible extensions to assumption (iv) have been suggested in the literature. Whilst a few

authors (e.g. Murchland (1975), Caldarola and Wickenhäuser (1977), Papazoglou and Gyftopoulos (1977) and Singh (1978))) work generally with multistate reliability models not restricted to a single direct physical analogue for the set of states, most either decompose the operating stage 1 (or equivalently the failed state 0) into an ordered set of states representing various degrees of partial operation or degradation (e.g. Lloyd and Lipow (1962), Derman (1963), Mine and Kawai (1974a, 1975, 1977), Proctor and Wang (1975), Singh (1976),Maruvada, Weise and Chamow (1978), Thomas, Derbalian and Bischel (1980)), or decompose the failed state 0 into states representing multiple failure modes.

That even the most simple equipment is often subject to a large number of distinct failure modes is well accepted in the engineering literature, as is the necessity to often consider various modes separately due to their differing system implications in terms of repair time, safety, etc.; e.g. Bøe (1974), Pau (1974), Mann, Schafer and Singpurwalla (1974), Hyun (M.Gen)(1975), Banfi, Garribba, Mussio, Naldi and Volta (1976), Dhillon (1976a,b,c,1977c,1978d), Proctor and Singh (1976a,b), McCool (1976), Barbour (1977), Dahiya (1977), Thomas (1977), Gopal, Aggarwal and Gupta (1978a), Legg (1978), Caldarola (1980a). But see also Codier (1968) and Fertig and Murthy (1978). Elsayed and Ziebib (1979) solve the general N-failure-mode Markov model, whilst Yamashiro (1980) extends the solution to general repair time distributions, and Yamashiro (1982) introduces standby units. Another extension using a mixture model is given by Muth (1980b). Mine and Nakagawa (1978) also employ a mixture formulation, whilst Annello (1968) discusses competing risks. Bendell and Samson (1981) employ rank-order distributions for the analysis of diverse failure

modes. Shooman (1968) classifies the failure modes according to the three regions of the bath-tub curve, and Gorg and Kumar (1977) classify them into various minor and major failures. The occurrence of distinct failure modes has been reported in amongst other systems marine equipment (Bøe (1974)), weapon systems (Gower (1975)), nuclear systems (Proctor and Singh (1976a), Dhillon (1976a)), mechanical systems (Martin (1980)), power transmission systems, electrical systems and aerospace equipment (Dhillon (1977c)).

An important special case of the multiple-failure-modes extension to the basic reliability model is the case of opposite failure modes for systems which are required to operate at certain times and idle at others (in violation of assumption (ii)). Codier (1968), Allen and De Oliveira (1977), Fertig and Murthy (1978), and Gopal, Aggarwal and Gupta (1978a) give some justification for grouping diverse failure modes into such opposite categories. There are many real systems for which such switching between the operating and idling states is necessary or desirable e.g. electronic equipment (Elburn and Knight (1975)), electrical distribution networks (Allen and De Oliveira (1977)), protective systems (Choy and Mazumdar (1975), Gibson and Knowles (1977), Kontoleon (1978b)), weapons systems and other emergency equipment (Nakagawa (1978)), nuclear power plants (Apostolakis and Bansal (1977)), computer hardware (e.g. Lewis (1964), Tasun (1977)), inertial navigation and ships power control systems (Kujawski and Rypka (1978)), and avionic equipment (Kern (1978)). See also Weiss (1961), Gaver (1964), Srinivasan (1966), Ueda (1972), Osaki (1972), Röde (1974), Nakagawa (1974, 1977), Kapur and Kapoor (1975, 1978a,b), Nakagawa, Sawa and Suzuki (1976), Sasaki and Hiramatsu (1976), Sasaki and Yanai (1977), Srinivasan and Bhaskar (1979),

Singh, Aggarwal and Kulkami (1979), Sasaski and Yokota (1980), Berg (1981).

The existence of opposite failure modes-failures to operate and failures to idle (or 'open' and 'short' or 'closed' failures, or 'passive' and 'active' failures, or 'fail-to-safe' and 'fail-to-danger') in such real systems is also well known. Green and Bourne (1972), for example, tabulate the proportions of total failures which are of each of the two types for common components such as fixed resistors, capacitors, coils and pneumatic and hydraulic components. Jordan (1978) considers an opposite failure mode model for iso-isolators. The literature discusses the design of systems of components subject to opposite failure modes quite extensively, having recently given particular emphasis to the identification of optimum redundancy; e.g. Moore and Shannon (1956), Gordon (1957), Barlow, Hunter and Proschan (1963), Barlow and Proschan (1965), Lomnicki (1973, 1977), Phillips (1976, 1977, 1980), Proctor and Proctor (1977), Kaufmann, Grouchko and Cruon (1977), Kontoleon (1978a), Nakagawa and Hattori (1980), Ben Dov (1980),Bendell and Humble (1981) and Ansell and Bendell (1982a).

Whilst components subject to opposite failure modes can be considered to be in any one of four states at any t - failing to operate, failing to idle, succeeding to operate, and succeeding to idle - the literature usually combines the two success states into a succeeding to operate or idle as required state, and thus produces a three-state representation. (e.g. Roberts (1964), Barlow and Proschan (1965), Shooman (1968), Lomnicki (1973), Allen and De Oliveira (1977)). However, Mathur and De Sousa (1975) amongst others do obtain a four-state representation, but by instead introducing the possibility

of indeterminate failures, for which it is not known whether they are failures to operate or to idle. See also Tasun (1977) and Dhillon (1977a). In contrast the four-state model of Berg (1981) employs states of succeeding to operate, failing to operate and being operational or failed when undemanded.

Three-state reliability models have appeared quite extensively in the literature since they can be obtained from a number of alternative extensions to the dichotomic reliability model as well as that of opposite failure modes, and in each case represent the simplest such extension. The general three-state Markov model was solved by Biggerstaff and Jackson (1969) in the context of power generation in which the three states considered represented full operation, derated operation and failure, so that the model corresponds to the simplest partial operation extension to the basic reliability model. The paper was subsequently overlooked in much of the three-state literature since the papers of Kontoleon and Kontoleon (1974) and Proctor and Singh (1976a) contain no more than the solutions to the reduced versions of the general three-state Markov model corresponding to the partial operation case with limited repair, and the opposite failure mode case respectively. Proctor and Singh (1975a) apparently independently re-solve the general three-state Markov model, whilst Dhillon (1976a) does not even get as far as deriving the explicit time dependent solutions of the opposite failure mode sub-model (although in the context of complete/ catastrophic failures).

Endrenyi (1970), Endrenyi, Maenhaut and Payre (1973), Grover and Billington (1974) and Allen and De Oliveira (1977) employ reduced forms of the general three-state Markov model

in the context of electrical networks. See also Billington, Allen and De Oliveira (1977). Regulinski (1980) also employs a reduced form of the general three-state Markov model in studying computer networks. See also Chan and Downs (1978) and Das, Hendry and Hong (1980) for reduced forms of the three-state model in the context of imperfect repair. Shenk (1977) considers the opposite failure mode submodel (in the sense that two of the states are not directly connected) but with the partial operation formulation. The two repair time distributions are allowed to be Erlangian or mixtures of exponentials. Kontoleon, Kontoleon and Chrysochoides (1975) analyse throw-away maintenance for modules subject to both partial and catastrophic failures, whilst Tumolillo (1974) has a three-state random stress model. Braff (1977) uses a three-state Markov chain model in which the states are operating, failed and pending failure (which is assumed obeervable) to analyse the relationship between failure rate and technician visitation. Phillips (1979) evaluates the reliability and MTTF of a three-state system in which, apart from full operation, the states correspond to the occurrence of revealed and unrevealed faults. Mine and Kawai (1974b) consider preventative replacement for a three-state unit with a wear-out state. Beichelt and Fischer (1979, 1980) allow for two types of failure; those removable by minimal repairs, and those needing complete replacement. See also Mendenhall and Hader (1958), Cox (1959), Fischer (1977), and Gorg and Kumar (1977).

Dhillon (1977c) discusses the steady state availability of parallel (and series) systems of components subject to two failure modes, whilst Singh and Proctor (1977) and Ksir (1979) consider series systems of two three-state components subject to opposite

failure modes and/or partial failures. Dhillon, Sambhi and Khan (1979) consider the analysis of a parallel network of components subject to opposite failure modes and common-cause failures, whilst Dhillon (1978c) consider a k-out-of-n system of three-state devices also subject to opposite failure modes and common-cause failures. Gupta and Sharma (1979) discuss a k-out-of-n system of three-state units but with states of operating, failed and being installed, whilst Dhillon (1979a) considers a four-unit redundant system with common-cause failures and units subject to opposite failure modes, and Chung (1979) extends this to an n-unit redundant system. Dhillon (1979b) considers a complex system subject to partial failures. Kumar andAggarwal (1978) analyse a two-unit warm standby system with two types of failures, whilst Khalil (1977) and Singh, Kapur and Kapoor (1979) consider a cold standby equivalent. See also Elsayed (1979). Dhillon (1978d) considers a system of n standby components subject to two failure modes. Takami, Inagaki, Sakino and Inoue (1978) and Kumar and Kapoor (1979) discuss the employment of fault detectors with opposite failure modes for series systems. See also Inagaki (1980).

Butler (1979a) discusses importance measures and rankings for three-state components in three-state systems, in which the states correspond to the partial operation formulation. This is also the case considered by Hatoyama (1979) who shows that the calculation of systems reliability can sometimes be reduced to that of a corresponding two-state system, and thus obtains methods of evaluation and bounds for the reliability of three-state systems. He also presents some reliability properties of systems with independent components, and some bounds for systems with associated components.

Dhillon (1977b) provides a limited bibliography on three-state models covering the period 1956 - 1976, whilst Virtanen (1977) gives a partial review of three-state models up to 1975. See also Sankaranarayanan and Usha (1980), Subramanian and Usha (1980), Locks (1980), Lee (1980) and the nominally unrelated models of e.g. Shooman (1968), Subramanian and Natarajan (1980), Allen and Billington (1980).

Whilst the general three-state model and its sub-models can thus have various plausible physical interpretations as extensions to the basic dichotomic reliability model, the opposite failure mode formulation does have the special feature that with it the expression for a systems probability of failure to idle in terms of the component probabilities of failing to idle (and failing to operate if the system is non-coherent) can be obtained as the dual of the expression for the systems probability of failing to operate in terms of the components probabilities of failing to operate (and to idle). See e.g. Lomnicki (1973). Thus apart from the study of specific systems corresponding to specific opposite failure mode models which were reviewed in the previous paragraphs, and apart from the literature on optimum redundancy for such components (reviewed previously), a number of general methodological papers appear in the literature on the reliability analysis of systems of three-state components subject to failures to operate and to idle; e.g. Proctor and Singh (1975b), Singh and Proctor (1976), Gupta and Sharma (1978a), Gopal, Aggarwal and Gupta (1978b), Nakagawa and Hattori (1980).

Just as it is possible to provide various physical interpretations of three-state reliability models, it is equally possible to do so for models with 4, 5, 6 or 7 states, even though the limited

models that have appeared in the literature have been generally presented in terms of specific physical systems. Proctor and Singh (1976b) and Dhillon (1976b) consider a four-state Markov model in the context of nuclear power systems and crucial industrial complexes in which there is a catastrophic failure state as well as the three states of the simplest partial operation model. Regulinski (1977) formulates a four-state Markov model for man-machine interaction, whilst Regulinski (1980) extends his three-state Markov model for computer networks to a four-state one. Billington, Medicherla and Sachdev (1978) consider four-and eight-state Markov models in the context of common-cause outages in multiple circuit transmission lines. Seitz (1980) discusses a four-state model with states generated by sequential operation and idling, and failure to operate. See also Mathur and De Sousa (1975) for an opposite failure modes formulation of a four-state model (described above), as well as the nominally unrelated model of Subramanian and Ravichandran (1980).

Kumar and Jain (1977) consider a two unit warm standby system of five-state units. The five-state model of Gopalan and Dharmadhikoi (1980) is nominally unrelated but in practice of interest. Maruvada, Weise and Chamow (1978) discuss, but do not solve, a five-state Markov model arising from the consideration of the derated states of a fossil-fired generating unit in order to improve the accuracy of system planning studies. They also extend this to a six-state formulation. Dhillon (1976c) analyses a six-state Markov model appropriate for electrical systems, in which as well as the three states of the failure to operate/failure to idle model there are states representing intermittent failures, maladjustment and drift-

out-of-tolerance. Dhillon (1977c) discusses what is described as a four-state Markov model although it is actually seven-state; there being one operating state and three failure modes for each of which repair facilities may or may not be available. The model is appropriate for power transmission systems, electrical systems and aerospace equipment. See also Gokcek, Bazovsky and Crellin (1979) and Pau (1979).

Apart from the case of opposite failure modes, the other violation of assumption (iv) that has received much attention in the literature is the formulation of the reliability model in terms of an ordered set of states representing various degrees of partial operation or degradation of operating function, and it is this that forms the main subject of this thesis. The published models for this phenomenon broadly divide into two categories. The first is those in which there is a finite set of states denoting various levels of partial operation or degradation; e.g. the 3, 5 and 14 state models of Biggerstaff and Jackson (1969), Maruvada, Weise and Chamow (1978) and Virtanen (1977), and the m-state models of Howard (1960), Lloyd and Lipow (1962), Flehinger (1962), Derman (1963), Hirsch, Meisner and Boll (1968), Mine and Kawai (1974a, 1975), Proctor and Wang (1975), Singh (1976), Ramanarayanan (1978), Barlow and Wu (1978), El-Neweihi, Proschan and Sethuraman (1978), Ross (1979), Butler (1979a) and Block and Savitz (1981b). The second category is of models which allow for the levels of partial operation or degradation to form a continuous scale; e.g. Mercer and Smith (1959), Mercer (1961), Derman and Sacks (1960), Morey (1966), Postelnicu (1970), A-Hameed and Proschan (1973, 1975), Esary, Marshall and Proschan (1973), Esary and Marshall (1974), Nakagawa and Osaki (1974a,b),

H.M. Taylor (1975), Abdel-Hameed (1975, 1977), Feldman (1976, 1977), Nakagawa (1976a,b), Aggarwal (1976), Cinlar (1977), Block (1977), Klaasen and Van Peppen (1977a,b), Dorgan and Emer (1978), Christer (1978), Bosch (1979), Zijlstra (1980) and Ansell, Bendell and Humble (1980a,b). Included in the later group are the additive-damage shock-models (see e.g. Buckland (1964) and Barlow and Proschan (1976) for reviews), whilst relevant to the former group is the increasing interest generally in multi-valued and fuzzy logic in both theoretical and practical terms, especially in the context of computer electronics. See e.g. the Proceedings of the 9th International Symposium on Multiple-Valued Logic, IEEE, 1979 . The extension of multi-valued logic to infinite-valued logic may be of interest for the second group; Lakoff (1975).

The use of additive or cumulative damage models to describe failures due to metal fatigue is long established in the literature (e.g. Buckland (1964)), and the physical justification for the existence of partially operating or degraded components and systems in the context of electrical power systems, communication systems and computer systems is also widely reported in the reliability literature. See e.g. Cavers (1975), Yost and Hall (1976), Tillman, Lie and Hwang (1976), Horwitz (1976), Dartois (1977), Livini and Bar-Ness (1978), Maruvada, Weise and Chamow (1978), Siu and Chan (1978), Moranda (1979) and Chou and Abraham (1980). Amesz, Garribba and Volta (1977) employ it in the context of nuclear power plants, whilst Virtanen (1977) adds production systems (see also Hay, Godbout and Brais (1979)), and Barlow (1978) and Smith (1980) add transportation networks and water systems to the list. Dorgan and Emer (1978) purpose-build a continuous degradation model for industrial exhaust systems, whilst Bosch (1979) and Zijlstra (1980) develop continuous models for the

degradation of electronic devices and lights respectively.

For practical applications, however, the finite number of states formulation has often been employed, often with a small number of states, as an improvement over the dichotomic reliability model. In some formulations (e.g. Hatoyama (1979), Butler (1979a)) the same number of states are available for the system and all components, but in others this is not the case, since e.g. the dichotomic failure of a component may cause the system to operate at reduced efficiency (e.g. Hirsch, Meisner and Boll (1968), Simon (1969, 1970, 1972), Bøe (1974), Singh (1976), Henley and Polk (1977), Nieuwhof (1978), Gupta and Sharma (1978b), Maruvada, Weise and Chamow (1978), Singh, Aggarwal and Kulkami (1979)). For system calculations in such cases, of course, one can just work with the maximum number of states amongst all the components and the system (e.g. Butler (1979b)). In certain cases, however, such as where components characteristics are subject to continuous drift whilst the system is considered to be at one of two (or more) levels, or where the system response can be measured continuously whilst component operation cannot, the system and/or some components may have a continuous scale of partial operation, whilst other components and/or the system can only take a finite number of levels. See e.g. Shooman (1968), Chapter 7 and Weiss and French (1975).

Thus, the assumption that components and systems can only be in one of two states at any point of time is not tenable in the light of the vast literature and physical evidence to the contrary. In the remainder of this thesis we develop a general partial operation model, and investigate associated problems in the enumeration of such systems, and in their optimal tuning and retuning.

CHAPTER 2

ENUMERATION OF MULTISTATE COHERENT SYSTEMS

2.1 Types of Multistate Coherent Systems

There is a recent and increasing interest in multilevel systems in which components and systems can be at any one of a finite number of strictly ordered levels ℓ $(\ell \geqslant 2)$, e.g. Kontoleon and Kontoleon (1974), Virtanen (1977), Barlow (1978), El-Neweihi, Proschan and Sethuraman (1978),Barlow and Wu (1978), Ross (1979), Butler (1979 a,b), Hatoyama (1979), Griffith (1980), Block and Savitz (1981b). In contrast, however, Hirsch, Meisner and Boll (1968) and Simon (1969, 1970, 1972) instead only allow systems to be at one of ℓ levels, whilst components remain dichotomic. Hochberg (1973), Fardis and Cornell (1981) and Hudson and Kapur (1982) allow the number of levels to vary between components and the system, and Caldarola (1980a) treats the ℓ component and system levels as unordered.

Since with ℓ -(ordered)- state components and systems there are even more possible systems than in the dichotomic case (ℓ^{ℓ^n}), there is great interest in finding a restricted subclass of systems of practical relevance. The main approach has been to generalise the class of coherent or monotonic systems from two-state systems to ℓ -state systems. However, the definition of a generalised coherent (or monotone) system varies between authors with almost no two authors the same, as Butler (1979b) points out. Hirsch, Meisner and Boll (1968), Postelnicu (1970), Simon (1969, 1970, 1972), Hochberg (1973) and Fardis and Cornell (1981) generalise the Birnbaum, Esary and Saunders (1961) definition (although in somewhat different contexts), El-Neweihi, Proschan and Sethuraman (1978), Hatoyama (1979),

Butler (1979a,b), Griffith (1980), Caldarola (1980a) and Hudson and Kapur (1982) generalise the Barlow and Proschan (1975) definition (although each differently) whilst Barlow (1977, 1978) and Barlow and Wu (1978), take a more restricted approach based upon the dichotomic set theoretic definition of coherency (and possibly applicable to communication and electrical power systems and water production and transportation system networks). The definition of coherency employed by Caldarola (1980b) is exceptional in that it is not based on monotonicity, but on the existence of a unique complete and irredundant base.

In this Chapter we consider the enumeration problem for generalised coherent systems, and define two generalisations of the original Birnbaum, Esary and Saunders (1961) definition which are of likely practical interest.

Since these two definitions do not include the restriction of relevancy (see Section 1.3), it follows that the least restrictive of these definitions (coherent systems in the wide sense) contains the generalisation of Butler (1979a,b)(to which it is identical apart from the requirement of relevancy), and consequently provides an upper bound for the number of such systems. We investigate the closeness of the number of systems under Butler's and our definitions below. Our wide-sense definition is identical to the monotone functions defined by Hochberg (1973). The coherent structures as defined by Postelnicu (1970) are in the ℓ -level case contained within our least restrictive class (coherent systems in the wide sense), but in turn contain our most restrictive class (coherent systems in the narrow sense). Thus, the numbers of these structures are between those based on our two classifications. The coherent

classes defined by El-Neweihi, Proschan and Sethuraman (1978) and Griffith (1980) are contained within our narrow-sense class, and consequently the number of such structures is smaller than the number of narrow sense coherent structures. El-Neweihi, Proschan and Sethuraman also show that their class contains the coherent systems of Barlow (1977, 1978) and Barlow and Wu (1978), so that the number of their structures exceeds the corresponding number due to these other authors.

In fact, the multi-state structures as defined by Barlow and his co-author correspond 1 to 1 (for fixed ℓ) to the dichotomic coherent structures as defined by Barlow and Proschan (1975). It follows that with this definition there is a unique ℓ-level coherent structure corresponding to each event network (see Section 1.3) and the number of Barlow (1977, 1978) and Barlow and Wu (1978)'s coherent structures is identical to the number of dichotomic coherent structures as defined by Barlow and Proschan (1975) and discussed in Section 1.3.

Griffith (1980) compares two sequentially weaker relevancy requirements with that of El-Neweihi, Proschan and Sethuraman (1978). Thus his <u>weakly coherent</u> systems contain his <u>coherent</u> systems which in turn contain his <u>strongly coherent</u> systems, which are themselves the coherent systems of El-Neweihi, Proschan and Sethuraman (1978). It follows that the number of weakly coherent structures is between the number of his own coherent structures and the number of our narrow sense coherent structures (to which they are identical apart from the relevancy requirement), whilst the number of his coherent structures in turn exceeds the numbers of those of El-Neweihi, Proschan and Sethuraman (1978).

Since the definition of Caldarola (1980a) includes relevancy

although it employs non-ordered logic, the class contains that of Butler (1979a, b) and in consequence there are more such structures. Finally, as the class defined independently by Hochberg (1973) and Fardis and Cornell (1981) allows for components and the system to have differing numbers of states, it is equivalent to our wide sense definition when the number of states are equal, and to the definition of Hirsch, Meisner and Boll (1968) and Simon (1969, 1970, 1972) for the case where the components can all only take two levels. In general, the number of such systems will be between the number of wide sense coherent systems corresponding to the minimum number of states of the components and the system, and the number of wide sense coherent systems corresponding to the maximum number of states.

The work of this section is largely based upon the joint paper in Appendix A1 which appeared in the Proceedings of the Royal Society of Edinburgh Series A (Mathematics) Vol. 89, 1981, and upon the paper which the author read to the Dundee Mathematical Association in February 1980. In the joint work, the recursive bounds and the structure of the results are due to the present author, whilst the proof of the theorems and the numerical evaluations are mainly due to J. Ansell and S. Humble, with the final (clarified) version of Theorem 2.1 being due to an anonymous referee.

2.2 Terminology and Notation

We suppose that each component and the system can be at any one of ℓ ordered levels ($\ell \geq 2$) but that the other assumptions of the basic reliability model apply. The state of all components of the system can be described by the state vector

$$\underline{s} = (s_1, s_2, \ldots, s_n)$$

where s_α, the level of the α^{th} component, may be any one of the levels

$$\lambda_1 < \lambda_2 < \ldots < \lambda_\ell \ .$$

For convenience we define (analogous to the dichotomic case)

$$\lambda_1 \equiv 0 \ , \ \lambda_\ell \equiv 1 \ ,$$

and for illustrative purposes, as well as later convenience.

(Chapter 4) we may take

$$\lambda_i = \frac{i-1}{\ell-1} \ , \ i = 1, 2, \ldots, \ell \qquad (2.1)$$

to obtain equally-spaced levels. Thus if $\ell = 3$ we have levels 0, ½, 1, whilst if $\ell = 4$ we have 0, ⅓, ⅔, 1 and if $\ell = 5$ we have 0, ¼, ½, ¾, 1.

The resulting state of the system can be described by the structure function $f(\underline{s})$ of the vector $\underline{s}$, with range $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. Introducing the notation

$$\underline{1} = (1, 1, \ldots, 1)$$

$$\underline{0} = (0, 0, \ldots, 0)$$

and $\underline{x} \geqslant \underline{y}$ if $x_\alpha \geqslant y_\alpha$ for all $\alpha = 1, 2, \ldots, n$, then by analogy with the dichotomic case, we define a <u>semi-coherent</u> system by

$$f(\underline{x}) \geqslant f(\underline{y}) \quad \text{for all } \underline{x} \geqslant \underline{y} \ . \qquad (2.2)$$

For dichotomic reliability a coherent system is defined by (2.2) together with

$$f(\underline{1}) = 1, \ f(\underline{0}) = 0 \qquad (2.3)$$

For the multi-level situation we shall say that (2.2) and (2.3) define a <u>coherent system in the wide sense</u> to distinguish it from a <u>coherent system in the narrow sense</u> which in addition to (2.2) we define to have the more restrictve requirement that if

$$\underline{\lambda_i} = (\lambda_i, \lambda_i, \ldots, \lambda_i)$$

then

$$f(\underline{x}_i) = x_i \text{ for all } i = 1, 2, \ldots, \ell \qquad (2.4)$$

In the dichotomic case, i.e. $\ell = 2$, these wide sense and narrow sense definitions are identical.

We define the state vector

$$\underline{x}_i = (x_i, x_i, \ldots, x_i)$$

as the i^{th} pivot of the system ($i = 1, 2, \ldots, \ell$). Between the i^{th} and j^{th} pivots ($j > i$) there exists a number of state vectors composed only of the levels $x_i, \ldots, x_j$. We say that the set of state vectors composed only of the elements $x_i, \ldots, x_j$ constitutes the $(i,j)^{th}$ lozenge of the system. Finally, for a system of n components we define the set of state vectors

$$\{(0, \ldots, 0, x_i) \mid i = 1, 2, \ldots, \ell\} \text{ as the } (0,n)^{th} \text{ chain}$$

of the system. Illustrations are shown in Figure 2.1.

2.3 Systems Enumeration

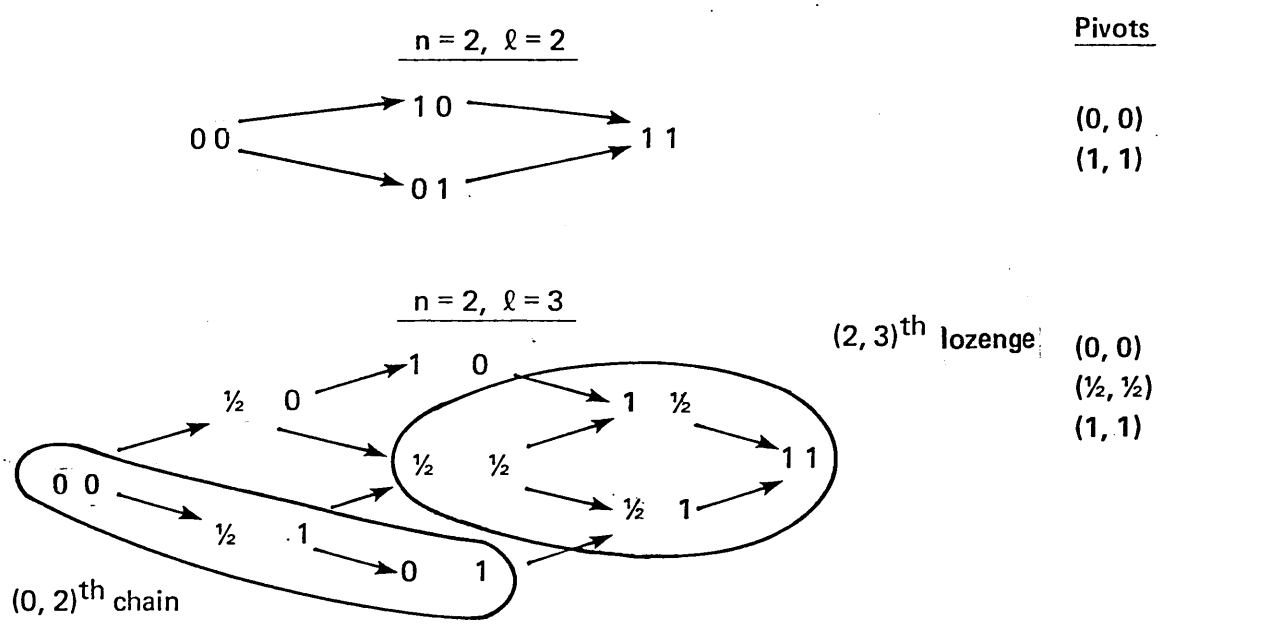
In Table 2.1 we show the numbers of coherent structure functions ${}^{n}N_{\ell}$, ${}^{n}W_{\ell}$ in the narrow and wide sense respectively corresponding to some low-n and low-ℓ values only, since even for these the results in the table represent a significant computer effort. We also show the numbers of coherent functions ${}^{n}B_{\ell}$, ${}^{n}G_{\ell}$ corresponding to the definition used by Butler (1979a,b) and the weakly coherent definition of Griffith (1980). These are obtained by generalising Lomnicki (1977) to give for $n > 1$,

$$\begin{aligned} {}^{n}B_{\ell} &= {}^{n}W_{\ell} - \binom{n}{n-1}{}^{n-1}B_{\ell} - \ldots \binom{n}{2}{}^{2}B_{\ell} - \binom{n}{1}{}^{1}B_{\ell} \\ {}^{n}G_{\ell} &= {}^{n}N_{\ell} - \binom{n}{n-1}{}^{n-1}G_{\ell} - \ldots \binom{n}{2}{}^{2}G_{\ell} - \binom{n}{1}{}^{1}G_{\ell} \end{aligned} \qquad (2.5)$$

The table indicates that the requirement of relevancy only has a significant effect on the number of systems for $\ell = 2$.

It is apparent from the table that the number of coherent functions in the narrow sense which can be constructed from two

Figure 2.1
Illustrations of partial orderings of component state vectors.



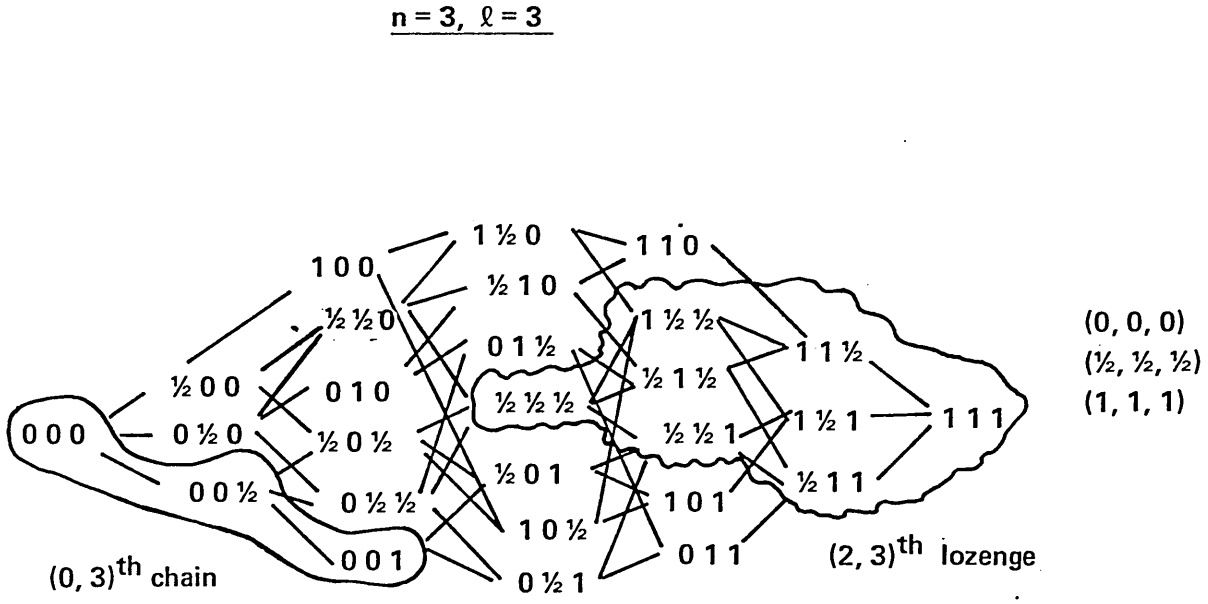


TABLE 2.1

Number of coherent systems in the narrow sense ($^{n}N_{\ell}$), wide sense ($^{n}W_{\ell}$), and as defined by Butler (1979a,b) ($^{n}B_{\ell}$) and Griffith (1980)($^{n}G_{\ell}$)

	$^{n}N_{\ell}$			$^{n}G_{\ell}$			$^{n}W_{\ell}$			$^{n}B_{\ell}$		
ℓ \ n	1	2	3	1	2	3	1	2	3	1	2	3
2	1	4	18	1	2	9	1	4	18	1	2	9
3	1	64	151,236	1	62	151,047	3	136	738,122	3	130	737,723
4	1	4,096	-	1	4,094	-	10	18,676	-	10	18,656	-
5	1	1,048,576	-	1	1,048,574	-	35	15,374,304	-	35	15,374,234	-

components with ℓ levels is given by

$$^{2}N_{\ell} = 2^{\ell(\ell-1)} \quad . \tag{2.6}$$

However, no such simple relationship appears available for more than two components, nor for wide sense coherent structures (nor Butler's, nor Griffith's definitions) except for $n = 1$, for which

$$^{1}W_{\ell} = {}^{1}B_{\ell} = \binom{2\ell-3}{\ell-2} . \tag{2.7}$$

(This will be proved below). Thus, as in the dichotomic case, the best one may do is obtain bounds on the number of coherent systems. First we generalise a theorem due to Birnbaum, Esary and Saunders (1961).

2.4 Bounds on the Number of Semi-Coherent Systems

Theorem 2.1:

For each positive integer n, let S_n denote the set of all semi-coherent functions of order n, and

$$G = \left\{(g_1,\ldots,g_\ell) : g_j \in S_n, g_1 \leq g_2 \leq \ldots \leq g_\ell, j = 1,\ldots,\ell\right\} . \tag{2.8}$$

Then there exists a bijection from G onto S_{n+1}.

Proof:

Let $(g_1,\ldots,g_\ell) \in G$. Define f and H by

$$f(\underline{s}, z_j) = f(s_1, s_2, \ldots, s_n, z_j) = g_j(\underline{s}), \; j = 1,2,\ldots,\ell \tag{2.9}$$

$$H(g_1, g_2, \ldots) = f \in S_{n+1}$$

Note that, since $g_j, j = 1,2,\ldots,\ell$ are semi-coherent and non-decreasing in j, $f \in S_{n+1}$ for $(\underline{s}, z_i) \leq (\underline{t}, z_j)$ implies

$$f(\underline{s}, z_i) \leq g_i(\underline{t}) \leq g_j(\underline{t}) = f(\underline{t}, z_j) .$$

That H is surjective follows from the observation that if

$f \in S_{n+1}$ and the functions g_j defined by

$$g_j(\underline{s}) = f(\underline{s}, z_j) \;, \quad j = 1, 2, \ldots, \ell$$

then $(g_1, g_2, \ldots, g_\ell) \in G$ and $H(g_1, g_2, \ldots, g_\ell) = f$.

It is clear H is injective.

To use Theorem 2.1 to obtain an upper bound for the number of semi-coherent systems, we first note that the number of solutions in positive integers of

$$1 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_r \leqslant m$$

is

$$\sum_{\mu_1=1}^{m} \sum_{\mu_2=1}^{\mu_1} \cdots \sum_{\mu_r=1}^{\mu_{r-1}} 1 = \binom{m+r-1}{r} \;. \tag{2.10}$$

Hence it follows from the theorem that if ${}^{n}S_{\ell}$ denotes the number of possible semi-coherent functions of n components and ℓ levels and <u>if</u> these functions were strictly ordered then by considering the number of ways these functions may be identified with g_i, $i = 1, 2, \ldots, \ell$, the number of possible semi-coherent functions of $(n+1)$ components and ℓ levels would be given by (2.10) with r and m replaced by ℓ and ${}^{n}S_{\ell}$ respectively; i.e. the number of such functions would be

$$\binom{{}^{n}S_{\ell} + \ell - 1}{\ell} \;. \tag{2.11}$$

However, the ${}^{n}S_{\ell}$ functions cannot be strictly ordered. In fact, the order in S_n as well as in the ℓ^n states of the system is partial not total. Nevertheless if $\leqslant$ is a partial ordering on a set P then there exists a total ordering $\leqslant_0$ on P such that

$$\leqslant \; \subset \; \leqslant_0 \;,$$

i.e. there exists an order preserving extension. Thus

$$\#\{g_1 \leq g_2 \leq \cdots \leq g_\ell\} \leq \#\{g_1 \leq_0 g_2 \leq_0 \cdots \leq_0 g_\ell\} = \binom{{}^{n}S_\ell + \ell - 1}{\ell} \tag{2.12}$$

since $g_1 \leq g_2 \leq \cdots \leq g_\ell$ implies $g_1 \leq_0 g_2 \leq_0 \cdots \leq_0 g_\ell$ and also every semi-coherent function whose domain is a totally ordered set $(P, \leq_0)$ is clearly also a semi-coherent function on any restriction $(P, \leq) \subset (P, \leq_0)$. Thus as the ${}^{n}S_\ell$ functions are not strictly ordered it follows that (2.11) represents an upper bound for the number of semi-coherent functions with $(n+1)$ components. Of course, we still have an upper bound even if we replace ${}^{n}S_\ell$ in (2.11) by the upper bound ${}^{n}U_\ell$. In this way we obtain a recursive upper bound for the number of semi-coherent functions of $(n+1)$ components and ℓ levels which is of the form

$${}^{n+1}U_\ell = \binom{{}^{n}U_\ell + \ell - 1}{\ell} \quad . \tag{2.13}$$

The proof of (2.7) follows immediately from (2.10) with $r = \ell - 2$ and $m = \ell$.

2.5 The Special Case of $\ell = 2$

For the case when $\ell = 2$, i.e. when components and systems can be only in one of two states (operational or failed), (2.13) becomes

$${}^{n+1}U_2 = {}^{n}U_2 \left({}^{n}U_2 + 1\right)/2 \quad , \tag{2.14}$$

which allows us to calculate an upper bound for the number of semi-coherent functions for $n+1$ components provided we are given an upper bound (or the actual value) for the number of such functions for n components.

It is of interest that starting with the actual value of 2,414,682,040,996 in Table 1.1 for $n = 7$, (2.14) provides upper bounds which are actually sharper than those obtainable from the best

published bound to date (1.12) for $7 < n < 15$.

See the numerical results of Table 2.2. For $n \geq 15$ Hansel's bound (1.12) is somewhat better than the bound obtained from (2.14). However, its recursive nature means that it can be used in conjunction with Hansel's bound to yield a systematic improvement to it for an even number of components. That is, if for n odd we take

$$ {}^{n}U_2 = 3^{M_n} $$

from (1.12), then we obtain from (2.14) that

$$ {}^{n+1}U_2 = 3^{M_n}\left(3^{M_n}+1\right)/2 \qquad (2.15) $$

which is less than Hansel's value of $3^{M_{n+1}}$ for all $n > 0$.

2.6 Bounds in the General Case

Whilst for $\ell = 2$ there are for any n only two functions which are semi-coherent but not coherent, for $\ell > 2$ the number of functions which are semi-coherent but not wide sense coherent (${}^{n}X_\ell$) rises rapidly. Thus in order to derive from (2.13) a useful upper bound on the number of coherent systems in the wide sense we must evaluate at least a lower bound for ${}^{n}X_\ell$. Such a bound can be obtained by supposing that the ℓ^n states of the system (in terms of the levels of its components) were strictly ordered, and again making use of (2.10). If a function is semi-coherent but not wide sense coherent it can at most take the $(\ell-1)$ values $1,\ldots,\ell-1$ or $2,\ldots,\ell$. Thus taking $r = \ell^n$, $m = \ell-1$ (say eliminating ℓ) and $r = \ell^n - 1$, $m = \ell-1$ respectively (say eliminating 1 and fixing $x_{\ell^n} = \ell$ to avoid double counting), one obtains as a lower bound for ${}^{n}X_\ell$,

$$ {}^{n}L_\ell = \binom{\ell^n+\ell-2}{\ell^n} + \binom{\ell^n+\ell-3}{\ell^n-1} . \qquad (2.16) $$

It follows that this is a lower bound since every semi-coherent

TABLE 2.2

Comparison of Hansel's and recursive bounds (Recursive bound based on Ω_7 from Table 1.1)

	$\log_{10} {}^n U_2$	
n	Hansel's bound (1.12)	Recursive bound (2.14)
8	33.39845	24.46468
9	60.11722	48.62831
10	120.23445	96.95558
11	220.42983	193.61012
12	440.85962	386.91919
13	818.73926	773.53735
14	1637.47876	1546.77368
15	3070.27222	3093.24634

function whose domain is a totally ordered set $(P, \leqslant_o)$ is, clearly, also a semi-coherent function on any restriction $(P, \leqslant) \subset (P, \leqslant_o)$. Thus the sharper upper bound for the number of wide-sense coherent structures ${}^{n}W_{\ell}$ is given by

$$ {}^{n}W_{\ell} \leqslant {}^{n}U_{\ell} - {}^{n}L_{\ell} \, . \tag{2.17} $$

Since ${}^{n}W_{\ell} \geqslant {}^{n}N_{\ell}$ and ${}^{n}W_{\ell} \geqslant {}^{n}B_{\ell}$, (2.17) is also an upper bound for the number of coherent systems in the narrow sense, ${}^{n}N_{\ell}$, and the number of coherent systems as defined by Butler (1979a,b), ${}^{n}B_{\ell}$ as well as by Postelnicu (1970), Hirsch, Meisner and Boll (1968), Simon (1969, 1970, 1972), El-Neweihi, Proschan and Sethuraman (1978), Barlow (1977, 1978) and Barlow and Wu (1978), and Griffith (1980).

Recursive lower bounds for ${}^{n}N_{\ell}$ and ${}^{n}W_{\ell}$ can also be obtained. The ℓ-level n-component configuration contains ℓ pivots, and defined on the $(\ell-1, \ell)^{th}$ lozenge there are ${}^{n}N_{2}$ possible narrow sense coherent structures. Further, the number of coherent structures allowable within this lozenge is not reduced by the particular structure existing in the $(1, \ell-1)^{th}$ lozenge on which can be defined ${}^{n}N_{\ell-1}$ possible structures. Consequently, we obtain a lower bound ${}^{n}T_{\ell}$ on ${}^{n}N_{\ell}$ by assuming that because of the coherency constraints corresponding to each state vector outside these two lozenges there is only one possible level of the system. Thus

$$ {}^{n}N_{\ell} > {}^{n}N_{\ell-1}{}^{n}N_{2} = {}^{n}T_{\ell} \, . \tag{2.18} $$

It follows that this is also a lower bound for the number of coherent systems based upon the definition of Postelnicu (1970).

A lower bound on the number of coherent structures in the wide sense can be obtained by taking a recursion over n.

Introducing a new component into a system of $(n-1)$ components corresponds to adding an entry of 0 to the previous state vectors of the system and adding further states. These new states include the $(0,n)^{th}$ chain. Following an argument akin to those above, defined on this chain there are

$$\binom{\ell+(\ell-1)-1}{\ell-1} = \binom{2\ell-2}{\ell-1}$$

possible structures unrestricted by the particular structure in the system of $(n-1)$ components. Thus assigning only one possible level to each remaining state vector, a lower bound for the number of coherent systems in the wide sense is obtained as

$$^{n}W_{\ell} > \binom{2\ell-2}{\ell-1} {}^{n-1}R_{\ell} = {}^{n}R_{\ell} \qquad (2.19)$$

where ${}^{n-1}R_{\ell}$ is a lower bound for ${}^{n-1}W_{\ell}$. Again (2.19) is also a lower bound for ${}^{n}B_{\ell}$.

Numerical illustrations of the bounds are given in Table 2.3 for some low n and ℓ values. It is seen that ${}^{n}R_{\ell}$ is sometimes below ${}^{n}T_{\ell}$ (for $n = 5$), and in such cases it is of course preferable to use ${}^{n}T_{\ell}$ as a lower bound on ${}^{n}W_{\ell}$.

2.7 Relationship of Event Networks to Structure Functions

We remarked in Section 1.3 on the equivalence of the structure function and logical event network for a dichotomic system, although the later may be hard to draw. For components which have $\ell > 2$ possible levels this equivalence no longer generally holds (although as previously noted using Barlow and Wu (1978)'s restricted definition of coherency the equivalence between the event network and structure function does remain). However, it may well be of interest to determine how many possible structure functions correspond to a

TABLE 2.3

Bounds on ${}^{n}S_{\ell}$, ${}^{n}X_{\ell}$, ${}^{n}N_{\ell}$ and ${}^{n}W_{\ell}$.

	$\log_{10}({}^{n}U_{\ell})$			
ℓ \ n	2	3	4	5
3	2.34242	6.25502	17.98690	53.18255
4	4.86814	18.09240	70.98940	282.57738
5	8.45674	40.20450	198.94331	992.63739

	$\log_{10}({}^{n}L_{\ell})$			
ℓ \ n	2	3	4	5
3	1.27875	1.74036	2.21219	2.68753
4	2.46090	3.62583	4.81987	6.02145
5	3.79246	4.82898	7.91364	10.00805

	$\log_{10}({}^{n}T_{\ell})$		
ℓ \ n	3	4	5
3	-	4.44022	7.75922
4	6.43492	6.66033	11.76388
5	7.69020	8.88043	15.51845

	$\log_{10}({}^{n}R_{\ell})$		
ℓ \ n	3	4	5
3	-	6.64628	7.42443
4	7.16916	7.94929	8.25032
5	9.31529	10.87700	11.72209

${}^{n}U_{\ell}$ calculated from (2.13) using ${}^{1}U_{\ell} = \binom{2\ell-1}{\ell}\,{}^{n}L_{\ell}$ and ${}^{n}T_{\ell}$ calculated from (2.16) and (2.18) respectively. ${}^{n}R_{\ell}$ calculated from (2.19) using values of ${}^{2}W_{\ell}$ and ${}^{3}W_{3}$ given in Table 2.1

single event network, because for example, a system may have originally been designed in terms of such a network. In fact, the (dichotomic) event network places dichotomic constraints on the structure function. For example, if A and B are two components in parallel (in the sense of an event network or of dichotomic reliability), the structure function $f(s_1, s_2)$ is such that

$$f(0,0) = 0, \quad f(0,1) = f(1,0) = f(1,1) = 1.$$

It follows that in general the event network reduces the number of states of the system to which levels have to be assigned from ℓ^n to $(\ell^n - 2^n)$. Corresponding to a single event network therefore there are $\ell^{(\ell^n - 2^n)}$ possible structure functions, and there are

$$\ell^{\ell^n} - \ell^{(\ell^n - 2^n)} 2^{2^n} \tag{2.20}$$

structure functions which do not correspond to event networks or systems defined in terms of the levels 0 and 1. Thus there are

$$\ell^{\ell^n} - \binom{\ell}{2} \ell^{(\ell^n - 2^n)} 2^{2^n} \tag{2.21}$$

structure functions which cannot be immediately deduced from event networks and two-level systems.

The determination of the number of coherent functions in the narrow or wide sense out of the $\ell^{(\ell^n - 2^n)}$ possible functions corresponding to a single coherent event network is again in general a difficult unsolved problem. However, for pure series or parallel event networks a relationship exists which may form a lower bound for the number of coherent structures in the narrow sense corresponding to any coherent event network. Unfortunately, no similar bound appears to exist for coherent structures in the wide sense.

Theorem 2.2:

The number of coherent structures in the narrow sense for a series or a parallel network with n components and ℓ levels is equal to the number of coherent structures in the narrow sense which can be constructed from n components and $\ell-1$ levels.

Proof:

Let ${}^{n}P_{\ell}$ be the set of narrow sense coherent functions for a parallel system with n components and ℓ levels. Hence if

$$\phi \in {}^{n}P_{\ell} \qquad \text{then}$$

$$\phi(s,s,\ldots,s) = s \qquad \text{for all } s = \lambda_1,\ldots,\lambda_\ell$$

$$\phi(\underline{x}) = 1 \qquad \text{if } x_\alpha = 1 \text{ for any } \alpha$$

$$\phi(\underline{x}) \geq \phi(\underline{y}) \qquad \text{if } \underline{x} > \underline{y}$$

and $$\phi(\underline{1}) = 1, \quad \phi(\underline{0}) = 0.$$

It also follows that if $\phi(\underline{x}) = 1$ then $x_\alpha = 1$ for some α, hence if we consider only the $(\ell-1)$ possible levels

$$0 = \lambda_1 < \lambda_2 < \cdots < \lambda_{\ell-1}$$

and ignore $\lambda_\ell = 1$, then $\phi(\underline{x}) \neq 1$.

Let ${}^{n}M_{\ell-1}$ be the set of functions which are narrow sense coherent for n components and $(\ell-1)$ levels, where these $(\ell-1)$ levels are denoted by

$$0 = \mu_1 < \mu_2 < \cdots < \mu_{\ell-1} = 1.$$

Then by introducing the mapping

$$g : (\lambda_1, \lambda_2, \ldots, \lambda_{\ell-1}) \rightarrow (\mu_1, \mu_2, \ldots, \mu_{\ell-1})$$

it is simple to show that for each $\phi \in {}^{n}P_{\ell}$ there exists one and only one function $\psi \in {}^{n}M_{\ell-1}$ and conversely. For the proof in the case of a series narrow sense coherent system we discount the level $\lambda_1 = 0$ for which the series system must fail.

It follows from this theorem therefore that if one can

evaluate the number of narrow sense coherent functions for n components and $(\ell-1)$ levels, or place a bound on this number, one immediately has the number of narrow sense coherent functions corresponding to a series or a parallel system of n components and ℓ levels, or has a bound for this number. (Thus, for example, from Table 2.1 we know that for 3 components there are 18 narrow sense coherent three-level systems, and 151,236 four-level systems, corresponding to a series event network). Moreover, the reduction in the number of levels one must consider for a series or parallel system, from ℓ to $(\ell-1)$, is unique to these event networks. Thus, it appears that the number of narrow sense coherent functions associated with a pure series or a pure parallel network is the minimum number of such functions associated with any type of coherent network of the same number of components and levels. Hence by this argument we might place a lower bound on the number of narrow sense coherent functions associated with any coherent event network. (Note, however, that the more restrictive definitions of series and parallel structures used by Barlow and Wu (1978), El-Neweihi, Proschan and Sethuraman (1978), Hatoyama (1979) and Hudson and Kapur (1982) differ from those used here and themselves contain elements of coherency. See Chapter 4.)

CHAPTER 3

THREE AND FIVE STATE MODELS

3.1 Introduction

Section 1.4 indicated that there exists an extensive literature on models with small numbers of states. A joint paper by the author to some extent extending and unifying part of this diverse literature was published within the period of registration for Ph.D, and this is briefly reviewed in the current Chapter. This paper in the IEEE Transactions on Reliability Vol. R-29 (1980) together with its supplement, NAPS document No. 03582-B (Microfiche Publications, New York), appears as Appendix item A2. The work was joint with my then colleagues J. Ansell and S. Humble and then technician at Sheffield City Polytechnic, C.S. Mudhar. The formulation and relationships to the previous literature is due to this author.

For the general three-state Markov model in which every transition between states is feasible (see Figure 3.1), the paper obtains the transitory probabilities of being in the various states. Whilst this model has been analysed previously in the literature, e.g. by Biggerstaff and Jackson (1969), we consider special cases and investigate the approach to the steady state. In addition, the three-state literature is extended by the analysis of a three-state model in which every transition is feasible and the transition rates from the full operation state (S_1) to the partial operation state (S_2) and failed state (S_3) are each sums of two Weibull hazards (one increasing and one decreasing) and consequently correspond to bath-tub shaped curves (see Section 1.2), whilst the remaining transition rates are constant. The five-state Markov model, also presented in the paper, models a unit subject to two mutually

Figure 3.1
General Three-State Reliability Model

transition diagram

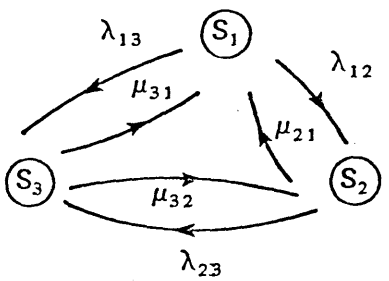
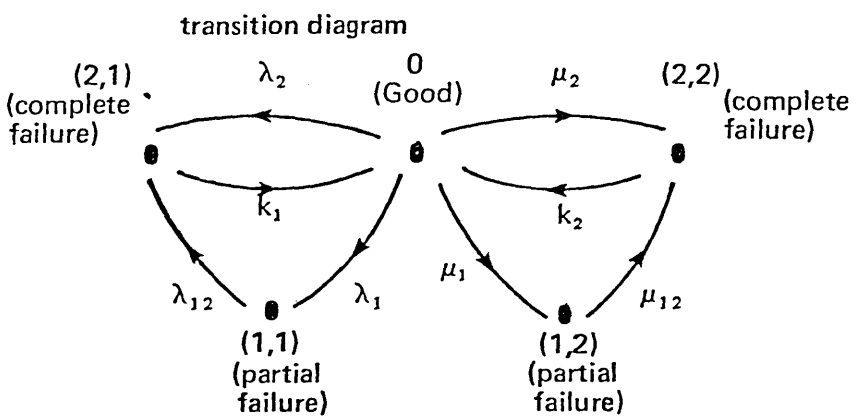


Figure 3.2
Five-State Reliability Model

transition diagram



exclusive failure modes, in each of which failure can be partial or complete (Figure 3.2).

3.2 Analysis of Three-State Models

Considering first the general three-state Markov model, we make the usual assumption that the device is initially in state S_1, and denote the probability of being in state S_i at time t by $P_i(t)$, $i = 1, 2, 3$. The constant transition rates between states are given in Figure 3.1. Then the set of differential equations corresponding to these time independent transition rates are, in matrix notation,

$$\frac{d}{dt}\begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix} = \begin{bmatrix} -(\lambda_{12}+\lambda_{13}) & \mu_{21} & \mu_{31} \\ \lambda_{12} & -(\lambda_{23}+\mu_{21}) & \mu_{32} \\ \lambda_{13} & \lambda_{23} & -(\mu_{32}+\mu_{31}) \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix}. \tag{3.1}$$

The solution of these equations are

$$P_1(t) = \sum_{i=1}^{3} \frac{(\lambda_{23}+\mu_{21}+r_i)(\mu_{32}+\mu_{31}+r_i)-\lambda_{23}\mu_{32}}{D_i} e^{r_i t}$$

$$P_2(t) = \sum_{i=1}^{3} \frac{\lambda_{12}(\mu_{32}+\mu_{31}+r_i)+\lambda_{13}\mu_{32}}{D_i} e^{r_i t} \tag{3.2}$$

$$P_3(t) = 1 - P_1(t) - P_2(t) ,$$

where

$$D_i = \prod_{j \neq i=1}^{3} (r_j - r_i)^{-1}$$

and $r_3 = 0$ and r_1, r_2 are the roots of the equation

$$\varphi(r) \equiv \left(r + \frac{6}{2}\right)^2 - \frac{1}{4}\left[(a+b+c)^2 - 4ab\right] = 0$$

with

$$6 \equiv \lambda_{23}+\lambda_{13}+\lambda_{12}+\mu_{32}+\mu_{31}+\mu_{21}$$

$$a \equiv \mu_{31}-\mu_{21}$$

$$b \equiv \lambda_{23}-\lambda_{13}$$

$$c \equiv \mu_{32}-\lambda_{12} .$$

The steady state availability of the device is

$$P_1(\infty) = \frac{(\lambda_{23}+\mu_{21})(\mu_{31}+\mu_{32})-\lambda_{23}\mu_{32}}{r_1 r_2} . \tag{3.3}$$

If $\Delta^2 \equiv 4ab - (a+b+c)^2 > 0$, r_1 and r_2 are complex and $P_1(\infty)$ is approached in a damped oscillatory manner with period $2\pi/\Delta$. These oscillations are very small such that

$$P_1(t) \geq P_1(\infty) - 0{\cdot}8\, e^{-2\pi}. \qquad (3.4)$$

In many situations the rate of repair or replacement of the device from S_2 and S_3 may be the same, i.e. $\mu_{31} = \mu_{21} = \mu$ (say); and the equation $\varphi(r) = 0$ has two real roots

$$\begin{aligned} r_1 &= -(\lambda_{12} + \lambda_{13} + \mu) \\ r_2 &= -(\lambda_{23} + \mu_{32} + \mu). \end{aligned} \qquad (3.5)$$

By setting $\mu_{32} = \mu_{21} = 0$ the model reduces to the partial/catastrophic failure model of Kontoleon and Kontoleon (1974), whilst by setting $\mu_{32} = \lambda_{23} = 0$ it reduces to the failure to operate/failure to idle model of Proctor and Singh (1976a), which was also considered in another context in the discussion of Biggerstaff and Jackson (1969). If instead $\lambda_{12} = \lambda_{23} = \mu_{31} = 0$, the model reduces to a three-state reliability model which has been considered in connection with electrical systems, e.g. Allen and De Oliveira (1977).

According to much of the literature (see e.g. Shooman (1968), Lomnicki (1973)) it is often the case that in general the hazard function for a device has a "bath-tub" shape, i.e. it is monotonically decreasing for small t (corresponding to burn-in failures), fairly constant at medium time values, and increasing for large t. Thus in order to model the whole of the bath-tub curve we now replace the constant degradation rates $\lambda_{12}, \lambda_{13}, \lambda_{23}$ by sums of Weibull hazards;

$$\begin{aligned} \lambda_{ij}(t) &= \theta_{ij} t^{\alpha_{ij}-1} + \varphi_{ij} t^{\beta_{ij}-1}, \\ (i,j) &= (1,2), (1,3), (2,3), \end{aligned} \qquad (3.6)$$

where $0 < \alpha_{ij} < 1$, $\beta_{ij} > 1$, $\theta_{ij} > 0$, $\phi_{ij} > 0$.

We also assume, as in the Markov model, that $\mu_{21} = \mu_{31} = \mu$.

In this case the differential equations are

$$\frac{d}{dt}\begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix} = \begin{bmatrix} -\lambda_{12}(t) - \lambda_{13}(t) & 0 & 0 \\ \lambda_{12}(t) & -\lambda_{23}(t) & 0 \\ \lambda_{13}(t) & \lambda_{23}(t) & 0 \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix} + \begin{bmatrix} 0 & \mu & \mu \\ 0 & -\mu & \mu_{32} \\ 0 & 0 & -\mu - \mu_{32} \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix} . \quad (3.7)$$

Their solution is

$$P_1(t) = \frac{1}{G(t)}\left[\mu \int_0^t G(x)\,dx + 1\right]$$

$$P_2(t) = \frac{1}{H(t)}\left[\int_0^t H(x)\{\mu_{32} - \mu_{32} P_1(x) + \lambda_{12}(x) P_1(x)\}\,dx - 1\right] \quad (3.8)$$

$$P_3(t) = 1 - P_1(t) - P_2(t) ,$$

where

$$G(t) \equiv \exp\left\{\int_0^t [\mu + \lambda_{13}(x) + \lambda_{12}(x)]\,dx\right\}$$

$$H(t) \equiv \exp\left\{\int_0^t [\mu + \mu_{32} + \lambda_{23}(x)]\,dx\right\} .$$

The steady-state availability for this model is

$$P_1(\infty) = 0 ,$$

i.e. constant rates for repair and replacements cannot keep up with increasing degradation and failure tendencies.

3.3 Analysis of Five -State Model

Now considering the five-state Markov model with states and (constant) transition rates given in Figure 3.2, we again assume that the device is initially good, and denote the probability that the device is in state 0 (good) at time t by $P_0(t)$, and the probability that the device is in state (i, j) at time t by Pij(t); i,j = 1,2. Then the differential equations for this device, in matrix notation are

$$\frac{d}{dt}\begin{bmatrix} P_0(t) \\ P_{11}(t) \\ P_{12}(t) \\ P_{21}(t) \\ P_{22}(t) \end{bmatrix} = \begin{bmatrix} -(\lambda_1+\lambda_2+\mu_1+\mu_2) & 0 & 0 & f_1 & f_2 \\ \lambda_1 & -\lambda_{12} & 0 & 0 & 0 \\ \mu_1 & 0 & -\mu_{12} & 0 & 0 \\ \lambda_2 & \lambda_{12} & 0 & -f_1 & 0 \\ \mu_2 & 0 & \mu_{12} & 0 & -f_2 \end{bmatrix} \begin{bmatrix} P_0(t) \\ P_{11}(t) \\ P_{12}(t) \\ P_{21}(t) \\ P_{22}(t) \end{bmatrix}. \tag{3.9}$$

These equations can be solved by use of Laplace transforms or by classical integration methods to yield

$$\begin{aligned}
P_0(t) &= \sum_{i=1}^{5} D_i(\lambda_{12}-r_i)(\mu_{12}-r_i)(f_1-r_i)(f_2-r_i)\exp(-r_i t) \\
P_{11}(t) &= \sum_{i=1}^{5} D_i(\mu_{12}-r_i)(f_1-r_i)(f_2-r_i)\lambda_1 \exp(-r_i t) \\
P_{12}(t) &= \sum_{i=1}^{5} D_i(\lambda_{12}-r_i)(f_1-r_i)(f_2-r_i)\mu_1 \exp(-r_i t) \\
P_{21}(t) &= \sum_{i=1}^{5} D_i(\mu_{12}-r_i)(f_2-r_i)[\lambda_2(\lambda_{12}-r_i)+\lambda_1\lambda_{12}]\exp(-r_i t) \\
P_{22}(t) &= \sum_{i=1}^{5} D_i(\lambda_{12}-r_i)(f_1-r_i)[\mu_2(\mu_{12}-r_i)+\lambda_1\lambda_{12}]\exp(-r_i t),
\end{aligned} \tag{3.10}$$

where

$$D_i \equiv \prod_{j\neq i=1}^{5} (r_j - r_i)^{-1}$$

and r_i, $i = 1, 2, 3, 4, 5$ are the five roots of the polynomial equation

$$\begin{aligned}
\varphi(r) \equiv\ & (r+f_1)(r+f_2)(r+\lambda_{12})(r+\mu_{12})(r+\lambda_1+\lambda_2+\mu_1+\mu_2) \\
& -(r+f_1)(r+\lambda_{12})f_2(\mu_1\mu_{12}+\mu_2\mu_{12}+r\mu_2) \\
& -(r+f_2)(r+\mu_{12})f_1(\lambda_1\lambda_{12}+\lambda_2\lambda_{12}+r\lambda_2) = 0.
\end{aligned} \tag{3.11}$$

Note that one of these roots r_1 (say) is zero. The steady-state availability of this device is

$$P_0(\infty) = \frac{\lambda_{12}\mu_{12}f_1f_2}{r_2 r_3 r_4 r_5} = \frac{N}{D}, \tag{3.12}$$

where

$$\begin{aligned}
N &\equiv \lambda_{12}\mu_{12}f_1f_2 \\
D &\equiv \lambda_{12}\mu_{12}f_1f_2 + \lambda_{12}f_1(\mu_1\mu_{12}+\mu_2\mu_{12}+\mu_1 f_2) \\
&\quad + \mu_{12}f_2(\lambda_1\lambda_{12}+\lambda_2\lambda_{12}+\lambda_1 f_1).
\end{aligned}$$

If $f_2 = \mu_1 = \mu_2 = \mu_{12} = 0$ the five-state model

also reduces to the model considered by Kontoleon and Kontoleon (1974), whilst if $\lambda_1 = \lambda_{12} = \mu_1 = \mu_{12} = 0$, the model reduces to the catastrophic failure model of Proctor and Singh (1976a).

In many situations the rate of repair or replacement of the device from state $(2,1)$ and state $(2,2)$ may be the same, i.e. $\kappa_1 = \kappa_2 = \kappa$. If also $\lambda_{12} = \mu_{12} = \beta$ only three of the five exponential factors determining $P_0(t)$ and $P_{ij}(t)$ in (3.10) (say those corresponding to r_1, r_2 and r_3) will have non-zero coefficients. As before $r_1 = 0$, and r_2, r_3 are the roots of the reduced polynomial equation

$$\varphi_1(r) \equiv r^2 + r(\lambda_1+\lambda_2+\mu_1+\mu_2+\beta+\kappa) + (\beta+\kappa)(\lambda_1+\mu_1) + \beta(\lambda_2+\mu_2+\kappa) = 0 . \qquad (3.13)$$

The steady-state availability in this case is

$$P_0(\infty) = \frac{\beta\kappa}{\beta(\lambda_1+\lambda_2+\mu_1+\mu_2)+\kappa(\lambda_1+\mu_1+\beta)} . \qquad (3.14)$$

If $\Delta^2 \equiv 4\kappa(\lambda_1+\mu_1) - (\lambda_1+\lambda_2+\mu_1+\mu_2+\kappa-\beta)^2 > 0$, r_2, r_3 are complex and $P_0(\infty)$ will be approached in a damped oscillatory manner with period $2\pi/\Delta$.

CHAPTER 4

A GENERAL PARTIAL OPERATION MODEL

4.1 Formulation

In the previous chapters we have treated the levels of operation of components and systems as forming a finite ordered discrete set on the range $[0,1]$. In formulating a general partial operation model in this Chapter we, like Postelnicu (1970), retain the range $[0,1]$, but instead largely concentrate upon a continuous state-space approach. Some of the material of this Chapter is based upon joint work with S. Humble, which is partly contained by the joint paper in Appendix A3. In this joint work the structure of the model and relationships are due to the present author, with some examples and the numerical evaluations due to S. Humble.

We define S(t) as the stochastic process indicating a component's level of performance (state) s at time $t \geq 0$, where S(t) is now assumed to take values in the interval $[0,1]$ instead of the couple $\{0,1\}$. As before we define '0' to be complete failure, '1' to be full operation, with values in between representing in some sense the proportion of operation achieved. Then for fixed t we may assume either that the random variable S(t) be discrete, so that it has a probability distribution with

$$p(s,t) > 0 \qquad \text{for} \quad s \in \mathcal{S},$$

with $\mathcal{S}$ a countable set, and

$$\sum_{s \in \mathcal{S}} p(s,t) = 1 \qquad \text{, for all t,} \tag{4.1}$$

or we may assume that S(t) is continuous, with probability density, say $p(s,t)$, on $0 < s < 1$. In the later case, the structure

of the model in fact requires that in general the distribution is a composite of discrete and continuous parts, so that

$$\int_{0+}^{1-} \varphi(s,t)\,ds + P_0(t) + P_1(t) = 1 \quad \text{for all } t$$

$$\varphi(s,t) \geqslant 0, \tag{4.2}$$

$$P_0(t) \geqslant 0, \quad P_1(t) \geqslant 0,$$

where $P_0(t)$ and $P_1(t)$ are atoms of probability at s = 0 and s = 1 respectively. (See below).

Assuming that transitions between states satisfy the (first-order) Markov property that the instantaneous transition rates at t depend only on the state at t and not on states previously passed through, we may define $\phi(s,r,t)$ as the transition rate probability function from state s to state r at time t ($r \neq s$), so that for the continuous case and $0 < r < 1, \quad r \neq s,$ $\varphi(s,r,t)\,\delta t\,\delta r + o(\delta t)$ is the instantaneous probability of a transition to states r to $(r + \delta r)$ at the end of the infinitesimal interval t to $(t+\delta t)$ given that the component is in state s at t. The corresponding instantaneous conditional probability of transition to state 1 given that the component is in state $s \neq 1$ is $\varphi(s,1,t)\,\delta t + o(\delta t)$, whilst that of transition to state 0 is for $s \neq 0$ $\varphi(s,0,t)\,\delta t + o(\delta t)$.

Thus

$$\varphi(s,r,t)\,\delta r = \lim_{\delta t \to 0} \frac{P[r \leqslant S(t+\delta t) \leqslant r+\delta r \,|\, S(t)=s]}{\delta t}$$

$$0 < r < 1, \; r \neq s$$

$$\varphi(s,1,t) = \lim_{\delta t \to 0} \frac{P[S(t+\delta t)=1 \,|\, S(t)=s]}{\delta t}$$

$$s \neq 1$$

$$\varphi(s,0,t) = \lim_{\delta t \to 0} \frac{P[S(t+\delta t)=0 \,|\, S(t)=s]}{\delta t}$$

$$s \neq 0 \quad .$$

Allowing, for the moment, all state transitions, the process can be considered as a special case of a general jump process, e.g. Feller (1940). [Feller (1971) Chapter X treats the stationary transition probabilities case.] It follows that for any infinitesimal interval t to $(t+\delta t)$ there will be a probability $P[S(t+\delta t)=s \,|\, S(t)=s]$ of the component ending the interval in the state s that it started it, corresponding to the non-occurrence of a jump in the interval apart from terms of $o(\delta t)$. For $0<s<1$, this probability is

$$P[S(t+\delta t)=s \,|\, S(t)=s] = 1-\delta t\left[\int_{\substack{0+\\ r\neq s}}^{1-} \varphi(s,r,t)\,dr + \varphi(s,1,t) + \varphi(s,0,t)\right] + o(\delta t),$$

whilst for s = 1 and s = 0 the probabilities are respectively

$$P[S(t+\delta t)=1 \,|\, S(t)=1] = 1-\delta t\left[\int_{0+}^{1-} \varphi(1,r,t)\,dr + \varphi(1,0,t)\right] + o(\delta t)$$

$$P[S(t+\delta t)=0 \,|\, S(t)=0] = 1-\delta t\left[\int_{0+}^{1-} \varphi(0,r,t)\,dr + \varphi(0,1,t)\right] + o(\delta t).$$

As is usual [e.g. Feller (1971) Chapter X] we assume the existence of the processes, and give below examples of practical processes of interest. Thus, for S(t) continuous we obtain the (time dependent) Chapman-Kolmogorov type equation for $0<s<1$,

$$\begin{aligned}\varphi(s,t+\delta t) = \varphi(s,t)\Big[1-\delta t\int_{\substack{0+\\ r\neq s}}^{1-} \varphi(s,r,t)\,dr - \delta t\,\varphi(s,1,t) - \delta t\,\varphi(s,0,t)\Big] \\ + \delta t\int_{\substack{0+\\ r\neq s}}^{1-} \varphi(r,t)\,\varphi(r,s,t)\,dr \qquad (4.3) \\ + \delta t\,P_1(t)\,\varphi(1,s,t) + \delta t\,P_0(t)\,\varphi(0,s,t) \\ + o(\delta t),\end{aligned}$$

whilst the integrals are replaced by sums if S(t) is discrete. Thus in the continuous case the $\varphi(s,t)$ and $\Phi(s,r,t)$ satisfy the following (forward) differential equations

$$\frac{\partial \varphi(s,t)}{\partial t} = -\varphi(s,t)\left[\int_{0+}^{1-}_{r\neq s} \Phi(s,r,t)\,dr + \Phi(s,1,t) + \Phi(s,0,t)\right] + \int_{\substack{0+ \\ r\neq s}}^{1-} \varphi(r,t)\,\Phi(r,s,t)\,dr + P_1(t)\,\Phi(1,s,t) + P_0(t)\,\Phi(0,s,t) \qquad (4.4)$$

$$0 < s < 1.$$

If S(t) is discrete the integrals are again replaced by sums.

A reasonable initial condition for the system of equations in most cases is

$$\varphi(s,0) = 0, \qquad 0 < s < 1$$
$$P_0(0) = 0, \qquad (4.5)$$
$$P_1(0) = 1,$$

analogous to the usual assumption for the basic dichotomic reliability model. An implication is that for any finite t a discrete atom of probability will in general remain at s = 1 corresponding to the dichotomic survivor function of the basic reliability model.

The evaluation of solutions for $\varphi(s,t)$ $(0<s<1)$, $P_0(t)$ and $P_1(t)$ can be simplified somewhat by the specification of restrictions on $\Phi(s,r,t)$ and $\varphi(s,t)$. For instance, in the absence of a repair, replacement or recovery mechanism one may assume (analogously to assumption (v) of Section 1.4) that

$$\Phi(s,r,t) = 0 \quad \text{for all t and all } r > s. \qquad (4.6)$$

Thus since state s = 0 is an absorbing state whilst all other states are transient, this implies in the continuous case that an absorptive sink with probability collected into a discrete atom will develop at s = 0, somewhat analogous to the declining discrete component in

the distribution of the backward recurrence time in renewal theory (e.g. Cox (1962)). Consequently by time infinity the distribution will be concentrated onto the failed state s = 0, so that

$$\begin{aligned} &\lim_{t\to\infty} \varphi(s,t) = 0, \quad 0<s<1 \\ &\lim_{t\to\infty} P_1(t) = 0, \\ &\lim_{t\to\infty} P_0(t) = 1. \end{aligned} \tag{4.7}$$

The differential equations for $P_0(t)$ and $P_1(t)$ are

$$\begin{aligned} \frac{dP_0(t)}{dt} &= \int_{0+}^{1-} \varphi(r,t)\,\varphi(r,0,t)\,dr + P_1(t)\,\varphi(1,0,t) \\ \frac{dP_1(t)}{dt} &= -P_1(t)\left[\int_{0+}^{1-} \varphi(1,r,t)\,dr + \varphi(1,0,t)\right]. \end{aligned} \tag{4.8}$$

The solution of the latter equation is

$$P_1(t) = \exp\left[-\int_0^t dy\left\{\int_{0+}^{1-} dr\,\varphi(1,r,y) + \varphi(1,0,y)\right\}\right] \tag{4.9}$$

which is a generalisation of the expression for R(t) in (1.4). Whilst the equation for $P_0(t)$ is not so readily soluble, we have from (4.2) that

$$P_0(t) = 1 - P_1(t) - \int_{0+}^{1-} \varphi(s,t)\,ds. \tag{4.10}$$

To illustrate the solutions to this model, we consider some special cases. Of course, in the case of dichotomic reliability $S = \{0,1\}$., and (4.8) reduces to (1.3).

Example 4.1.1

If we suppose that $\varphi(1,0,t) = 0$ for all t, and that for $0<s<1$ (so that $t>0$),

$$\begin{aligned} \varphi(s,r,t) &= 2, \quad 0<r<s \\ \varphi(s,0,t) &= 2[(1-s) - \ln(1-s)] + s/t \\ \varphi(s,t) &= 2t(1-s)^{2t-1} e^{-2t}, \end{aligned} \tag{4.11}$$

then substituting into (4.4) we obtain

$$\varphi(1,s,t)P_1(t) = 2\lambda s(1-s)^{\lambda t-1} e^{-\lambda t} . \qquad (4.12)$$

Using the initial conditions (4.5) we have upon substituting into (4.4)

$$\varphi(1,s,0) = \left[\frac{d\varphi(s,t)}{dt}\right]_{t=0} = \frac{\lambda}{1-s} .$$

Substituting (4.12) into (4.8) we obtain

$$\frac{dP_1(t)}{dt} = \frac{-2e^{-\lambda t}}{(\lambda t+1)t} .$$

Employing the initial conditions (or noting the analogue to dichotomic reliability) and using (4.10) we thus find

$$\begin{aligned} P_1(t) &= 2\{E_1(\lambda t) - eE_1(\lambda t+1)\} \\ P_0(t) &= 1-2\{E_1(\lambda t) - eE_1(\lambda t+1)\} - e^{-\lambda t}, \end{aligned} \qquad (4.13)$$

where $E_1(a)$ is the exponential integral,

$$E_1(a) = \int_a^\infty x^{-1} e^{-x} dx .$$

It follows from (4.13) that for λ sufficiently large

$$\begin{aligned} P_1(t) &\doteq \delta(t) \\ P_0(t) &\doteq 1-e^{-\lambda t} . \end{aligned} \qquad (4.14)$$

Example 4.1.2

Another special case of the solution of (4.4), which corresponds to a semi-Markov shock-model, is of interest. The formulation is similar to that of Ansell, Bendell and Humble (1980a,b) and Bendell and Scott (1982).

For this

$$\varphi(s,r,t) = \frac{\sum_{k=1}^{\infty} g^k(1-s)\, g_1(s-r,s) * f^k(t)}{\sum_{k=0}^{\infty} g^k(1-s)\int_0^t F(t-\tau) f^k(\tau)\,d\tau}, \qquad 0 \leq r < s \leq 1, \tag{4.15}$$

which gives as a special case

$$\varphi(1,r,t) = \frac{f(t) g_1(1-r,1)}{F(t)}, \quad 0 \leq r < 1,$$

where $f^k(t)$ is the k^{th} convolution of $f(t)$, $f^0(t) = \delta(t)$, $f^1(t) = f(t)$, $F(t) = \int_t^\infty f(\tau)\,d\tau$, $f(t)$ is the density function of time between state transitions (or shocks), $g^k(x) = \int_0^x g^{k-1}(y)\, g_1(x-y, 1-y)\,dy$, $k = 1,2,\ldots$, $g^1(x) = g(x)$, $g^0(x) = \delta(x)$, and $g_1(x,s)$ is the density function of amount of degradation or loss of operative ability x associated with a transition or shock to a unit at level s, where $0 \leq x \leq s$. An advantage of this formulation is that explicit solutions can be obtained for (4.4) by direct means. That is, for the continuous part of $\varphi(s,t)$ we can write

$$\varphi(s,t) = \sum_{k=0}^{\infty} g^k(1-s)\int_0^t f^k(t-\tau) F(\tau)\,d\tau. \tag{4.16}$$

The discrete parts of the distribution are

$$P_0(t) = \sum_{k=1}^{\infty} g^k(1)\int_0^t f^k(\tau)\,d\tau$$
$$P_1(t) = F(t). \tag{4.17}$$

As a simple example of this shock-model formulation, suppose that S(t) is discrete with N levels denoted $(i-1)/(N-1)$, $i = 1,2,\ldots,N$, and that f(t) is exponential with density

$$f(t) = \mu e^{-\mu t}, \quad t \geq 0. \tag{4.18}$$

Then if

$$g_1(x,y) = \delta\left(x - \frac{1}{N-1}\right), \text{ for all } y \tag{4.19}$$

the process forms a truncated Poisson process, so that its (discrete)

distribution is

$$\varphi\left(\left[\frac{i-1}{N-1}\right],t\right)=\begin{cases}1-\sum_{k=0}^{N-2}\frac{e^{-\mu t}(\mu t)^k}{k!}, & i=1\\ \frac{e^{-\mu t}(\mu t)^{N-i}}{(N-i)!}, & i=2,\ldots,N\end{cases} \tag{4.20}$$

Example 4.1.3

As a final example of our partial operation formulation, we consider the relationship to more conventional cumulative damage processes, which have been studied by various authors (see e.g. Buckland (1964) and Barlow and Proschan (1976) for reviews). In general for such processes we have as the direct analogue of (4.4), (4.5) and (4.6) that

$$\begin{aligned}\frac{\partial q(z,t)}{\partial t} &= -q(z,t)\int_{z+}^{\infty}\psi(z,w,t)\,dw\\ &\quad+\int_{0+}^{z-}q(w,t)\psi(w,z,t)\,dw\\ &\quad+Q_0(t)\psi(0,z,t)\end{aligned} \tag{4.21}$$

$$q(z,0)=0,\quad 0<z<\infty,\quad Q_0(0)=1,$$

where the terms are defined analogously to before, with $q(z,t)$ giving the density of damage $z\,(z>0)$ at time t and $Q_0(t)$ is the atom of probability corresponding to zero damage at t. However, these models differ somewhat from the one introduced above in that there we are particularly concerned with the reduction in the operation levels of components, and consequently systems, with time. Thus, particular importance is given by the partial operation model to the absorbing state representing complete failure s = 0, and to the maximum (and initial) state s = 1. In contrast, in the absence of replacement policies the state spaces for cumulative damage processes are usually unbounded (to the right) with states in the range $[0,\infty)$.

However, it is apparent that associated with the accumulation of damage on $[0,\infty)$ (which may be described by (4.21)), there will be a reduction of operation level of the unit with magnitude in $[0,1]$. Assuming that for any specific unit a unique level of operation is associated with a specified accumulation of damage, we can thus search for an appropriate transformation from cumulative damage to level of operation. Since the transformation is onto the range $[0,1]$, one set of transformations which are appropriate in the continuous case are the probability integral transformations given by

$$s = \int_z^\infty h(u)\,du = H(z) \tag{4.22}$$

where $h(z)$ is some specified probability density function defined on $[0,\infty)$, and $H(z)$ is the corresponding survivor function. Note that transformations of this type are monotonically decreasing and map infinite damage onto level of operation zero, and zero damage onto level of operation unity. The choice of $h(u)$ of course depends on the physical relationship between damage and partial operation. If s and z are both discrete the exact analogue of (4.22) applies with the density function replaced by a discrete probability function, and the integral by a sum.

One possible choice for $h(u)$ is the uniform distribution on (a,b), $0 \leq a < b$, for which

$$s = \begin{cases} 0, & z \geq b \\ \frac{b-z}{b-a}, & a < z < b \\ 1, & 0 \leq z \leq a \end{cases} . \tag{4.23}$$

Thus, in this case the component can sustain damage up to an amount a without experiencing a reduced level of operation. As damage accumulates further, the level of operation reduces linearly until

the component has failed completely by accumulated damage b .

As $(b-a)$ tends to zero, dichotomic reliability is obtained.

Alternatively, if $h(u)$ is Weibull

$$s = \exp\left[-\left(\frac{z}{\theta}\right)^{\beta}\right] \tag{4.24}$$

and this again reduces to dichotomic reliability as $\beta \to \infty$, whilst if $\beta = 1$ an exponential relationship is obtained. For finite β this transformation is 1 to 1, with s = 1 only for $z = 0$, and s = 0 only for $z = \infty$.

As an example of the use of the transformation from cumulative damage, we consider the cumulative damage model of Mercer (1961) in which there is a constant probability $\mu \delta t$ that a shock (state transition) occurs in any infinitesimal interval $(t, t+\delta t)$, and a probability $r(z)\delta z$ that the damage resulting from a shock lies in $(z, z+\delta z)$. In addition, Mercer's model includes a dichotomic failure intensity $\lambda(t)$ which we take as associated 1 to 1 with the occurrence of infinite damage (and hence with s = 0). Thus for finite z and $w > z$ we have in our notation

$$\psi(z, w, t) = \begin{cases} \mu r(w-z), & z < w < \infty \\ \mu r(\infty) + \lambda(t), & w = \infty \end{cases} \tag{4.25}$$

and as previously,

$$\psi(z, w, t) = 0, \qquad w < z . \tag{4.26}$$

$\psi(z, \infty, t)$ is the age-wear-specific failure rate in the terminology of Cox (1962).

Mercer studies the case where $r(z)$ is gamma (n, γ), and considers the diffusion-type limit of this extended Poisson process obtained as $\mu \to \infty, n \to 0, E(z) = \frac{n}{\gamma} \to 0$

such that $\left(\frac{\mu n}{\gamma}\right) = \alpha$ remains finite and non-zero. He consequently obtains the limiting distribution with continuous part

$$q(\zeta,t) = \frac{\zeta^{\alpha\gamma t-1}}{\Gamma(\alpha\gamma t)} \exp\left[-\gamma\zeta - \alpha\gamma t - \int_0^t \lambda(y)\,dy\right] \tag{4.27}$$

In this case there is no discrete part at $\zeta = 0$ (except at t = 0), and analogous to (4.10) the distribution has a discrete atom at $\zeta = \infty$ of

$$Q_\infty(t) = 1 - \int_0^\infty q(\zeta,t)\,d\zeta \quad , t > 0. \tag{4.28}$$

Applying the transformation (4.23) to the above distribution we obtain as the continuous part of the distribution over s

$$\varphi(s,t) = \frac{[b-(b-a)s]^{\alpha\gamma t-1}}{\Gamma(\alpha\gamma t)} \exp\left\{-\gamma[b-(b-a)s] - \alpha\gamma t - \int_0^t \lambda(y)\,dy\right\}, \quad t > 0, \tag{4.29}$$

and

$$P_1(t) = \int_0^a q(\zeta,t)\,d\zeta = \frac{\Gamma(a\gamma, \alpha\gamma t)}{\gamma^{\alpha\gamma t}\,\Gamma(\alpha\gamma t)} \exp\left[-\alpha\gamma t - \int_0^t \lambda(y)\,dy\right] \tag{4.30}$$

$$P_0(t) = 1 - \frac{[\Gamma(b\gamma, \alpha\gamma t) + (b-a-1)\,\Gamma(a\gamma, \alpha\gamma t)]}{(b-a)\,\gamma^{\alpha\gamma t}\,\Gamma(\alpha\gamma t)} \cdot \exp\left[-\alpha\gamma t - \int_0^t \lambda(y)\,dy\right], \tag{4.31}$$

where

$$\Gamma(\alpha, x) = \int_0^\alpha u^{x-1} e^{-u}\,du.$$

If instead the transformation (4.24) is employed then

$$\varphi(s,t) = \frac{\theta^{\alpha\gamma t} \exp\left[-\alpha\gamma t - \int_0^t \lambda(y)\,dy\right]}{\beta\,\Gamma(\alpha\gamma t)} \left[\ln s^{-1}\right]^{\frac{\alpha\gamma t}{\beta}-1} \cdot s^{\gamma\beta\theta^\beta - 1} \tag{4.32}$$

and

$$P_0(t) = 1 - \frac{\Gamma\left(\frac{\alpha\gamma t}{\beta}\right)}{\beta\gamma^{\alpha\gamma t}\,\Gamma(\alpha\gamma t)} \exp\left[-\alpha\gamma t - \int_0^t \lambda(y)\,dy\right] \tag{4.33}$$

$$P_1(t) = 0, \quad t > 0.$$

The probabilities $P_0(t)$ and $P_1(t)$ and the probability that the level at t equals or exceeds s ($0 < s < 1$),

$$R(s,t) = \int_s^{1-} \varphi(\sigma,t)\, d\sigma + P_1(t) \qquad (4.34)$$

(with the direct analogue in the discrete case) will often be of physical interest, as will the mean time till the level drops below s (or MTTLS) $\overline{T}(s)$ = E(T[s]) where T[s] is the time till the level drops below s. We also define

$$R(1,t) = P_1(t)$$

and

$$R(0,t) = 1 - P_0(t) .$$

By treating s as a dichotomic failure point, the MTTLS can be written following (1.6) as

$$\overline{T}(s) = \int_0^{\infty} R(s,t)\, dt , \quad 0 \leq s < 1. \qquad (4.35)$$

Of course, $P_0(t)$ is monotonically non-decreasing in t, $P_1(t)$ is monotonically non-increasing in t, and $R(s,t)$ and $\overline{T}(s)$ are monotonically non-increasing in s. Higher moments of the time to level s will also be of interest, in particular the variance,

$$\mathrm{var}[T(s)] = \overline{T^2}(s) - [\overline{T}(s)]^2 \qquad (4.36)$$

where $\overline{T^2}(s) = E[\{T(s)\}^2]$.

The mean time at level s (MTAS) $s > 0$, may also be of interest. In the continuous case since there are an infinite number of states prior to any state $s < 1$, all of which may be sojourned in, it follows that in general for the MTTLS to remain finite the MTAS($0 < s < 1$) should be infinitesimal. Thus we denote the mean time in the infinitesimal range of levels (s, s + δs) by $\overline{V}(s)\delta s$ $(s \neq 0, s+\delta s \neq 1)$ and interest focuses on the mean time intensity function, $\overline{V}(s)$, which we shall assume is continuous. This can be conveniently obtained by defining for $0 < s < 1$,

$$\Psi_s(t) = \begin{cases} 1, \text{if } S(t) = s \\ 0, \text{ otherwise} \end{cases} . \tag{4.37}$$

Then

$$E[\Psi_s(t)] = \varphi(s,t) , \tag{4.38}$$

and the time intensity at s, V(s) is

$$V(s) = \int_0^\infty \Psi_s(t)\, dt \tag{4.39}$$

so that

$$\overline{V}(s) = E[V(s)] = \int_0^\infty \varphi(s,t)\, dt . \tag{4.40}$$

Also, by analogy to (1.6) in the dichotomic case, the mean time at s = 1 is obtained as

$$\overline{V_1} = \int_0^\infty P_1(t)\, dt . \tag{4.41}$$

Thus

$$\overline{T}(s) = \int_s^{1-} \overline{V}(\sigma)\, d\sigma + \overline{V_1} . \tag{4.42}$$

Another useful measure of the reliability of a component or system is provided by the expected level of operation at t (ELO),

$$\overline{S}(t) = E[S(t)] .$$

In the continuous case

$$\overline{S}(t) = \int_{0+}^{1-} s\, \varphi(s,t)\, ds + P_1(t) \tag{4.43}$$

with the usual analogue in the discrete case. The higher moments of S(t) are also of interest, in particular the variance of S(t)

$$\mathrm{var}[S(t)] = \overline{S^2}(t) - [\overline{S}(t)]^2 \tag{4.44}$$

where

$$\overline{S^2}(t) = \int_{0+}^{1-} s^2\, \varphi(s,t)\, ds + P_1(t) . \tag{4.45}$$

It follows that $\overline{S}(t)$ is monotonically non-increasing in t whilst from (4.5) and (4.6),

$$\mathrm{var}[S(0)] = \mathrm{var}[S(\infty)] = 0. \tag{4.46}$$

Also, a single summary measure of the expected performance of a component or system over its lifetime is provided by the expected

lifetime coefficient of operation (ELCO),

$$\bar{\bar{S}} = \int_0^\infty \bar{S}(t)\,dt. \tag{4.47}$$

A useful alternative representation of $\bar{\bar{S}}$ is obtained from (4.43) by interchanging the order of integration,

$$\bar{\bar{S}} = \int_0^\infty \left[\int_{0+}^{1-} s\,\varphi(s,t)\,ds + P_1(t)\right]dt$$
$$= \int_{0+}^{1-} ds.\, s \int_0^\infty \varphi(s,t)\,dt + \int_0^\infty P_1(t)\,dt = \int_{0+}^{1-} s\bar{V}(s)\,ds + \bar{V}_1,$$

from (4.40) and (4.41).

In the case of dichotomic reliability there are only two levels of operation s = 1 and s = 0, and $\bar{S}(t)$ and $R(0,t) = P_1(t) = 1 - P_0(t)$ are equivalent to the dichotomic reliability R(t), whilst $\bar{\bar{S}}$ and $\bar{T}(0)$ are equivalent to the mean time to failure, E(t).

To illustrate the above measures we consider the examples previously introduced.

Example 4.1.1

For the case of (4.11) and (4.13) we obtain

$$R(s,t) = (1-s)^{2t} e^{-2t} + 2\{E_1(2t) - e\,E_1(2t+1)\}, \tag{4.48}$$

$$\bar{T}(s) = \lambda^{-1}\{2e\,E_1(1) + [1 - \ln(1-s)]^{-1}\}$$
$$\simeq \frac{2.1927 - 1.1927\ln(1-s)}{2[1-\ln(1-s)]}, \tag{4.49}$$

$$\bar{V}(s) = \{2(1-s)[1-\ln(1-s)]^2\}^{-1}, \tag{4.50}$$

$$\bar{S}(t) = \frac{e^{-2t}}{2t+1} + 2\{E_1(2t) - e\,E_1(2t+1)\}, \tag{4.51}$$

$$\bar{S^2}(t) = \frac{2e^{-2t}}{(2t+1)(2t+2)} + 2\{E_1(2t) - e\,E_1(2t+1)\}, \tag{4.52}$$

and

$$\bar{\bar{S}} = 3e\,E_1(1)\lambda^{-1} \simeq 1.7890\,\lambda^{-1}, \tag{4.53}$$

where $E_1(a)$ is the exponential integral, and $E_1(1) \doteq 0.219384$.

For the case where λt is large so that (4.14) holds

$$\mathrm{Var}\left[S(t)\right] = \frac{2e^{-2\lambda t}}{(\lambda t+1)(\lambda t+2)} - \left(\frac{e^{-\lambda t}}{\lambda t+1}\right)^2 . \qquad (4.54)$$

Example 4.1.2

For the shock-model example (4.20)

$$R\left(\left[\frac{i-1}{N-1}\right], t\right) = \sum_{j=i}^{N} \varphi\left(\left[\frac{j-1}{N-1}\right], t\right) = \sum_{j=0}^{N-i} \frac{e^{-\mu t}(\mu t)^j}{j!} \qquad (4.55)$$

$$\bar{T}\left(\frac{i-1}{N-1}\right) = \frac{N-i+1}{\mu} , \qquad (4.56)$$

$$\mathrm{Var}\left[T\left(\frac{i-1}{N-1}\right)\right] = \frac{N-i+1}{\mu^2} , \qquad (4.57)$$

$$\bar{V}\left(\frac{i-1}{N-1}\right) = \frac{1}{\mu} , \qquad (4.58)$$

$$\bar{S}(t) = \frac{e^{-\mu t}(\mu t)^{N-2}}{(N-2)!} + \left(1-\frac{\mu t}{N-1}\right)\sum_{k=0}^{N-3} \frac{e^{-\mu t}(\mu t)^k}{k!} , \qquad (4.59)$$

$$\bar{\bar{S}} = \frac{N^2-3N+4}{2\mu(N-1)} . \qquad (4.60)$$

Example 4.1.3

For the transformation of Mercer's model, the measures are rather complicated, although the Weibull transformation does tend to give somewhat more convenient results than the uniform transform. Thus, for the model of (4.29) to (4.31) we have

$$R(s,t) = \frac{\exp\left[-\alpha\gamma t - \int_0^t \lambda(y)dy\right]}{(b-a)\gamma^{\alpha\gamma t}\,\Gamma(\alpha\gamma t)}\left\{\Gamma\left([b-(b-a)s],\alpha\gamma t\right)\right.$$
$$\left. + (b-a-1)\,\Gamma(a\gamma,\alpha\gamma t)\right\}, \qquad (4.61)$$

$$\overline{S}(t) = P_1(t) + \frac{\left[\Gamma(b\gamma,\alpha\gamma t+1)-\Gamma(a\gamma,\alpha\gamma t+1)\right]-b\gamma\left[\Gamma(b\gamma,\alpha\gamma t)-\Gamma(a\gamma,\alpha\gamma t)\right]}{(b-a)^2\gamma^{\alpha\gamma t+1}\Gamma(\alpha\gamma t)\exp\left[\alpha\gamma t+\int_0^t\lambda(y)dy\right]}$$

$$= P_1(t) + \frac{(\alpha t-b)\left[\Gamma(b\gamma,\alpha\gamma t)-\Gamma(a\gamma,\alpha\gamma t)\right]+a(a\gamma)^{\alpha\gamma t-1}e^{-a\gamma}-b(b\gamma)^{\alpha\gamma t-1}e^{-b\gamma}}{(b-a)^2\gamma^{\alpha\gamma t}\Gamma(\alpha\gamma t)\exp\left[\alpha\gamma t+\int_0^t\lambda(y)dy\right]} \qquad (4.62)$$

(See e.g. Gradshteyn and Ryzhik (1980)).

$$\overline{S^2}(t) = P_1(t) + \frac{(b\gamma)^2\left[\Gamma(b\gamma,\alpha\gamma t)-\Gamma(a\gamma,\alpha\gamma t)\right]-2b\gamma\left[\Gamma(b\gamma,\alpha\gamma t+1)-\Gamma(a\gamma,\alpha\gamma t+1)\right]}{(b-a)^3\gamma^{\alpha\gamma t+2}\Gamma(\alpha\gamma t)\exp\left[\alpha\gamma t+\int_0^t\lambda(y)dy\right]}$$
$$+ \frac{\left[\Gamma(b\gamma,\alpha\gamma t+2)-\Gamma(a\gamma,\alpha\gamma t+2)\right]}{(b-a)^3\gamma^{\alpha\gamma t+2}\Gamma(\alpha\gamma t)\exp\left[\alpha\gamma t+\int_0^t\lambda(y)dy\right]}$$

$$= P_1(t) + \frac{\left[(b\gamma)^2-2b\alpha\gamma^2 t+(\alpha\gamma t+1)\alpha\gamma t\right]\gamma^{-\alpha\gamma t}\left[\Gamma(b\gamma,\alpha\gamma t)-\Gamma(a\gamma,\alpha\gamma t)\right]}{(b-a)^3\gamma^2\Gamma(\alpha\gamma t)\exp\left[\alpha\gamma t+\int_0^t\lambda(y)dy\right]} \qquad (4.63)$$
$$+ \frac{e^{-b\gamma}b^{\alpha\gamma t}(\alpha\gamma t+1-b\gamma)-e^{-a\gamma}a^{\alpha\gamma t}(\alpha\gamma t+1+a\gamma-2\gamma b)}{(b-a)^3\gamma^2\Gamma(\alpha\gamma t)\exp\left[\alpha\gamma t+\int_0^t\lambda(y)dy\right]}.$$

For the model of (4.32) and (4.33) we instead obtain

$$R(s,t) = \frac{\exp\left[-\alpha\gamma t-\int_0^t\lambda(y)dy\right]\Gamma\left(-\gamma^\beta\theta^\beta\ln s,\frac{\alpha\gamma t}{\beta}\right)}{\beta\gamma^{\alpha\gamma t}\,\Gamma(\alpha\gamma t)}, \qquad (4.64)$$

$$\overline{S}(t) = \frac{\theta^{\alpha\gamma t}\Gamma\left(\frac{\alpha\gamma t}{\beta}\right)\exp\left[-\alpha\gamma t-\int_0^t\lambda(y)dy\right]}{\beta\left(\gamma^\beta\theta^\beta+1\right)^{\alpha\gamma t/\beta}\Gamma(\alpha\gamma t)}, \qquad (4.65)$$

$$\overline{S^2}(t) = \frac{\theta^{\alpha\gamma t}\Gamma\left(\frac{\alpha\gamma t}{\beta}\right)\exp\left[-\alpha\gamma t-\int_0^t\lambda(y)dy\right]}{\beta\left(\gamma^\beta\theta^\beta+2\right)^{\alpha\gamma t/\beta}\Gamma(\alpha\gamma t)}. \qquad (4.66)$$

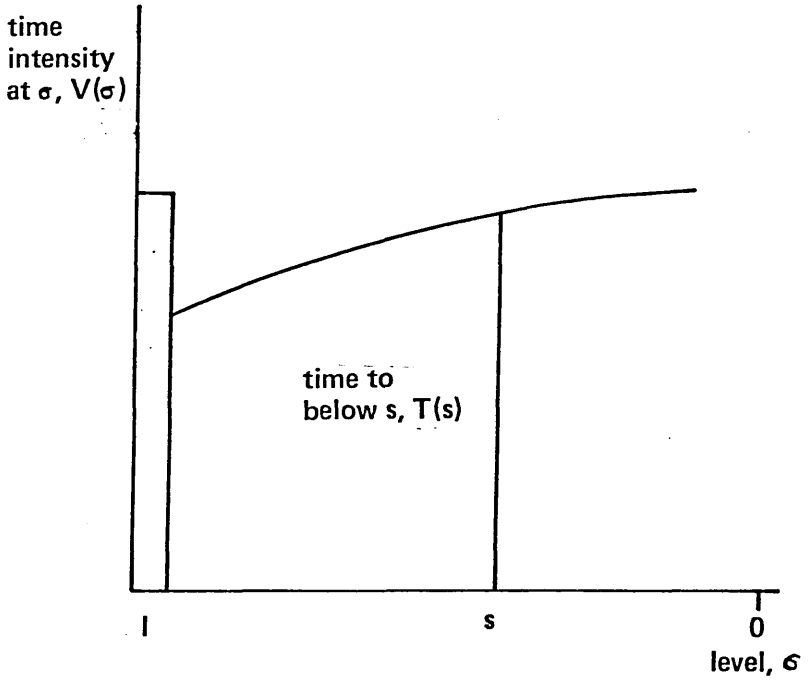
The form of $\lambda(t)$ is of particular importance for the measures defined in terms of integration over t. For example, if $\lambda(t) = \lambda$ for all t, then for the exponential special case of the Weibull transformation ($\beta = 1$) we obtain

$$\bar{\bar{S}} = \left[\alpha\gamma + \lambda - \alpha\gamma \ln \left(\gamma + \frac{1}{\alpha}\right)\right]^{-1} . \qquad (4.67)$$

4.3 Component Independence

The above measures are useful for studying and comparing the characteristics of systems of partially operating components, particularly since various authors have suggested different standard system structure functions as a basis for systems performance. Generally, however, the same basic underlying assumption of independence is made in each of these approaches (e.g. Postelnicu (1970), Barlow and Wu (1978), El-Neweihi, Proschan and Sethuraman (1978), Hatoyama (1979), Butler (1979b), Ross (1979)). In the dichotomic reliability model independence of times to failure of components is usually assumed, but in the partial operation model it is apparent from the various summary measures that there are three distinct families of distributions of interest for each component. These are the distribution of the level S(t) at a specified point of time t, $\varphi(s,t)$, the density of the time T(s) till the level drops below a specified level s $(s>0)$, say $\omega(t,s)$, and the density of the time intensity V(s) at a specified level s $(s>0)$, say $u(v,s)$. We assume that V(s) is continuous (see Figure 4.1). For convenience we also define $\omega(t,0)$ as the density of the time to level 0. Typically, previous authors have assumed the independence between components of the levels at any specified time, ignoring the other two component characteristics. Intuitively, however, if any one of

Figure 4.1
Relationship of T(s) and V(s) [equation (4.68)]



these three component characteristics is independent of the corresponding variable for other components then the other two characteristics should also be independent between components. We prove this result below, and also consider the less restricted property of zero correlation.

Of course, considering a single component,

$$T(s) = \int_s^{1-} V(\epsilon)\, d\epsilon + V_1 \quad , \quad 0 < s < 1 \qquad (4.68)$$

where V_1 is the finite time at s = 1, and we define $T(1) = V_1$.

Analogous to (1.2) in the dichotomic case we have for all $0 \leq s \leq 1$,

$$R(s,t) = \int_t^{\infty} w(\tau, s)\, d\tau \,. \qquad (4.69)$$

Equation (4.69) can be inverted to give

$$w(t,s) = -\frac{\partial R(s,t)}{\partial t}$$

in order to evaluate the density $w(t,s)$ which is of interest in its own right. Thus, for example, for the simple example of (4.11) to (4.13) we have

$$w(t,s) = 2(1-s)^{2t}\left[1-\ln(1-s)\right]e^{-2t} + \frac{2e^{-2t}}{t(2t+1)} \quad , \qquad (4.70)$$

whilst for the discrete example of (4.20) we have

$$w\left(t, \frac{i-1}{N-1}\right) = \mu\left[\frac{e^{-\mu t}(\mu t)^{N-i}}{(N-i)!}\right] \,. \qquad (4.71)$$

We can also derive a relationship for $u(v,s)$ in terms of $\varphi(s,t)$, which is again of interest in its own right. Supposing that the component reaches some level S $\quad 0 < s < 1$ at time t, then

$$\int_{v+\delta v}^{\infty} u(y,s|t)\, dy = \left[\int_v^{\infty} u(y,s|t)\, dy\right] \cdot \left[1 - \delta v \int_{o+}^{s-} d\tau\, \varphi(s,v,t) - \delta v\, \varphi(s,0,t)\right] \qquad (4.72)$$

where $u(v,s|t)$ is the conditional density of the component

having time intensity ν at state s given it reaches s at time t.

It follows that taking the limit as $\delta r \to 0$ then analogous to (1.3)

$$\frac{u(\nu,s/t)}{\int_{\nu}^{\infty} u(y,s/t)\,dy} = \int_{0+}^{s-} \varphi(s,r,t)\,dr + \varphi(s,0,t)\,, \tag{4.73}$$

and thus analogous to the expression for f(t) in (1.4)

$$u(\nu,s|t) = \left[\int_{0+}^{s-} \varphi(s,r,t)\,dr + \varphi(s,0,t)\right] \cdot \exp\left[-\nu\int_{0+}^{s-} \varphi(s,\epsilon,t)\,d\epsilon - \nu\,\varphi(s,0,t)\right], \tag{4.74}$$

which is a simple negative exponential in ν. (This corresponds to the Markovian nature assumed for $\varphi(s,\nu,t)$). Thus the unconditional density of time intensity at level s is

$$u(\nu,s) = \int_0^{\infty} u(\nu,s/t)\left[P_1(t)\varphi(1,s,t) + \int_{s+}^{1-} \varphi(r,t)\varphi(r,s,t)\,dr\right]dt, \tag{4.75}$$

and since analogous to (1.6)

$$\int_0^{\infty} \nu\, u(\nu,s/t)\,d\nu = \int_0^{\infty} d\nu \int_{\nu}^{\infty} u(y,s/t)\,dy\,,$$

it follows from (4.40), (4.73) and (4.75) that

$$\overline{V}(s) = \int_0^{\infty} \left[\frac{P_1(t)\varphi(1,s,t) + \int_{s+}^{1-} \varphi(r,t)\varphi(r,s,t)\,dr}{\int_{0+}^{s-} \varphi(s,r,t)\,dr + \varphi(s,0,t)}\right] dt\,.$$

For example, in the discrete example of (4.20) we obtain

$$u(\nu,s) = u(\nu,s/t) = \mu e^{-\nu\mu}\,, \tag{4.76}$$

as expected.

In what follows i subscripts on the functions $\varphi(s,t)$ $w(t,s)$, $u(\nu,s)$ etc. denote the component, i = 1,...,n. Writing $\underline{t} = (t_1,\ldots,t_n)$, $\underline{s} = (s_1,\ldots,s_n)$, $\underline{\nu} = (\nu_1,\ldots,\nu_n)$, the corresponding n-component (multivariate) functions to $\varphi(s,t)$, $w(t,s)$ and $u(\nu,s)$ may now be denoted $\varphi(\underline{s},\underline{t})$, $w(\underline{t},\underline{s})$ and $u(\underline{\nu},\underline{s})$ respectively, and represent the probability density that for all i = 1,..., n component i is in state s_i at time t_i, the joint density that for every components i the time till the level drops below level s_i is t_i $(s_i > 0)$, and the joint density that for all i the time intensity at state s_i is ν_i $(s_i > 0)$.

As in the univariate case, if any $s_i = 0$ in $w(\underline{t},\underline{s})$ then the joint density is defined as containing the time till component i enters level 0. Where subsets of the components are considered, these are identified by appropriate subscripts on the functions.

For notational convenience, and without loss of generality, we now treat $\varphi(\underline{s},\underline{t})$ and the associated univariate densities $\varphi_i(s_i,t_i)$ as composite distributions which include the discrete atoms of probability analogous to $p_0(t)$ and $p_1(t)$ in the univariate case, as well as the continuous density analogous to the univariate $\varphi(s,t)$. In consequence, integrations of these densities over s_i are interpreted as including the addition of these discrete atoms of probability where they are included in the range of integration. $u(v,s)$ is treated similarly.

Thus, analogous to (4.34) in the univariate case, we define for $0 \leq \underline{s} \leq \underline{1}$,

$$R(\underline{s},\underline{t}) = \int_{s_1(+)}^{1} d\sigma_1 \cdots \int_{s_n(+)}^{1} d\sigma_n \, \varphi(\underline{\sigma},\underline{t}) \qquad (4.77)$$

where

$$s_i(+) = \begin{cases} s_i, & 0 < s_i < 1 \\ 0+, & s_i = 0 \\ 1-, & s_i = 1 \end{cases}, \quad i = 1,\ldots,n.$$

It follows that for $0 < s_i \leq 1$, $i = 1,\ldots,n$, $R(\underline{s},\underline{t})$ is the probability that the level of component i at time t_i equals or exceeds s_i, for all i = 1, ..., n. On the other hand, if any $s_i = 0$, the atom of probability at zero is excluded from $R(\underline{s},\underline{t})$, so that the lower limit of integration is treated as 0+. With this definition, for $\underline{t} = \underline{0}$ we obtain $R(\underline{s},\underline{0}) = 1$ for all s, and for $\underline{t} = \underline{\infty}$ $R(\underline{s},\underline{\infty}) = 0$ for all s.

We also introduce

$$\psi_{s_i}^{i}(t_i) = \begin{cases} 1, \text{ if } S_i(t_i) = s_i \\ 0, \text{ otherwise} \end{cases} \tag{4.78}$$

where $S_i(t_i)$ is the stochastic process for component i.

Generalising (4.69) and (4.38) we now have

$$R(\underline{s},\underline{t}) = \int_{t_1}^{\infty} d\tau_1 \cdots \int_{t_n}^{\infty} d\tau_n \, w(\underline{\tau},\underline{s}), \quad \underline{0} \leqslant \underline{s} \leqslant \underline{1}, \tag{4.79}$$

and

$$E\left[\prod_{i=1}^{n} \psi_{s_i}^{i}(t_i)\right] = \varphi(\underline{s},\underline{t}). \tag{4.80}$$

Theorem 4.1

Independence between components of levels of operation S(t) for all time t, independence of the times T(s) till the levels drop below specified levels for all levels s(s>0), and independence of time intensities V(s) at specified levels for all levels s(s>0), are equivalent.

Proof:

Let $$\varphi(\underline{s},\underline{t}) = \prod_{i=1}^{n} \varphi_i(s_i,t_i) \tag{4.81}$$

for all values of s_i and t_i,

$i = 1, \ldots, n$,

then by (4.77)

$$R(\underline{s},\underline{t}) = \prod_{i=1}^{n} \left[\int_{s_i(t)}^{1} \varphi_i(u,t_i)\,du\right] = \prod_{i=1}^{n} R_i(s_i,t_i). \tag{4.82}$$

Therefore, from (4.79) and (4.82)

$$w(\underline{t},\underline{s}) = \frac{(-1)^n d^n R(\underline{s},\underline{t})}{dt_1\, dt_2 \cdots dt_n} = (-1)^n \prod_{i=1}^{n} \frac{dR_i(s_i,t_i)}{dt_i} = \prod_{i=1}^{n} w_i(t_i,s_i), \quad \underline{0} \leqslant \underline{s} \leqslant \underline{1}. \tag{4.83}$$

Thus the times till the component levels drop below specified levels are independent if the levels at specified times are, and reversing the steps of the proof the converse also follows.

Independence of time intensities at specified levels follows from the independence of times till the component levels drop below specified levels due to the triangular structure of the $T_i(s_i)$ in terms of the $V_i(\delta_i)$, $s_i \le \delta_i < 1$ in (4.68). Thus, for any $0 < s_i < 1$,

$$V_i(s_i+\delta s_i)\delta s_i = T_i(s_i) - T_i(s_i+\delta s_i). \quad (4.84)$$

If (4.83) holds for all s_i, then for all $\underline{s}$ both the $T_i(s_i)$ $i=1,\ldots,n$ and the $T_i(s_i+\delta s_i)$ $i = 1, \ldots, n$ are indepdndent. Also, from (4.84),

$$u(\underline{v},\underline{s}) = \int_{v_1}^{\infty} dt_1 \cdots \int_{v_n}^{\infty} dt_n \quad (4.85)$$
$$\cdot w(\underline{t}-\underline{v},\underline{s}+\delta\underline{s})\, w^*(\underline{t},\underline{s}\,|\,\underline{t}-\underline{v},\underline{s}+\delta\underline{s}),$$

where $w^*(\underline{t},\underline{s}\,|\,\underline{t}-\underline{v},\underline{s}+\delta\underline{s})$ is the conditional joint density for all components $i = 1, \ldots, n$ of the time t_i till component i drops below level s_i, given the time $t_i - v_i$ till it drops below level $s_i + \delta s_i$. (As before, if any $s_i = 0$ this conditional joint density is defined as containing the time till component i enters level 0). Thus if (4.83) holds, the joint conditional density decomposes into a product by Bayes Theorem and

$$u(\underline{v},\underline{s}) = \prod_{i=1}^{n}\Big[\int_{v_i}^{\infty} dt_i\, w_i(t_i-v_i, s_i+\delta s_i)$$
$$\cdot w_i^*(t_i, s_i\,|\,t_i-v_i, s_i+\delta s_i)\Big] \quad (4.86)$$
$$= \prod_{i=1}^{n} u_i(v_i, s_i),$$

where $\omega_i^*(t_i, s_i | t_i - v_i, s_i + \delta s_i)$ is the conditional density for component i of the time t_i till it drops below level s_i $(s_i > 0)$ given the time $t_i - v_i$ till it drops below level $s_i + \delta s_i$, with the usual redefinition if $s_i = 0$.

If any $s_i = 1$, then $s_i + \delta s_i$ on the right-hand-side of (4.85) is replaced by $1+$, $t_i - v_i$ by 0, and t_i by v_i, and the integration over t_i is deleted. Thus we can eliminate the i^{th} component from $\omega(\underline{t} - \underline{v}, \underline{s} + \delta \underline{s})$, and the conditioning in $\omega^*(\underline{t}, \underline{s} | \underline{t} - \underline{v}, \underline{s} + \delta \underline{s})$ on the time for the i^{th} component can also be removed. Otherwise the proof is unchanged.

To prove that (4.86) implies (4.83), we commence with

$$s_i + \delta s_i = 1, \quad i = 1, \ldots, n$$

and proceed recursively using

$$T_i(1) = V_i(1)$$

and $$T_i(s_i) = T_i(s_i + \delta s_i) + V_i(s_i + \delta s_i)\delta s_i,$$

where $V_i(1)$ is the finite time at $s_i = 1$ for component i. A similar argument to that which leads to (4.86) then yields the result.

Proposition 4.1.1

Pairwise independence between components of levels of operation for all time t, pairwise independence of the times till the levels drop below specified levels for all levels $s(s>0)$, and pairwise independence of time intensities at specified levels for all levels $s(s>0)$ are equivalent.

Proof:

The Proposition is proved by restricting the set of components considered in proving Theorem 4.1 to any two. (We note that the Proposition extends immediately to independence between any subset of the components).

Theorem 4.2

The following two conditions are equivalent:

(i) $$E\left[\prod_{i=1}^{n} V_i(s_i)\right] = \prod_{i=1}^{n} \overline{V}_i(s_i), \text{ for all } \underline{s} \qquad (4.87)$$

(ii) $$E\left[\prod_{i=1}^{n} T_i(s_i)\right] = \prod_{i=1}^{n} \overline{T_i}(s_i), \text{ for all } \underline{s}. \quad (4.88)$$

Proof:

Let (4.87) hold. Then for $s_i < 1$ for all $i = 1, \ldots, n$, integrating both sides with respect to $s_1, \ldots, s_n$ and interchanging integration and expectation, we obtain

$$E\left[\prod_{i=1}^{n} \int_{s_i}^{1} V_i(\theta)\, d\theta\right] = \prod_{i=1}^{n} E\left[\int_{s_i}^{1} V_i(\theta)\, d\theta\right].$$

Thus by (4.68) it follows that (4.88) holds. If any $s_i = 1$ then $V_i(1) = T_i(1)$ and integration is with respect to all $s_1, \ldots, s_n$ not equal to unity.

Differentiation together with the triangular structure of the $T_i(s_i)$ in terms of the $V_i(\theta)$ $s_i \le \theta \le 1$ in (4.68), provides the converse.

Proposition 4.2.1

For any two components, zero correlation of times till the levels drop below specified levels for all levels s_i, s_j, and zero correlation of time intensities at all pairs of specified levels are equivalent.

Proof:

Again, the Proposition follows immediately by restricting the components considered in proving Theorem 4.2 to any two.

Theorem 4.3

Any of the conditions (4.87), (4.88), or

$$E\left[\prod_{i=1}^{n} S_i(t_i)\right] = \prod_{i=1}^{n} \overline{S_i}(t_i), \text{ for all } \underline{t}, \quad (4.89)$$

imply that

$$E\left[\prod_{i=1}^{n} \int_0^{\infty} S_i(t)\, dt\right] = \prod_{i=1}^{n} \overline{\overline{S_i}}. \quad (4.90)$$

Proof:

That (4.89) implies (4.90) follows immediately by integrating over $t_1, \ldots, t_n$, interchanging the order of integration and expectation, and applying (4.47).

It remains to prove that (4.87), or equivalently (4.88), imply (4.90). To do this we first establish a useful equality for the multivariate process. From (4.39)

$$E\left[\prod_{i=1}^{n} V_i(s_i)\right] = E\left[\prod_{i=1}^{n} \int_0^\infty \psi_{s_i}^{i}(t_i)\, dt_i\right],$$

and interchanging integration and expectation we have using (4.80) that

$$E\left[\prod_{i=1}^{n} V_i(s_i)\right] = \int_0^\infty dt_1 \cdots \int_0^\infty dt_n\, E\left[\prod_{i=1}^{n} \psi_{s_i}^{i}(t_i)\right]$$

$$= \int_0^\infty dt_1 \cdots \int_0^\infty dt_n\, \varphi(\underline{s}, \underline{t}),$$

analogous to (4.40).

Hence multiplying through by $\left(\prod_{i=1}^{n} s_i\right)$ and integrating over $s_1, \ldots, s_n$ we obtain after interchanging the order of integration,

$$\int_0^1 ds_1 \cdots \int_0^1 ds_n \left(\prod_{i=1}^{n} s_i\right) E\left[\prod_{i=1}^{n} V_i(s_i)\right] \quad (4.91)$$

$$= E\left[\prod_{i=1}^{n} \int_0^\infty s_i(t)\, dt\right].$$

Suppose now that (4.87) holds. Then by the alternative form for $\bar{\bar{S}}$ in (4.47), the left-hand-side of (4.91) is equivalent to $\left(\prod_{i=1}^{n} \bar{\bar{s}}_i\right)$, and thus (4.91) reduces to (4.90).

Theorem 4.4

In the dichotomic case, (4.89) implies independence of levels of operation, and consequently of times to failure. Zero correlation between the levels at specified times implies pairwise independence.

Proof:

As standard, define the two operation levels as 1 (operation) and 0 (failed). Then

$$E[s_i(t_i)] = \varphi_i(1,t_i),$$
$$E\left[\prod_{i=1}^{n} s_i(t_i)\right] = \varphi(\underline{1},\underline{t}).$$

Thus

$$\varphi(\underline{1},\underline{t}) = \prod_{i=1}^{n} \varphi_i(1,t_i) \quad \text{for all } t_1,\ldots,t_n \quad (4.92)$$

iff (4.89) holds.

Also, in the dichotomic case (4.69) and (4.79) become

$$\varphi_i(1,t_i) = \int_{t_i}^{\infty} w_i(u,0)\,du$$
$$\varphi(\underline{1},\underline{t}) = \int_{t_1}^{\infty} dv_1 \cdots \int_{t_n}^{\infty} dv_n\, w(\underline{v},\underline{0}),$$

so that iff (4.89) holds

$$w(\underline{t},\underline{0}) = (-1)^n \frac{\partial^n \varphi(\underline{1},\underline{t})}{\partial t_1 \cdots \partial t_n} = (-1)^n \prod_{i=1}^{n}\left[\frac{\partial \varphi_i(1,t_i)}{\partial t_i}\right]$$
$$= \prod_{i=1}^{n} w_i(t_i,0).$$

We also note that we have immediately,

$$\varphi(\underline{0},\underline{t}) = \int_0^{t_1} dv_1 \cdots \int_0^{t_n} dv_n\, w(\underline{v},\underline{0})$$
$$= \prod_{i=1}^{n}\left[\int_0^{t_i} w_i(v,0)\,dv\right]$$
$$= \prod_{i=1}^{n} \varphi_i(0,t_i).$$

Further,

$$P_{1,\ldots,i-1,i+1,\ldots,n}(\underline{1},\underline{t}) = P_{1,\ldots,n}(\underline{1},\underline{t}) + P^{(i)}_{1,\ldots,n}(\underline{1},\underline{t}),$$

where $P^{(i)}_{1,\ldots,n}(\underline{1},\underline{t})$ represents the probability that at the times t_1, ..., t_n respectively all of the components are operating except the i^{th} ($1 \le i \le n$) which is failed, $P_{1,\ldots,n}(\underline{1},\underline{t})$ represents the probability that all the n components are operating, and $P_{1,\ldots,i-1,i+1,\ldots,n}(\underline{1},\underline{t})$ represents the probability that the group of (n-1) components excluding the i^{th} are all operating. It follows from (4.92) that

$$P^{(i)}_{1,\ldots,n}(\underline{1},\underline{t}) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \varphi_j(1,t_j) - \prod_{j=1}^{n} \varphi_j(1,t_j)$$

$$= \left[1 - \varphi_i(1,t_i)\right] \prod_{\substack{j=1 \\ j\neq i}}^{n} \varphi_j(1,t_j) .$$

Similarly,

$$P^{(i,j)}_{1,\ldots,n}(\underline{1},\underline{t}) = P^{(j)}_{1,\ldots,i-1,i+1,\ldots,n}(\underline{1},\underline{t}) - P^{(j)}_{1,\ldots,n}(\underline{1},\underline{t}),$$

where $P^{(i,j)}_{1,\ldots,n}(\underline{1},\underline{t}),\ i\neq j,$ represents the probability that at the times $t_1, \ldots, t_n$ respectively all of the components are operating except i and j which are failed. Thus,

$$P^{(i,j)}_{1,\ldots,n}(\underline{1},\underline{t}) = \left[1 - \varphi_i(1,t_i)\right]\left[1 - \varphi_j(1,t_j)\right] \prod_{\substack{l=1 \\ l\neq i,j}}^{n} \varphi_l(1,t_l).$$

In general,

$$P^{(J,i)}_{1,\ldots,n}(\underline{1},\underline{t}) = P^{(J)}_{1,\ldots,i-1,i+1,\ldots,n}(\underline{1},\underline{t}) - P^{(J)}_{1,\ldots,n}(\underline{1},\underline{t}),$$

where J is any non-empty subset of $j = 1, \ldots, i-1, i+1, \ldots, n$, and $i \notin J$, $P^{(J)}_{1,\ldots,n}(\underline{1},\underline{t})$ represents the probability that at the times $t_1, \ldots, t_n$ respectively all of the components are operating except the members of J which are failed. Thus the independence of the levels of operation follows immediately by induction.

To establish pairwise independence, again restrict the components considered to any two.

We now provide two counter examples to prove respectively that Theorem 4.4 does not generalise to more than two levels of performance, and that (4.88) and (4.87) do not, in the dichotomic case, imply independence of times to failure.

To verify that Theorem 4.4 does not directly generalise to more than two levels ofperformance, we consider the following simple example of three-level units.

Example 4.3.1

Suppose that

$$\varphi_i(0,t_i) = \varphi_i\left(\tfrac{1}{2},t_i\right) = \varphi_i(1,t_i) = \tfrac{1}{3}, \text{ for } i=1,2.$$

Then

$$E\left[S_1(t_1)\right] = E\left[S_2(t_2)\right] = \tfrac{1}{2},$$

and if the levels are uncorrelated

$$E\left[S_1(t_1)S_2(t_2)\right] = \tfrac{1}{4}.$$

But

$$\begin{aligned} E\left[S_1(t_1)S_2(t_2)\right] = {} & \tfrac{1}{4}\varphi\left(\tfrac{1}{2},\tfrac{1}{2},t_1,t_2\right) \\ & + \varphi(1,1,t_1,t_2) \\ & + \tfrac{1}{2}\varphi\left(\tfrac{1}{2},1,t_1,t_2\right) \\ & + \tfrac{1}{2}\varphi\left(1,\tfrac{1}{2},t_1,t_2\right), \end{aligned}$$

so that if say $\varphi(s_1,s_2,t_1,t_2)$ is given by

$\varphi(s_1,s_2,t_1,t_2)$		s_2 0	$\frac{1}{2}$	1	$\varphi_1(s_1,t_1)$
s_1	0	7/36	0	5/36	$\frac{1}{3}$
	$\frac{1}{2}$	0	8/36	4/36	$\frac{1}{3}$
	1	5/36	4/36	3/36	$\frac{1}{3}$
$\varphi_2(s_2,t_2)$		$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

then the levels are uncorrelated but not independent.

Example 4.3.2

For our second example we consider two components with a joint distribution of failure times given by a uniform distribution on the circumference of a circle with center (1,1) and radius r, $0<r\leq 1$. Thus

$$w(t_1,t_2,0,0) = \begin{cases} \dfrac{1}{2\pi r}, & (t_1-1)^2+(t_2-1)^2 = r^2 \\ 0, & \text{otherwise} \end{cases}.$$

It is obvious that t_1 and t_2 are not independent, because knowing the value of t_1 gives information about t_2. It also follows by symmetry that

$$E[T_1(0)] = E[T_2(0)] = 1,$$

and

$$E[T_1(0)T_2(0)] = 1,$$

so that the times to failure are uncorrelated, although not independent.

The last example provides a verification that the property (4.89) is not equivalent to properties (4.88) and (4.87), since in the dichotomic case (4.89) implies independence of times to failure whilst (4.87) and (4.88) do not.

As a final point in this section we note that in the dichotomic case property (4.90) is equivalent to (4.87) and (4.88) since

$$\int_0^\infty s_i(t)\,dt = T_i(0) .$$

Thus

$$E\left[\prod_{i=1}^{n} \int_0^\infty s_i(t)\,dt\right] = E\left[\prod_{i=1}^{n} T_i(0)\right] ,$$

and

$$\overline{s_i} = \int_0^\infty E[s_i(t_i)]\,dt_i = \int_0^\infty \varphi_i(1,t_i)\,dt_i$$

$$= \overline{T_i}(0) , \qquad \text{by (4.41).}$$

4.4 Series/Parallel Connections

Whether components characteristics are independent or not, there remains the selection of appropriate structure functions $f(s_1, \ldots, s_n)$, specifying the level of operation of the system in terms of the level of operation of the components. For a coherent

system Barlow and Wu (1978) define $f(\underline{s})$ by

$$f(\underline{s}) = \max_{1 \le r \le p} \min_{i \in P_r} s_i = \min_{1 \le u \le k} \max_{i \in K_u} s_i$$

where $\{P_1, \ldots, P_p\}$ are the minimum path sets and $\{K_1, \ldots, K_k\}$ the minimum cut sets of a corresponding (one-to-one) binary coherent system of the Barlow and Proschan (1975) type. The above definition is an obvious generalisation of the dichotomic case, and Barlow and Wu use this to define a multi-levelled <u>series system</u> and a multi-levelled <u>parallel system</u> by

$$f(\underline{s}) = \min_{i=1,\ldots,n} s_i \qquad (4.93)$$

and

$$f(\underline{s}) = \max_{i=1,\ldots,n} s_i \ ,$$

respectively.

These are also the definitions of series and parallel multi-levelled systems employed by El-Neweihi, Proschan and Sethuraman (1978) and Hatoyama (1979), but not Hudson and Kapur (1982).

With so many possible system structure functions and coherent structure functions in the multi-level case, and infinite numbers in the continuous case, it is not unnatural to look particularly at the relatively simple series and parallel systems. However, the definitions of these structures employed by the above authors have many possible alternatives, e,g. Zijlstra (1980), Fardis and Cornell (1981), Hudson and Kapur (1982). They are contained within our categories of series and parallel systems employed in Theorem 2.2 which only impose dichotomic restrictions on the system's structure function. In fact, the definitions in (4.93) are of some physical interest, since El-Neweihi, Proschan and Sethuraman (1978) show that for their

restricted definition of coherent systems the level of the system is always between or equal to the series and parallel system levels as given in (4.93). This proof, in fact, generalises immediately to our wider class of narrow-sense coherent systems, as is shown by Griffith (1980), since for these

$$s_{min} = f(s_{min}, \ldots, s_{min}) \leq f(s_1, \ldots, s_n) \leq f(s_{max}, \ldots, s_{max}) = s_{max}, \tag{4.94}$$

where

$$s_{min} = \min_{i=1,\ldots,n} s_i \, , \quad s_{max} = \max_{i=1,\ldots,n} s_i \, .$$

Griffith (1980) shows that the class of monotone functions satisfying (4.94) is in fact equivalent to our narrow-sense class. Thus, the above proof cannot be generalised further to include coherent systems in the wide sense.

Physical considerations, however, may suggest alternative restrictions on the classes of multi-level series and parallel systems. For example, in certain applications we may regard as somewhat pathological a parallel system whose level exceeds the sum of the levels of the components, or a series system the level of which is less than the product of the component levels. Of course, in applying these structures, the sum of the levels must be truncated above at unity, and for the discrete case the product of the levels will not always exist as a level and a rounding mechanism will be necessary. It is apparent that

$$\begin{aligned} \prod_{i=1}^{n} s_i &\leqslant s_{min} \\ \sum_{i=1}^{n} s_i &\geqslant s_{max}, \end{aligned} \tag{4.95}$$

and that these systems are outside the class of narrow sense coherent

systems although within the wide sense class. It is also apparent that whilst the series and parallel definitions as used by the previous authors are appropriate for certain systems such as some communication, transportation and water systems, for others even within these applications our alternative definitions may be more appropriate. For example, the parallel lighting units considered by Zijlstra (1980) correspond to our truncated sum definition. (See also Fardis and Cornell (1981) and Hudson and Kapur (1982)).

For a system of n independent components ($n \geq 2$) the following expressions are obtained for the distribution of systems states at time t, where the suffixes denote the component, and the superfix n denotes that the system is of the first n components. In each case the distribution is, of course, again a composite of a continuous density $\varphi^{n}_{system}(s,t)$ over the range (0,1), and discrete atoms of probability at s = 0 and 1, $p^{n}_{system(0)}(t)$ and $p^{n}_{system(1)}(t)$.

Firstly, considering the minimum rule we have

$$\varphi^{n}_{min}(s,t) = \sum_{\{I\}} \prod_{i \in I} \varphi_i(s,t) \cdot \prod_{j \in I'} R_j(s+,t), \qquad (4.96)$$

where $\{I\}$ represents every non-empty subset of i = 1,2,...,n and $\{I'\}$ represents the complements of these subsets (i.e. the values of j = 1,2,..., n not included in $\{i : i \in I\}$). Alternatively, $\varphi^{n}_{min}(s,t)$ is expressable by the Inclusion - Exclusion Theorem (e.g. Feller (1968)). Also we have

$$p^{n}_{min(0)}(t) = \sum_{\{I\}} \prod_{i \in I} p_{i(0)}(t) \cdot \prod_{j \in I'} \left[1 - p_{j(0)}(t)\right], \qquad (4.97)$$

$$p_{\min(1)}(t) = \prod_{i=1}^{n} p_{i(1)}(t) . \qquad (4.98)$$

With the same notation for the maximum rule we obtain

$$\varphi^{n}_{\max}(s,t) = \sum_{\{I\}} \prod_{i\in I} \varphi_i(s,t) \cdot \prod_{j\in I'} [1 - R_j(s,t)] , \qquad (4.99)$$

$$p^{n}_{\max(0)}(t) = \prod_{i=1}^{n} p_{i(0)}(t) , \qquad (4.100)$$

$$p^{n}_{\max(1)}(t) = \sum_{\{I\}} \prod_{i\in I} p_{i(1)}(t) \cdot \prod_{j\in I'} [1 - p_{j(1)}(t)] . \qquad (4.101)$$

Alternatively, for the product rule,

$$\varphi^{n}_{\text{prod}}(s,t) = \int_{s+}^{1-} \varphi_n\left(\left[\frac{s}{\sigma}\right],t\right) \varphi^{n-1}_{\text{prod}}(\sigma,t) \, d\sigma + p_{n(1)}(t)\, \varphi^{n-1}_{\text{prod}}(s,t) + \varphi_n(s,t) \prod_{i=1}^{n-1} p_{i(1)}(t) , \qquad (4.102)$$

$$p^{n}_{\text{prod}(0)}(t) = \sum_{\{I\}} \prod_{i\in I} p_{i(0)}(t) \prod_{j\in I'} [1 - p_{i(0)}(t)] , \qquad (4.103)$$

$$p^{n}_{\text{prod}(1)}(t) = \prod_{i=1}^{n} p_{i(1)}(t) . \qquad (4.104)$$

Finally, for the case of the truncated sum,

$$\varphi^{n}_{\text{sum}}(s,t) = \int_{0+}^{s-} \varphi_n([s-\sigma],t)\, \varphi^{n-1}_{\text{sum}}(\sigma,t)\, d\sigma + p_{n(0)}(t)\, \varphi^{n-1}_{\text{sum}}(s,t) + \varphi_n(s,t) \prod_{i=1}^{n-1} p_{i(0)}(t) , \qquad (4.105)$$

$$p^{n}_{\text{sum}(0)}(t) = \prod_{i=1}^{n} p_{i(0)}(t) , \qquad (4.106)$$

$$P^{n}_{sum(1)}(t) = p_{n(1)}(t)\left[1 - P^{n-1}_{sum(1)}(t)\right] + P^{n-1}_{sum(1)}(t) + \int_{1}^{(1+\delta)-} dr \int_{0+}^{1-} d\delta\, \varphi_n\left([r-\delta],t\right)\varphi^{n-1}_{sum}(\delta,t). \tag{4.107}$$

As may be expected, the conventional definitions of series and parallel systems tend to lead to simpler expressions for the systems probability distribution than do the alternatives. For example, for two identical components, with densities corresponding to the exponential transformation of Mercer's model we obtain for $t>0$, for the continuous part of the distributions

$$\varphi^{2}_{min}(s,t) = \left[\frac{2\Gamma(-\gamma\theta \ln s, \alpha\gamma t)\exp\{-\alpha\gamma t - \int_0^t \lambda(y)dy\}}{\gamma\,\alpha\gamma t\,\Gamma(\alpha\gamma t)}\right] \cdot \varphi(s,t) \tag{4.108}$$

$$\varphi^{2}_{max}(s,t) = 2\varphi(s,t) - \varphi^{2}_{min}(s,t) \tag{4.109}$$

$$\varphi^{2}_{prod}(s,t) = \sqrt{\Pi}\,\Theta^{2\alpha\gamma t}\exp\{-2\alpha\gamma t - 2\int_0^t \lambda(y)dy\} \cdot I_{\alpha\gamma t - 1/2}\left(-\tfrac{1}{2}\ln s\right) s^{\gamma\theta - \frac{1}{2}}(-\ln s)^{\alpha\gamma t - 1/2}, \tag{4.110}$$

where $I_{\nu}(x)$ is the Bessel function of imaginary argument. (See e.g. Gradshteyn and Ryzhik (1980)). In this case $\varphi^{2}_{sum}(s,t)$ does not appear to simplify.

The various measures of component operation have their direct analogues for system operation, and these are of use for comparison purposes. One useful relationship immediately to hand is that under component independence, the expected level of the product system at time t is

$$\overline{S^{n}_{prod}}(t) = \prod_{i=1}^{n} \overline{S_i}(t). \tag{4.111}$$

Of course, in the limit as t tends to infinity, the distribution of levels concentrates towards state 0. Thus, for t large the expected

level of the truncated sum system is given approximately by

$$\overline{S^{n}_{sum}}(t) \simeq \sum_{i=1}^{n} \overline{S_i}(t), \qquad (4.112)$$

and this also provides an upper bound for $\overline{S^{n}_{sum}}(t)$ for all t. Therefore, whilst the evaluation of the system probability distribution is easier for the minimum and maximum rules, this does not extend to the measures of performance.

To illustrate the differing series and parallel connections we show in Figure 4.2 the ELO at time t, $\overline{S}(t)$ for a single component with distribution given by (4.32) and (4.33) with $\beta = 1$, $\alpha = 5$, $\gamma = 0.2$, $\theta = 10$ and $\lambda(t) = 0.6$, and also the corresponding series and parallel systems of two components. The corresponding values of the ELCO, , are given in Table 4.1. Also, for all the possible (dichotomic) three-identical component series-parallel systems shown in Figure 4.3, we show in Figure 4.4 the expected levels of operation corresponding to various series-parallel connections, for the components described above. It is seen that substantive differences in system characteristics can be obtained depending on the definitions of series and parallel connections employed. In particular, comparisons between system configurations are very dependent in magnitude upon the definitions employed (although not in direction).

In the case of dichotomic reliability, it is well known that the advantages to be gained from the use of parallel redundancy depend greatly upon the particular failure distribution which is appropriate and the values of the parameters.(See e.g. Lomnicki (1973) Table 1). This is the case too with our generalised model, although now the

Figure 4.2
Expected Levels of Operation for different series and parallel systems of two components with distribution (4.32), (4.33) and $\beta = 1, \alpha = 5, \gamma = 0.2, \theta = 10, \lambda(t) = 0.6$.

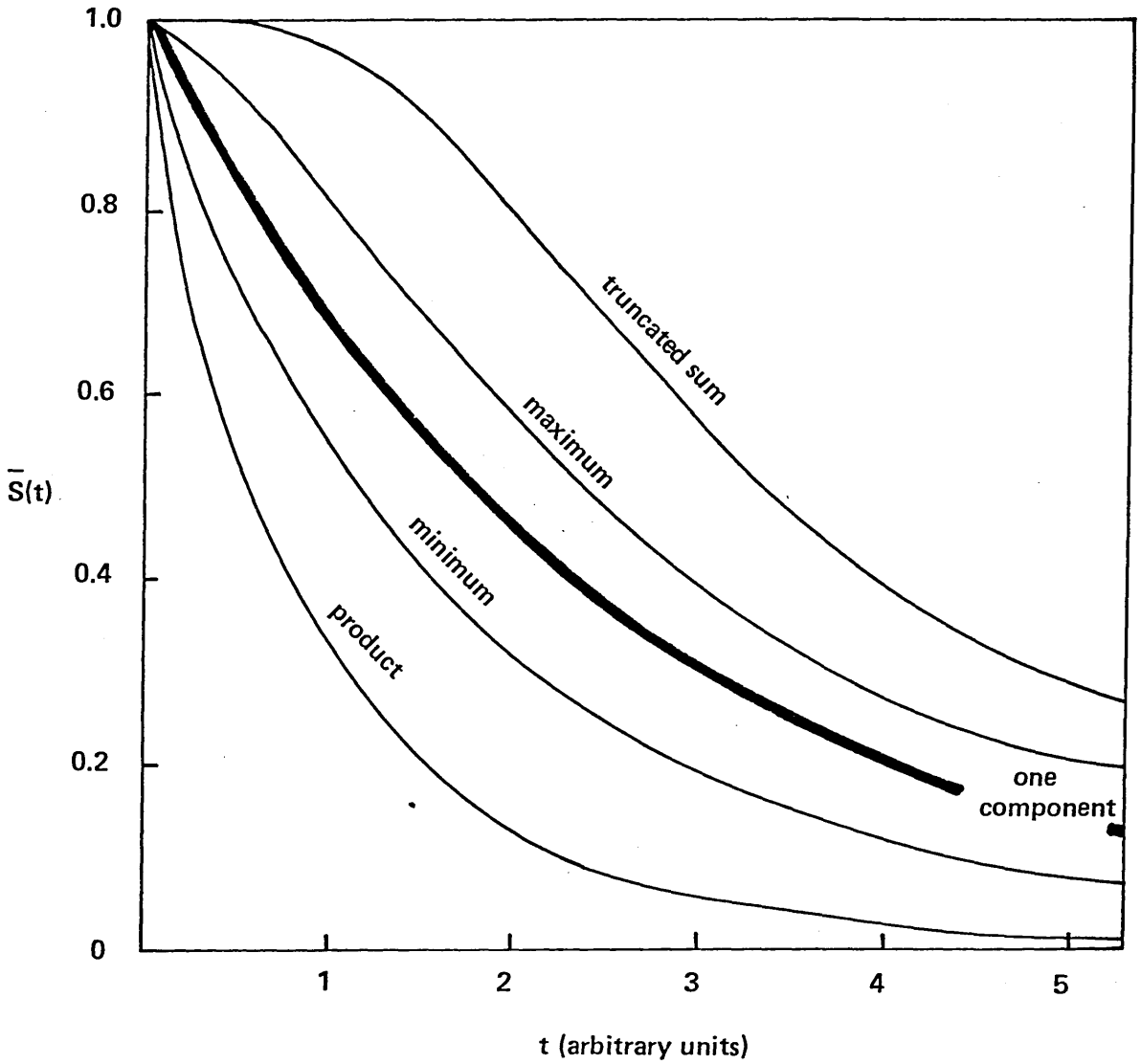


TABLE 4.1

Expected Lifetime Coefficient of Operation for the two-component systems shown in Figure 4.2.

Connection	ELCO
1 component	1.25
truncated sum	2.13
maximum	1.64
product	0.45
minimum	0.91

FIGURE 4.3

Series-parallel systems of three identical components.

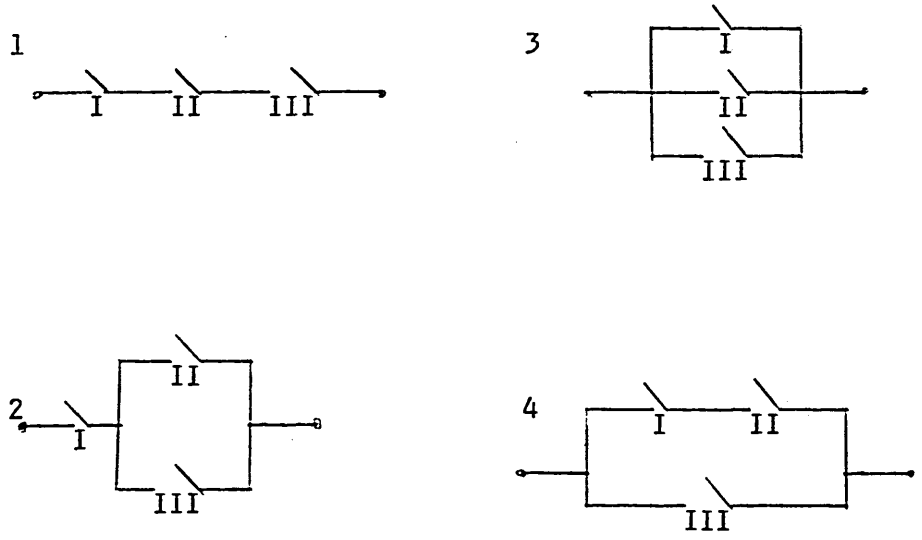
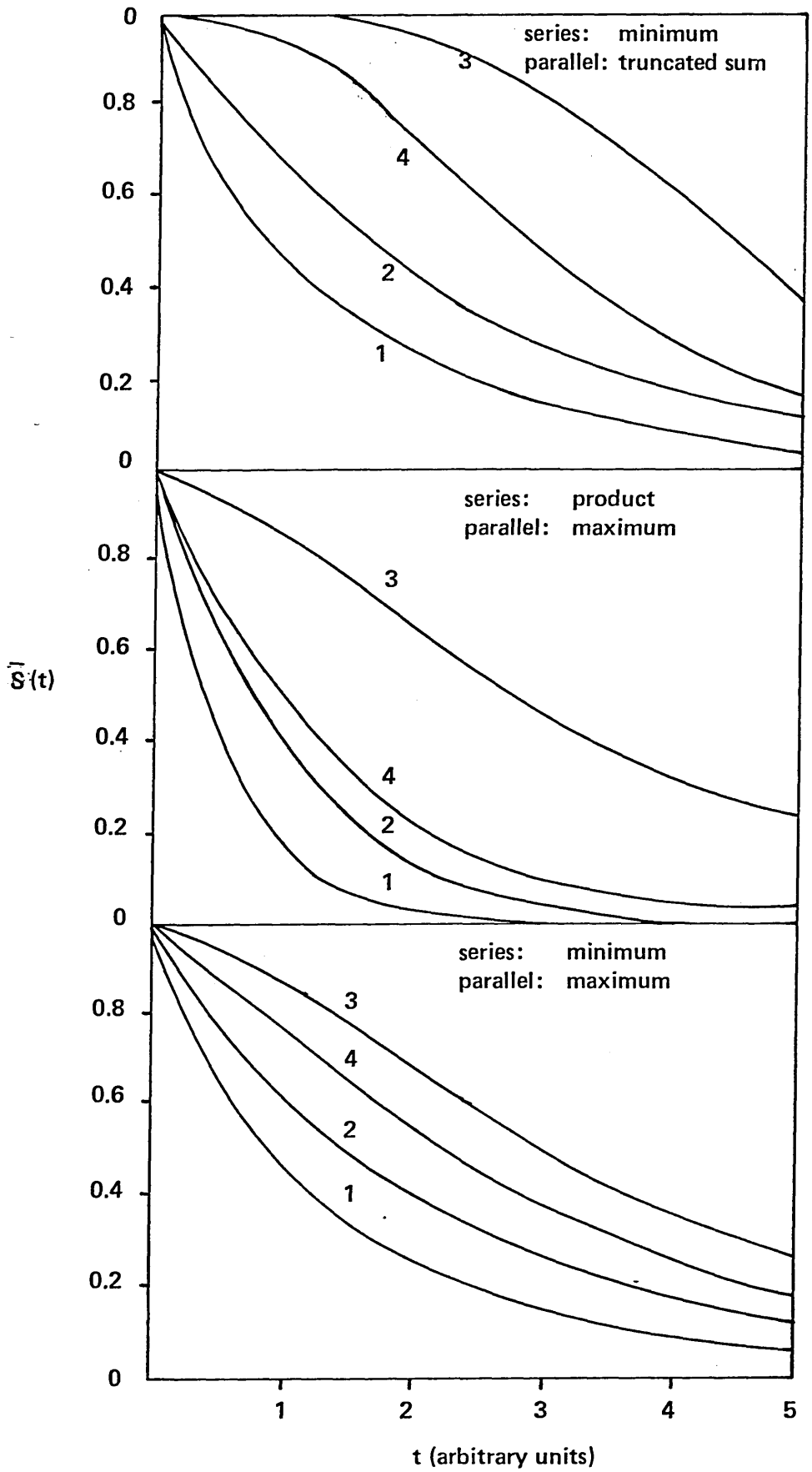


Figure 4.4
Expected Levels of Operation for series-parallel systems of three identical components



advantages are also dependent upon the particular combination rules employed. Table 4.2 shows the improvement ratios indicating the value of $\overline{S}(t)$ as a proportion of that for a single component which can be obtained by replacing a single component with the above characteristics by a parallel configuration of two or three such identical components. Whilst the truncated-sum connection of course provides the greatest improvement, with it for small t using a parallel configuration of three rather than two components offers little advantage. For each combination rule the advantage of redundancy is seen to be increasing with time.

4.5 Replacement

The model considered so far in this Chapter ignores the possibility of replacement. The simplest renewal theory for this process would associate instantaneous replacement to the full operating state 1 with entry to the failed state 0, and reinitiation of the time count from this point. In this case the continuous part of the distribution of level of operation s at time t is

$$\mathfrak{z}(s,t) = \int_0^t \varphi(s,\tau) \sum_{k=0}^{\infty} f^{k}(t-\tau)\, d\tau \qquad (4.113)$$

where $f(t) = \dfrac{dP_0(t)}{dt}$,

and $f^{k}(t)$ represents the k^{th} convolution of f(t), and $f^{0}(t) = \delta(t), f^{1}(t) = f(t)$. The probability of being in state 1 at time t, $Z_1(t)$ is given by (4.113) with $\varphi(s,\tau)$ replaced by $P_1(\tau)$. Measures for this process may be defined analogously to those obtained previously, with $\mathfrak{z}(s,t)$ replacing $\varphi(s,t)$ and $Z_1(t)$ replacing $P_1(t)$ in (4.34), (4.40), (4.41), (4.43) and (4.45), and the form of (4.44) and (4.47) remaining unchanged.

TABLE 4.2

Improvement ratios for systems of n identical components arranged in parallel (using truncated sum or maximum rule).

$\beta = 1, \quad \alpha = 5, \quad \gamma = 0.2, \quad \theta = 10, \quad \lambda(t) = 0.6.$

	Truncated sum		Maximum	
t \ n	2	3	2	3
1	1.45	1.50	1.20	1.33
2	1.82	2.14	1.32	1.57
3	1.93	2.83	1.38	1.73
4	1.97	2.95	1.42	1.80
5	2.00	3.01	1.45	1.86

If $R(s,t)$ is instead defined as previously (4.35) now gives the mean first passage time to level s and this is related to the mean time intensity at level s for one cycle by (4.42). Now, of course, the monotonicity of the various measures no longer holds, except for the mean first passage times.

As an example of the renewal formulation, consider the simple limiting case of (4.14). Then

$$g(s,t) = \frac{\left[1-(1-s)^{2t}e^{-2t}\left\{1-2t\left[\ln(1-s)-1\right]\right\}\right]}{(1-s)\left[\ln(1-s)-1\right]^2} + 2t(1-s)^{2t-1}e^{-2t}, \quad t>0, \tag{4.114}$$

and the steady-state distribution for the process is

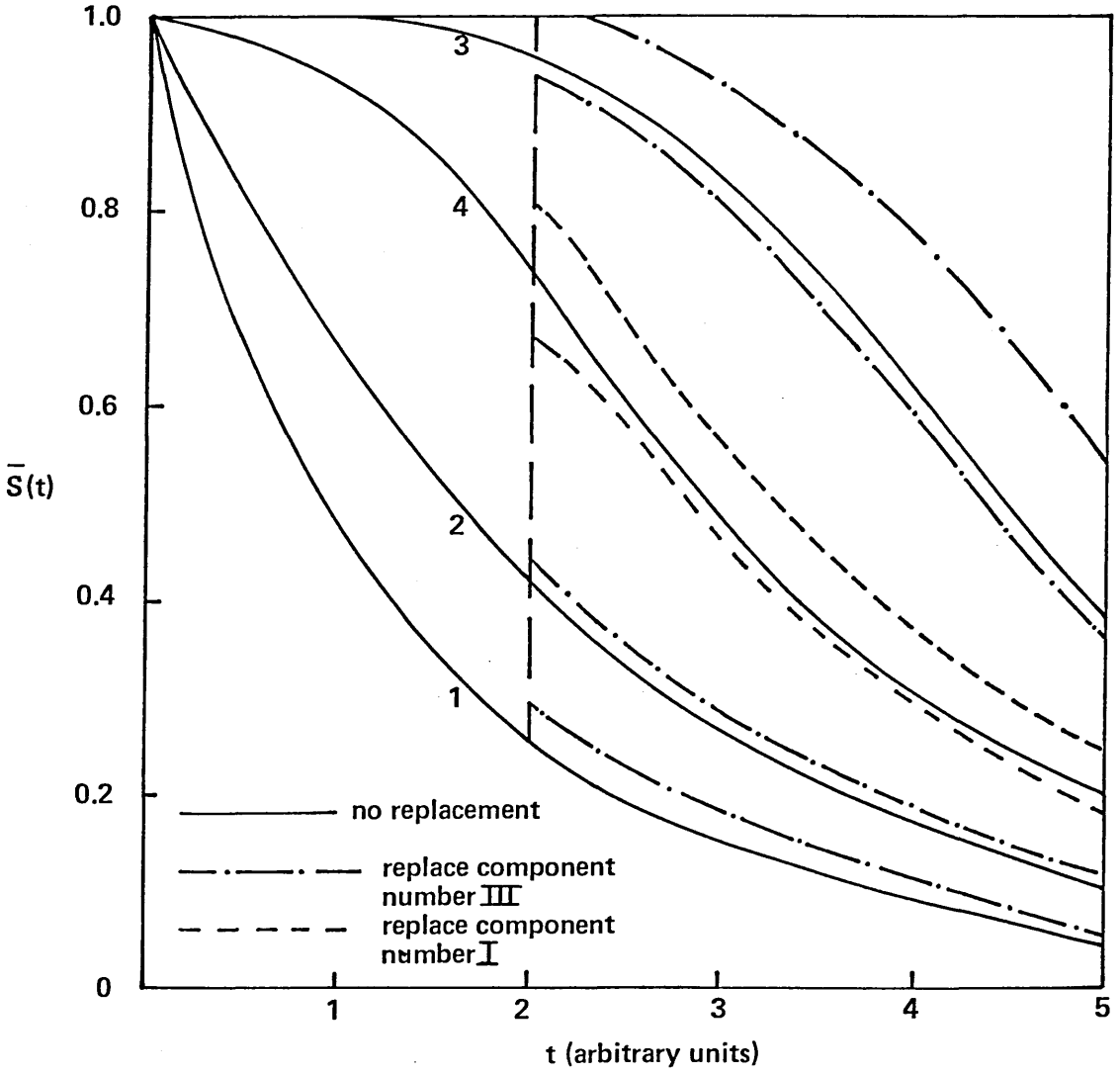
$$\lim_{t\to\infty} g(s,t) = (1-s)^{-1}\left[1-\ln(1-s)\right]^{-2}, \tag{4.115}$$

which is independent of λ. It is unimodal with a peak at

$$s = 1-e^{-1} \doteq 0.6321.$$

Apart from replacement upon failure, preventative replacement is of interest for deteriorating units. Optimal preventative replacement for such units was considered in another formulation in the author's joint paper, Ansell, Bendell and Humble (1980b), whilst in Chapter 5 and 6 we consider the connected tuning and retuning of partially operating units to optimal levels to maximise expected work. However, even if we for the moment ignore these optimisation problems, it is of interest to consider the effect of preventative replacement upon the performance of the system in this extended reliability model. In Figure 4.5 we show the effect of replacement of a single component at arbitrary time $t = 2$, for the three-component systems of Figure 4.3 with identical components again with distributions given by (4.32) and (4.33) with $\beta = 1$, $\alpha = 5$,

Figure 4.5
Expected Levels of Operation under replacement at t = 2 for the systems of three identical components in Figure 4.3 with component distributions given by (4.32) and (4.33) with $\beta = 1$, $\alpha = 5$, $\gamma = 0.2$, $\theta = 10$, λ (t) = 0.6, and minimum and truncated sum connections.



$\gamma = 0.2$, $\Theta = 10$ and $\lambda(t) = 0.6$. The series and parallel connections employed are the minimum and truncated sum rules. As expected, the results are again direct generalisations of the dichotomic case, with replacement offering the greatest potential gains for t small in the mixed series-parallel systems. It follows from (4.112) that for both systems 3 and 4 the maximum asymptotic improvement which is available by replacement of component III at time t_0, is $\overline{S}(t-t_0) - \overline{S}(t)$. This result generalises to all <u>essentially parallel</u> systems, i.e. to all systems whose most outward link is a parallel one.

<u>4.6 Multiple Time Scales and a Random Environment</u>

In multilevel, as in dichotomic reliability, auxiliary time scales such as elapsed operating time or time at risk may be important in accounting for the failure or degradation pattern. See e.g. Isham (1974), Farewell and Cox (1979), Para and Garribba (1980) and Bendell and Humble (1981). A small amount of joint work on the overlap between multilevel reliability and multiple time scales was undertaken by the author and S. Humble within the period of registration for Ph.D., and this forms part of the paper in the <u>IEEE Transactions on Reliability</u> Vol. <u>R-27</u>, 1978which appears as Appendix A3, and is referenced by Para and Garribba. A brief resume of the rather specific formulation of this model is given in this section, together with a brief treatment of the effects of a random environment, also from that paper.

We consider partially operating units deteriorating in current operating time t_f, elapaed operating time t_0, and elapsed calendar time t. In the corresponding dichotomic case, Bendell and

Humble (1978, 1981), the three failure mechanisms were taken as independent and competing, so that the cumulative hazards were added, and the distribution of time to first failure corresponded to a system of three hypothetical units in series, one corresponding to each time scale. If we assume a similar series mechanism in the partial operation case with the product rule for component levels, we obtain for the continuous part of the distribution of level at time (t_f, t_o, t),

$$\begin{aligned} q(s, t_f, t_o, t) = & \int_{s+}^{1-} d\delta \int_{\delta+}^{1-} d\varepsilon \, \varphi_f\left(\left[\tfrac{s}{\delta}\right], t_f\right) \\ & \cdot p_o\left(\left[\tfrac{\delta}{\varepsilon}\right], t_o\right) \varphi(\varepsilon, t) \\ & + p_{f(1)}(t_f)\left[p_{o(1)}(t_o)\,\varphi(s,t)\right. \\ & \left. + \int_{s+}^{1-} \varphi_o\left(\left[\tfrac{s}{\varepsilon}\right], t_o\right) \varphi(\varepsilon, t)\, d\varepsilon\right] \\ & + p_{o(1)}(t_o)\left[\varphi_f(s, t_f)\, p_{(1)}(t)\right. \\ & \left. + \int_{s+}^{1-} \varphi_f\left(\left[\tfrac{s}{\delta}\right], t_f\right) \varphi(\delta, t)\, d\delta\right] \\ & + p_{(1)}(t)\left[p_{f(1)}(t)\,\varphi_o(s, t_o)\right. \\ & \left. + \int_{s+}^{1-} \varphi_f\left(\left[\tfrac{s}{\delta}\right], t_f\right) \varphi_o(\delta, t_o)\, d\delta\right], \end{aligned} \tag{4.116}$$

$0 < s < 1$.

If we now employ the simple limiting case (4.14) for each of the hypothetical units, we obtain for $t_f, t_o, t > 0$,

$$\begin{aligned} q(s, t_f, t_o, t) = & (\lambda_f t_f \lambda_o t_o \lambda t)\, e^{-\lambda_f t_f - \lambda_o t_o - \lambda t} \\ & \cdot (1-s)^{\lambda_f t_f + \lambda_o t_o + \lambda t - 1} \\ & \cdot X(s, t_f, t_o, t), \end{aligned} \tag{4.117}$$

where

$$\begin{aligned} X(s, t_f, t_o, t) = & B(\lambda t, \lambda_o t_o) \int_0^1 dy\, (1-y)^{\lambda_f t_f - 1} \\ & \cdot \left[1 - (1-s)y\right]^{1 - \lambda_f t_f} \\ & \cdot y^{\lambda_o t_o + \lambda t - 1} \\ & \cdot F\left[\lambda t, \lambda_o t_o - 1; \lambda_o t_o + \lambda t; (1-s)y\right], \end{aligned} \tag{4.118}$$

$B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Euler beta function,

and $F(a,b;c;z)$ is a hypergeometric function. It follows that near s = 1,

$$X(s,t_f,t_o,t) \simeq 1, \tag{4.119}$$

whilst near s = 0,

$$X(s,t_f,t_o,t) \simeq \frac{\Gamma(\lambda_f t_f)\Gamma(\lambda_o t_o)\Gamma(\lambda t)}{\Gamma(\lambda_f t_f + \lambda_o t_o + \lambda t)} . \tag{4.120}$$

In the limit that $\lambda t >> \lambda_o t_o >> \lambda_f t_f$, which may be of physical interest

$$X(s,t_f,t_o,t) \simeq \lambda t \left(\frac{\lambda_o t_o}{\lambda t}\right)^{\lambda_o t_o - \frac{1}{2}} \left(\frac{\lambda_f t_f}{\lambda t}\right)^{\lambda_f t_f - \frac{1}{2}} . \tag{4.121}$$

The unconditional distribution of level of operation at time t is obtained analogously to the dichotomic case as

$$\Upsilon(s,t) = \int_o^t dt_o \int_o^{t_o} dt_f \, \gamma(s,t_f,t_o,t) \cdot g_f(t_f|t_o,t)\, g_o(t_o|t) . \tag{4.122}$$

Thus, for example, if t_o has a truncated gamma distribution on (0,t) whilst t_f has a uniform distribution on $(0,t_0)$ independent of t

$$g_o(t_o|t) = (t_o/T)^{\alpha-1} e^{-t_o/T} / [T\,\Gamma(t/T,\alpha)]$$
$$g_f(t_f|t_o,t) = \begin{cases} 1/t_o, & 0 \le t_f \le t_o \\ 0, & \text{otherwise} \end{cases} , \tag{4.123}$$

then near s = 1

$$\Upsilon(s,t) = \frac{\lambda_f \lambda_o \lambda t (1-s)^{\lambda t - 1} e^{-\lambda t}}{T^\alpha \Gamma(\frac{t}{T},\alpha)\, c_s} \left[\frac{\Gamma(a_s t,\alpha)}{a_s{}^\alpha c_s} + \frac{\Gamma(b_s t,\alpha+1)}{b_s{}^{\alpha+1}} + \frac{\Gamma(b_s t,\alpha)}{b_s{}^\alpha c_s}\right]$$
$$= \frac{\lambda_f \lambda_o \lambda t (1-s)^{\lambda t - 1} e^{-\lambda t}}{T^\alpha \Gamma(t/T,\alpha)\, c_s} \left[\frac{\Gamma(a_s t,\alpha)}{a_s{}^\alpha c_s} + \left(\frac{\alpha}{b_s} + \frac{1}{c_s}\right)\frac{\Gamma(b_s t,\alpha)}{b_s{}^\alpha} - \frac{t^\alpha e^{-b_s t}}{b_s}\right] , \tag{4.124}$$

where $a_s = [T^{-1} - \lambda_0 \ln(1-s)]$,

$b_s = [\lambda_s + \lambda_0 + T^{-1}] - [\lambda_0 + \lambda_s] \ln(1-s)$,

and $c_s = \lambda_s [\ln(1-s) - 1]$.

For t close to zero this reduces to

$$\varphi(s,t) = \frac{\lambda_0 \lambda T (a_s + b_s) t}{\lambda_s (1-s)[1 - \ln(1-s)]^2} \quad . \tag{4.125}$$

Following Gaver (1963), Harris and Singpurwalla (1968) and many later authors, the above analysis may be extended to take account of a random environment by making the transition parameter λ itself a random variable. A good justification of such an approach in reliability work is provided by the preface to Tsokos and Shimi (1977). With this approach (4.11) and (4.13) must be interpreted as the conditional distribution of S(t) given λ , and we may obtain the unconditional distribution based on a single time scale as

$$\varphi(s,t) = \int_0^\infty \varphi(s,t|\lambda)\, dL(\lambda) . \tag{4.126}$$

Thus, if for example, λ has a uniform distribution on (a,b) then

$$\varphi(s,t) = \frac{(1-s)^{bt-1}[bt\ln(1-s) - bt - 1]e^{-bt} - (1-s)^{at-1}[at\ln(1-s) - at - 1]e^{-at}}{(b-a)t[1 - \ln(1-s)]^2} \tag{4.127}$$

and $$P_1(t) = \left(\frac{2}{b-a}\right)[e(t^{-1}+a)E_1(at+1) - e(t^{-1}+b)E_1(bt+1) - a E_1(at) + b E_1(bt)] . \tag{4.128}$$

CHAPTER 5

OPTIMAL TUNING

5.1 Introduction and Specialisation of Model

Even in the absence of a replacement mechanism, associated optimisation problems arise in the design and management of systems with partially operating units. In particular, interest may be focussed on the initial tuning, and subsequent retuning, of units to optimal levels of operation. It is clear that in many cases components capable of partial operation can be so tuned to appropriate levels. For example, generators can be deliberately operated at various power outputs, and conveyor belts at various speeds. In general, the higher is the level of activity at which a unit is operated, the greater is the amount and value of work produced by the unit per unit time. However, this does not necessarily imply that units should be run at the maximum level of activity of which they are capable, even if there is an unlimited demand for the work of the units, since there may be advantages to be obtained by operating them at some lower level. One important case of this is where the value of the additional work per unit time achieved by increasing the level of operation of a partially operating unit is more than offset by the cost of the increased wear or deterioration to the unit caused by the increase in level.

In this Chapter we consider a simple formulation of the problem of tuning such units to some optimal level in order to obtain optimal performance, whilst subsequent retuning is considered in Chapter 6. The work of both of these Chapters is again joint with S. Humble, and again the present author is responsible for the major element of

the formulation and analysis, with S. Humble contributing some of the numerical evaluations and examples. Zijlstra (1980) discusses the related but distinct problem of optimising maintenance times for partially operating units, for which maintenance increases the operation level. See also Khandelwal, Sharma and Roy (1979).

In order to simplify the analysis, but to still treat the model at the greatest level of geneality conveniently available, we now both specialise and extend the partial operation model of the previous Chapter. A specialisation that we introduce is that for any level s, $0 < s \leq 1$, state transitions are only possible to the adjacent state below and to the zero state (s = 0). Since for convenience we also take a continuous state space, we refer to these two mechanisms by which the failed state (0) may be reached as drift and catastrophic failures respectively. The catastrophic failure mechanism thus corresponds to the dichotomic failure of a partially operating unit tuned to a level of operation, s_0. Many partially operating units are also subject to some type of drift failure mechanism whereby the level of operation of a unit is gradually degraded from the tuned level, s_0 through intermediary levels to the lowest or failed level, 0. Often and most simply this drift degradation takes the form of a continuous downward transition through all states or levels $s, s_0 \geq s \geq 0$. For example, a hydraulic system with a small leak only gradually loses its ability to convey power, and a drill experiences an almost continuous drop in its rate of cutting as its cutting edges are eroded. Bosch (1979) considers such a drift degradation phenomena in electronic devices, although the model he uses for its analysis differs somewhat from the one we use here; in

particular his drift is deterministic.

In our current context we shall be primarily concerned with the densities of times/time intensities at particular levels $u(v,s)$, rather than $\varphi(s,t)$ and $w(t,s)$. For the model of the previous Chapter these were deduced as infinite mixtures of exponential densities in (4.74) and (4.75). In this Chapter we shall assume by analogy with the dichotomic case, and in order to work at the greatest convenient level of generality available, that the $u(v,s)$ are Weibull, or the minimum of two competing Weibull drift and catastrophic failure mechanisms. The exponential special case of the Weibull is consequently directly consistent with the approach of the previous Chapter. More generally, for most of the states it is reasonable to suppose that the Weibull densities and minimum of two Weibull densities may, in some cases, correspond to appropriate mixtures of exponentials. However, since the mixing distribution in (4.75) is the time of entry to the state s this assumption is not tenable for the initial state s_0 which is known to be entered. at $t = 0$. In this sense the model that we develop in this Chapter is a partial extension of that of Chapter 4.

In general, the mean time intensity for a particular level of operation will depend on what the level is. Transfer from that level will be achieved by catastrophic failure or downward drift through reduced operation levels, and a unit which is working at a higher level of operation than another identical one, will often be expected to drift or catastrophically fail more quickly. On the other hand, the higher is the level of operation s the greater is the work achieved by the unit per time, g(s). Hence, the problem of optimal tuning

discussed in this Chapter arises from the balancing of these two effects. Here, we assume that the optimisation criterion is to maximise the expected amount of work done by the unit in its lifetime. Whilst this is only one of many reasonable optimisation criteria, with it analysis is relatively simple.

If a unit is of such a nature that it can only leave the level to which it is originally tuned by catastrophic failure, then since in this case only one level is sojourned in prior to failure, the mean time in this level is, of course, treated as finite. Thus, the tuning problem consists simply of choosing that value of s, s_0, which will maximise the mean work achieved

$$M_1(s_0) = q(s_0)\,\overline{V_1}(s_0) \quad , \tag{5.1}$$

where $\overline{V_1}(s)$ is the mean time at level s,

$$\overline{V_1}(s) = \int_0^\infty v\,u(v,s)\,dv \, , \tag{5.2}$$

$$\overline{V_1}(s) = \int_0^\infty v\,u(v,s)\,dv \, , \tag{5.2}_e$$

and $u(v,s)$ now corresponds to the time to (catastrophic) failure

On the other hand, if the unit is subject to drift failure, as well as or instead of catastrophic failure, then an infinite number of states will be passed through in progressing from the initial state s_0 to any lower state $s(s \neq 0)$, so that for the MTTLS to remain finite, the mean time at any s must be infinitesimal, as in Chapter 4. In general, the mean time intensity at s may depend on the level to which the unit was originally tuned, s_0, as well as the current level s. For the case where the unit is subject to drift but not catastrophic failure, the expected amount of work done by the unit in its lifetime is the integral of the expected intensities of work done at all the levels of operation s, $s_0 \geqslant s > 0$, through which the unit drifts.
Thus, in this case our optimisation

criterion is to choose s_0 to maximise

$$M_2(s_0) = \int_{0+}^{s_0} g(s)\,\overline{V_2}(s,s_0)\,ds\,, \tag{5.3}$$

where the mean time intensity at s, $\overline{V_2}(s,s_0)$ is given by

$$\overline{V_2}(s,s_0) = \int_0^\infty v\,u_2(v,s,s_0)\,dv \tag{5.4}$$

and where $u_2(v,s,s_0)$ is the density for the time intensity at level s given an initial level of s_0.

Suppose now that the unit suffers from both kinds of failure mechanisms. Then, the expected amount of work done by the unit during its lifetime can be obtained as in the case of drift alone, except that we must now take into consideration that if catastrophic failure does occur at some level s_1, $s_0 \geqslant s_1 > 0$, then the levels s, $s_1 > s > 0$ will not be entered. Thus, our optimisation criterion will now be to choose s_0 to maximise

$$M_3(s_0) = \int_{0+}^{s_0} g(s)\,\overline{V_3}(s,s_0)\,P(s,s_0)\,ds\,, \tag{5.5}$$

where $P(s,s_0)$ is the probability that the unit enters level s given an initial level of s_0, where the mean time intensity at s is given by

$$\overline{V_3}(s,s_0) = \int_0^\infty v\,u_3(v,s,s_0)\,dv \tag{5.6}$$

and where $u_3(v,s,s_0)$ is the density of time intensity at s given an initial level s_0. Of course, $P(s_0,s_0) = 1$.

If we define $h(s,s_0)\delta s, 0 < s < s_0$, to be the probability that catastrophic failure occurs in the infinitesimal interval $(s+\delta s, s)$ given that the unit was initially tuned to level s_0 and has not catastrophically failed by $s+\delta s$, then

$$P(s,s_0) = P(s+\delta s,s_0)\left[1 - h(s,s_0)\delta s\right]\,, \tag{5.7}$$

so that $P(s,s_0)$ satisfies the differential equation

$$\frac{dP(s,s_0)}{ds} = P(s,s_0)\,h(s,s_0)\,.$$

It follows that analogous to (1.3)

$$h(s,s_0) = \frac{\partial \ln P(s,s_0)}{\partial s} \quad , \tag{5.8}$$

so that analogous to the expression for R(t) in (1.4),

$$P(s,s_0) = \exp\left[-\int_s^{s_0} h(y,s_0)\,dy\right] \quad . \tag{5.9}$$

Note, however, that this hazard function $h(s,s_0)$ is a function over the state space $0 \leq s \leq s_0$ rather than over time.

By analogy with the dichotomic case, we shall suppose that the time/time intensity for catastrophic failure at s, and the time intensity for drift at s, each follow a Weibull distribution of the form (1.8). For such a distribution

$$\begin{aligned} \overline{V}(s,s_0) = E\left[V(s,s_0)\right] &= \Theta\,\Gamma(1+\beta^{-1}) \\ &= \Theta\,\Gamma(\beta^{-1})/\beta \quad , \end{aligned} \tag{5.10}$$

(where the conditioning on s_0 is suppressed in the case of catastrophic failure alone). Some justification for the use of the Weibull in such circumstances is provided by the direct analogy of the catastrophic failure case with accelerated life testing under different stresses, for which Nelson (1970) amongst others, provides examples justifying the Weibull.

The assumption that the level of operation or stress applied to a unit effects only the parameters of the time to failure distribution, and not the distributional form, is an assumption that has been made by many authors and for which there is substantial physical evidence (see e.g. Bazovsky (1961), Mann (1972) and Hahn and Nelson (1974)). A number of authors (e.g. Charles (1961), Saunders (1966), Mann (1968), Nelson (1970) and Singpurwalla and Al-Khayyal (1977)) have pointed out

that in such dichotomic life-testing situations the Weibull scale parameter Θ is often a function of level of stress applied to the unit under life test, whilst the shape parameter β is not. Hence for a partially operating unit at level s subject to a catastrophic failure time/time intensity and/or a drift time intensity, each of the form (1.8), Θ will be a function of present level s and initial level s_0, whilst β remains constant. Thus, from (5.1) and (5.3),

$$M_1(s_0) = \Gamma(\beta^{-1})\, q(s_0)\, \theta(s_0) / \beta \,, \qquad (5.11)$$

$$M_2(s_0) = \Gamma(\beta^{-1}) \int_{0+}^{s_0} q(s)\, \theta(s, s_0)\, ds / \beta. \qquad (5.12)$$

The evaluation of (5.5) is more complex. We suppose in this case that at any level s the Weibull drift and catastrophic failure mechanisms are competing and independent, and that the failure mechanisms renew themselves at each new state. If the catastrophic failure mechanism is exponential, then this is equivalent to the case where the catastrophic failure density is an exponential over elapsed calendar time since the start of the process, and for the Weibull case with shape parameter close to unity it may also approximate well to a similar Weibull distribution over elapsed calendar time. More generally, such a mechanism may be physically reasonable if the unit is continually readjusted (perhaps automatically) as it drifts downwards through the states. Further, the formulation has the desirable feature that with it the catastrophic failure mechanism can be taken to depend upon the current level s, which is not so conveniently available if, for example, it was taken as a function of elapsed calendar time.

Thus, denoting the scale and shape parameters of the Weibull drift density of time intensity at s by $\theta(s, s_0)$ and β respectively,

and the scale and shape parameters of the Weibull catastrophic failure density of time intensity at s by $\mu(s,s_0)$ and λ, the hazard with which catastrophic failure occurs at level s, $h(s,s_0)$, is obtained as the hazard with which catastrophic failure takes place before drift occurs,

$$h(s,s_0) = \int_0^\infty \frac{\lambda}{\mu}\left(\frac{v}{\mu}\right)^{\lambda-1} \exp\left[-\left(\frac{v}{\mu}\right)^{\lambda} - \left(\frac{v}{6}\right)^{\beta}\right] dv. \quad (5.13)$$

Further, the density of time intensity at s is the minimum of two Weibull lifetimes,

$$u_3(v,s,s_0) = \left[\frac{\beta}{6}\left(\frac{v}{6}\right)^{\beta-1} + \frac{\lambda}{\mu}\left(\frac{v}{\mu}\right)^{\lambda-1}\right] \exp\left[-\left(\frac{v}{6}\right)^{\beta} - \left(\frac{v}{\mu}\right)^{\lambda}\right]. \quad (5.14)$$

It follows from (5.6) and (5.14) that for general Weibull densities it will not be possible to write out $\overline{V_3}(s,s_0)$ explicitly as a simple function of $6(s,s_0)$ and $\mu(s,s_0)$. However, if $\beta=\lambda$, which may be physically realisable in many cases, $u_3(v,s,s_0)$ is also Weibull with shape parameter β and scale parameter

$$\left[6^{-\beta} + \mu^{-\beta}\right]^{-1/\beta}, \quad (5.15)$$

so that

$$\overline{V_3}(s,s_0) = \left[6^{-\beta} + \mu^{-\beta}\right]^{-1/\beta} \Gamma\left(\beta^{-1}\right)/\beta. \quad (5.16)$$

In this case

$$h(s,s_0) = \frac{6(s,s_0)^{\beta}}{6(s,s_0)^{\beta} + \mu(s,s_0)^{\beta}}, \quad (5.17)$$

and

$$M_3(s_0) = \frac{\Gamma(\beta^{-1})}{\beta} \int_{0+}^{s_0} g(s)\left[6(s,s_0)^{-\beta} + \mu(s,s_0)^{-\beta}\right]^{-1/\beta} \exp\left[-\int_s^{s_0} \frac{6(y,s_0)^{\beta}}{6(y,s_0)^{\beta} + \mu(y,s_0)^{\beta}} dy\right] ds. \quad (5.18)$$

It is worth noting from (5.18) that in contrast to the situation in (5.11) and (5.12) for catastrophic failures and drift failures alone, the optimum value of s_0 in the presence of both failure modes is dependent upon the value of β (and of λ). For the exponential special case it follows that

$$\overline{V_3}(s,s_0) = \frac{\sigma(s,s_0)\mu(s,s_0)}{\sigma(s,s_0)+\mu(s,s_0)} \quad , \tag{5.19}$$

and

$$h(s,s_0) = \frac{\sigma(s,s_0)}{\sigma(s,s_0)+\mu(s,s_0)} \quad . \tag{5.20}$$

For the rest of this Chapter we for the sake of definiteness take

$$g(s) \equiv s \, . \tag{5.21}$$

The resultant models correspond to the situation in which the work achieved by the device per unit time is directly proportional to its level of operation, and this will be appropriate in many circumstances. Doubling the speed of a conveyor belt or a drill will, provided this is within the tolerance of the equipment, double the work achieved by the unit per unit time. Similarly, the assumption may be appropriate for production equipment. For these M_1, M_2 and M_3 may be interpreted as the expected production in the units lifetime.

It remains to specify possible functional forms for θ, σ and μ. Physical considerations suggest that these should be monotonically non-increasing functions of s (and s_0 where appropriate) such that

$\theta, \sigma, \mu = 1$ (in arbitrary time units) when s = 1

and

$$\theta, \sigma, \mu \to \infty \quad \text{as} \quad s \to 0 \ . \tag{5.22}$$

If this is the case then for the Weibull distribution whilst the mean drift time intensity and the mean catastrophic failure time intensity will be finite at s = 1, they will become infinite as $s \to 0$. Further, the requirement that θ, σ and μ are monotonically non-increasing in s and s_0 means that the mean drift time intensity and mean catastrophic failure time intensity of a Weibull unit now at level s_1 (but originally at s_0) will be smaller than or equal to the corresponding mean time intensities for an identical unit now at s_2 (but also originally at s_0) for all $s_1 > s_2$. It also follows that the mean drift time intensity and the mean catastrophic failure time intensity for a Weibull unit now at level s but originally at s_{01} will be smaller than or equal to the corresponding mean time intensities for an identical unit now also at s but originally at $s_{02} < s_{01}$. We would expect such properties to be valid in most real situations.

The form of the relationship between stress level and expected lifetime (or equivalently failure rate) has been considered by many authors, e.g. Bazovsky (1961), Mann (1972), Hahn and Nelson (1974). The Inverse Power Law Model, discussed e.g. by Singpurwalla (1971), Nelson (1972, 1975), Singpurwalla and Al-Khayyal (1977) and Kahn (1979), has the advantages of wide applicability, simplicity and smoothness. This states that the Weibull scale parameter θ is an inverse power function of the stress s,

$$\theta(s) = A s^{-\alpha} \ , \tag{5.23}$$

where A and α are positive scale and shape parameters characteristic of the unit being stressed. The model has been applied extensively

in the accelerated testing of, for example, insulating fluids, capacitors, bearings and electronic devices (e.g. Endicott and Starr (1961), Endicott and Zoellner (1961) and Endicott, Hatch and Schmer (1965)).

Another model with similar characteristics and advantages is the simple logarithmic relationship

$$\Theta(s) = B(1 - \gamma \ln s), \qquad (5.24)$$

where B and γ are also positive scale and shape parameters characteristic of the unit being stressed. Similar logarithmic models have also been employed (although less widely) in the accelerated testing of products, (e.g. Tomlinson, Andrew and Fitzgerald (1970)), and in the current context (5.24) has the additional advantage of leading to somewhat simpler analysis.(See below).

Here we shall suppose that Θ, σ and μ are linear combinations of functions of the form (5.23) and (5.24) where as before the level s is standardised to the range $0 \leq s \leq 1$. Of course, the results obtained from our models below are conditional upon the functional forms being valid. However, these are more general than either (5.23) or (5.24) alone, and anyway the methodology of this Chapter essentially remains unaltered if instead other functional forms are used, if these are known to be more appropriate. In any case, the functional forms for Θ, σ and μ employed provide reasonable approximations to many physical situations, and are likely to be particularly appropriate if there is physical evidence to suggest that the relationship between Θ and s is convex to the origin and smooth.

5.2 Analysis of Catastrophic Failure

We consider first the case of a Weibull unit subject to catastrophic failure alone, so that (5.11) holds, and for which $\Theta(s)$ is a linear combination of functions of the form (5.23) and (5.24). In particular, to ensure (5.22) holds we take

$$\Theta(s_0) = A s_0^{-\alpha} + (1-A)(1-\gamma \ln s_0) , \qquad (5.25)$$

where

$$\alpha > 0, \ \gamma > 0, \ 0 \leq A \leq 1, \ 0 \leq s_0 \leq 1.$$

Since Θ represents the Weibull scale parameter (5.25) must be positive for all s_0. However, it is not necessary in (5.25) unlike (5.23) and (5.24) for $A > 0, (1-A) > 0$ in order to ensure that $\Theta(s_0) > 0$ for all s_0. Hence we relax these conditions in order to gain increased generality.

Of course, if A = 0 or A = 1 (5.25) reduces to (5.23) or (5.24) respectively, if $0 < A < 1$ (5.25) represents a sum of terms of these types, and if $A < 0$ or $A > 1$ it represents a difference of such terms. Hence (5.25) represents a more generally valid relationship than either (5.23) or (5.24). It is necessary, however, if $A > 1$ to impose the condition that $\gamma \leq \alpha A/(A-1)$, and if $A < 0$ to impose the condition that $\gamma \geq \alpha A/(A-1)$, in order to ensure that $\Theta(s_0)$ is a monotonically non-increasing function of s_0. In addition, in order to ensure that $\Theta(s_0) > 0$ it is necessary if $A > 1$ that

$$s_0^{\alpha} < \frac{A}{(A-1)(1-\gamma \ln s_0)} \qquad (5.26)$$

for all s_0, and if $A < 0$ that

$$s_0^{\alpha} > -\frac{A}{(1-A)(1-\gamma \ln s_0)} \qquad (5.27)$$

for all s_0 .

From (5.11) and (5.25),

$$M_1(s_0) = \Gamma(\beta^{-1})\left[A s_0^{1-\alpha} + s_0(1-A)(1-\gamma \ln s_0)\right] / \beta \, . \tag{5.28}$$

It follows that for $\alpha > 1$, the optimum policy to maximise the expected production is to produce nothing, since $M_1(s_0)$ will be a maximum (infinite) when $s_0 = 0$. Thus, if the mean lifetime decreases too rapidly with increasing level, a work per unit time function of the form (5.21) will be insufficient to justify any production.

The optimum value of s_0 can now be obtained by differentiating $M_1(s_0)$ with respect to s_0, and equating the differential to zero, to give

$$s_0 = \left[\frac{(\alpha-1)A}{(1-A)(1-\gamma-\gamma \ln s_0)}\right]^{1/\alpha} \, . \tag{5.29}$$

The second order condition for the solution to (5.29) to maximise $M_1(s_0)$ is

$$s_0^{\alpha} < \alpha(1-\alpha)A / [\gamma(A-1)], \text{ for } A > 1 \tag{5.30}$$

$$s_0^{\alpha} > -\alpha(1-\alpha)A / [\gamma(A-1)], \text{ for } A < 1 \, . \tag{5.31}$$

The optimum value of s_0 can be found by solving (5.29) numerically. However, the relatively simple structure of this non-linear equation enables us to obtain a simple graphical method of solution. The method is similar to that used elsewhere for the graphical solution of optimum replacement problems; see Ansell and Bendell (1982b) and Ansell, Bendell and Humble (1982). Rewriting (5.29) we have that

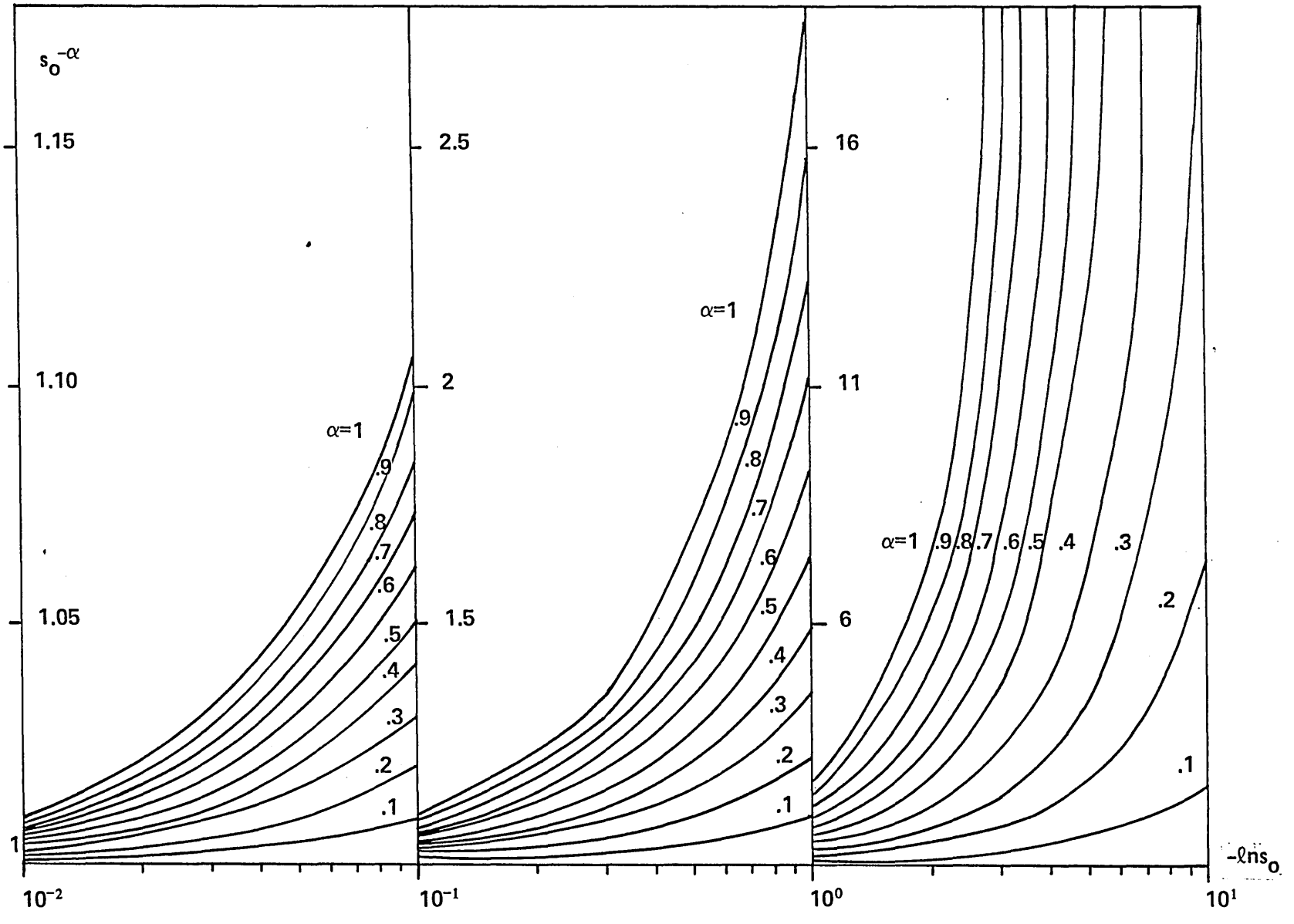
$$\left[\frac{(1-A)(1-\gamma)}{(\alpha-1)A}\right] - \left[\frac{(1-A)\gamma}{(\alpha-1)A}\right] \ln s_0 = s_0^{-\alpha} \, . \tag{5.32}$$

Thus constructing the graph of $S_o^{-\alpha}$ against $\ln S_o$ as shown in Figure 5.1, the solution (s) to (5.29) can be obtained, if any exist, by drawing the straight line represented by the left-hand-side of (5.32) onto the graph paper and identifying the intersection with the appropriate α curve. Since $S_o^{-\alpha}$ rises so rapidly, a number of graphs corresponding to differing scales are shown and the appropriate one(s) must first be identified. An example of the application of this method will be given below.

Figure 5.1 also demonstrates the convexity of $S_o^{-\alpha}$ (from s_0=0), so that it is clear that the straight line of (5.32) can at most intersect any α curve at two points. Thus, there are at most two solutions to (5.29) and consequently at most one analytic maximum for $M_1(s_0)$. When there are two solutions, it follows from (5.28) that since $\lim_{S_o \to 0} M_1(S_o) = 0$, the smaller solution is the analytic maximum, unless it is a point of inflection. However, since we are considering $M_1(s_0)$ in a bounded range, it is possible that the local maximum given by (5.29) in the case where it has two solutions does not necessarily correspond to the maximum value of M_1 in the range. In such cases the maximum will instead occur at the end of the range at s_0 = 1, as it will when there are no solutions to (5.29) in (0, 1). When (5.29) has a single solution in the range, it follows from (5.28) that it is either a point of inflection or a maximum, in this case a global maximum. For $0<A<1$ there is in fact a single solution which is a maximum.

Useful bounds for the optimum level of operation can also be obtained from the first and second order conditions for a maximum. Substituting (5.29) in (5.30) gives for $A>1$ an upper bound for

Figure 5.1
Graphs of $s_o^{-\alpha}$ against $-\ell n(s_o)$



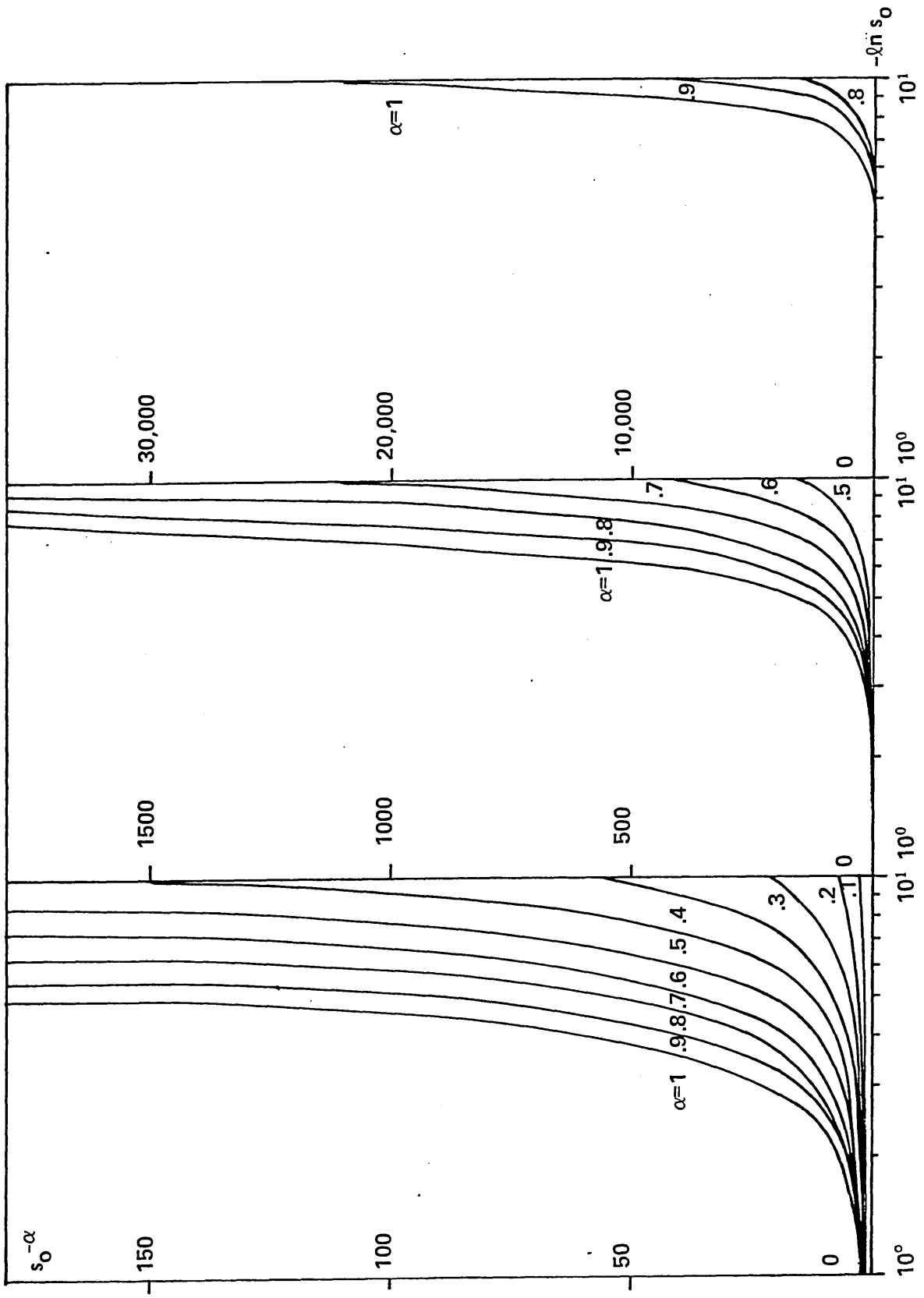


Figure 5.1 (cont.)

the level of operation which will maximise the expected production, $M_1(s_0)$,

$$s_0 < \exp\left[\frac{1}{\gamma} - 1 - \frac{1}{\alpha}\right] . \qquad (5.33)$$

For $0<A<1$ the bound in (5.31) is negative so that (5.29) must merely be positive, which leads to the lower bounds,

$$s_0 > \exp\left[\frac{1}{\gamma} - 1\right] > e^{-1} \simeq 0.3679 \qquad (5.34)$$
$$(0 < A < 1).$$

Finally, for $A < 0$ substituting (5.29) in (5.31) gives as a lower bound the upper bound of (5.33). In order for the bounds (5.33) and (5.34) to be of use they must be in the range (0,1), which will be the case provided that

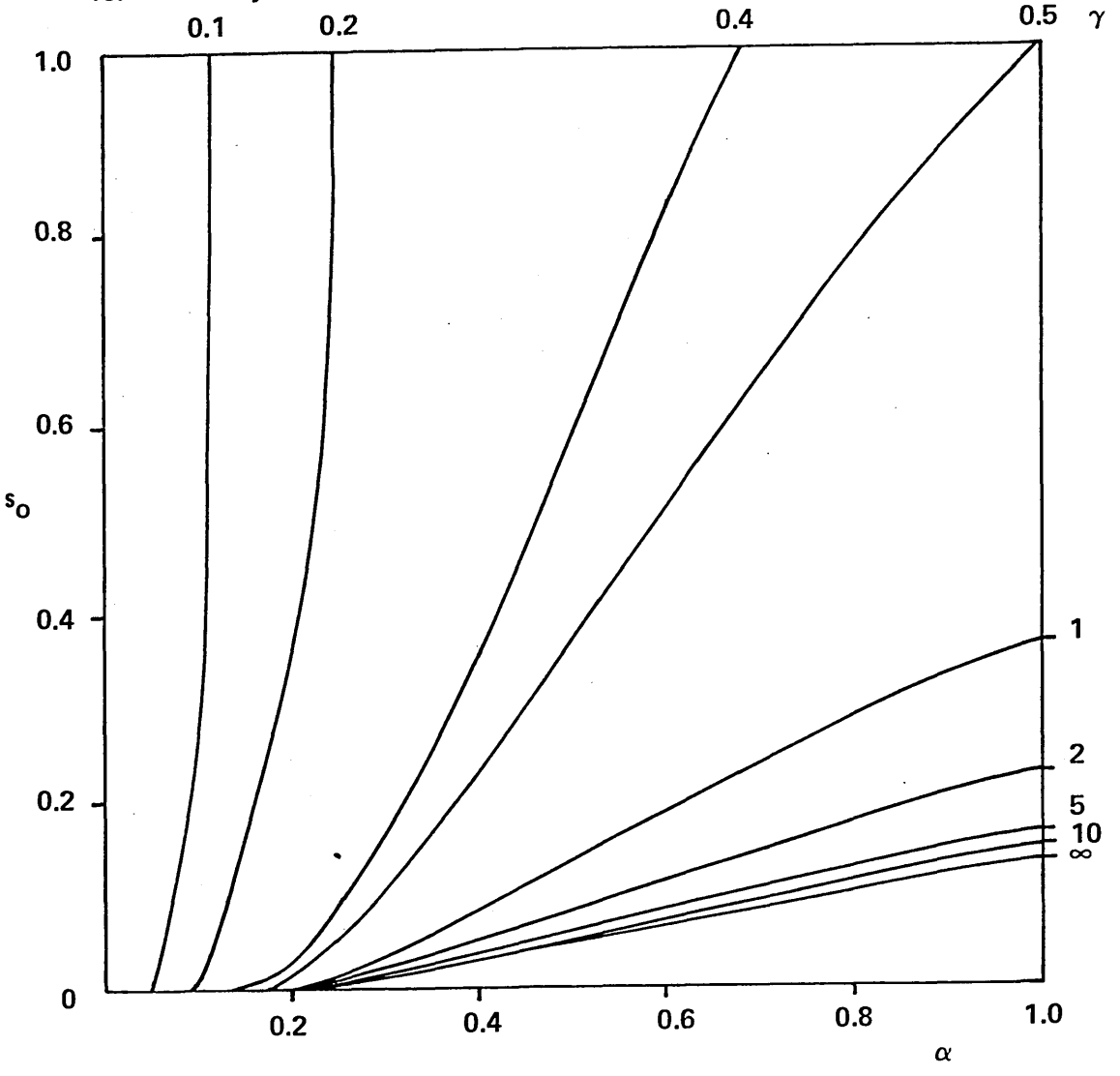
$$\gamma > \alpha/(\alpha+1) \text{ , if } A > 1 \text{ or } A < 0$$
$$\gamma > 1 \text{ , if } 0 < A < 1. \qquad (5.35)$$

Where in the permissible range from the bound to the edge of the (0,1) interval the optimum s_0 occurs depends on the weighting parameter A. For A = 1 the maximum value of $M_1(s_0)$ occurs at $s_0 = 1$, whilst for A = 0 the optimum value is identical to the bound of (5.34),

$$s_0 = \exp\left[\frac{1}{\gamma} - 1\right] . \qquad (5.36)$$

The upper bound (5.33) and the equivalent lower bound for $A < 0$ are shown in Figure 5.2 When γ is large, α small and $A > 1$, the range of possible optimum s_0 is small so that knowledge of the exact value of A is not too critical in the selection of the optimum level of operation. This is also the case when γ is small, α large (i.e. almost 1) and $A < 0$. On the other hand, if γ is small, α large and $A > 1$ or γ large, α small and $A < 0$ great choice of level is available. For $0 < A < 1$ knowledge of the exact values of

Figure 5.2
Upper Bounds (for A>1) and Lower Bounds (for A<0) for optimum s_o for a unit subject to catastrophic failure.



A and α are most important in the selection of the optimum level of operation when γ is large, although the optimum level can never be less than 0.3679.

For the limiting case where $\alpha = 1$ and $A < 1$ the optimum value of s_0 is also given by (5.36), whilst for $\alpha = 0$ and $A < 1$ the optimum value of s_0 is

$$s_0 = \exp\left[\frac{1}{\gamma(1-A)} - 1\right] . \tag{5.37}$$

In each case there is no analytic maximum for $A > 1$.

For $\gamma = 0$ and $A > 1$ the optimum value of s_0 is

$$s_0 = \left[\frac{(1-\alpha)A}{A-1}\right]^{1/\alpha} , \tag{5.38}$$

whilst if $A < 1$ there again is no analytic maximum. If $\gamma = 1$ and $0 < A < 1$ an analytic maximum is always obtained.

By requiring $0 \leq s_0 \leq 1$ in (5.29) it is possible to deduce additional bounds on s_0, some of which are stricter than those obtained above. In particular, for $A < 0$ we now obtain an upper bound on s_0 to supplement our previous lower bound,

$$s_0 \leq \exp\left[\frac{1}{\gamma} - 1 + \frac{(1-\alpha)A}{(1-A)\gamma}\right] , \tag{5.39}$$

whilst for $A > 1$ we obtain the same upper bound and for

$$\gamma < A\alpha(1-\alpha)/(A-1)$$

this is sharper than the upper bound (5.33). This bound will lie in the range (0,1) provided that

$$\gamma \leq 1 + \frac{(1-\alpha)A}{A-1} , \quad \text{for} \quad A > 1 \tag{5.40}$$

and

$$\gamma \geq 1 + \frac{(1-\alpha)A}{A-1} , \quad \text{for} \quad A < 0 .$$

Note that for the case $A < 0$, with suitable parameter values, the

optimum level of production is confined within a range which contains neither 0 nor 1.

5.3 Analysis of Drift Failure

We now consider a unit subject to drift failures. To obtain simple expressions for $\Theta(s, s_0)$ which are monotonically non-increasing in both s and s_0 and which obey (5.22) we shall for the drift failure case (5.12) consider sums of two functions of the types (5.23) and (5.24) with weighting parameters A and (1-A) respectively; one function being a function of s_0 and the other of s,

$$\Theta(s, s_0) = \Theta_1(s) + \Theta_2(s_0) \quad . \tag{5.41}$$

This sum of two functions satisfies (5.22) since the definition of drift implies that if the current level is s = 1 the originally tuned level must also be $s_0 = 1$. It is, however, necessary that the weighting parameters A and (1-A) are between 0 and 1 inclusive in order for $\Theta(s, s_0)$ to be non-increasing in both s and s_0, and non-negative. The resultant expressions for $\Theta(s, s_0)$ consequently represent generalisations of the well established relationships (5.23) and (5.24) and possess many of their properties. It is, of course, true that the simple additive relationship of (5.41) is probably inappropriate in certain situations, but the methodology of this Chapter will essentially remain unchanged if instead other functional forms are used. For the case where $\Theta_1(s)$ takes the form (5.24) and $\Theta_2(s_0)$ the form (5.23), it follows from (5.12) that the optimum initial level to which to tune the unit to is always $s_0 = 1$ provided $0 \leq \alpha \leq 2$, or $s_0 = 0$ if $\alpha > 2$.

If instead $\Theta_1(s)$ is given by (5.23) and $\Theta_2(s_0)$ by

((5.24) we have

$$\Theta(s, s_0) = A s^{-\alpha} + (1-A)(1-\gamma \ln s_0), \qquad (5.42)$$

where $\alpha > 0, \gamma > 0, 0 \leq s \leq s_0 \leq 1.$

Equation (5.12) now becomes

$$M_2(s_0) = \Gamma(\beta^{-1})\left[\left(\frac{A}{2-\alpha}\right)s_0^{2-\alpha} + \left(\frac{1-A}{2}\right)s_0^2(1-\gamma \ln s_0)\right]/\beta. \qquad (5.43)$$

Thus for $\alpha > 2$ the optimum policy is again to tune the unit to produce nothing. For $\alpha < 2$, equating $M_2'(s_0)$ to zero we obtain

$$s_0 = \left[\frac{A}{(1-A)(\gamma/2 - 1 + \gamma \ln s_0)}\right]^{1/\alpha}. \qquad (5.44)$$

Again, the solution to this equation can be found graphically using Figure 5.1. Rewriting (5.44) we have

$$\left[\frac{(1-A)(\gamma-2)}{2A}\right] + \left[\frac{(1-A)\gamma}{A}\right] \ln s_0 = s_0^{-\alpha}, \qquad (5.45)$$

so that again the equation can be solved by drawing the straight line represented by the left-hand-side of (5.45) onto the graph paper of Figure 5.1 and identifying the intersection with the appropriate α curve. It follows that again there are at most two intersections for given parameter values, so that there is at most one analytic maximum for $M_2(s_0)$. Also since from (5.43) $M_2(0) = 0$, if there is a single solution to (5.44) it is a maximum or a point of inflection, whilst if there are two solutions it is the smaller one which is the maximum unless it is a point of inflection. Considering the second differential of $M_2(s_0)$ we see that a point of inflection cannot arise if $1 < \alpha < 2$.

When there are no analytic solutions, the maximum is at $s_0 = 1$, which may also be the case when there are two analytic turning values. If $0 < \alpha < 1$ and $s_0 < e^{1/\gamma - 3/2}$ there is no analytic solutions, whilst if $1 < \alpha < 2$ and $s_0 > e^{1/\gamma - 3/2}$ there is a single analytic turning value which is a maximum.

The requirement that the solution(s) to (5.44) are below unity leads us to eliminate the possibility of the optimum s_0 being in the range from

$$\exp\left[\frac{1}{\gamma} - \frac{1}{2}\right] \text{ to } \exp\left[\frac{A}{\gamma(1-A)} + \frac{1}{\gamma} - \frac{1}{2}\right]. \qquad (5.46)$$

For the lower value to be in (0,1), we must have $\gamma > 2$, whilst for the upper to be in this range we require the condition $\gamma > \frac{2}{1-A}$. As $\gamma \to \infty$ the eliminated range concentrates onto $e^{-.5} \simeq 0.6065$ and finally disappears. The eliminated ranges (5.46) are shown in Figure 5.3. The smaller A is and the larger γ is, the larger is the range of possible optimum s_0, so that the more important is knowledge of the exact value of α.

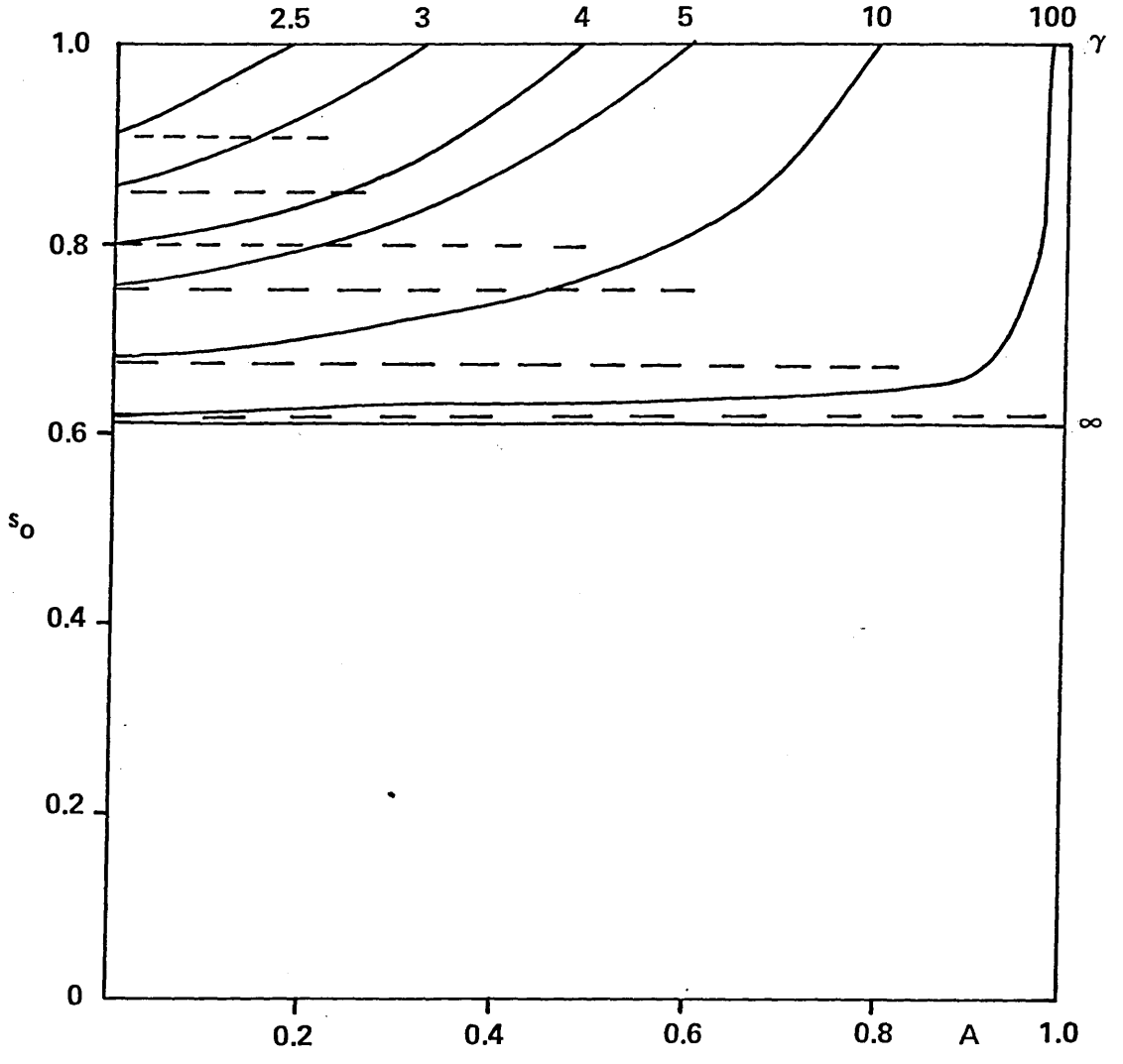
The second order condition for the solution to (5.44) to maximise $M_2(s_0)$ is quite complicated, being

$$s_0^{\alpha} > \frac{A(1-\alpha)}{(1-A)\left(\frac{3\gamma}{2} - 1 + \gamma \ln s_0\right)}, \text{ if } s_0 > e^{1/\gamma - 3/2} \qquad (5.47)$$

$$s_0^{\alpha} < \frac{A(1-\alpha)}{(1-A)\left(\frac{3\gamma}{2} - 1 + \gamma \ln s_0\right)}, \text{ if } s_0 < e^{1/\gamma - 3/2}.$$

However, substituting the first order condition (5.44) into the

Figure 5.3
Eliminated ranges for optimum s_o for drift failure model (5.42)



The function graphed is $\exp\left[\frac{A}{\gamma(1-A)} + \frac{1}{\gamma} - \frac{1}{2}\right]$. The eliminated range is the interval from the height of the curve to the height of that curve at A = 0.

left-hand-side of (5.47), we find for $s_0 > e^{1/\gamma - 3/2}$ that in order for (5.47) to hold a necessary (not sufficient) condition is that either

$$s_0 < e^{1/\gamma - 1/2 - 1/\alpha} \quad \text{and} \quad 1 < \alpha < 2 \qquad (5.48)$$

(so that there is a single analytic maximum), or $s_0 > e^{1/\gamma - 1/2}$ (so that by (5.46) s_0 exceeds $\exp\left[\frac{A}{\gamma(1-A)} + \frac{1}{\gamma} - \frac{1}{2}\right]$).

For $s_0 < e^{1/\gamma - 3/2}$ (which implies $s_0 < e^{\frac{1}{\gamma} - \frac{1}{2}}$) we find by a similar argument that it is necessary that $1 < \alpha < 2$, so that a point of inflection cannot arise.

If we now suppose instead that $\Theta_1(s)$ and $\Theta_2(s_0)$ both take the form (5.24) we obtain a somewhat simpler model for drift failures, since the physical requirement (5.22) results in the number of parameters in the relationship between Θ and s and s_0 being effectively reduced from three to two. That is if

$$\Theta(s, s_0) = A(1 - \eta \ln s) + (1-A)(1 - \psi \ln s_0)$$
$$\eta > 0,\ \psi > 0,\ 0 \le A \le 1,\ 0 \le s \le s_0 \le 1, \qquad (5.49)$$

then writing

$$\delta = A\eta \ , \quad \xi = (1-A)\psi$$

we obtain

$$\Theta(s, s_0) = 1 - \delta \ln s - \xi \ln s_0 \qquad (5.50)$$

where $\delta > 0,\ \xi > 0,\ 0 \le s \le s_0 \le 1.$

Substituting (5.50) into (5.12) leads us to

$$M_2(s_0) = \Gamma(\beta^{-1})\left[0.25(2+\delta)s_0^2 - 0.5(\xi+\delta)s_0^2 \ln s_0\right]/\beta, \qquad (5.51)$$

so that equating $M_2'(s_0)$ to zero gives the explicit

solution

$$s_0 = \exp\left[\frac{2-\xi}{2(\xi+\delta)}\right] \quad . \tag{5.52}$$

The second order condition for $M_2(s_0)$ to be maximised is automatically satisfied by (5.52) and the optimum level will lie in (0,1), provided that $\xi \geqslant 2$.

The optimum levels are shown in Figure 5.4 for $\delta \leq 100$. For all δ and ξ the optimum level always exceeds $e^{-.5} \simeq 0.6065$, which is approached as $\xi \to \infty$ with δ remaining finite. As $\delta \to \infty$ with ξ remaining finite the optimum level to tune to tends to 1, whilst for $\delta \to \infty$, $\xi \to \infty$, the optimum level is $e^{-.25} \simeq 0.7788$. The smaller the value of δ and the larger the value of ξ the lower is the optimum level to which to tune. Also, when δ is small, the larger the value of ξ the less important is it to know the exact value of δ or ξ in order to locate the optimum value of s_0. Similarly, the smaller is ξ the less important is it to know the exact value of δ or ξ when δ is large.

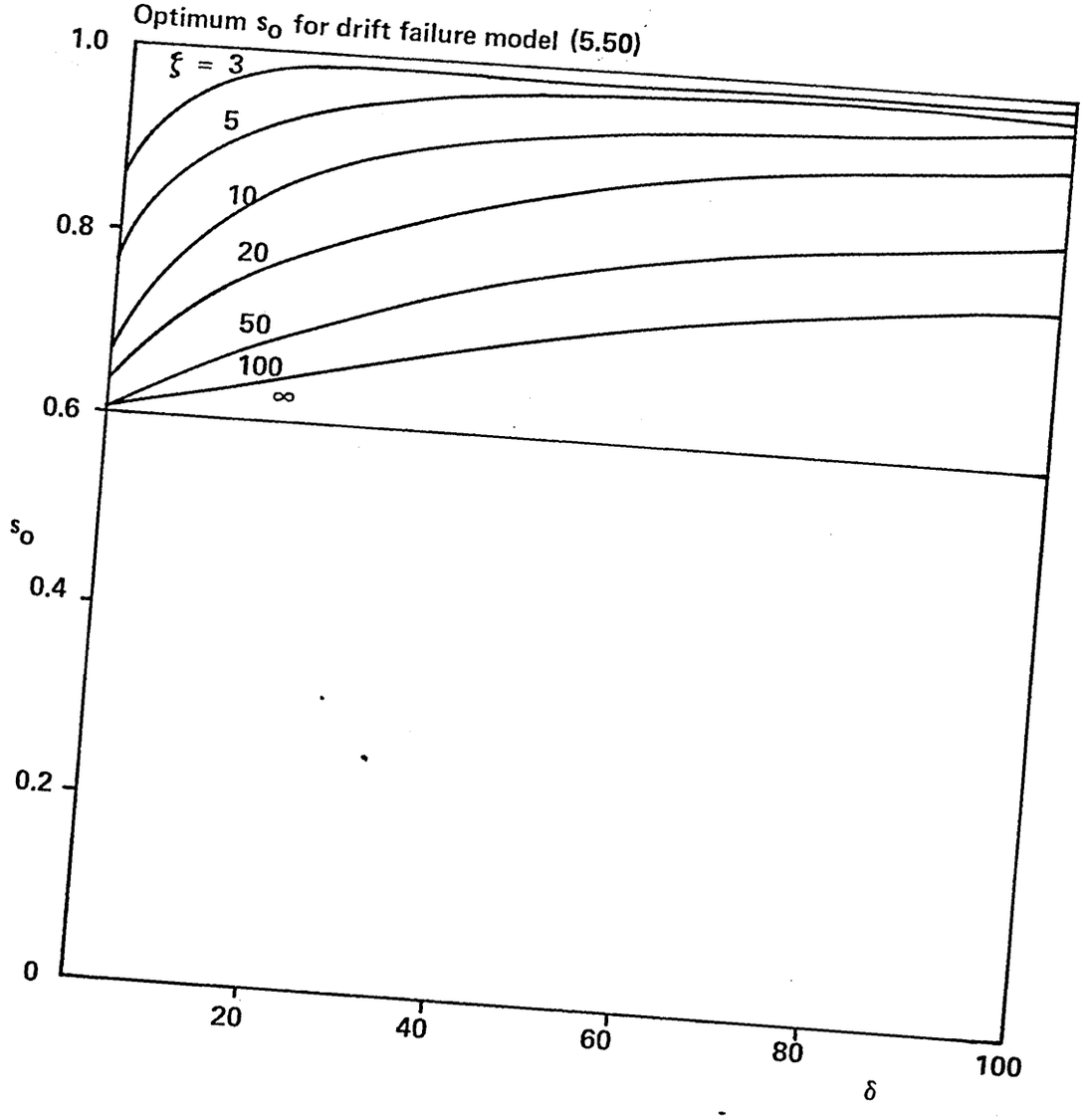
5.4 Drift and Catastrophic Failure

The case where the unit is subject to both catastrophic and drift failures is of course the most difficult to deal with, so that we have already restricted attention to the situation where both the failure mechanism densities have a common shape parameter. Even in this case it is impossible to obtain general analytic results for optimum s_0, so that instead we content ourselves here with considering one of the simplest examples of the combined problem.

As for the case of drift alone, we in general suppose that the scale parameters σ and μ have the form (5.41),

$$\sigma(s, s_0) = \sigma_1(s) + \sigma_2(s_0)$$
$$\mu(s, s_0) = \mu_1(s) + \mu_2(s_0) \quad , \tag{5.53}$$

Figure 5.4
Optimum s_o for drift failure model (5.50)



and take the case where $\sigma_1(s)$ and $\mu_1(s)$ are each of the form (5.23) with scale parameters A and B respectively, and shape parameters (α) in each case 1. This represents the simplest case of (5.23) except for the degenerate forms where $\alpha = 0$ or $A = B = 0$, and implies a simple inverse relationship between the level of operation s and the mean time intensity at s (so that it is likely to be physically realisable in certain cases). The restriction on the values of α , however, does represent a substantial loss of generality, and serves to emphasise the substantial increase in complexity which is involved when a unit is subject to both failure mechanisms. To simplify evaluation we further restrict attention to the exponential case ($\beta = 1$). With these assumptions we obtain from (5.9) and (5.20) that

$$P(s,s_0) = \left[\frac{(A+B)+sW(s_0)}{(A+B)+s_0W(s_0)}\right]^{\left[A\mu_2(s_0)-B\sigma_2(s_0)\right]/W(s_0)^2} \cdot \exp\left[\frac{(s-s_0)\sigma_2(s_0)}{W(s_0)}\right], \qquad (5.54)$$

where $W(s_0) = \sigma_2(s_0) + \mu_2(s_0)$.

However, the evaluation of $M_3(s_0)$ still remains difficult even in this very simplified case.

The values of s_0 which optimise $M_3(s_0)$ can be found numerically. Some solutions are shown in Figure 5.5 for the exponential case with

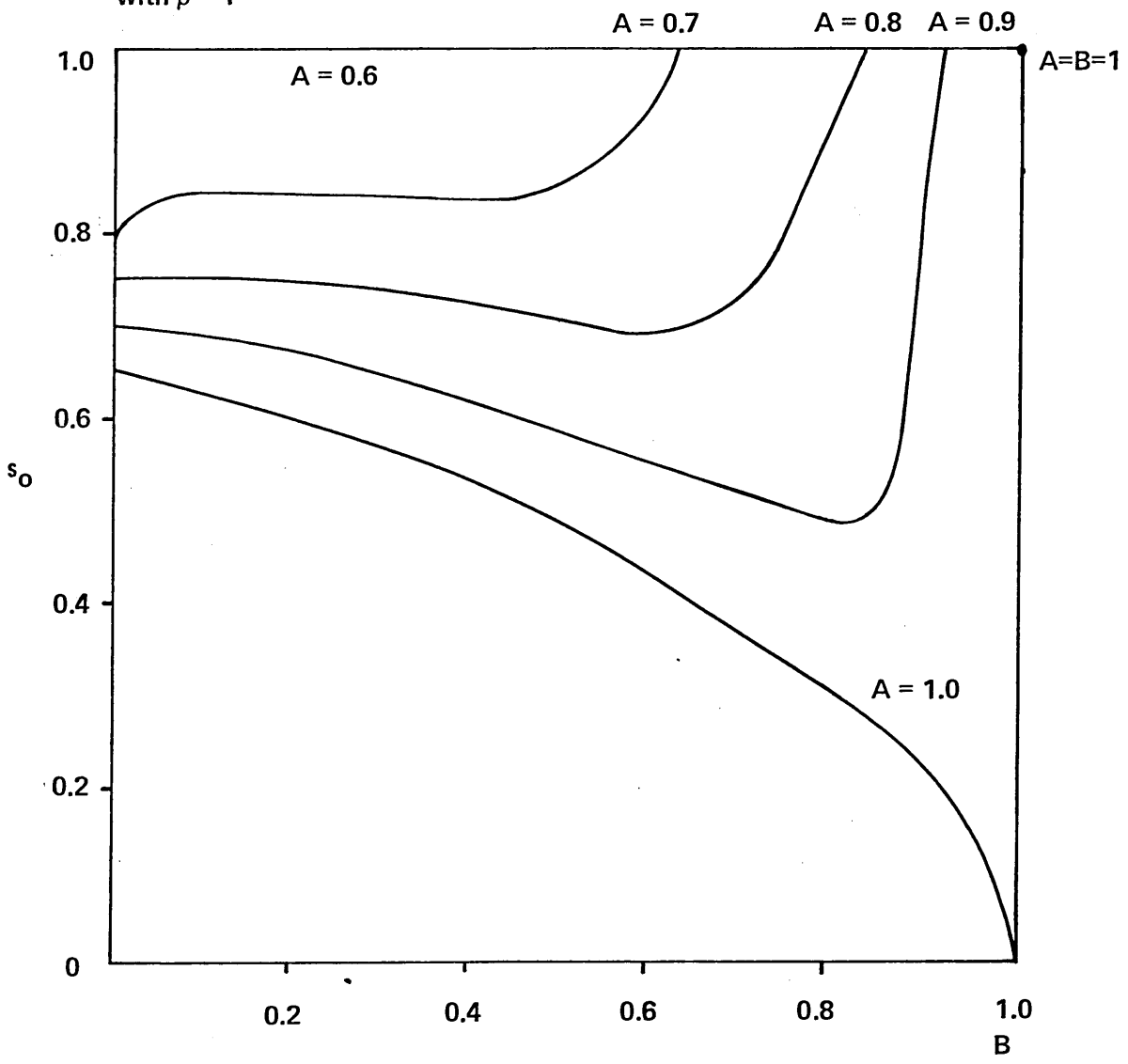
$$\sigma_2(s_0) = (1-A)\,s_0^{-1.9} \quad \text{and} \quad \mu_2(s_0) = (1-B)\,s_0^{-1.9},$$

so that

$$\sigma(s,s_0) = As^{-1} + (1-A)\,s_0^{-1.9}$$

$$\mu(s,s_0) = Bs^{-1} + (1-B)\,s_0^{-1.9} \qquad (5.55)$$

Figure 5.5
Optimum solutions for drift and catastrophic failure model (5.55), with $\beta = 1$



In order for $M_3(s_0)$ to be non-increasing in both s and s_0 and non-negative, it is again necessary to specify that $0 \leq A \leq 1$, $0 \leq B \leq 1$. The figure shows that (for these parameter values) when B is small the optimum values of s_0 vary less with A than when B is large. Of course, for A = B = 1 for which the time intensities are independent of the initial level s_0,

$$M_3(s_0) = 2^{\frac{(\beta-1)}{\beta}} \Gamma(\beta^{-1})(1-e^{-s_0/2})/\beta,$$

so that the optimum value of s_0 is unity. It is also of interest to note that for the case considered, the function $M_3(s_0)$ is always relatively flat, although this flatness reduces as B increases.

5.5 An Example

Finally, in this Chapter we consider the simplest application of the failure models we have introduced in the evaluation of the optimum levels to tune to, s_0, for actual equipment. In Table 5.1 we show the mean lifetimes under various stress conditions of electrical and electronic components assumed to have exponential failure densities and operating at 100°C. This data is taken from Bazovsky (1961), Figure 15.3. Such components may be treated as subject to catastrophic failure alone, and previous experience with these types of devices as well as an investigation of the shape of the relationship between Θ and s suggest that (5.25) provides an appropriate description of the data.

The parameters of the model were estimated by numerical minimisation of

$$\sum_{i=1}^{7} \left[\Theta_i - 0.0011\left\{A\, s_{(i)}^{-\alpha} - (1-A)(1-\gamma \ln s_{(i)})\right\}\right]^2, \qquad (5.56)$$

where Θ_i is the mean lifetime corresponding to stress level $s_{(i)}$

TABLE 5.1

Mean Lifetimes, Θ_i (10^6 hours) at 100°C

Stress

i	Percentage of Nominal Stress	Relative Stress Level, $s_{(i)}$	Θ_i
1	20	0.14	0.1389
2	40	0.29	0.05
3	60	0.43	0.0154
4	80	0.57	0.0071
5	100	0.71 (rated level)	0.0037
6	120	0.86	0.0019
7	140	1	0.0011

TABLE 5.2

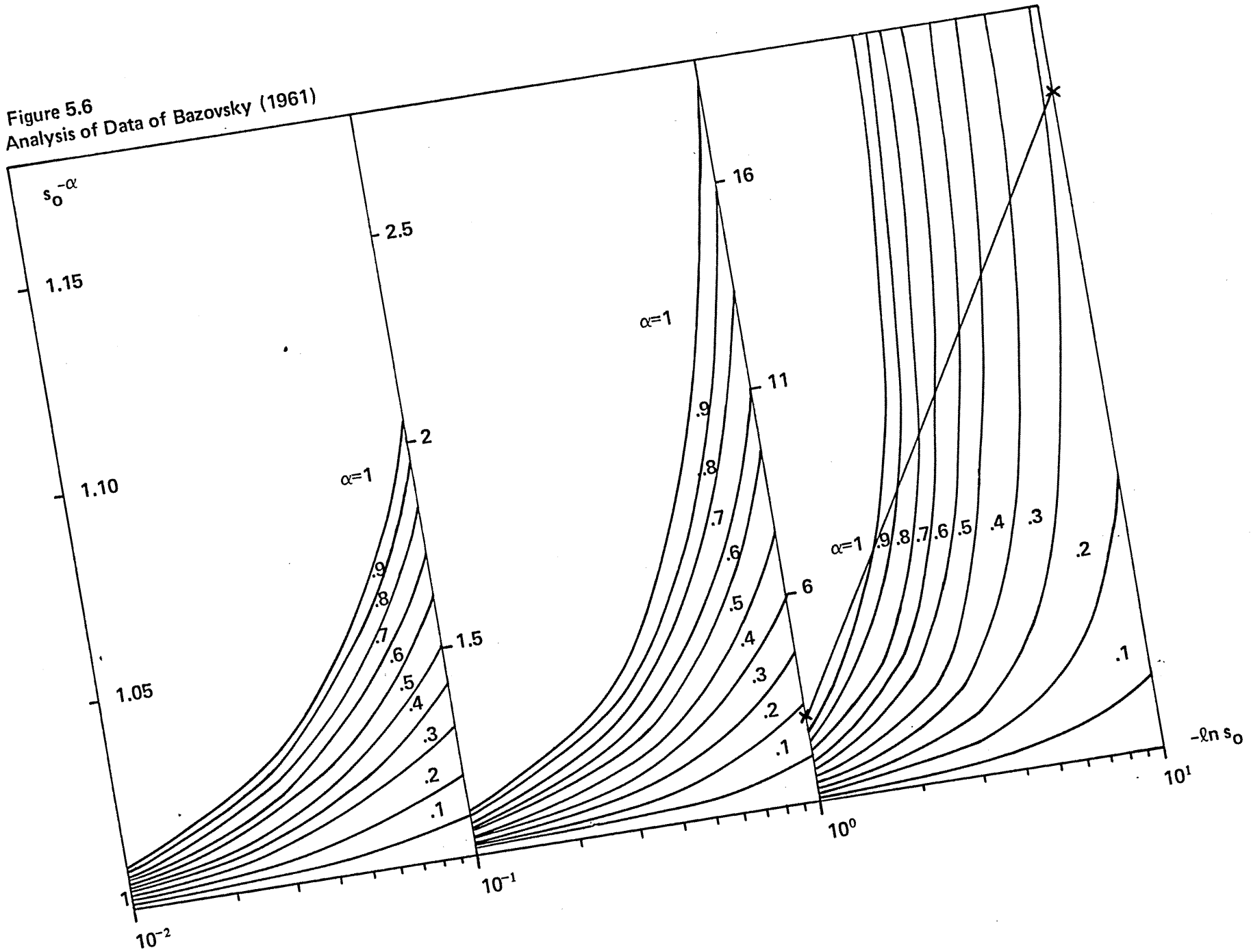
Minimum values of work necessary to justify production at each level relative to value at rated level, 0.71.

$s_{(i)}$	Minimum Relative Values of $g(s_{(i)})/g(0.71) = \Theta_5/\Theta_i$
0.14	0.03
0.29	0.07
0.43	0.24
0.57	0.52
0.71 (rated level)	1
0.86	1.95
1	3.36

and 0.0011 is the length in real time obtained from Table 5.1 of the arbitrary unit time interval corresponding to s = 1 in (5.22) and (5.25). The least squares estimates obtained were $\hat{A} = 38.631$, $\hat{\alpha} = 0.6832$ and $\hat{\gamma} = 0.5031$. Using (5.33) this gives an upper bound for optimum s_0 of 0.6213. Plotting the line $y = 1.528 - 1.547 \ln s_0$ onto the graph paper of Figure 5.1 we obtain Figure 5.6, from which it is apparent that there is one analytic turning value. Corresponding to $\alpha = .7$ we obtain from the graph $-\ln s_0 = 3.45(10^0)$, and for $\alpha = .6$ we obtain $-\ln s_0 = 4.2(10^0)$. Consequently interpolation yields $-\ln s_0 = 3.53(10^0)$, so that the optimum level is $s_0 = 0.029$. This satisfies the second order condition (5.30).

The above solutions are based on the value of the work achieved by a component per unit time having the simple form (5.21). If this should be inappropriate in a particular environment, similar results may be obtained by paralleling the above procedures using an alternative appropriate function g(s). In any event we show in Table 5.2 the minimum values of the work done at each recorded stress level, relative to the value at the rated level 0.71, necessary to justify production at that level. That is, Table 5.2 shows the minimum values of $g(s_{(i)}) / g(0.71)$ necessary to ensure that $M_1(s_{(i)})$ obtained from (5.1) exceeds $M_1(0.71)$. It is apparent from this table that a substantive increase in the value of work with level is necessary in comparison to (5.21) in order to justify production above the rated level. For example, in order for an increase in level from 0.71 to 0.86 to be worthwhile, the value of work must almost double between these two levels. In contrast, operation below the rated level is certainly worthwhile with the current value of work, since for example a reduction of level of about 4/5ths from 0.71 to 0.14 is worthwhile as long as

Figure 5.6
Analysis of Data of Bazovsky (1961)



CHAPTER 6

RETUNING

6.1 Introduction

It may be advantageous to subsequently retune a partially operating unit to some level of operation, $s_{(2)}$ different from that to which it was originally tuned, $s_{(1)}$. One reason for this is that units which survive for long periods of time are often then subject to a rapidly increasing failure intensity or hazard. This is the case for instance for units subject to Weibull failure distributions with shape parameters exceeding unity. Consequently it might be worthwhile to retune units which survive long periods of time to a lower level of operation, thus reducing the failure intensity, although at the expense of also reducing the work achieved by the unit per unit time. Thus a new optimisation problem arises in determining whether, and to what level, we should retune a partially operating unit. It is this problem which is discussed in this Chapter. Throughout it is assumed that the optimisation criterion, as in the previous Chapter, is to maximise the expected amount of work achieved by the unit in its lifetime, and that (5.21) holds. Attention is restricted to units subject to catastrophic failure alone.

Where the hazard function is a monotonically non-decreasing function of time ($\beta > 1$ in the Weibull), as occurs in practice for components which have been successfully burnt-in (e.g. Lomnicki (1973)), no advantage would usually be obtained by retuning units to a higher level of operation than that to which they were originally optimally tuned. This is because if it is worthwhile increasing the level for large time, t, it should also have been worthwhile to do so at t = 0,

since the increase in the value of the work achieved per unit time will be the same whilst the increase in the failure intensity would usually be less. Conversely, for units for which the hazard function is a monotonically non-increasing function of time ($\beta<1$) retuning to lower levels of operation usually gives no advantages, whilst retuning to a higher level may result in an increase in the expected amount of work achieved in the units lifetime. For units with exponential time to failure distributions the hazard is constant, and assuming that although the hazard and expected lifetime are changed by altering the level of operation of the unit the distributional form remains exponential, retuning will never be worthwhile if the unit was originally optimally tuned. In this case the constancy of the hazard over time means that the optimal level at t = 0 remains optimal for all t.

For a Weibull distribution of the form (1.8), the expected residual lifetime given that the unit has already survived for time T is

$$E_T(t) = \Theta \exp\left[\left(\frac{T}{\Theta}\right)^{\beta}\right] \gamma\left[\frac{1}{\beta}+1, \left(\frac{T}{\Theta}\right)^{\beta}\right] - T \qquad (6.1)$$

where $\gamma(\alpha,x) = \int_x^{\infty} e^{-t} t^{\alpha-1}\, dt$.

As indicated previously, in many circumstances the Weibull scale parameter Θ is a function of the level of stress to which the unit is subjected, whilst the shape paremeter β is unaffected by the level of stress applied. This means that for a unit tuned to $s_{(1)}$, Θ should be a function of $s_{(1)}$. If the unit survives a period at $s_{(1)}$ and is subsequently retuned to level $s_{(2)}$, then the Weibull failure distribution to which the unit is subject once it has been

retuned has a scale parameter Θ which in general is a function of both $s_{(1)}$ and $s_{(2)}$. We denote these functions by $\Theta^{+}(s_{(1)})$ and $\Theta^{+}(s_{(2)},s_{(1)})$ respectively.

We know from (1.8) that for the failure distribution of the unit prior to retuning, $\Theta^{+}(s_{(1)})$ should be a monotonically non-increasing function of $s_{(1)}$ in order to ensure that the hazard is a monotonically non-decreasing function, and the expected lifetime is a monotonically non-increasing function, of the level of operation or stress $s_{(1)}$. For the failure distribution after retuning we would require by a similar argument that $\Theta^{+}(s_{(2)},s_{(1)})$ is a monotonically non-increasing function of both $s_{(1)}$ and $s_{(2)}$. (Compare our treatment of $\Theta(s,s_0)$ in Chapter 5). This means that the hazard rate of a unit retuned to level $s_{(2A)}$ (but originally tuned to $s_{(1)}$) will be greater than or equal to the corresponding hazard rate for an identical unit retuned to level $s_{(2B)}$ (but also originally tuned to $s_{(1)}$) for all $s_{(2A)} > s_{(2B)}$. It also means that the hazard rate of a unit retuned to level $s_{(2)}$ but originally tuned to $s_{(1A)}$ will be greater than or equal to the corresponding hazard rate for an identical unit retuned to $s_{(2)}$ but originally at $s_{(1B)} < s_{(1A)}$.

As well as requiring monotonicity, physical conditions suggest that $\Theta^{+}(s_{(1)})$ and $\Theta^{+}(s_{(2)},s_{(1)})$ should be such that

$$\Theta^{+}(s_{(1)}) = \begin{cases} 1 \text{ (in arbitrary time units), when } s_{(1)} = 1 \\ \infty \text{ , when } s_{(1)} = 0 \end{cases}$$

$$\Theta^{+}(s_{(2)},s_{(1)}) = \begin{cases} 1 \text{ (in arbitrary time units), when } s_{(2)} = s_{(1)} = 1 \\ \infty \text{ , when } s_{(2)} = 0 \end{cases} \qquad (6.2)$$

These conditions mean that whilst the expected lifetime at $s_{(1)}$ will, by (1.8), be a finite period when $s_{(1)} = 1$, it will become infinite

as $s_{(1)} \to 0$. Similarly, the expected residual lifetime upon retuning from $s_{(1)}$ to $s_{(2)}$ will be finite for the limiting case $s_{(2)} = s_{(1)} = 1$, but infinite if $s_{(2)} = 0$.

In the previous Chapter, $\Theta(s)$ [or $\Theta(s_{(1)})$] was taken to be a sum of functions of the forms in (5.23) and (5.24) with various parameter restrictions. By an analogous argument to those for $\Theta(s_0)$ and $\Theta(s,s_0)$ we may now consider $\Theta^{+}(s_{(2)},s_{(1)})$ to be a linear combination of two functions with such forms. Now, however, one of these functions is a function of $s_{(1)}$ and the other of $s_{(2)}$. Thus,

$$\Theta^{+}(s_{(2)},s_{(1)}) = \Theta_1^{+}(s_{(1)}) + \Theta_2^{+}(s_{(2)}) \quad , \qquad (6.3)$$

where $\Theta_1^{+}(s_{(1)})$ and $\Theta_2^{+}(s_{(2)})$ may each be defined by (5.23) or (5.24). Consequently, the resultant expressions for $\Theta^{+}(s_{(2)},s_{(1)})$ again represent simple generalisations of the well-established relationships (5.23) and (5.24), and process many of their properties.

6.2 Optimal Retuning

We now suppose that it has been decided that at some time after energising, T, a unit will be retuned from its initial level of operation $s_{(1)}$ to some other level $s_{(2)}$. The time period T may be determined by managerial policy, scheduling considerations or some optimality argument. For a given T and $s_{(1)}$, we desire to select $s_{(2)}$ such that the expected amount of work achieved by the unit in its lifetime will be a maximum. This implies that the choice of $s_{(2)}$ should be such that the expected amount of work to be achieved by the unit in its residual lifetime following T is also maximal. Thus denoting the expected residual lifetime at T given that at that time we retune from $s_{(1)}$ to $s_{(2)}$ by $E_T(s_{(2)},s_{(1)})$ we are by (6.1)

choosing $s_{(2)}$ to maximise

$$M(s_{(2)}, s_{(1)}, T) = s_{(2)} E_T(s_{(2)}, s_{(1)})$$
$$= s_{(2)}\left\{\theta^+(s_{(2)},s_{(1)}) \exp\left[\frac{T}{\theta^+(s_{(2)},s_{(1)})}\right]^\beta \gamma\left[\frac{1}{\beta}+1, \left(\frac{T}{\theta^+(s_{(2)},s_{(1)})}\right)^\beta\right] - T\right\}. \quad (6.4)$$

This has a first differential with respect to $s_{(2)}$ of

$$\frac{\partial M(s_{(2)},s_{(1)},T)}{\partial s_{(2)}} = \left[\theta^+(s_{(2)},s_{(1)}) + s_{(2)}\theta^{+\prime}(s_{(2)},s_{(1)}) - s_{(2)}\theta^{+\prime}(s_{(2)},s_{(1)})\beta\left(\frac{T}{\theta^+(s_{(2)},s_{(1)})}\right)^\beta\right] \exp\left(\frac{T}{\theta^+(s_{(2)},s_{(1)})}\right)^\beta \gamma\left[\frac{1}{\beta}+1, \left(\frac{T}{\theta^+(s_{(2)},s_{(1)})}\right)^\beta\right] + s_{(2)}\theta^{+\prime}(s_{(2)},s_{(1)})\beta\left(\frac{T}{\theta^+(s_{(2)},s_{(1)})}\right)^{\beta+1} - T, \quad (6.5)$$

where

$$\theta^{+\prime}(s_{(2)}, s_{(1)}) = \frac{\partial \theta^+(s_{(2)}, s_{(1)})}{\partial s_{(2)}}.$$

The second derivative is rather complicated, but for certain values of the parameters, values of $s_{(2)}$ exist which make (6.5) zero and the second derivative negative. These optimum values can be found numerically, or in special cases analytically. For values of the parameters where this is not the case no analytic maxima exist, and $M(s_{(2)},s_{(1)},T)$ will be a maximum at $s_{(2)} = 0$ or $s_{(2)} = 1$ (or possibly at $s_{(2)} = s_{(1)}$ if the level to which the unit is retuned is constrained to be $s_{(2)} \leqslant s_{(1)}$ or $s_{(2)} \geqslant s_{(1)}$). The objective function (6.4) may have more than one turning value, so that inspection may be necessary to locate the global maximum.

If we allow $T \to 0$, then (6.4) reduces to (5.11) i.e.

$$\lim_{T\to 0} M(s_{(2)}, s_{(1)}, T) = s_{(2)}\theta^+(s_{(2)},s_{(1)})\Gamma(\beta^{-1})/\beta \quad (6.6)$$
$$= s_0\theta(s_0)\Gamma(\beta^{-1})/\beta,$$

since in the limit as $T\to 0$ the initial level $s_{(1)}$ and the retuned level $s_{(2)}$ become the same (s_0). Thus, in the limit as $T\to 0$, the problem

of optimal retuning is identical to that of optimal tuning, discussed in the previous Chapter. On the other hand, if for general T, $\beta = 1$ so that the Weibull failure distribution reduces to an exponential, then (6.4) becomes

$$M(s_{(2)},s_{(1)},T) = s_{(2)}\overset{+}{\Theta}(s_{(2)},s_{(1)}) \quad , \tag{6.7}$$

which is independent of T, so that the optimum $s_{(2)}$ is also independent of T. Thus, since for $T = 0$, $s_{(2)} = s_{(1)} = s_0$ it follows that in this case (6.7) again reduces to (5.11), and that provided that the unit was originally optimally tuned no advantage is obtained by retuning at any T.

In other simple special cases analytic solutions can be obtained for the optimum level to which to retune. For example, if $\beta = 0.5$, then

$$M(s_{(2)},s_{(1)},T) = 2s_{(2)}\left[\Theta^{+}(s_{(2)},s_{(1)})+\sqrt{T}\sqrt{\Theta^{+}(s_{(2)},s_{(1)})}\right] \quad , \tag{6.8}$$

whilst if $\beta = 2$ then

$$M(s_{(2)},s_{(1)},T) = s_{(2)}\Theta^{+}(s_{(2)},s_{(1)}) \cdot \exp\left[\left(\frac{T}{\Theta^{+}(s_{(2)},s_{(1)})}\right)^2\right]\int_{T/\Theta^{+}(s_{(2)},s_{(1)})}^{\infty} e^{-t^2}dt \quad . \tag{6.9}$$

For T small compared to Θ, (6.8) approximates to twice (6.7). Thus for $\beta = 0.5$ retuning for small T is not worthwhile if the unit was originally optimally tuned. However, for T large compared to Θ,

$$M(s_{(2)},s_{(1)},T) \simeq 2\sqrt{T}s_{(2)}\sqrt{\Theta^{+}(s_{(2)},s_{(1)})} \quad , \tag{6.10}$$

from which the optimum value of $s_{(2)}$ may be evaluated for given $\overset{+}{\Theta}(s_{(2)},s_{(1)})$. We note that since T only appears via the multiple $\sqrt{T}$, the optimum value of $s_{(2)}$ is independent of T for T large. For $\beta = 2$ and T small compared to Θ, so that $T/\Theta \simeq 0$, (6.9)

reduces to

$$M(s_{(2)}, s_{(1)}, T) = s_{(2)}\left[0.5\sqrt{\pi}\,\theta^{+}(s_{(2)}, s_{(1)}) - T\right], \quad (6.11)$$

from which the optimum value of $s_{(2)}$ may again be evaluated for given $\theta^{+}(s_{(2)}, s_{(1)})$. (See below).

6.3 Bounds

Despite the complexity of (6.5) and $\dfrac{d^2M(s_{(2)}, s_{(1)}, T)}{d\,s_{(2)}^2}$, and hence of the numerical evaluation of the optimal $s_{(2)}$ in general, relatively simple bounds can be derived for the optimal level to which to retune. Since $\theta^{+}(s_{(2)}, s_{(1)})$ is a monotonically non-increasing function of both $s_{(1)}$ and $s_{(2)}$ and equals unity when $s_{(1)}$ and $s_{(2)}$ take their maximum values, it follows that it is always non-negative (as required). As $T \geqslant 0$, it also follows that $\gamma\left[\frac{1}{\beta}+1, \left(\frac{T}{\theta^{+}(s_{(2)}, s_{(1)})}\right)^{\beta}\right]$ is non-negative, and in addition since $\theta^{+}(s_{(2)}, s_{(1)})$ is monotonically non-increasing in $s_{(2)}$, $\theta^{+\prime}(s_{(2)}, s_{(1)})$ is non-positive. Hence, since $s_{(1)} \geqslant 0$ and $s_{(2)} \geqslant 0$, it is necessary that in order for a turning value to exist

$$\theta^{+}(s_{(2)}, s_{(1)}) + s_{(2)}\,\theta^{+\prime}(s_{(2)}, s_{(1)}) - s_{(2)}\,\theta^{+\prime}(s_{(2)}, s_{(1)})\,\beta\left(\frac{T}{\theta^{+}(s_{(2)}, s_{(1)})}\right)^{\beta} > 0. \quad (6.12)$$

This in turn implies that

$$T > \theta^{+}(s_{(2)}, s_{(1)})\,\beta^{-1/\beta} \cdot \left[1 + \frac{\theta^{+}(s_{(2)}, s_{(1)})}{s_{(2)}\,\theta^{+\prime}(s_{(2)}, s_{(1)})}\right]^{1/\beta}. \quad (6.13)$$

Substituting the functional forms for $\theta^{+}(s_{(2)}, s_{(1)})$ and $\theta^{+\prime}(s_{(2)}, s_{(1)})$ into (6.13) bounds can be obtained for optimal $s_{(2)}$ for the cases where analytic maxima exist.

Thus if for example

$$\Theta^{+}(s_{(2)},s_{(1)}) = \Theta_1^{+}(s_{(1)}) + (1-A)\,s_{(2)}^{-\delta} \quad , \tag{6.14}$$

where $\Theta^{+}(s_{(1)})$ may be given by (5.23) or by (5.24) with B replaced by A, so that (6.2) holds, then

$$\Theta^{+\prime}(s_{(2)},s_{(1)}) = -\delta\,(1-A)\,s_{(2)}^{-(\delta+1)} \quad . \tag{6.15}$$

It is again necessary that $0 \leq A \leq 1$, so that $\Theta^{+}(s_{(2)},s_{(1)})$ is monotonically non-increasing in both $s_{(1)}$ and $s_{(2)}$ and always positive. Substituting into (6.13) we obtain

$$T > \left[\Theta_1^{+}(s_{(1)}) + (1-A)\,s_{(2)}^{-\delta}\right]\beta^{-1/\beta}\left[\frac{(\delta-1)(1-A) - \Theta_1^{+}(s_{(1)})\,s_{(2)}^{\delta}}{\delta(1-A)}\right]^{1/\beta} , \tag{6.16}$$

and since $s_{(2)}^{-\delta} \gtrless 1$ as δ is positive, upon eliminating $s_{(2)}^{-\delta}$ in the first square bracket we obtain the lower bound

$$s_{(2)} > \left\{\frac{(\delta-1)(1-A)}{\Theta_1^{+}(s_{(1)})} - \frac{\delta(1-A)\,\beta T^{\beta}}{\left[\Theta_1^{+}(s_{(1)}) + (1-A)\right]^{\beta}\Theta_1^{+}(s_{(1)})}\right\}^{1/\delta} . \tag{6.17}$$

If instead we eliminate $s_{(2)}^{\delta}$ in the second square bracket of (6.16) it is necessary for the bound so obtained to be of any use that $(\delta-1)(1-A) > \Theta_1^{+}(s_{(1)})$, which in turn implies that $\delta > 1$. In this case we obtain that

$$s_{(2)} > \left\{\frac{(1-A)\left[(\delta-1)(1-A) - \Theta_1^{+}(s_{(1)})\right]^{1/\beta}}{\left[\delta(1-A)\beta\right]^{1/\beta}T - \Theta_1^{+}(s_{(1)})\left[(\delta-1)(1-A) - \Theta_1^{+}(s_{(1)})\right]^{1/\beta}}\right\}^{1/\delta} \tag{6.18}$$

provided that

$$T > \frac{\Theta_1^{+}(s_{(1)})\left[(\delta-1)(1-A) - \Theta_1^{+}(s_{(1)})\right]^{1/\beta}}{\left[\delta(1-A)\beta\right]^{1/\beta}} . \tag{6.19}$$

For smaller values of T the inequality in (6.18) is reversed, but in this case the right-hand-side is then negative. Thus, for $(\delta-1)(1-A) > \Theta_1^{+}(s_{(1)})$, retuning to $s_{(2)}$ ($s_{(2)} \neq 0$) at any time prior to the bound on T in (6.19) is suboptimal.

For the above bounds to be useful they must of course lie in the range (0, 1). For (6.17) to be greater than or equal to zero and less than or equal to unity, T must lie in the interval

$$T \leqslant \frac{[\Theta_1^+(s_{(1)})+(1-A)](\delta-1)^{1/\beta}}{(\delta\beta)^{1/\beta}} \quad , \tag{6.20}$$

$$T > [\Theta_1^+(s_{(1)})+(1-A)]\left[\frac{(\delta-1)(1-A)-\Theta_1^+(s_{(1)})}{\delta(1-A)\beta}\right]^{1/\beta} . \tag{6.21}$$

Similarly, for (6.18) not to be less than zero we require that

$$\delta \geqslant 1+\Theta_1^+(s_{(1)}) \,/\, (1-A) \quad , \tag{6.22}$$

whilst for it not to exceed unity we again require that T is subject to the bound (6.21) which is more restrictive than (6.19).

If retuning is only possible to lower levels of operation, or only to higher ones, it is possible to place further restrictions upon the usefulness of (6.17) and (6.18). If $s_{(2)}$ must be smaller than or equal to $s_{(1)}$ we find that for (6.17) to be of use

$$T > [\Theta_1^+(s_{(1)})+(1-A)]\left[\frac{(\delta-1)(1-A)-s_{(1)}^{\delta}\Theta_1^+(s_{(1)})}{\delta(1-A)\beta}\right]^{1/\beta} \quad , \tag{6.23}$$

which is in turn more restrictive than (6.21). Similarly, for (6.18) to be of use

$$T > \left[\frac{s_{(1)}^{\delta}\Theta_1^+(s_{(1)})+(1-A)}{s_{(1)}^{\delta}}\right]\left[\frac{(\delta-1)(1-A)+\Theta_1^+(s_{(1)})}{\delta(1-A)\beta}\right]^{1/\beta} \quad , \tag{6.24}$$

which is also more restrictive than (6.21). Alternatively, if $s_{(2)}$ must be greater than or equal to $s_{(1)}$ then for (6.17) and (6.18) to be useful the inequalities in (6.23) and (6.24) must be reversed. In the case of (6.17) this implies a further restriction compared to (6.20). Thus for (6.17) or (6.18) to be applicable, it is necessary

if $s_{(2)}$ is constrained to be greater than or equal to $s_{(1)}$ that T falls within an interval defined by (6.21) and the reversal of (6.23), or (6.21) and the reversel of (6.24) respectively. If $s_{(2)}$ must be less than or equal to $s_{(1)}$, it is necessary instead for (6.17) to be of use that T is between the bound in (6.23) and that of (6.20), and for (6.18) to be of use that T must be larger than the bound in (6.24).

From (6.11) we can obtain for T small the approximate optimum $s_{(2)}$ for the special case of $\beta = 2$. Substituting (6.14) in (6.11) and differentiating with respect to $s_{(2)}$ yields

$$s_{(2)} \simeq \left[\frac{2T-\sqrt{\pi}\,\Theta_1^+(s_{(1)})}{\sqrt{\pi}\,(1-A)\,(1-\delta)}\right]^{-1/\delta} , \qquad (6.25)$$

which maximises $M(s_{(2)},s_{(1)},T)$ provided $\delta < 1$. A necessary condition for (6.25) to be in the range (0,1) is that

$$0.5\sqrt{\pi}\left[\Theta_1^+(s_{(1)}) + (1-\delta)(1-A)\right] \geq T \geq 0.5\sqrt{\pi}\,\Theta_1^+(s_{(1)}) . \qquad (6.26)$$

For $\delta > 1$ there is for $\beta = 2$ no analytic maximum for small T. For $\beta = 0.5$ and T large there is also no analytic maximum.

Suppose now that instead of (6.14) we consider

$$\Theta^+(s_{(2)},s_{(1)}) = \Theta_1^+(s_{(1)}) + (1-A)(1-\delta \ln s_{(2)}) , \qquad (6.27)$$

with $\Theta_1^+(s_{(1)})$ defined as previously and $0 \leq A \leq 1$.

Thus

$$\Theta^{+\prime}(s_{(2)},s_{(1)}) = -\delta\,(1-A)\,s_{(2)}^{-1} , \qquad (6.28)$$

and substituting into (6.13) we obtain

$$T > \left[\Theta_1^+(s_{(1)}) + (1-A)(1-\delta \ln s_{(2)})\right]\beta^{-1/\beta} \cdot \left[\frac{(1-A)(\delta+\delta \ln s_{(2)}-1) - \Theta_1^+(s_{(1)})}{\delta\,(1-A)}\right]^{1/\beta} . \qquad (6.29)$$

Eliminating $\ln s_{(2)}$ in the first square bracket gives the bound

$$s_{(2)} < \exp\left[\frac{\beta T^{\beta}}{[\theta_1^{+}(s_{(1)})+(1-A)]^{\beta}} - \frac{(\delta-1)(1-A)-\theta_1^{+}(s_{(1)})}{\delta(1-A)}\right], \tag{6.30}$$

whilst the elimination of $\ln s_{(2)}$ in the second square bracket is not feasible due to its positive sign. For (6.30) to be useful it must again lie in the range (0, 1) which will be true provided that

$$T \leq [\theta_1^{+}(s_{(1)})+(1-A)]\left[\frac{(\delta-1)(1-A)-\theta_1^{+}(s_{(1)})}{\delta(1-A)\beta}\right]^{1/\beta}, \tag{6.31}$$

which is the reversal of (6.21). This time bound must be non-negative for the bound on $s_{(2)}$ to be of use. This will occur if the parameters satisfy the condition (6.22).

If $s_{(2)}$ must be smaller than or equal to $s_{(1)}$, then for (6.30) to be of use

$$T \leq [\theta_1^{+}(s_{(1)})+(1-A)]\left[\frac{(\delta-1)(1-A)-\theta_1^{+}(s_{(1)})+\delta(1-A)\ln s_{(1)}}{\delta(1-A)\beta}\right]^{1/\beta}, \tag{6.32}$$

which is a more severe restriction on T than (6.31). On the other hand if $s_{(2)}$ cannot be less than $s_{(1)}$ then for (6.30) to be useful the inequality in (6.32) is reversed, so that T is constrained to lie in the range from the reversal of (6.32) to (6.31).

Again, for the special case of $\beta = 2$ we can obtain explicitly an approximation to the optimum $s_{(2)}$ for T small. Substituting (6.27) into (6.11) and differentiating with respect to $s_{(2)}$ now yields the analytic maximum

$$s_{(2)} \simeq \exp\left\{\left[\frac{2}{\delta(1-A)\sqrt{\pi}}\right]\left[\frac{\sqrt{\pi}\,\theta_1^{+}(s_{(1)})+(1-A)}{2} - T\right]-1\right\}. \tag{6.33}$$

This will be in the range (0, 1) if

$$T \geq \frac{\sqrt{\pi}\left[\Theta_1^+(s_{(1)}) - \delta(1-A)\right] + (1-A)}{2} \quad . \tag{6.34}$$

For $\beta = 0.5$ and T large we similarly obtain the analytic maximum

$$s_{(2)} \simeq \exp\left[\frac{(2-\delta)(1-A) + 2\Theta_1^+(s_{(1)})}{2\delta(1-A)}\right] \quad , \tag{6.35}$$

and for this to be in the range (0, 1) we require that

$$\delta > 2\left[1 + \frac{\Theta_1^+(s_{(1)})}{1-A}\right] \quad . \tag{6.36}$$

To illustrate the bounds on optimal $s_{(2)}$ we show in Table 6.1 the ranges of T over which the bounds (6.17), (6.18) and (6.30) are applicable, as well as the values of the bounds for various values of T, for the case

$$\Theta_1^+(s_{(1)}) = A\, s_{(1)}^{-\alpha} \text{ and } \quad \beta = 2, A = 0.5, \delta = 4, \quad \alpha = 0.5, \; s_{(1)} = 0.7. \tag{6.21}$$

The ranges of applicability were obtained from (6.20) and (6.21), (6.21), and (6.31) respectively. The bounds get wider as T increases, as is apparent from their definitions. In this example the bound in (6.18) is always stricter than the bound in (6.17), but it is also of interest to observe that the intervals of optimum $s_{(2)}$ obtained from (6.17) are relatively narrow over most of the values of T for which the bound is valid. To illustrate the approximation to the optimum solution for $\beta = 2$ we may evaluate (6.33) for $T = 0.1$. This yields $s_{(2)} \simeq 0.5397$. (Since $\delta > 1$ there is no analytic maximum in the inverse power law model).

Finally, we note that for the orthogonal problem of interest in which $s_{(1)}$ and $s_{(2)}$ are known but T is to be optimised, no such simple bounds are available.

TABLE 6.1

Bounds on optimal $s_{(2)}$ for $\Theta_1^+(s_{(1)}) = A\, s_{(1)}^{-\alpha}$ and $\beta = 2$, $A = .5$, $\delta = 4$, $\alpha = .5$, $s_{(1)} = .7$

	Inverse Power Law Model		Logarithmic Model
Bound	(6.17)	(6.18)	(6.30)
Range of T for which bound is applicable	0.5213 - 0.6721	0.5213 - ∞	0 - 0.5213
Bounds on $s_{(2)}$ for various T			
T			
0.1	-	-	0 - 0.6479
0.2	-	-	0 - 0.6806
0.4	-	-	0 - 0.8312
0.5	-	-	0 - 0.9647
0.55	0.9543 - 1	0.9595 - 1	-
0.6	0.8450 - 1	0.9190 - 1	-
0.65	0.6351 - 1	0.8602 - 1	-
0.8	-	0.8131 - 1	-
1	-	0.7492 - 1	-
2	-	0.6021 - 1	-
5	-	0.4677 - 1	-
10	-	0.3904 - 1	-

SUMMARY OF ADVANCES IN KNOWLEDGE ACHIEVED AND CONCLUSIONS

This thesis is concerned with the construction of a generalised model of reliability. Chapter 1 reviews the literature and basic model for component and systems reliability. The implicit assumptions of the basic reliability model are identified and their potential for generalisation investigated.

In Chapter 2 the enumeration of multi-state coherent systems is considered and several recursive bounds derived. In the special case of the usual reliability model a new upper bound is shown to be superior to the best explicit and non-asymptotic upper bound previously derived. The relationship of structure functions to event networks is also considered and a theorem proved for pure series and pure parallel systems.

Chapter 3 briefly considers certain three-state and five-state systems and derives explicit state probabilities.

In Chapter 4 a generalised model of reliability is constructed, in which components and systems can take any values in an ordered discrete or continuous state-space representing various levels of partial operation. Discrete and continuous examples of the generalised model of reliability are investigated, and properties of the model derived. Various forms of independence between components are shown to be equivalent, but this equivalence does not completely generalise to the property of zero-covariance. Alternative forms of series and parallel connections are compared, together with the effects of replacement. Multiple time scales are incorporated into the formulation.

The above generalised reliability model is specialised in Chapter 5 so as to facilitate the study of the optimal tuning of partially operating units. Simple drift and catastrophic failure

mechanisms are considered. Explicit and graphical solutions are derived, together with several bounds. In Chapter 6 the optimal retuning of such units is also studied and bounds are again obtained, together with some explicit solutions.

The overall conclusion of the thesis is that it is feasible and desirable to construct more general models of reliability then available henceto. The thesis has implemented this in the context of partial operation. The construction of a reliability model at the level of greatest generality feasible, which was the original aim, still requires further investigation.

REFERENCES

ABDEL-HAMEED, M. (1975). A gamma wear process. IEEE Trans.Rel., R-24, 152-154.

ABDEL-HAMEED, M. (1977). Optimal replacement policies for devices subject to a gamma wear process. In The Theory and Applications of Reliability with emphasis on Bayesian and Nonparametric. Methods, C.P. Tsokos and I.N. Shimi eds., Academic Press, New York, 397-412.

AGGARWAL, K. K. (1976). Reliability of probabilistic logic circuits with random inputs. Microelectron. Reliab., 15, 627-628.

AGGARWAL, K. K., MISRA, K.B. and GUPTA, J.S. (1975a). Reliability evaluation; a comparative study of different techniques. Microelectron. Reliab., 14, 49-56.

AGGARWAL, K. K., MISRA, K.B. and GUPTA, J.S. (1975b). A fast algorithm for reliability evaluation. IEEE Trans.Rel. R24, 83-85.

AGGARWAL, K. K., MISRA,K.B. and GUPTA, J.S. (1975c). Computational time and absolute error comparison for reliability expressions derived by various methods. Microelectron. Reliab., 14, 465-467.

AGGARWAL, K.K. and RAI, S. (1978). Symbolic reliability evaluation using logical signal relations. IEEE Trans. Rel., R-27, 202-205.

A.G.R.E.E. (Advisory Group on Reliability of Electronic Equipment) (1957). Reliability of Military Electronic Equipment. Office of the Assistant Secretary of Defence, U.S.A.

A-HAMEED, M.S. and PROSCHAN, F. (1973). Non-stationary shock models. Stoch. Proc. Applic., 1, 383-404.

A-HAMEED, M.S. and PROSCHAN, F. (1975). Shock models with underlying birth processes. J. Appl. Probability, 12, 18-29.

ALEKSEEV, V.B. (1973). On the number of k-valued monotonic functions. Soviet Math. Dokl., 14, 87-91.

ALLEN, R.N. and BILLINGTON, R. (1980). Effect of common mode failures on the availability of systems. Proceedings of the Sixth Advances in Reliability Technology Symposium, Bradford, Vol. 2. National Centre of Systems Reliability, N.C.S.R. R23, 171-190.

ALLEN, R.N. and DE-OLIVEIRA, M.F. (1977). Three-state reliability modelling of electrical networks. Proceedings of the Advances in Reliability Technology Symposium, Bradford, April 1976, National Centre of Systems Reliability (U.K.A.E.A.), N.C.S.R. R12, ART/3/1-16.

AMENDOLA, A. and CONTINI, S. (1980). About the definition of coherency in binary system reliability analysis. In Synthesis and Analysis Methods for Safety and Reliability Studies, G. Apostolakis, S. Garribba and G. Volta eds., Plenum Press, New York, 79-84.

AMESZ, J., GARRIBBA, S. and VOLTA, G. (1977). Probabilistic analysis of accidental transients in nuclear power plants. In Nuclear Systems Reliability Engineering and Risk Assessment, J.B. Fussell and G.R. Burdick eds., SIAM, 465-487.

ANDERSON, Ø. (1979). Failure characteristics of oil and gas pipelines. Proceedings of the Second National Reliability Conference, Birmingham, Vol. 2, National Centre of Systems Reliability (U.K.A.E.A.), 6B/2/1-13.

ANNELLO, C. (1968). On the Maximum-Likelihood Estimation of Failure Probabilities in the Presence of Competing Risks. Research Analysis Corportion, McLean, Virginia, Technical Paper RAC-TP-291.

ANSELL, J. and BENDELL, A. (1982a). On the optimality of k-out-of-n:G systems. IEEE Trans.Rel., R-31.

ANSELL, J. and BENDELL, A. (1982b). The effect of alternative cost criteria on optimal replacement. Proceedings of the 7th Advances in Reliability Technology Symposium, Bradford, National Centre of Systems Reliability (U.K.A.E.A.), 1B/1/1-11.

ANSELL, J., BENDELL, A. and HUMBLE, S. (1980a). Nested renewal processes. Adv. Appl. Prob., 12, 880-892.

ANSELL, J., BENDELL, A. and HUMBLE, S. (1980b). Optimisation within nested renewal processes. In Analysis and Optimisation of Stochastic Systems, O.L.R. Jacobs., M.H.A. Davis, M.A.H. Dempster, C.J. Harris and P.C. Parks, eds., Academic Press, 197-213.

ANSELL, J., BENDELL, A. and HUMBLE, S. (1981). Generalisation of Dedekind's problem of the enumeration of coherent structures. Proceedings of the Royal Society of Edinburgh, Series A, 89, 239-248.

ANSELL, J., BENDELL, A. and HUMBLE, S. (1982). Age replacement under alternative cost criteria. Submitted to Management Science.

ANSELL, J., BENDELL, A., HUMBLE, S. and MUDHAR, C.S. (1980). 3-state and 5-state reliability models. IEEE Trans.Rel., R-29, 176-177. Supplement; NAPS document No. 03582-B, Microfiche Publications, New York.

APOSTOLAKIS, G.E. and BANSEL, P.P. (1977). Effect of human error on the availability of periodically inspected redundant systems. IEEE Trans.Rel., R-26, 220-225.

APOSTOLAKIS, G., GARRIBBA, S. and VOLTA, G., eds. (1980). Synthesis and Analysis Methods for Safety and Reliability Studies. Plenum Press, New York.

ARCHER, N.P. (1980). A computational technique for maximum likelihood estimation with Weibull models. IEEE Trans.Rel., R-29, 57-62.

ARNBORG, S. (1978). Reduced state enumeration - another algorithm for reliability evaluation. IEEE Trans.Rel., R-27, 101-105.

BANFI, A., GARRIBBA, S., MUSSIO, P., NALDI, F. and VOLTA, G. (1976). Use of three-valued logic for describing reliability of safety systems. Am. Nucl. Soc. Trans., 24, 326-327.

BARBOUR, G. L. (1977). Failure modes and effects analysis by matrix methods. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 114-133.

BARLOW, R.E. (1977). Coherent systems with multi-state components. University of California Operations Research Center Technical Report ORC 77-5.

BARLOW, R.E. (1978). Multistate network reliability theory. 11th European Meeting of Statisticians, Oslo.

BARLOW, R.E., FUSSELL, J.B. and SINGPURWALLA, N.D., Eds. (1975). Discussion by the editors (Topic 3) in Reliability and Fault Tree Analysis, SIAM, 205-212.

BARLOW, R.E., HUNTER, L.C. and PROSCHAN, F. (1963). Optimum redundancy when components are subject to two kinds of failure. J. Soc. Ind. Appl. Math., 11, 64-73.

BARLOW, R.E., MARSHALL, A.W. and PROSCHAN, F. (1963). Properties of probability distributions with monotone hazard rate. Ann. Math. Statist., 34, 375-389.

BARLOW, R.E. and PROSCHAN, F. (1965). Mathematical Theory of Reliability. Wiley, New York.

BARLOW, R.E. and PROSCHAN, F. (1969). A note on tests for monotone failure rate based on incomplete data. Ann. Math. Statist., 40, 595-600.

BARLOW, R.E. and PROSCHAN, F. (1975). Statistical Theory of Reliability and Life Testing: Probability Models. Holt, Rinehart and Winston, New York.

BARLOW, R.E. and PROSCHAN, F. (1976). Some current academic research in system reliability theory. IEEE Trans.Rel., R-25, 198-202.

BARLOW, R.E. and WU, A.S. (1978). Coherent systems with multi-state components. Maths. of Optns. Res. , 3, 275-281.

BAZOVSKY, I. (1961). Reliability Theory and Practice. Prentice Hall, Englewood Cliffs, N.J.

BAZOVSKY, I. Jr. (1977). Fault trees, block diagrams and Markov graphs. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 134-141.

BEICHELT, F. and FISCHER, K. (1979). On a basic equation of reliability theory. Microelectron. Reliab., 19, 367-369.

BEICHELT, F. and FISCHER, K. (1980). General failure model applied to preventative maintenance policies. IEEE Trans. Rel, R-29,39-41.

BELL TELEPHONE LABORATORIES (1961). Launch Control Safety Study, Sect. VII, Vol. I.

BENDELL, A. and HUMBLE, S. (1978). Operating history and failure and degradation tendencies. IEEE Trans.Rel., R-27, 75-76.

BENDELL, A. and HUMBLE, S. (1981). Optimum redundancy with equipment that operates and idles. J. Op. Res. Soc., 32, 875-884.

BENDELL, A., HUMBLE, S. and MUDHAR, C.S. (1979). When is Weibull estimation worthwhile? Proceedings of the Second National Reliability Conference, Birmingham, Vol. 2., National Centre of Systems Reliability (U.K.A.E.A.), 4D/1/1-8.

BENDELL A. and SAMSON, W.B. (1981). The use of rank order distributions for the estimation of the probabilities of rare events. Proceedings of the Third National Reliability Conference, Birmingham, Vol. 1., National Centre of Systems Reliability (U.K.A.E.A.), 2B/1/1-11.

BENDELL, A. and SCOTT, N.H. (1982). Nested renewal processes with Special Erlangian densities. Operations Research, to appear.

BEN-DOV, Y. (1980). Optimal reliability design of k-out-of-n systems subject to two kinds of failure. J.Opl. Res. Soc., 31, 743-748.

BENNETT, G.K. (1977). Basic concepts of Empirical Bayes methods with some results for the Weibull distribution. In The Theory and Applications of Reliability with emphasis on Bayesian and Nonparametric Methods, Vol. II, C.P. Tsokos and I.N. Shimi eds., Academic Press, 181-202.

BENNETTS, R.G. (1975). On the analysis of fault trees. IEEE Trans.Rel., R-24, 175-185.

BERG, M. (1981). A stochastic model for the reliability of a power unit with deliberate shutdowns. Europ. J. of O.R., 6, 309-314.

BICKEL, P.J. (1969). Tests for monotone failure rate II. Ann. Math. Statist., 40, 1250-1260.

BICKEL, P.J. and DOKSUM, K.A. (1969). Tests for monotone failure rate based on normalized spacings. Ann. Math. Statist., 40, 1216-1235.

BIGGERSTAFF, B.E. and JACKSON, T.M. (1969). The Markov process as a means of determining generating-unit state probabilities for use in spinning reserve applications (with discussion). IEEE Trans. Power Apparatus and Systems, PAS-88, 423-430.

BILLINGTON, R., ALLEN, R.N. and DE OLIVEIRA, M.P. (1977). Reliability modelling of electrical auxiliary systems in nuclear power stations. In Nuclear Systems Reliability Engineering and Risk Assessment, J.B. Fussell and G.R. Burdick eds., SIAM, 219-241.

BILLINGTON, R., MEDICHERLA, T.K.P. and SACHDEV, M.S. (1978). Common-cause outages in multiple circuit transmission lines. IEEE Trans. Rel., R-27, 128-131.

BIRNBAUM, Z.W., ESARY, J.D. and MARSHALL, A.W. (1966). A stochastic characterization of wear-out for components and systems. Ann. Math. Statist., 37, 816-825.

BIRNBAUM, Z.W., ESARY, J.D. and SAUNDERS, S.C. (1961). Multi-component systems and structures and their reliability. Technometrics, 3, 55-77.

BLIN, A., CARNINO, A., DUCHEMIN, B., HERVY, A., JUBAULT, G., KOEV, B.V. and LANORE, J.M. (1977). The state-of-the-art of PATREC: a computer code for the evaluation of reliability and availability of complex systems. Proceedings of the National Reliability Conference, Nottingham, National Centre of Systems Reliability (U.K.A.E.A.), NRC3/10/1-9.

BLOCK, H.W. (1977). A family of bivariate life distributions. In The Theory and Applications of Reliability with emphasis on Bayesian and Nonparametric methods, Vol. II, C.P. Tsokos and I.N. Shimi eds., Academic Press, 349-371.

BLOCK, H.W. and SAVITS, T.H. (1980). Multivariate IFRA distributions. Ann. Probability, 8, 793-801.

BLOCK, H.W. and SAVITS, T.H. (1981a). Multidimensional IFRA processes. Annals of Prob., 9, 162-166.

BLOCK, H.W. and SAVITS, T.H. (1981b). A decomposition for multistate monotone systems. J. Appl. Probability, to appear.

BØE, C. (1974). The failure mode concept in reliability engineering. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 281-284.

BOFFEY, T.B. and WATERS, R.J.M. (1979). Calculation of system reliability by algebraic manipulation of probability expressions. IEEE Trans.Rel., R-28, 358-363.

BOSCH, G. (1979). Model for failure rate curves. Microelectron. Reliab., 19, 371-375.

BOURNE, A.J. (1973). Reliability assessment of technological systems, U.K.A.E.A. (S.R.S.)

BRAFF, R. (1977). Impact of the frequency of technician visits on facility failure rate. IEEE Trans.Rel., R-26, 245.

BREWER, R.A. (1977). Desirable developments in reliability. Proceedings of the National Reliability Conference, Nottingham, National Centre of Systems Reliability (U.K.A.E.A.) NRC 6/2/1-10.

BRINDLEY, E.C. and THOMPSON, W.A.Jr.(1972). Dependence and aging aspects of multivariate survival. J. Amer.Statist.Ass., 67, 822-830.

BRYSON, M.C. and SIDDIQUI, M.M. (1969). Some criteria for aging. J.Amer.Statist.Assoc., 64, 1472-1483.

BUCHANAN, W.B. and SINGPURWALLA, N.D. (1977). Some stochastic characterisations of multivariate survival. In The Theory and Applications of Reliability with emphasis on Bayesian and Nonparametric Methods, Vol. II, C.P. Tsokos and I.N. Shimi eds., Academic Press, 329-348.

BUCKLAND, W.R. (1964). Statistical Assessment of the Life Characteristic. Griffin, London.

BUTLER, D.A. (1979a). A complete importance ranking for components of binary coherent systems, with extensions to multi-state systems. Naval Res. Log. Qtly., 26, 565-578.

BUTLER, D.A. (1979b). Bounding the Reliability of Multistate Systems. Dept. of O.R. and Dept. of Statistics, Stanford, California, Technical Report No. 193.

BUZACOTT, J.A. (1967). Finding the MTBF of repairable systems by reduction of the reliability block diagram. Microelectron. Reliab., 6, 105-112.

BUZACOTT, J.A. (1970). Markov approach to finding failure times of repairable systems. IEEE Trans.Rel., R-19, 128-134.

CALDAROLA, L. (1980a). Coherent systems with multi-state components. Nucl. Eng. and Design, 58, 127-139.

CALDAROLA, L. (1980b). Fault-tree analysis with multi-state components. In Synthesis and Analysis Methods for Safety and Reliability Studies, G.Apostolakis, S. Garribba and G. Volta, eds., Plenum Press, New York, 199-248.

CALDAROLA, L. and WICKENHÄUSER, A. (1977). Recent advancements in fault tree methodology at Karlsruhe. In Nuclear Systems Reliability

Engineering and Risk Assessment, J.B. Fussell and G.R. Burdick eds., SIAM, 518-542.

CANNON, A.G. and JONES, G. (1977). The collection and assembly of reliability data for components in chemical plants. Proceedings of the National Reliability Conference, Nottingham, National Centre of Systems Reliability (U.K.A.E.A.), NRC5/8/1-16.

CARLITZ, L. and RIORDAN, J. (1956). The Number of labelled two-terminal series-parallel networks. Duke Math. J., 23, 435-446.

CAVERS, J.K. (1975). Cutset manipulations for communication network reliability estimation. IEEE Trans.Commun., COM-23, 569-573.

CHAN, P.K.W. and DOWNS, T. (1978). Two criteria for preventive maintenance. IEEE Trans.Rel., R-27, 272-273.

CHARLES, R.J. (1961). A review of glass strength. In Progress in Ceramic Science, J.E. Burke ed., Pergamon Press, New York, 1-38.

CHOU, T.C.K. and ABRAHAM, J.A. (1980). Performance/availability model of shared resource multiprocessors. IEEE Trans.Rel., R-29, 70-74.

CHOY, S.C. and MAZUMDAR, M. (1975). Determination of test intervals in certain repairable standby protective systems. IEEE Trans. Rel., R-24, 201-205.

CHRISTER, A.H. (1978). Refined asymptotic costs for renewal reward processes. J. Opl. Res. Soc., 29, 577-583.

CHUNG, W.K. (1979). A N-unit redundant system with common-cause failures. Microelectron.Reliab., 19, 377-378.

CHURCH, R. (1940). Numerical analysis of certain free distributive structures. Duke Math. J., 6, 732-734.

CHURCH, R. (1965). Enumeration by rank of the elements of the free distributive lattice with seven generators. Notices Amer. Math. Soc., 12, 724.

CINLAR, E. (1977). Shock and wear models and Markov additive processes. In The Theory and Applications of Reliability, Vol. I, C.P. Tsokos and I.N. Shimi eds., Academic Press, New York, 193-214.

CODIER, E.O. (1968). Reliability growth in real life. Proceedings of the Annual Symposium on Reliability, IEEE.

COHEN, A.C. Jr. (1965). Maximum likelihood estimation in the Weibull distribution based on complete and on censored samples. Technometrics, 7, 579-588.

COHN, P.M. (1977). Algebra, Vol. 2. Wiley, New York.

COLOMBO, A.G. and JAARSMA, R.J. (1980). Statistical methods for the European reliability data bank. Proceedings of the Third Reliability Data Bank Seminar, Bradford, National Centre for Systems Reliability, NCSR, R-24, 157-173.

COTTRELL, D.F. (1977). Dormancy effects on Non-electronics. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 7-11.

COX, D.R. (1959). The analysis of exponentially distributed life-times with two types of failure. J.R. Statist. Soc., B, 21, 411-421.

COX, D.R. (1962). Renewal Theory. Methuen, London.

COX, D.R. (1972). Regression models and life-tables (with discussion). J.R. Statist. Soc., B., 34, 187-220.

CRAN, G.W. (1976). Graphical estimation methods for Weibull distributions. Microelectron. Reliab., 15, 47-52.

DAHIYA, R.C. (1977). Estimation of reliability after corrective action. IEEE Trans. Rel., R-26, 348-351.

DARTOIS, J.P. (1977). Grade-of-service and service quality concepts in public telephone exchanges. Elec. Commun., 52, 266.

DAS, S., HENDRY, A. and HONG, S. (1980). The impact of imperfect repair on system reliability. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 393-402.

DEDEKIND, R. (1897). Über Zerlegungen von Zahlen durch ihre grössten gemeinsamen Teiler. Festschrift Technische Hochschule Braunschweig und Gesammelte Werke, II, 103-148.

DERMAN, C. (1963). On optimal replacement rules when changes of state are Markovian. Chapter 9 in Mathematical Optimisation Techniques, R. Bellman ed., University of California Press, Berkley, 201-210.

DERMAN, C. and SACKS, J. (1960). Replacement of periodically inspected equipment. Naval Res. Logist. Quart., 7, 597-607.

DHILLON, B.S. (1976a). Analysis of a three-state system with two types of components. Microelectron. Reliab., 15, 245-246.

DHILLON, B.S. (1976b). A note on a four-state system. Microelectron. Reliab., 15, 491-492.

DHILLON, B.S. (1976c). A six-state system model. Microelectron. Reliab., 15, 139.

DHILLON, B.S. (1977a). A four-unit redundant system with common-cause failures. IEEE Trans. Rel., R-26, 373-374.

DHILLON, B.S. (1977b). Literature survey on three-state device reliability systems. Microelectron. Reliab., 16, 601-602.

DHILLON, B.S. (1977c). A multi-state system model. Microelectron. Reliab., 16, 55-56.

DHILLON, B.S. (1978c). A k-out-of-N three-state devices system with common-cause failures. Microelectron. Reliab., 18, 447-448.

DHILLON, B.S. (1978d). A two failure modes system with cold standby units. Microelectron. Reliab., 18, 251-252.

DHILLON, B.S. (1979a). A four-unit redundant system with common-cause failures. IEEE Trans. Rel., R-28, 267.

DHILLON, B.S. (1979b). Partial and catastrophic failure analysis of a complex system. Microelectron. Reliab., 19, 349-350.

DHILLON, B.S. (1979c). New hazard rate functions. Microelectron. Reliab., 18, 531-532.

DHILLON, B.S. , SAMBHI, A. and KHAN, M.R. (1979). Common-cause failure analysis of a three-state device system. Microelectron. Reliab., 19, 345-348.

DHILLON, B.S. and SINGH, C. (1978). Bibliography of literature on fault trees. Microelectron. Reliab., 501-503.

DOREY, J. (1979). Reliability data derived from Electricite de France operating experience: example of the pumps. Proceedings of the Second National Reliability Conference, Birmingham, Vol. 1, National Centre of Systems Reliability (U.K.A.E.A.), 3A/2/1-8.

DORGAN, C.E. and EMER, D.S. (1978). A degradation reliability model. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 510-518.

DYER, D. and KEATING, J. (1980). Estimation of the guarantee time in an exponential failure model. IEEE Trans.Rel., R-29, 63-5.

EASTON, M.C. and WONG, C.K. (1980). Sequential destruction method for Monte Carlo evaluation of system reliability. IEEE Trans. Rel., R-29, 81-84.

EDWARDS, G.T. and WATSON, I.A. (1979). A Study of Common-Mode Failures. U.K.A.E.A. Safety and Reliability Directorate, SRD R 146.

ELBURN, A.J. and KNIGHT, D.J. (1975). The Effect of Power Supply Switching on the Reliability of Electronic Equipment. Procurement Executive, Ministry of Defence, MOD(PE) No. GR/77/070-01, Tech. Report No. RTP/1300/01.

EL-NEWEIHI,E., PROSCHAN, F. and SETHURAMAN, J. (1978). Multi-state coherent systems. J. Appl. Probability, 15, 675-688.

ELSAYED, E.A. (1979). Repairable systems with one standby unit. Microelectron. Reliab., 19, 243-245.

ELSAYED, E.A. and ZIEBIB, A. (1979). A repairable multi-state device. IEEE Trans. Rel., R-28, 81-82.

ENDICOTT, H.S., HATCH, B.D. and SCHMER, R.G. (1965). Application of the Eyring Model to capacitor aging data. IEEE Trans. Component Parts, CP-12, 34-41.

ENDICOTT, H.S. and STARR, W.T. (1961). Progressive stress - a new accelerated approach to voltage endurance. AIEE Trans. Power App. Syst, 80, 515-522.

ENDICOTT, H.S. and ZOELLNER, J.A. (1961). A preliminary investigation of the steady and progressive stress testing of mica capacitors. Proceedings of the Seventh National Symposium on Reliability and Quality Control, 229-235. Reprinted in General Elec. Co. TIS Rep. GIGL 223.

ENDRENYI, J. (1970). Three state models in power system reliability evaluations. IEEE Trans. Pas., PAS-89, 1909-1916.

ENDRENYI, J., MAENHAUT, P.C. and PAYRE, L.E. (1973). Reliability evaluation of transmission systems with switching after faults - approximations and a computer program. IEEE Trans. Pas., PAS-92, 1863-1875.

ESARY, J.D. and MARSHALL, A.W. (1974). Some classes of distributions closed under the formation of coherent systems. In Reliability and Biometry, F. Proschan and R.J. Serfling eds., SIAM, Philadelphia, 9-30.

ESARY, J.D. and MARSHALL, A.W. (1979). Multivariate distributions with increasing hazard rate averages. Ann. Probability, 7, 359-370.

ESARY, J.D., MARSHALL, A.W. and PROSCHAN, F. (1973). Shock models and wear processes. Ann. Probability, 1, 627-649.

ESARY, J.D. and PROSCHAN, F. (1963). Coherent structures of non-identical components. Technometrics, 5, 191-209.

EVANS, R.A. (1971). Data we will never get (editorial). IEEE Trans. Rel., R-20, 2.

EVANS, R.A. (1974). Data, data, oh where art thou data? (editorial). IEEE Trans. Rel., R-23, 289.

EVANS, R.A. (1976). Analysis of logic diagrams: a quandary (editorial). IEEE Trans. Rel., R-25, 65.

EVANS, R.A. (1977). Theory and/v's practise (editorial). IEEE Trans. Rel., R-26, 145.

EVANS, R.A. (1978). Review of 'Statistical Theory of Reliability and Life Testing' by R.E. Barlow and F. Proschan. IEEE Trans. Rel., R-27, 307.

FARDIS, M.N. and CORNELL, C.A. (1981). Analysis of coherent multistate systems. Submitted to IEEE Trans. Rel.

FAREWELL, V.T. and COX, D.R. (1979). A note on multiple time scales in life testing. Appl. Statist., 28, 73-75.

FELDMAN, R.M. (1976). Optimal replacement with Semi-Markov shock models. J. Appl. Probability, 13, 108-117.

FELDMAN, R.M. (1977). The maintenance of systems governed by Semi-Markov shock models. In The Theory and Application of Reliability with emphasis on Bayesian and Nonparametric Methods, Vol. I, C.P. Tsokos and I.N. Shimi eds., Academic Press, New York, 215-226.

FELLER, W. (1940). On the integro-differential equations of purely discontinuous Markoff processes. Trans. Amer. Math. Soc., 48, 488-515.

FELLER, W. (1968). An Introduction to Probability Theory and its Applications, Vol. 1, Third edition. Wiley, New York.

FELLER, W. (1971). An Introduction to Probability Theory and its Applications, Vol. 2, Second edition. Wiley, New York.

FERTIG, K.W. and MURTHY, V.K. (1978). Models for reliability growth during burn-in: theory and applications. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 504-509.

FISCHER, K. (1977). The Markov reliability model in railway safety engineering. (In German). Die Eisenbahntechnik, 25, 189-192.

FISHER, R.A. and TIPPETT, L.H.C. (1928). Limiting forms of the frequency distribution of the largest or smallest member of a sample. Proc. Cambridge Phil. Soc., 24, 180-190.

FLEGG, H.G. (1971). Boolean Algebra, Macdonald, London.

FLEHINGER, B.J. (1962). A Markovian model for the analysis of the

effects of marginal testing on system reliability. Ann. Math. Statist., 33, 754-766.

FUSSELL, J.B. (1975). How to hand-calculate system reliability and safety characteristics. IEEE Trans. Rel., R-24, 169-174.

FUSSELL, J.B., ABER, E.F. and RAHL, R.G. (1976). On the quantitative analysis of priority-AND failure logic. IEEE Trans. Rel., R-25, 324-326.

GAERTNER, W.W., ELDERS, D.S., ELLINGHAM, D.B., KOSTNING, J.A. and SCHREYER, W.M. (1977). Automatic electronic component failure-rate prediction with MIL-HDBK-217B. IEEE Trans.Rel., R-26,214-219.

GARRIBBA, S., MUSSIO, P. and NALDI, F. (1980). Multivalued logic in the representation of engineering systems. In Synthesis and Analysis Methods for Safety and Reliability Studies, G. Apostolakis, S. Garribba and G. Volta eds., Plenum Press, New York, 183-198.

GAVER, D.P. (1963). Random hazard in reliability problems. Technometrics, 5, 211-226.

GAVER, D.P. (1964). Failure time for a redundant repairable system of two dissimilar units. IEEE Trans. Rel., R-13, 14-22.

GEORGE, L.L. (1978). Re-evaluation of the Air Force Actuarial System. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 7-10.

GIBSON, M.R. (1979). Field data from the chemical industry. Proceedings of the Second National Reliability Conference, Birmingham, Vol. 1., National Centre of Systems Reliability, 2B/2/1-8.

GIBSON, M.R. and KNOWLES, G. (1977). Design and maintenance of instrument trip systems. Proceedings of the National Reliability Conference, Nottingham, National Centre of Systems Reliability (U.K.A.E.A.), NRC4/11/1-16.

GILBERT, E.N. (1954). Lattice theoretic properties of frontal switching functions. J. Math. Phys., 33, 57-67.

GNEDENKO, B.W., BIELAJEV, J.K. and SOLOVIER, A.D. (1965), English edition 1970; Mathematical Methods of Reliability Theory. Academic Press, New York.

GOKCEK, O., BAZOVSKY, I. and CRELLIN, G.L. (1979). Markov analyses of nuclear plant failure dependencies. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 104-109.

GOPAL, L., AGGARWAL, K.K. and GUPTA, J.S. (1978a). Reliability optimization in systems with many failure modes. Microelectron. Reliab., 18, 423-425.

GOPAL, K., AGGARWAL, K.K. and GUPTA, J.S. (1978b). Reliability analysis of multistate device networks. IEEE Trans.Rel., R-27. 233-236.

GOPALAN, M.N. and DHARMADHIKOI, A.D. (1980). Two-unit system with delayed repair facility. IEEE Trans.Rel., R-29, 187.

GORDON, R. (1957). Optimum component redundancy for maximum system reliability. Operations Res., 5. 229-243.

GORG, R.C. and KUMAR, A. (1977). A complex system with two types of failure and repair. IEEE Trans. Rel., R-26, 299-300.

GOWER, I.M. (1975). Reliability modelling for an air-to-air guided missile. I.M.A. Conference on the Mathematics and Statistics of Reliability, London; Bull, I.M.A., 12, 258-262.

GRADSHTEYN, I.S. and RYZHIK, I.M. (1980). Table of Integrals, Series and Products. Academic Press, New York.

GREEN, A.E. (1977). A view of the reliability scene. Proceedings of the First National Reliability Conference, Nottingham; Keynote Papers and Discussion, National Centre of Systems Reliability. (U.K.A.E.A.), 1978.

GREEN, A.E. and BOURNE, A.J. (1972). Reliability Technology. Wiley, New York.

GRIFFITH, W.S. (1980). Multistate reliability models. J. Appl. Probability, 17, 735-744.

GROSS,A.J. and LURIE, D. (1977). Monte Carlo comparisons of parameter estimates of the two-parameter Weibull distribution. IEEE Trans. Rel., R-26, 356-363.

GROVER, M.S. and BILLINGTON, R. (1974). A computerised approach to substation and switching station reliability evaluation. IEEE Trans. Power Apparatus and Systems, PAS-93, 1488-1497.

GUPTA, H. and SHARMA, J. (1978a). A delta-star transformation approach for reliability evaluation. IEEE Trans. Rel., R-27, 212-214.

GUPTA, H. and SHARMA, J. (1978b). Complex system reliability with imperfect switching under preemptive-resume repair. IEEE Trans. Rel., R-27, 68-9.

GUPTA, H. and SHARMA, J. (1979). A method symbolic steady-state availability evaluation of k-out-of-n: G system. IEEE Trans. Rel., R-28. 56-57.

HAAS, P.M. and BATT, T.F. (1980). Criteria for safety-related nuclear plant operator actions: a preliminary assessment of available data. Proceedings of the Sixth Advances in Reliability Technology Symposium, Bradford, Vol. 2., National Centre of Systems Reliability, NCSR R23, 198-217.

HAGER, H.W., BAIN, L.J. and ANTLE, C.E. (1971). Reliability estimation for the generalized gamma distribution and robustness of the Weibull model. Technometrics, 13, 547-557.

HAHN, G.J. and NELSON, W. (1974). A comparison of methods for analysing censored life data to estimate relationships between stress and product life. IEEE Trans. Rel., R-23, 2-11.

HAINES, A.L. and SINGPURWALLA, N.D. (1974). Some contributions to the stochastic characterisation of wear. In Reliability and Biometry, F. Proschan and R.J. Serfling eds., SIAM, 47-80.

HANISH, H., HILTON, P.J. and HIRSCH, W.M. (1969). Algebraic and combinational aspects of coherent structures. Trans. New York Acad. Sci., 31, 1024-1037.

HANSELL, G. (1966). Sur le nombres des fonctions booléenes monotones de n variables. C.R. Acad. Sci. Paris, Ser. A-B, 262, A1088-A1090.

HARRIS, R. (1970). A multivariate definition for increasing hazard rate distribution functions. Ann. Math. Statist., 41, 713-717.

HARRIS, C.M. and SINGPURWALLA, N.D. (1968). Life distributions derived from stochastic hazard functions. IEEE Trans. Rel., R-17, 70-79.

HARTER, H.L. and MOORE, A.H. (1965). Maximum - likelihood estimation of the parameters of Gamma and Weibull populations from complete and from censored samples. Technometrics, 7, 639-643.

HARTER, H.L. and MOORE, A.H. (1967). Asymptotic variances and covariances of maximum likelihood estimators, from censored samples, of the parameters of Weibull and Gamma populations. Ann. Math. Statist., 38, 557 - 570.

HARTER, H.L. and MOORE, A.H. (1976). An evaluation of exponential and Weibull test plans. IEEE Trans. Rel., R-25, 100-104.

HATOYAMA, Y. (1979). Reliability analysis of three-state systems. IEEE Trans. Rel., R-28, 386-393.

HAY, D.R., GODBOUT, P. and BRAIS, A. (1979). Behavioural prediction of materials degrading in load-bearing structures. Microelectron. Reliab., 19, 127-132.

HENLEY, E.J. and KUMAMOTO, H. (1981). Reliability Engineering and Risk Assessment. Prentice Hall, Englewood Cliffs, N.J.

HENLEY, E.J. and POLK, R.E. (1977). A risk/cost assessment of administrative time restrictions on nuclear power plant operations. In Nuclear Systems Reliability Engineering and Risk Assessment., J.B. Fussell and G.R. Burdick eds., SIAM, 495-517.

HIGGINS, J.J. and TSOKOS, C.P. (1977). Sensitivity of Bayes estimators of reciprocal MTBF and reliability to an incorret failure model. IEEE Trans. Rel., R-26, 286-289.

HINDS, P.R., NEWTON,D.W. and JARDINE, A.K.S. (1977). Problems of Weibull parameter estimation from small samples. National Reliability Conference, Nottingham, National Centre of Systems Reliability (U.K.A.E.A.), NRC5/3/1-32.

HIRSCH, W.M., MEISNER, M. and BOLL, C.H. (1968). Cannibalization in multicomponent systems and the theory of reliability. Naval Res. Logist. Quart., 15, 331-360.

HOCHBERG, M. (1973). Generalized multicomponent systems under cannibalization. Naval Res. Logist. Quart., 20, 585-605.

HOHN, F.E. (1962). Applied Boolean Algebra: An Elementary Introduction. Macmillan, New York.

HOLLANDER, M. (1978). Testing whether more failures occur later. Proceedings of the Annual Reliability and Maintainability Symposium. IEEE., 103-106.

HOLLANDER, M. and PROSCHAN, F. (1972). Testing whether new is better than used. Ann. Math. Statist., 43, 1136-1146.

HOLLANDER, M. and PROSCHAN, F. (1974). A test for superadditivity of the mean value function of a non-homogeneous Poisson process. Stoch. Proc. Applic., 2, 195-209.

HOLMBERG, G. and MARKLING, G. (1980). From failure report to part failure rate, reliability data collection and evaluation for monitoring equipment. Proceedings of the Third European Data Bank Seminar, Bradford, National Centre of Systems Reliability, NCSR R24, 271-293.

HORWITZ, D. (1976). General method for incipient fault detection. IEEE Trans. Rel., R-25, 299-301.

HOWARD, R.A. (1960). Dynamic Programming and Markov Processes. Technology Press, MIT., Cambridge, Mass. and Wiley, New York.

HUDSON, J.C. and KAPUR, K.C. (1982). Reliability theory for multistate systems with multistate components. Microelectron. Reliab., 22, 1-7.

HYUN, (M.Gen), K-N. (1975). Reliability optimisation by 0-1 programming for a system with several failure modes. IEEE Trans.Rel., R-24, 206-210.

INAGAKI, T. (1980). Optimization problems in system supervision. In Synthesis and Analysis Methods for Safety and Reliability Studies, G. Apostolakis, S. Garribba and G. Volta eds., Plenum Press, New York, 329-340.

ISHAM, V. (1974). The effect of intermittent exposure to risk on failure time distributions. J. Appl. Prob., 11, 310-319.

JACOBI, G.T. (1968). Reliability physics. IEEE Trans. Rel., R-17, 2-4.

JORDAN, A.S. (1978). Reliability of two dissimilar devices degrading in a periodic alternating sequence-application to opto-isolators. Microelectron. Reliab., 18, 357-366.

KAHN, H.D. (1979). Least squares estimation for the inverse power law for accelerated life tests. Appl. Statist., 28, 40-46.

KAIO, N. and OSAKI, S. (1980. Comparisons of point estimation methods in the two-parameter Weibull distribution. IEEE Trans. Rel, R-29, 21.

KAO, J.H.K. (1959). A graphical estimation of mixed Weibull parameters in life testing of electron tubes. Technometrics. 1, 389-407.

KAO, J.H.K. (1960). A summary of some new techniques on failure analysis. Proceedings of the Sixth National Symposium on Reliability and Quality Control, USA, 190-201.

KAPUR, P.K. and KAPOOR, K.R. (1975). Intermittent complex systems. IEEE Trans. Rel., R-24, 276-277.

KAPUR, P.K. and KAPOOR, K.R. (1978a). Intermittently used redundant systems. Microelectron. Reliab., 17, 593-596.

KAPUR, P.K. and KAPOOR, K.R. (1978b). Stochastic behaviour of some two-unit redundant systems. IEEE Trans. Rel., R-27, 382-385.

KAUFMANN, A. (1969). La Confiance Technique. Dunod, Paris. English edition 1972; Reliability, a Mathematical Approach. Transworld, London.

KAUFMANN, A., GROUCHKO, D. and CRUON, R. (1977). Mathematical Models for the Study of the Reliability of Systems. Academic Press, New York.

KAY, E. (1973). Estimates of renewal probabilities from Weibull 'life to failure' distributions. The Quality Engineer, 37, 237-245.

KERN, G.A. (1978). Operational influences on avionics reliability. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 231-242.

KHALIL, Z.S. (1977). On the reliability of a two-unit redundant system with random switchover time and two types of repair. Microelectron. Reliab., 16, 159-160.

KHANDELWAL, D.N., SHARMA, J. and ROY L.M. (1979). Optimal periodic maintenance policy for machines subject to deterioration and random breakdown. IEEE Trans. Rel., R-28, 328-330.

KIMBALL, E.W. (1980). Accelerated versus real time aging tests. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 51-54.

KING, J.R. (1971). Probability Charts for Decision Making. Industrial Press Inc., New York.

KLAASEN, K.B. and VAN-PEPPEN, J.C.L. (1977a). Majority and similarity voting in analogue redundant systems. Microelectron. Reliab., 16, 47-54.

KLAASEN, K.B. and VAN-PEPPEN, J.C.L. (1977b). Reliability improvement by redundancy voting in analogue electronic systems. Microelectron. Reliab., 16, 593-600.

KLEITMAN, D. (1969). On Dedekind's problem:the number of monotone Boolean functions. Proc. Amer. Math. Soc., 21, 677-682.

KLEITMAN, D. and MARKOWSKY, G. (1975). On Dedekind's problem: the number of isotone Boolean functions II. Trans.Amer. Math.Soc., 213, 373-390.

KNÖDEL, W. (1950). Über Zerfallungen. Monats.Math., LV, 20-27.

KONARSKI, D. and EVERT, K.P. (1975). Acquisition of reliability statements. (In German). Nachrichtentechnik: Elektronik, 25. 367-383.

KONTOLEON, J.M. (1978a). Optimum allocation of components in a special two-port network. IEEE Trans.Rel., R-27. 112-115.

KONTOLEON, J.M. (1978b). Analysis of a dynamic redundant system. IEEE Trans.Rel., R-27, 116-119.

KONTOLEON, J.M. and KONTOLEON, N. (1974). Reliability analysis of a system subject to partial and catastrophic failures. IEEE Trans.ReL, R-23, 277-278.

KONTOLEON, N., KONTOLEON, J.M. and CHRYSOCHOIDES, N.G. (1975). The throw-away maintenance philosophy versus repair. International Symposium on Reliability of Nuclear Power Plants, I.A.E.A., Austria.

KOOI, C.F. (1967). Relation of a physical process to the reliability of electronic components. IEEE Trans.Rel., R-16, 113-116. (Correction: R-17, 179.)

KOROBKOV, V.K. (1963). Estimation of the number of monotonic functions of the algebra of logic and of the complexity of the algorithm for finding the resolvent set for an arbitrary monotonic function of the algebra of logic. Soviet Math. Dokl., 4, 753-756.

KORŠUNOV, A.D. (1977). Solution of Dedekind's problem on the number of monotonic Boolean functions. Soviet Math. Dokl., 18, 442-445.

KOUL, H.L. (1977). A test for new better than used. Commun. Statis - Theory and Meths., 6, 563-573.

KSIR, B. (1979). Reliability analysis of systems composed of elements with three states. Microelectron. Reliab., 19, 229-232.

KUCHII, S., KAIO, N. and OSAKI, S. (1979). Simulation comparisons of point estimation methods in the two-parameter Weibull distribution. Microelectron. Reliab., 19, 333-336.

KUJAWSKI, G.F. and RYPKA, E.A. (1978). Effects of 'on-off' cycling on equipment reliability. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 225-230.

KUMAMATO, H. and HENLEY, E.J. (1978). Top-down algorithm for obtaining prime implicant sets of non-coherent fault trees. IEEE Trans. Rel., R-27, 242-249.

KUMAR, A. and AGGARWAL, M. (1978). Analysis of a two-unit standby redundant system with two types of failure. IEEE Trans.Rel., R-27, 301-302.

KUMAR, A. and JAIN, J.L. (1977). On a two-unit standby system with repair specification. IEEE Trans.Rel., R-26, 369-370.

KUMAR, A. and KAPOOR, V.B. (1979). Analysis of a two-unit maintained system with imperfect detection of failures. Microelectron. Reliab., 19, 329-331.

KURATOWSKI, K. (1972). Introduction to Set Theory and Topology (translated from Polish by L.F. Boron), second English edition. Pergamon, London.

LAKOFF, G. (1975). Hedges. a study in meaning criteria. In Contemporary Research in Philosophical Logic and Linguistics, D. Hockney, W. Harper and B. Freed eds., D. Reidel.

LAPP, S.A. and POWERS, G.J. (1977). Computer-aided synthesis of fault trees. IEEE Trans. Rel., R-26, 2-13.

LAVIRON, A., BÉRARD, C. and QUENEE, R. (1979). ESCAF - Failure simulation and reliability calculation device. Proceedings of the Second National Reliability Conference, Birmingham, Vol. 2., National Centre of Systems Reliability (U.K.A.E.A.), 6C/4/1-10.

LEE, S.H. (1980). Reliability evaluation of a flow network. IEEE Trans. Rel., R-24, 24-26.

LEGG, J.M. (1978). Computerized approach for matrix-form FMEA. IEEE Trans. Rel., R-27. 254-257.

LEHTINEN, E.A. (1979). On estimation for three-parameter Weibull distribution. Proceedings of the Second National Reliability Conference, Birmingham, Vol. 2., National Centre of Systems Reliability (U.K.A.E.A.), 4D/3/1-13.

LEMON, G.H. (1975). Maximum likelihood estimation for the three parameter Weibull distribution based on censored samples. Technometrics, 17, 247-254.

LEVINE, S. and VESELY, W.E. (1977). Prospects and problems in risk analysis: some viewpoints. In Nuclear Systems Reliability Engineering and Risk Assessment, J.B. Fussell and G.R. Burdick eds., SIAM, 5-21.

LEWIS, P.A.W. (1964). A branching Poisson process model for the analysis of computer failure patterns (with discussion). J. Roy.Statist. Soc. B., 26, 398-456.

LIN, P.M., LEON, B.J. and HWANG, T.C. (1976). A new algorithm for symbolic system reliability analysis. IEEE Trans.Rel, R-25, 2-15.

LIVINI, H. and BAR-NESS, Y. (1978). Reliability analysis of telecommunication networks using cutset approach. Microelectron. Reliab., 18, 285-289.

LLOYD, D.K. and LIPOW, M. (1952). Reliability: Management Methods and Mathematics. Prentice-Hall, London.

LOCKS, M.O. (1978). Reliability analysis: a tutorial. Microelectron. Reliab., 18, 335-345.

LOCKS, M.O. (1979a). The Lapp and Powers fault-tree model: statistical noncoherence. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 44-46.

LOCKS, M.O. (1979b). Evaluating the KTI Monte Carlo Method for system reliability calculations. IEEE Trans. Rel., R-28, 368-372.

LOCKS, M.O. (1979c). Inverting and minimizing Boolean functions, minimal paths and minimal cuts: noncoherent system analysis. IEEE Trans. Rel., R-28, 373-375.

LOCKS, M.O. (1980). The fail-safe feature of the Lapp and Powers fault tree. IEEE Trans. Rel., R-29, 10-11.

LOMNICKI, Z.A. (1966). A note on the Weibull renewal process. Biometrika, 53, 375-381.

LOMNICKI, Z.A. (1972). Two-terminal series-parallel networks. Adv. Appl. Prob., 4, 109-150.

LOMNICKI, Z.A. (1973). Some aspects of the statistical approach to reliability (with discussion). J. Roy. Statist. Soc., B, 136, 395-419.

LOMNICKI, Z.A. (1977). The role of various types of coherent structures in reliability studies. Proceedings of the Advances in Reliability Technology Symposium, Bradford, April 1976, National Centre of Systems Reliability (U.K.A.E.A.), NCSR R12, ART/9/1-12.

MACLANE, S. and BIRKHOFF, G. (1967). Algebra. Collier-McMillan, London.

MACMAHON, P.A. (1892). The combination of resistances. The Electrician, 28, 601-602.

MANN, N.R. (1968). Point and interval estimation procedures for the two-parameter Weibull and extreme-value distributions. Technometrics, 10, 231-256.

MANN, N.R. (1972). Design of over-stress life-test experiments when failure times have the two-parameter Weibull distribution. Technometrics, 14, 437-451.

MANN, N.R., SCHAFER, R.E. and SINGPURWALLA, N.D. (1974). Methods for Statistical Analysis of Reliability and Life Data. Wiley, New York.

MARSHALL, A.W. (1975). Multivariate distributions with monotone hazard rate. In Reliability and Fault Tree Analysis, R.E. Barlow, J.B. Fussell and N.O. Singpurwalla eds., SIAM, 259-284.

MARSHALL, A.W. and PROSCHAN, F. (1965). Maximum likelihood estimation for distributions with monotone failure rate. Ann.Math.Statist., 36, 69-77.

MARSHALL, A.W. and PROSCHAN, F. (1972). Classes of distributions applicable in replacement, with renewal theory implications. Proceedings of the Sixth Berkley Symposium on Probability and Statistics, Vol. I., 395-415.

MARTIN, P. (1980). Consequential failures in mechanical systems. Proceedings of the Sixth Advances in Reliability Technology Symposium, Bradford, April 1980, Vol. I, National Centre of Systems Reliability (U.K.A.E.A.), NCSR R23, 253-278.

MARTZ, H.F. Jr. and LIAN, M.G. (1977). Bayes and Empirical Bayes point and interval estimation of reliability for the Weibull model. In The Theory and Applications of Reliability with Emphasis on Bayesian and Nonparametric Methods, Vol. II., C.P. Tsokos and I.N. Shimi eds., Academic Press, New York, 203-233.

MARUVADA, S.N., WEISE, K.E. and CHAMOW, M.F. (1978). Failure rate as a design parameter: possibilities and limitations. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 342-346.

MATHUR, F.P. and de SOUSA, P.T. (1975). Reliability models of NMR systems. IEEE Trans. Rel., R-24, 108-113.

McCOOL, J.I. (1976). Estimation of Weibull parameters with competing-mode censoring. IEEE Trans. Rel., R-25, 25-31.

MENDENHALL, W. and HADER, R.J. (1958). Estimation of parameters of mixed exponentially distributed failure time distributions from censored life test data. Biometrika, 45, 504-520.

MERCER, A. (1961). Some simple wear-dependent renewal processes. J. Royal Statist., Soc., B, 23, 368-376.

MERCER, A. and SMITH, C.S. (1959). A random walk in which the steps occur randomly in time. Biometrika, 46, 30-35.

MINE, H. and KAWAI, H. (1974a). An optimal maintenance policy for a two-unit parallel system with degraded states. IEEE Trans.Rel, R-23, 81-86.

MINE, H. and KAWAI, H. (1974b). Preventative replacement of a one unit system with a wearout state. IEEE Trans.Rel., R-23, 24-29.

MINE, H. and KAWAI, H. (1975). An optimal inspection and replacement policy. IEEE Trans. Rel., R-24, 305-309.

MINE, H. and KAWAI, H. (1977). Optimal ordering and replacement for a one-unit system. IEEE Trans. Rel., R-26, 273-276.

MINE, H. and NAKAGAWA, T. (1978). Age replacement model with mixed failure times. IEEE Trans. Rel., R-27, 173.

MISRA, K.B. (1970). An algorithm for the reliability evaluation of redundant networks. IEEE Trans. Rel., R-19, 146-151.

MISRA, R.B. (1979). Symbolic reliability evaluation of reducible networks. Microelectron. Reliab., 19, 253-257.

MOORE, E.F. and SHANNON, C.E. (1956). Reliable circuits using less reliable relays. J. Franklin Inst., 262, 191-208. 281-297.

MORANDA, P.B. (1979). Event-altered rate models for general reliability analysis. IEEE Trans. Rel., R-28, 376-381.

MOREY, R.C. (1966). Some stochastic properties of a compound renewal damage model. Operations Res., 14, 902-908.

MOSS, R.Y. (1978). Modelling variable hazard life data. Proceedings of the 28th Electronic Components Conference, Anaheim, IEEE, 16-22.

MURCHLAND, J.D. (1975). Fundamental concepts and relations for reliability analysis of multi-state systems. In Reliability and Fault Tree Analysis, R.E. Barlow, J.B. Fussell and N.D. Singpurwalla, eds., SIAM, Philadelphia, 581-618.

MUTH, E.J. (1980a). Memory as a property of probability distributions. IEEE Trans. Rel., R-29, 160-165.

MUTH, E.J. (1980b). Comment on "A repairable multistate device". IEEE Trans. Rel., R-29, 275.

NAKAGAWA, T. (1974). Off time distributions in an alternating renewal process with reliability applications. Microelectron. Reliab., 13, 181-184.

NAKAGAWA, T. (1976a). On a replacement problem cumulative damage model. Opl. Res. Quart., 27, 895-900.

NAKAGAWA, T. (1976b). On a cumulative damage model with N different components. IEEE Trans. Rel., R-25. 112-114.

NAKAGAWA, T. (1977). Optimum preventive maintenance policies for repairable systems. IEEE Trans. Rel., R-26, 168-173.

NAKAGAWA, T. (1978). Reliability analysis of standby repairable systems when an emergency occurs. Microelectron. Reliab., 17, 461-4.

NAKAGAWA, Y. and HATTORI, Y. (1980). Reliability of all possible series-parallel redundant structures of m i.i.d. units with two failure modes. IEEE Trans. Rel., R-29, 320-323.

NAKAGAWA, T. and OSAKI, S. (1974a). Some aspects of damage models. Microelectron. Reliab., 13, 253-257.

NAKAGAWA, T. and OSAKI, S. (1974b). Combining drift and catastrophic failure modes. IEEE Trans. Rel., R-23, 278-279.

NAKAGAWA, T., SAWA, Y. and SUZUKI, Y. (1976). Reliability analysis of intermittently used systems when failures are detected only during a usage period. Microelectron. Reliab., 15, 35-38.

NAKAGAWA, T. and YASUI, K. (1978). Approximate calculation of block replacement with Weibull failure times. IEEE Trans. Rel., R-27. 268-269.

NAKAZAWA, H. (1977). A decomposition method for computing system reliability by a Boolean expression. IEEE Trans. Rel., R-26, 250-252.

NATIONAL CENTRE OF SYSTEMS RELIABILITY (1978). National Conference on Reliability, September 1977, Nottingham; Keynote papers and Discussion, U.K.A.E.A.

NELSON, W.B. (1970). Statistical Methods for Accelerated Life Test Data - the Inverse Power Law Model. General Electric Research and Development Centre, Schenectady, New York: Technical Information Series Report No. 71-C-011.

NELSON, W. (1972). Graphical analysis of accelerated life test data with the inverse power law model. IEEE Trans. Rel., R-21, 2-11.

NELSON, W. (1975). Graphical analysis of accelerated life test data with a mixture of failure modes. IEEE Trans. Rel., R-24, 230-237.

NIEUWHOF, G.W.E. (1978). The time dependent failure rate of a system consisting of three 50% units in active parallel. Microelectron. Reliab., 17, 53-58.

O'CONNOR, P.D.T. (1977). Practical reliability assessment using Weibull analysis. Proceedings of the National Reliability Conference, Nottingham, National Centre of Systems Reliability, (U.K.A.E.A.), NRC3/3/1-11.

OGUNBIYI, E.I. and HENLEY, E.J. (1981). Irredundant forms and prime implicants of a function with multistate variables. IEEE Trans. Rel., R-30, 39-42.

OSAKI, S. (1972). An intermittently used system with preventive maintenance. J. Operat. Res. Soc., Japan, 15, 102-111.

PAPAZOGLOU, I.A. and GYFTOPOULOS, E.P. (1977). Markov processes for reliability analysis of large systems. IEEE Trans. Rel., R-26, 232-237.

PARA, A.F. and GARRIBBA, S. (1980). Reliability analysis of a repairable single unit under general age dependence. In Synthesis and Analysis Methods for Safety and Reliability Studies, G. Apostolakis, S. Garribba and G. Volta eds., Plenum Press, New York, 251-268.

PAU, L.F. (1974). Diagnosis of equipment failures by pattern recognition. IEEE Trans. Rel., R-23, 202-208.

PAU, L.F. (1979). Specification of an automatic test system v's an optimum maintenance policy and equipment reliability. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 142-146.

PHILLIPS, M.J. (1976). The reliability of two-terminal parallel-series networks subject to two kinds of failure. Microelectron. Reliab., 15, 535-549.

PHILLIPS, M.J. (1977). The choice of networks subject to two kinds of failure. Proceedings of the Advances in Reliability Technology Symposium, Bradford April 1976, National Centre of Systems Reliability (U.K.A.E.A.), NCSR R12, ART/20/1-6.

PHILLIPS, M.J. (1979). The reliability of a system subject to revealed and unrevealed faults. Microelectron. Reliab., 18, 495-503.

PHILLIPS, M.J. (1980). k-out-of-n:G systems are preferable. IEEE Trans. Rel., R-29, 166-169.

POLOVKO, A.M. (1964). Osnovy Teorii Nadezhnasti, Nauka Press, Moscow. English edition 1968; Fundamentals of Reliability Theory, Academic Press, New York.

POSTELNICU, V. (1970). Nondichotomic multi-component structures. BULL. MATH. de la Soc. Sci. Math. de la R.S. de Roumanie, 14, 209-217.

POSTEN, H.O. (1973). The robustness of reliability predictions for series systems of identical components. IEEE Trans. Rel., R-22, 213-218.

POWERS, W.A. and POSTEN, H.O. (1975). The robustness of reliability predictions for parallel systems of identical components. IEEE Trans. Rel., R-24, 126-128.

PROCTOR, C.L. II and PROCTOR, C.L. (1977). Multistate dependent system modelling. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 401-403.

PROCTOR, C.L. and SINGH, B. (1975a). A three-state system Markov model. Microelectron. Reliab., 14, 463-464.

PROCTOR, C.L. and SINGH, B. (1975b). The graphical reliability evaluation of three-state device networks. Microelectron. Reliab., 14, 203-214.

PROCTOR, C.L. and SINGH, B. (1976a). A repairable three-state device. IEEE Trans. Rel., R-25, 210-211.

PROCTOR, C.L. and SINGH, B. (1976b). The analysis of a four-state system. Microelectron. Reliab., 15, 53-55.

PROCTOR, C.L. and WANG, Y.T. (1975). Optimal maintenance policy for systems that experience state degradations. Microelectron. Reliab., 14, 199-202.

PROSCHAN, F. and PYKE, R. (1967). Tests for monotone failure rate. Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, Vol. III, University of California Press, Berkeley, California, 293-312.

PROSCHAN, F. and SERFLING, J.R. (1974). Classes of life distributions and their properties - discussion by the editors. In Reliability and Biometry, F. Proschan and R.J. Serfling eds., SIAM, Philadelphia, 1-6.

RAI, S. and AGGARWAL, K.K. (1978). An efficient method for reliability evaluation of a general network. IEEE Trans.Rel., R-27, 206-211.

RAMANARAYANAN, R. (1978). Reliability and availability of two general multi-unit systems. IEEE Trans. Rel., R-27, 70-72.

REGULINSKI, T.L. (1977). Time-continuous,space-discrete Markovian model of man-machine system. Proceedings of the Advances in Reliability Technology Symposium, Bradford April 1976, National Centre of Systems Reliability (U.K.A.E.A.), NCSR R12, ART/12/1-13.

REGULINSKI, T.L. (1980). Availability function for communicating computers network. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 141-143.

RINGER, L.J. and SPRINKLE, E.E. III (1972). Estimation of the parameters of the Weibull distribution from multicensored samples. IEEE Trans. Rel., R-21, 46-51.

RIORDAN, J. and SHANNON, C.E. (1942). The number of two-terminal series-parallel networks. J. Math. & Physik., 21, 83-92.

ROBERTS, N.H. (1964). Mathematical Methods in Reliability Engineering. McGraw-Hill, New York.

ROCKETTE, H., ANTLE, C. and KLIMKO, L.A. (1974). Maximum likelihood estimation with the Weibull model. J.Amer. Statist. Assoc., 69, 246-249.

RODE, L. (1974). A reliability problem treated by the additional event method. Microelectron. Reliab., 13, 483-485.

ROSS, S. (1979). Multivalued state component systems. Annals Prob., 7, 379-383.

ROSENTHAL, A. (1975). A computer scientist looks at reliability computations. In Reliability and Fault Tree Analysis, R.E. Barlow, J.B. Fussell and N.D. Singpurwalla eds., SIAM, Philadelphia, 133-152.

SANKARANARAYANAN, G. and USHA, K. (1980). Two-unit standby system and alertness of the worker. IEEE Trans. Rel., R-29, 282-3.

SASAKI, M. and HIRAMATSU, T. (1976). Reliability of intermittently used systems. IEEE Trans. Rel., R-25, 208-9.

SASAKI, M. and YANAI, T. (1977). Availability of a fixed periodic system. IEEE Trans. Rel., R-26, 300-302.

SASAKI, M. and YOKOTA, T. (1980). Improvement of instantaneous availabilities by decreasing delay time. IEEE Trans. Rel., R-29, 178-181.

SATYANARAYANA, A. and PRABHAKAR, A. (1978). New topological formula and rapid algorithm for reliability analysis of complex networks. IEEE Trans. Rel., R-27, 82-100.

SAUNDERS, S.C. (1966). Lower tolerance intervals for life distributions. Annual Meeting of the American Statistical Association.

SAYLOR, F. (1977). A Monte Carlo comparison of the method of moments to the maximum-likelihood estimates of Weibull parameters for CAS data. In The Theory and Applications of Reliability

with Emphasis on Baysian and Nonparametric Methods, Vol. II, C.P. Tsokos and I.N. Shimi eds., Academic Press, New York, 517-529.

SCHUELLER, G.I. and SCHWARTZ, F.F. (1976). Some aspects of the reliability-based design of reactor containment structures. J. Nucl. Engr. Res., 37, 299-305.

SEITZ, N.B. (1980). Measuring communicating availability with Federal Standard 1033. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 403-409.

SHARMA, J. (1976). Algorithm for reliability evaluation of a reducible network. IEEE Trans. Rel., R-25, 337-339.

SHENK, K.F. (1977). Reliability model and analysis of a system with non-exponential down-time distributions including a derated state. Microelectron. Reliab., 16, 35-40.

SHOOMAN, M.L. (1968). Probabilistic Reliability: An Engineering Approach. McGraw-Hill, New York.

SHOOMAN, M.L. and SINKAR, S. (1977). Generation of reliability and safety data by analysis of expert opinion. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 186-193.

SILBERBERG, S. (1979). American nuclear power plant operational data compilation system - use of the system by Electricite de France. Proceedings of the Second National Reliability Conference, Birmingham, Vol. I., National Centre of Systems Reliability (U.K.A.E.A.), 2B/1/1-6.

SIMON, R.M. (1969). The Reliability of Multi-State Systems subject to Cannibalization. Washington University School of Engineering and Applied Science, Rep. AM-69-1.

SIMON, R.M. (1970). Optimal cannibalization policies for multicomponent systems. SIAM J. Appl. Math., 19, 700-711.

SIMON, R.M. (1972). The reliability of multicomponent systems subject to cannibalization. Naval Research Logist. Quart., 19, 1-14.

SINGH, C. (1976). Reliability modelling algorithms for a class of large repairable systems. Microelectron. Reliab., 15, 159-162.

SINGH, C. (1978). Generalised conditional frequency formula. Microelectron. Reliab., 18, 351-352.

SINGH, C. (1979). A matrix approach to calculate the failure frequency and other related indices. Microelectron. Reliab., 19, 395-398.

SINGH, I.P., AGGARWAL, N.N. and KILKAMI, S.V. (1979). Three-component intermittent system. IEEE Trans. Rel., R-28, 415-6.

SINGH, S.P., KAPUR, P.K. and KAPOOR, K.R. (1979). Two-unit redundant system with random switchover time and two types of repair. Microelectron. Reliab., 19, 325-328.

SINGH, B. and PROCTOR, C.L. (1976). Reliability analysis of multi-state device networks. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 31-35.

SINGH, B. and PROCTOR, C.L. (1977). Reliability analysis of repairable three-state device Markov models. In The Theory and Applications of Reliability with Emphasis on Bayesian and Nonparametric Methods, Vol. II, C.P. Tsokos and I.N. Shimi eds., Academic Press, New York, 457-470.

SINGPURWALLA, N.D. (1971). A problem in accelerated life testing. J. Amer. Statist. Ass., 66, 841-845.

SINGPURWALLA, N.D. and AL-KHAYYAL, F.A. (1977). Accelerated life tests using the inverse power law model for the Weibull distribution. In The Theory and Applications of Reliability with Emphasis on Bayesian and Nonparametric Methods, Vol. II, C.P. Tsokos and I.N. Shimi eds., Academic Press, New York, 381-399.

SIU, T.K. and CHAN, W.C. (1978). A mathematical approach to reliability evaluation for static generating capacity. IEEE Trans. Rel., R-27, 132-3.

SMITH, F.C. (1980). Transit system acceptance by sequential sampling. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 129-135.

SMITH, W.L. and LEADBETTER, M.R. (1963). On the renewal function for the Weibull distribution. Technometrics, 5, 393-6.

SNAITH, E.R. (1979). Can reliability predictions be validated? Proceedings of the Second National Reliability Conference, Birmingham, Vol. 2., National Centre of Systems Reliability (U.K.A.E.A.), 6B/1/1-10.

SRINIVASAN, V.S. (1966). The effect of standby redundancy in system's failure with repair maintenance. Operations Research, 14, 1024-1036.

SRINIVASAN, S.K. and BHASKAR, D. (1979). Analysis of intermittently used two-unit redundant systems with a single repair facility. Microelectron. Reliab., 19, 247-252.

STOLL, R.R. (1961). Sets, Logic and Axiomatic Theories. Freeman, London.

SUBRAMANIAN, R. and NATARAJAN, R. (1980). Availability analysis of a two-unit redundant system. IEEE Trans. Rel., R-29, 182-3.

SUBRAMANIAN, R. and RAVICHANDRAN, N. (1980). A two-unit priority redundant system with preemptive resume repair. IEEE Trans. Rel., R-29, 183-4.

SUBRAMANIAN, A. and USHA, K. (1980). 1-out-of-2:F systems exposed to a damage process. IEEE Trans. Rel., R-29, 284-285.

TAKAMI, I., INAGAKI, T., SAKINO, E. and INOUE, K. (1978). Optimal allocation of fault detectors. IEEE Trans. Rel., R-27, 360-362.

TASUN, O. (1977). Dynamic decision elements for three-unit systems. IEEE Trans. Rel., R-26, 335-338.

TAYLOR, H.M. (1975). Optimal replacement under additive damage and other failure models. Naval Res. Logist. Quart, 22, 1-18.

THOMAS, P.G. (1977). Reliability analysis of complex electronic systems. Proc. Symp. IEE, Heriot-Watt Univ., Edinburgh March 1977, 3.

THOMAS, J.M., DERBALIAN, G. and BISCHEL, K. (1980). Parametric sensitivity of rocket motor reliability. Proceedings of the Annual Reliability and Maintainability Symposium, IEEE, 121-128.

TILLMAN, F.A., LIE, C.H. and HWANG, C.L. (1976). Analysis of pseudo-reliability of a combat tank system and its optimal design. IEEE. Trans. Rel., R-25, 239-242.

TOMLINSON, L.N., ANDREW, J.E. and FITZGERALD, F. (1970). Optimisation of Parametric Constants for Creep Data, British Steel Corporation, Special Steels Division, S.5316/8/70/A.

TSOKOS, C.P. and SHIMI, I.N. eds.. (1977). The Theory and Application of Reliability with emphasis on Bayesian and Nonparametric Methods. Academic Press, New York.

TUMOLILLO, T.A. (1974). Methods for calculating the reliability function for systems subjected to random stresses. IEEE Trans. Rel., R-23, 256-262.

UEDA, T. (1972). Reliability of system when two states occur alternately (in Japanese). Keiei-Kogaku, 16, 216-232.

VENTON, A.O.F. (1977). Data requirements for mechanical reliability. Proceedings of the National Reliability Conference, Nottingham, National Centre of Systems Reliability (U.K.A.E.A.), NRC5/5/1-14.

VIRTANEN, I. (1977). On the concepts and derivation of reliability in stochastic systems with states of reduced efficiency; an application of supplementary variables and discrete transforms. Publications of the Turku School of Economics, Series A-4.

VON NEUMAN, J. (1956). Probabilistic logics and the synthesis of reliable organisms from unreliable components. Automata Studies, Ann. Maths. Studies, 34, Princeton University Press, 43-98.

WARD, M. (1946). Note on the order of free distributive lattices. Bull. Amer. Math. Soc., 52, Abstract 52-5-13.

WEIBULL, W. (1939). A statistical theory of the strength of materials. Ing. Vestenskaps Akad. Handl., No. 151. The phenomenon of rupture in solids, Ibid., No. 153.

WEIBULL, W. (1951). A statistical distribution function of wide applicability. J. Applied Mechanics, 18, 293-7.

WEISS, G.H. (1961). The reliability of components exhibiting cumulative damage effects. Technometrics, 3, 413-422.

WEISS, J.R. and FRENCH, R.X. (1975). A reliability model for interconnected electrical power systems. IEEE Trans. Rel., R-24, 129-132.

WINGO, D.R. (1972). Maximum likelihood estimation of the parameters of the Weibull distribution by modified quasilinearization. IEEE Trans. Rel., R-21, 89-93.

WINGO, D.R. (1973). Solution of the three-parameter Weibull equations by constrained modified quasilinearization (progressively censored samples). IEEE Trans. Rel., R-22, 96-102.

WOODCOCK, E.R. (1971). The Calculation of Reliability of Systems: the Program NOTED. United Kingdom Atomic Energy Authority, AHSB(S) R153.

YAMASHIRO, M. (1980). A repairable multistate device with general repair time. IEEE Trans. Rel., R-29, 276.

YAMASHIRO, M. (1982). A repairable system with N failure modes and K standby units. Microelectron. Reliab., 22, 53-57.

YASUDA, J. (1977). Correlation between laboratory test and field part failure rates. IEEE Trans. Rel., R-26, 82-84.

YOST, F.G. and HALL, I.J. (1976). Failure by the processes of nucleation and growth. IEEE Trans. Rel., R-25, 45-48.

ZANAKIS, S.H. (1977). Computational experience with some nonlinear optimization algorithms in deriving maximum likelihood estimates for the three-parameter Weibull distribution. TIMS Studies in the Management Sciences, 7, 63-7.

ZANAKIS, S.H. (1979a). Extended pattern search with transformations for the three-parameter Weibull maximum likelihood estimation problem. Mgmt. Sci., 25, 1149-1161.

ZANAKIS, S.H. (1979b). A simulation study of some simple estimators for the three-parameter Weibull distribution. J. Statist. Comp. Simul., 9, 101-116.

ZELEN, M. ed. (1963). Statistical Theory of Reliability. University of Wisconsin Press, Wisconsin.

ZELEN, M. and DANNEMILLER, M.C. (1961). The robustness of life testing procedures derived from the exponential distribution. Technometrics, 3, 29-49.

ZIJLSTRA, M. (1980). Optimum replacement and maintenance policies for units with a decreasing performance level. Europ. J. Operational Res., 5, 267-275.

APPENDIX

PUBLISHED PAPERS ON THE WORK OF THE THESIS*

A1. Generalisation of Dedekind's problem of the enumeration of coherent structures.

by

J. Ansell, A. Bendell and S. Humble. Proceedings of the Royal Society of Edinburgh, Series A, Vol. 89 (1981), 239-248.

* Other papers by the author, published within the period of registration for Ph.D. but of more peripheral interest to the work of the thesis, appear in the reference list.

Proceedings of the Royal Society of Edinburgh, **89A**, 239–248, *1981*

Generalisation of Dedekind's problem of the enumeration of coherent structures

J. Ansell
University of Keele

A. Bendell
Dundee College of Technology

and

S. Humble
Royal Military College of Science, Shrivenham

(MS received 1 May 1980, Revised MS received 16 January 1981)

Synopsis

In the context of reliability theory, two definitions are given for coherent functions of n variables, where both function and variables can take any of l possible levels. The enumeration problem for such functions is discussed and several recursive bounds are derived. In the case of $l=2$ (the *Dedekind problem*) a recursive upper bound is derived which is better than the previous best explicit upper bound for $n<15$, and also provides a systematic improvement on this bound for larger values of n.

1. Introduction

In the past 25 years, inspired by the pioneering work of von Neumann [**20**], much work has been done in the development of a reliability theory for complex structures composed of a number of components [**4**, **16**, **17**]. Most of this work has concentrated on *dichotomic reliability*, i.e. the assumption that at each moment of time, each component as well as the system as a whole is in one of two possible states: either it is operational or it is not. Thus the states of all n components of a system at any instant can be specified by a vector $\mathbf{v}=(s_1, s_2, \ldots, s_n)$ where $s_\alpha = 1$ if component α is operational and $s_\alpha = 0$ otherwise. Hence the design of a system determines a Boolean function, the structure function of the system, $f\colon V_n \rightarrow V_1$, where V_n is the unit cube in Euclidean n-space. Although there are 2^{2^n} such Boolean functions on V_n, it is clear that some will represent rather unrealistic "machines". For example it is unlikely that one would obtain a structure such that $f(\mathbf{v})=0$ when $s_\alpha = 1$ all α, or one where replacement of a failed component by an operational one actually degrades the system performance. Discounting these unacceptable situations the question arises; how many realistic functions are there? The enumeration of these functions, the so-called coherent or monotonic functions (which we shall define more precisely in the next section) has received much attention in the reliability literature, e.g. Lomnicki [**16**], partly because the amount of information necessary to identify the appropriate systems structure will

be inversely proportional to the number of possible structures. In general the enumeration problem remains unsolved. It is in fact identical to the problem posed by Dedekind in 1897 [**6**] on the cardinality of the free distributive lattice generated by the symbols $s_1, \ldots, s_n$. Numerical results have been obtained up to $n = 7$, but for larger numbers of components only certain upper bounds have so far been established.

Although this enumeration problem is already difficult enough, the simplifying assumption of dichotomic reliability is, however, only applicable to a very limited range of situations. In general, components will not simply be *operational* or *failed* but will be in one of a number of states of partial operation. That is to say, it will often be more realistic to suppose that components and systems can be in any one of a finite number of levels l ($l \geqq 2$). There is an increasing interest in the reliability literature in such multilevel systems, e.g. [**2**, **3**, **11**, **18**, **19**]. Hence the physical problem of interest is to enumerate the number of (generalised) coherent functions which can be formed when each component can take any of l possible levels.

In this paper therefore we introduce two possible definitions of generalised coherent functions and derive several useful upper and lower bounds for their enumeration. A major by-product of our approach is to deduce an upper bound for the Dedekind problem (i.e. with $l = 2$) which is better than existing bounds for $n < 15$ and also provides a means of systematically improving these bounds for larger n values.

The outline of the paper is as follows. In the next two sections we provide some necessary formalism and definitions, and give some numerical results. In Section 4 we prove a theorem which allows us in the following two sections to deduce useful upper bounds on the number of coherent structures. In particular, in Section 5 we discuss the advantages of our approach in obtaining a systematic improvement on existing bounds to the Dedekind enumeration problem; while in Section 6 we concentrate on obtaining several upper and lower bounds in the general case. Finally in Section 7, we discuss some results concerning the connection between the structure functions and the event network.

2. Formalism and definitions

The state of all components of an n-component system can be described by a state vector

$$\mathbf{s} = (s_1, s_2, \ldots, s_n)$$

where s_α, the level of the α-th component, may be any one of the levels,

$$\lambda_1 < \lambda_2 < \cdots < \lambda_l$$

and where for convenience we define

$$\lambda_1 = 0, \quad \lambda_l = 1.$$

The resulting state of the system will be described by the structure function $f(\mathbf{s})$ of the vector $\mathbf{s}$, with range $\{\lambda_1, \lambda_2, \ldots, \lambda_l\}$.

Introducing the notation

$$\mathbf{1} = (1, 1, \ldots, 1)$$
$$\mathbf{0} = (0, 0, \ldots, 0)$$

and $\mathbf{x} \geqq \mathbf{y}$ if $x_\alpha \geqq y_\alpha$ for all $\alpha = 1, 2, \ldots, n$, then by analogy with the dichotomic case, we define a *semi-coherent* system by

$$f(\mathbf{x}) \geqq f(\mathbf{y}) \quad \text{for all} \quad \mathbf{x} \geqq \mathbf{y}. \tag{1}$$

For dichotomic reliability a coherent system is defined by (1) together with

$$f(\mathbf{1}) = 1, \quad f(\mathbf{0}) = 0. \tag{2}$$

For the multi-level situation we shall say that (1) and (2) define a *coherent system in the wide sense* to distinguish it from a *coherent system in the narrow sense* which in addition to (1) has the more restrictive requirement that if

$$\boldsymbol{\sigma} = (\sigma, \sigma, \ldots, \sigma) \quad \text{then} \quad f(\boldsymbol{\sigma}) = \sigma, \quad \text{for all} \quad \sigma = \lambda_1, \lambda_2, \ldots, \lambda_l. \tag{3}$$

In the dichotomic case, i.e. $l = 2$, these wide sense and narrow sense definitions are identical.

We define the state vector

$$\boldsymbol{\lambda}_i = (\lambda_i, \lambda_i, \ldots, \lambda_i) \quad \text{for all} \quad i = 1, 2, \ldots, l$$

as the *i-th pivot* of the system. Between the i-th and j-th pivots ($j > i$) there exists a number of state vectors composed only of the levels $\lambda_i, \ldots, \lambda_j$. We say that the set of state vectors composed only of the elements $\lambda_i, \ldots, \lambda_j$ constitutes the *(i, j)-th lozenge* of the system. Finally, for a system of n components we define the set of state vectors $\{(0, \ldots, 0, \lambda_i) \mid i = 1, \ldots, l\}$ as the *(0, n)-th chain* of the system.

3. Some numerical results

In Table 1 we show the number of coherent structure functions ${}^nW_l$, ${}^nN_l$, in the wide and narrow sense respectively, corresponding to some low-n and low-l values. Even in the dichotomic case the numerical evaluation problem is extremely complex since a general analytic expression for the number of coherent structures is still lacking, and values of ${}^nW_2 = {}^nN_2$ have only been established for $n \leqq 7$ (see Lomnicki [**16**], Church [**5**]). For multi-level, multi-component functions

TABLE 1

	${}^nN_l$		${}^nW_l$	
l \ n	2	3	2	3
2	4	18	4	18
3	64	151,236	136	738,122
4	4,096	—	18,676	—
5	1,048,576	—	15,374,304	—

the general enumeration problem is of course even more difficult and the results shown in the table represent a significant computer effort.

It may be conjectured from Table 1 that the number of coherent functions in the narrow sense which can be constructed from two components with l levels is given by

$$^2N_l = 2^{l(l-1)}. \tag{4}$$

In general, however, the computational difficulties involved are such that, just as in the dichotomic case, it is necessary to construct bounds for the number of possible multi-level structures. In order to do this we generalise a theorem due to Birnbaum *et al.* [**4**].

4. Theorem 1

For each positive integer n, let S_n denote the set of all semi-coherent functions or order n, and

$$G = \{(g_1, g_2, \ldots, g_l): g_j \in S_n, g_1 \leqq g_2 \leqq \cdots \leqq g_l, j = 1, 2, \ldots, l\}.$$

Then there exists a bijection from G onto S_{n+1}.

Proof. Let $(g_1, g_2, \ldots, g_l) \in G$. Define functions f and H by

$$f(\mathbf{s}, \lambda_j) = f(s_1, s_2, \ldots s_n, \lambda_j) = g_j(\mathbf{s}), \quad j = 1, 2, \ldots l$$

$$H(g_1, g_2, \ldots g_l) = f.$$

Note that, since $g_j, j = 1, 2, \ldots, l$ are semi-coherent and non-decreasing in j, $f \in S_{n+1}$ for $(\mathbf{s}, \lambda_i) \leq (\mathbf{t}, \lambda_j)$ implies $f(\mathbf{s}, \lambda_i) = g_i(\mathbf{s}) \leqq g_i(\mathbf{t}) \leqq g_j(\mathbf{t}) = f(\mathbf{t}, \lambda_j)$. That H is surjective follows from the observation that if $f \in S_{n+1}$ and the functions g_j defined by

$$g_j(\mathbf{s}) = f(\mathbf{s}, \lambda_j), \quad j = 1, 2, \ldots l$$

then $(g_1, g_2 \cdots g_l) \in G$ and $H(g_1, g_2, \ldots, g_l) = f$. It is clear H is injective.

To use this theorem to derive an upper bound for the number of semi-coherent functions of $n+1$ components and l levels let us first note that the number of solutions in integers of $1 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_r \leqq m$ is

$$\sum_{\beta_1=1}^{m} \sum_{\beta_2=1}^{\beta_1} \cdots \sum_{\beta_r=1}^{\beta_{r-1}} 1 = \binom{m+r-1}{r}. \tag{5}$$

Hence it follows from Theorem 1 that if ^nS_l denotes the number of possible semi-coherent functions of n components and l levels and *if* these functions were strictly ordered then by considering the ways in which these functions may be identified with $g_i, i = 1, 2, \ldots, l$, the number of possible semi-coherent functions of $n+1$ components and l levels would be given by (5) with r and m replaced by l and ^nS_l respectively, i.e.

$$\binom{^nS_l + l - 1}{l}. \tag{6}$$

However the ${}^nS_l$ functions cannot be strictly ordered. In fact the order in S_n as well as in the l^n states of the system is partial not total. Nevertheless if $\leqq$ is a partial ordering in a set P then there exists a total ordering $\leqq_0$ on P such that

$$\leqq \subset \leqq_0,$$

i.e. there exists an order preserving extension. Thus

$$\#\{g_1 \leqq g_2 \leqq \cdots \leqq g_l\} \leqq \#\{g_1 \leqq_0 g_2 \leqq_0 \cdots \leqq_0 g_l\} = \binom{{}^nS_l + l - 1}{l} \tag{7}$$

since $g_1 \leqq g_2 \leqq \cdots \leqq g_l$ implies $g_1 \leqq_0 g_2 \leqq_0 \cdots \leqq_0 g_l$ and also every semicoherent function whose domain is a totally ordered set $(P, \leqq_0)$ is clearly also a semi-coherent function on any restriction $(P, \leqq) \subset (P, \leqq_0)$. Hence (6) represents an upper bound for the number of semi-coherent functions with $n+1$ components. Of course we still have an upper bound even if we replace ${}^nS_l$ in (7) by its upper bound ${}^nU_l$. In this way we obtain a recursive formula for the upper bound for the number of semi-coherent functions of $n+1$ components and l levels which is of the form

$${}^{n+1}U_l = \binom{{}^nU_l + l - 1}{l} \tag{8}$$

5. The special case of $l = 2$

For the case when $l = 2$, i.e. when components and systems can be in only one of two states (operational or failed) (8) becomes

$${}^{n+1}U_2 = {}^nU_2({}^nU_2 + 1)/2 \tag{9}$$

which allows us to calculate an upper bound for the number of semi-coherent functions for $n+1$ components provided we are given an upper bound (or the actual value) for the number of such functions for n components.

Methods for obtaining sharp upper bounds for the number of semi-coherent functions of n components with this dichotomic behaviour has long been of interest; see for example Dedekind [**6**], Gilbert [**8**], Korobkov [**14**], Hansel [**10**] Kleitman [**12**], Hanish *et al.* [**9**], Alekseev [**1**], Kleitman and Markowsky [**13**] and Kurshunov [**15**].

The sharpest explicit and non-asymptotic bound to data is due to Hansel who proved that

$${}^nS_2 \leqq 3^{M_n} \tag{10}$$

where M_n is the middle binomial coefficient, i.e.

$$M_n = \begin{cases} \dfrac{n!}{(n/2)!\,(n/2)!} & \text{if } n \text{ even} \\ \dfrac{n!}{\left(\dfrac{n+1}{2}\right)!\left(\dfrac{n-1}{2}\right)!} & \text{if } n \text{ odd.} \end{cases}$$

It is of interest to note that starting with the actual value of 2, 414, 682, 040, 996 obtained by Church for ${}^7S_2$, (9) provides upper bounds which are actually

sharper than those given by Hansel for $7<n<15$. See the numerical results of Table 2. For $n \geqq 15$ the bound (10) is somewhat better than the bound obtained from (9). However the recursive nature of (9) means it can be used in conjunction with (10) to obtain a systematic improvement on Hansel's bound for $(n+1)$ even. That is, if for n odd we take

$$^{n}U_2 = 2^{(\log_2 3)M_n}$$

with

$$M_n = \frac{n!}{\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!}$$

we obtain from (9)

$$^{n+1}U_2 = 2^{2(\log_2 3)M_n - 1} + 2^{(\log_2 3)M_n - 1} \tag{11}$$

which is less than Hansel's value of $2^{(\log_2 3)M_{n+1}}$ for all $n>0$.

TABLE 2
Recursive bound calculated from (9) using the value for $^{7}S_2$ obtained by Church [5]. Hansel's bound calculated from (10).

	$\log_{10}(^{n}U_2)$	
n	Recursive bound	Hansel's bound
8	24·46468	33·39845
9	48·62831	60·11722
10	96·95558	120·23445
11	193·61012	220·42983
12	386·91919	440·85962
13	773·53735	818·73926
14	1546·77368	1637·47876
15	3093·24634	3070·27222

6. Bounds in the general case

When there are only two possible levels it is clear from definitions (1), (2) that there are only two functions which are semi-coherent, but not coherent. However for $l>2$ the number, $^{n}X_l$, of functions which are semi-coherent, but not wide-sense coherent rises rapidly, and in order to derive from (8) a useful upper bound on the number of coherent structures in the wide sense we must evaluate at least a lower bound for $^{n}X_l$. Such a lower bound can be obtained by assuming that the l^n states of the system (in terms of the levels of its components) are ordered. An argument analogous to that in Section 4 then yields the following lower bound for $^{n}X_l$,

$$^{n}L_l = \binom{l^n+l-2}{l^n} + \binom{l^n+l-3}{l^n-1} \tag{12}$$

so that the sharper upper bound for the number of wide-sense coherent structures, $^{n}W_l$ is given by

$$^{n}W_l \leqq {}^{n}U_l - {}^{n}L_l. \tag{13}$$

Since ${}^nW_l \geqq {}^nN_l$, (13) must also be an upper bound for the number of coherent systems in the narrow sense.

Finally in this section we introduce two lower bounds. The l-level, n-component configuration contains l pivots and defined on the $(l-1, l)$-th lozenge there are ${}^nN_2$ possible narrow-sense coherent structures. Further, the number of coherent structures allowable within this lozenge is not reduced by the particlular structure existing in the $(1, l-1)$-th lozenge on which can be defined ${}^nN_{l-1}$ possible structures. Consequently we obtain a lower bound ${}^nT_l$ on ${}^nN_l$ by

TABLE 3

${}^nU_l$ calculated from (8) using ${}^nU_l = \binom{2l-1}{l} \cdot {}^nL_l$ and ${}^nT_l$ calculated from (12), (14) respectively. ${}^nR_l$ calculated from (15) using values of ${}^2W_l$ and ${}^3W_3$ given in Table 1.

$\log_{10}({}^nU_l)$

l \ n	2	3	4	5
3	2·34242	6·25502	17·98690	53·18255
4	4·86814	18·09240	70·98940	282·57738
5	8·45674	40·20450	198·94331	992·63739

$\log_{10}({}^nL_l)$

l \ n	2	3	4	5
3	1·27875	1·74036	2·21219	2·68753
4	2·46090	3·62583	4·81987	6·02145
5	3·79246	4·82898	7·91364	10·00805

$\log_{10}({}^nT_l)$

l \ n	3	4	5
3	—	4·44022	7·75922
4	6·43492	6·66033	11·76388
5	7·69020	8·88043	15·51845

$\log_{10}({}^nR_l)$

l \ n	3	4	5
3	—	6·64628	7·42443
4	7·16916	7·94929	8·25032
5	9·31529	10·87700	11·72209

assuming that because of the coherency constraints, corresponding to each state vector outside these two lozenges there is only one possible level of the system. Thus

$$ {}^{n}N_l \geqq {}^{n}N_{l-1} \,.\, {}^{n}N_2 = {}^{n}T_l. \tag{14} $$

A lower bound on the number of coherent structures in the wide sense can be obtained by taking a recursion over n. Introducing a new component into a system of $(n-1)$ components corresponds to adding an entry of 0 to the previous state vectors of the system and adding further states. These new states include the $(0, n)$-th chain. Following an argument akin to that in Section 4, defined on this chain there are

$$ \binom{l+(l-1)-1}{l-1} = \binom{2l-2}{l-1} $$

possible structures unrestricted by the particular structure in the system of $(n-1)$ components. Thus assigning only one possible level to each remaining state vector, a lower bound for the number of coherent systems in the wide sense is obtained as

$$ {}^{n}W_l \geqq \binom{2l-2}{l-1} \,.\, {}^{n-1}R_l $$

where ${}^{n-1}R_l$ is a lower bound for ${}^{n-1}W_l$.

Numerical illustrations of the bounds are given in Table 3 for some low n and l values.

7. Connection between structure functions and event networks

In dichotomic reliability it is well known that the structure function is determined by the logical event network (e.g. Flegg [7]). This is no longer the case for components which have $l > 2$ possible levels. However, it may well be of interest to determine how many possible structure functions correspond to a single event network, for example because a system may have originally been designed in terms of such a network.

The event network places dichotomic constraints on the structure function. For example, if A and B are two components in parallel (in the sense of an event network or of dichotomic reliability), the structure function $f(s_1, s_2)$ is such that

$$ f(0,0) = 0, \quad f(0,1) = f(1,0) = f(1,1) = 1. $$

It follows that in general the event network reduces the number of states of the system to which levels have to be assigned from l^n to $l^n - 2^n$. Corresponding to a single event network therefore there are $l^{(l^n-2^n)}$ possible structure functions, and there are

$$ l^{l^n} - l^{(l^n-2^n)} \,.\, 2^{2^n} \tag{16} $$

structure functions which do not correspond to event networks or systems defined in terms of the levels 0 and 1.

Thus there are

$$ l^{l^n} - {}^{l}C_2 \,.\, l^{(l^n-2^n)} \,.\, 2^{2^n} \tag{17} $$

structure functions which cannot be immediately deduced from event networks and two level systems.

The determination of the number of coherent functions in the narrow or wide sense out of the $l^{(l^n-2^n)}$ possible functions corresponding to a single event network is again a difficult, unsolved problem, except for some numerical computations for small n, small l systems. However as we show below for pure series or parallel event networks a relationship exists which forms a lower bound for the number of coherent structures in the narrow sense corresponding to any event network. Unfortunately no similar bound appears to exist for coherent structures in the wide sense.

8. Theorem 2

The number of coherent structures in the narrow sense for a series or a parallel network with n components and l levels is equal to the number of coherent structures in the narrow sense which can be constructed from n components and $l-1$ levels.

Proof. Let ${}^nP_l$ be the set of narrow-sense coherent functions for a parallel system with n components and l levels. Hence if $\phi \in {}^nP_l$,

$$\begin{aligned} &\phi(s, s, \ldots, s) = s \quad \text{for all} \quad s = \lambda_1, \ldots, \lambda_l \\ &\phi(\mathbf{x}) = 1 \qquad\quad \text{if} \quad x_\alpha = 1 \text{ for any } \alpha \\ &\phi(\mathbf{x}) > \phi(\mathbf{y}) \qquad \text{if} \quad \mathbf{x} > \mathbf{y} \end{aligned}$$

and

$$\phi(\mathbf{1}) = 1, \quad \phi(\mathbf{0}) = 0.$$

It also follows that if $\phi(\mathbf{x}) = 1$ then $x_\alpha = 1$ for some α, hence if we consider only the $l-1$ possible levels $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_{l-1}$ and ignore $\lambda_l = 1$, then $\phi(\mathbf{x}) \neq 1$.

Let ${}^nN_{l-1}$ be the set of functions which are narrow-sense coherent for n components and $l-1$ levels, where these $l-1$ levels are denoted by

$$0 = \mu_1 < \mu_2 < \ldots < \mu_{l-1} = 1.$$

Then by introducing the mapping

$$g\colon (\lambda_1, \lambda_2, \ldots, \lambda_{l-1}) \to (\mu_1, \mu_2, \ldots, \mu_{l-1})$$

it is simple to show that for each $\phi \in {}^nP_l$ there exists one and only one function $\psi \in {}^nN_{l-1}$ and conversely. For the proof in the case of a series narrow-sense coherent system we discount the level $\lambda_1 = 0$ for which the series system must fail.

It follows from this theorem, therefore, that if one can evaluate the number of narrow-sense coherent functions for n-components and $l-1$ levels, or place a bound on this number, one immediately has the number of narrow-sense coherent functions corresponding to a series or parallel system of n components and l levels, or has a bound for this number. Moreover, the reduction in the number of levels one must consider for a series or parallel system, from l to $l-1$, is unique to these event networks. The number of narrow-sense coherent functions associated with a pure series or a pure parallel network, therefore, is the minimum number of such functions associated with any type of network of the same number

of components and levels. Hence by this argument we can place a lower bound on the number of narrow-sense coherent functions associated with any event network.

Acknowledgment

The authors are grateful to a referee for the clarification of Theorem 1 and other helpful comments.

References

1 V. B. Alekseev. On the number of k-valued monotonic functions. *Soviet Math. Dokl.* **14** (1973), 87–91.
2 R. E. Barlow. Multistate Network Reliability Theory. *11th European Meeting of Statisticians, Oslo, 1978,* unpublished.
3 D. A. Butler. A complete importance ranking for components of binary coherent systems, with extensions to multi-state systems. *Naval Res. Logist. Quart.* **26** (1979), 565–578.
4 Z. W. Birnbaum, J. D. Esary and S. C. Saunders. Multi-component systems and structures and their reliability. *Technometrics* **3** (1961), 55–77.
5 R. Church. Enumeration by rank of the elements of the free distributive lattice with 7 generators. *Notices Amer. Math. Soc.* **12** (1965), 724.
6 R. Dedekind. Über Zerlegungen von Zahlen durch ihre grössten gemeinsamen Teiler. *Festschrift Technische Hochschule Braunschweig und Gesammelte Werke* **II** (1897), 103. 148.
7 G. Flegg. *Boolean Algebra and its Application* (London: Blackie, 1964).
8 E. N. Gilbert. Lattice theoretic properties of frontal switching functions. *J. Math. Phys.* **33** (1954), 57–67.
9 H. Hanisch, P. J. Hilton and W. M. Hirsch. Algebraic and combinational aspects of coherent structures. *Trans. New York Acad. Sci.* **31** (1969), 1024–1037.
10 G. Hansel. Sur le nombres des fonctions booléenes monotones de n variables. *C.R. Acad. Sci. Paris. Sér. A-B* **262** (1966), A1088–A1090.
11 Y. Hatoyama. Reliability analysis of 3-state systems. *IEEE Trans. Reliability* **R-28** (1979), 386–393.
12 D. Kleitman. On Dedekind's problem: the number of monotone Boolean functions. *Proc. Amer. Math. Soc.* **21** (1969), 677–682.
13 D. Kleitman and G. Markowsky. On Dedekind's problem: the number of isotone Boolean functions II. *Trans. Amer. Math. Soc.* **213** (1975), 373–390.
14 V. K. Korobkov. Estimation of the number of monotonic functions of the algebra of logic and of the complexity of the algorithm for finding the resolvent set for an arbitrary monotonic function of the algebra of logic. *Soviet Math. Dokl.* **4** (1963), 753–756.
15 A. D. Kurshunov. On Dedekind's problem of the number of monotone Boolean functions. *Soviet Math. Dokl.* **18** (1977), 442–445.
16 Z. Lomnicki. Some aspects of the statistical approach to reliability (with discussion). *J. Roy. Statist. Soc. Ser. A* **136** (1973), 395–419.
17 E. F. Moore and C. E. Shannon. Reliable circuits using less reliable relays. *J. Franklin Inst.* **262** (1956), 191–208.
18 B. Singh and C. L. Proctor. Reliability analysis of multistate devices networks. *Proceedings Annual Reliability and Maintainability Symposium,* 31–35 (New York: I.E.E.E., 1976).
19 I. Virtanen. On the concepts and derivation of reliability in stochastic systems with states of reduced efficiency. *Publ. Turku School Economics* **A4** (1977).
20 J. von Neumann. Probabilistic logics and the synthesis of reliable organisms from unreliable components. *Automata Studies,* pp. 43–98. *Ann. Math. Studies* 34 (Princeton, N.J.; Princeton Univ. Press, 1956).

(*Issued* 8 *October* 1981)

A2. Three-state and five-state reliability models.

by

J. Ansell, A. Bendell, S. Humble and C.S. Mudhar.

(i) IEEE Transactions on Reliability, Vol. R-29 (1980), 176-177.

(ii) Supplement: NAPS document No. 03582-B, Microfiche Publications, New York.

3-State and 5-State Reliability Models

J. Ansell
Keele University, Keele
A. Bendell
Dundee College of Technology, Dundee
S. Humble
Sheffield City Polytechnic, Sheffield
C.S. Mudhar
Hughes Aparel Systems, Weybridge

Key Words—3-state device, 5-state device, Partial failure, Failure to operate, Failure to idle, Repair.
Reader Aids—
Purpose: Report of derivations
Special math needed for explanation: Markov processes
Special math needed for results: None
Results useful to: Reliability theoreticians
Abstract—This note considers models for devices subject to 1) partial and catastrophic failure, repair and replacement 2) each of two types of partial and catastrophic failures.

INTRODUCTION

Reliability analysis in terms of Markov processes has been widely reported in the literature, e.g. [1]. In this note we consider generalisations of models previously reported.

3-STATE MODEL

Assumptions:

1. The 3 states of the device are S_1 (full operation), S_2 (partial operation), S_3 (failed).
2. The probability of more than one transition between the states occurring during a short interval of time is negligibly small.
3. The entry point to any state is a regeneration point.
4. The device is initially in state S_1.

Notation

$S_i(t)$ event that the device is in state S_i at time t
$\lambda_{ij}(t)$ transition rate from S_i to S_j (degradation or failure) $j > i$; $i = 1, 2,$; $j = 2, 3$
$\mu_{ij}(t)$ transition rate from S_i to S_j (repair) $j < i$; $i = 2, 3$; $j = 1, 2$

The reliability transition diagram is shown in Figure 1. In the Supplement [2] the transitory probabilities of the device being in the various states are derived, and the approach to the steady state availability is considered both for the case where the pdf's for times to degradation and repair are exponential, and for the case where the degradation rates are of the 'bath tub' shape and correspond to the sum of 2 Weibulls. Previously reported models are derived as special cases.

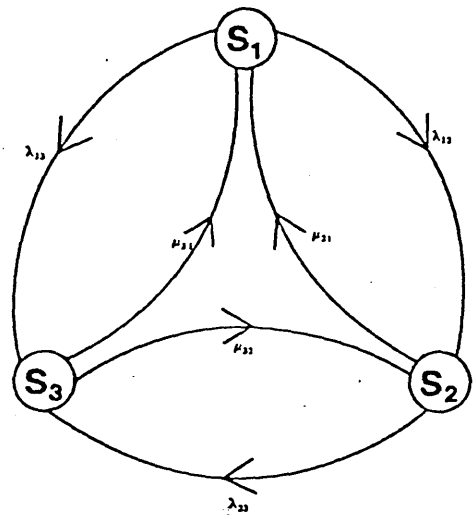


Fig. 1. 3-state model.

5-STATE MODEL

Assumptions:

1. The states are (0) . . . good; (1, j) . . . partially failed in mode j, $j = 1, 2$; (2, j) . . . catastrophically failed in mode j, $j = 1, 2$.
2. Direct transition from (1, 1) or (1, 2) to (0) is impossible.
3. The other transition rates are constant.
4. The probability of more than one transition occurring during a short interval of time is negligibly small.

Notation

λ_i transition rate from state (0) to state (i, 1); $i = 1, 2$
λ_{12} transition rate from state (1, 1) to (2, 1)
μ_i transition rate from (0) to (i, 2); $i = 1, 2$
μ_{12} transition rate from (1, 2) to (2, 2)
k_j transition rate from (2, j) to (0); $j = 1, 2$ (repair)

The model is shown diagrammatically in Figure 2. The transitory probabilities of being in the various states and the steady state availability are given in the Supplement [2]. Various special cases of this model are also considered there.

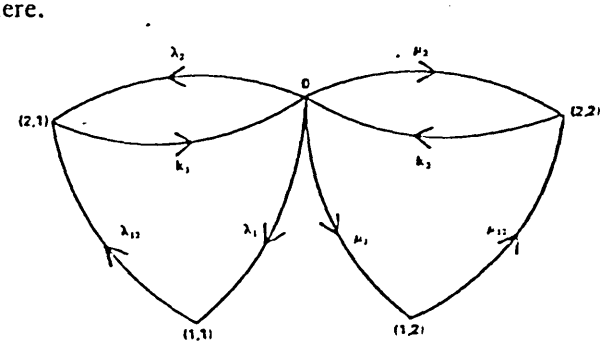


Fig. 2. 5-state model.

0018-9529/80/0600-0176$00.75

REFERENCES

[1] B.V. Gnedenco, Y.K. Delyayev, A.S. Soloviyev, *Mathematical Methods of Reliability Theory,* New York: Academic Press, 1969.

[2] Supplement: NAPS document No. 03582-B, 5 pages in this Supplement. For current ordering information, see "Information for Readers & Authors" in a current issue. Order NAPS document No. 03582, 9 pages. ASIS-NAPS; Microfiche Publications; P.O. Box 3513, Grand Central Station; New York, NY 10017 USA.

AUTHORS

Jake Ansell; Department of Mathematics and Statistics; Keele University; Keele ENGLAND.

Jake Ansell was born in Rotherham England on December 29, 1953. He received the BSc in Mathematics from University of Leicester in 1975 and the MSc in Statistics from the University of Newcastle in 1976. He was a lecturer at Sheffield City Polytechnic until 1978, and now holds a similar post at Keele University.

Tony Bendell; Department of Mathematics and Computer Studies; Dundee College of Technology; Dundee DD1 1HG SCOTLAND.

Tony Bendell: For biography, see vol R-27, 1978 April, p 76.

Stephen Humble; Department of Mathematics and Statistics; Sheffield City Polytechnic; Sheffield S1 1WB; ENGLAND.

Stephen Humble: For biography, see vol R-27, 1978 April, p 76.

Charter S. Mudhar; Hughes Aparel Systems; Queens Road; Weybridge, Surrey, ENGLAND.

Charter Singh Mudhar was born in Uganda in 1947. He received the HND in Mathematics, Statistics and Computing in 1973 and the Polytechnic Associateship in Mathematics (Grad.I.M.A.) in 1974, both from Sheffield Polytechnic. In 1978 he received the Postgraduate Diploma in Industrial Systems from Bradford University. He has worked as a technician in the Mathematics and Statistics Department of Sheffield Polytehnic from 1974-1978, and is currently Systems Analyst with Hughes Aparel Systems.

Manuscript TR79-65 received 1979 May 3; revision received 1979 June 12, 1979 September 10. ★★★

lement to 3-State and 5-State Reliability Models

ısell
endell
umble
Mudhar

ATE MODEL

tion:

$P_i(t)$ $\quad$ pr$\{S_i(t)\}$

1 : Exponential pdf's for the Times to Degradation and Repair

tional Assumption:- $\lambda_{ij}(t) \equiv \lambda_{ij}$, $\mu_{ij}(t) \equiv \mu_{ij}$, λ_{ij}, μ_{ij} constant

set of differential equations, corresponding to time independent transition rates in matrix notation,

$$\frac{d}{dt}\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} = \begin{bmatrix} -(\lambda_{13}+\lambda_{12}) & \mu_{21} & \mu_{31} \\ \lambda_{12} & -(\lambda_{23}+\mu_{21}) & \mu_{32} \\ \lambda_{13} & \lambda_{23} & -(\mu_{32}+\mu_{31}) \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix}$$

solution of these equations are

$$p_1(t) = \sum_{i=1}^{3} \frac{(\lambda_{23}+\mu_{21}+r_i)(\mu_{32}+\mu_{31}+r_i) - \lambda_{23}\mu_{32}}{D_i} e^{r_i t}$$

$$p_2(t) = \sum_{i=1}^{3} \frac{\lambda_{12}(\mu_{32}+\mu_{31}+r_i) + \lambda_{13}\mu_{32}}{D_i} e^{r_i t}$$

$$p_3(t) = 1 - p_1(t) - p_2(t)$$

$$D_i \equiv \prod_{j\neq i=1}^{3} (r_j - r_i)^{-1}$$

re $r_3 = 0$ and r_1, r_2 are the roots of the equation

$$p(r) \equiv (r + \tfrac{\sigma}{2})^2 - \tfrac{1}{4}[(a + b + c)^2 - 4ab] = 0$$

:h

$$\sigma \equiv \lambda_{23}+\lambda_{13}+\lambda_{12}+\mu_{32}+\mu_{31}+\mu_{21}$$

$$a \equiv \mu_{31}-\mu_{21}$$

$$b \equiv \lambda_{23}-\lambda_{13}$$

$$c \equiv \mu_{32}-\lambda_{12}$$

The steady state availability of the device is

$$p_1(\infty) = \frac{(\lambda_{23}+\mu_{21})(\mu_{32}+\mu_{31}) - \lambda_{23}\mu_{32}}{r_1 r_2}$$

If $\Delta^2 = 4ab - (a + b + c)^2 > 0$, r_1 and r_2 are complex and $p_1(\infty)$ is approached in a damped oscillatory manner with period $2\pi/\Delta$. These oscillations are very small such that

$$p_1(t) \gtrsim p_1(\infty) - 0.8e^{-2\pi}$$

repair or
In many situations the rate of replacement of the device from S_2 or S_3 will be the same, i.e. $\mu_{31} = \mu_{21} = \mu$ (say); and the equation $p(r) = 0$ has two real roots

$$r_1 = -(\lambda_{13}+\lambda_{12}+\mu)$$

$$r_2 = -(\lambda_{23}+\mu_{32}+\mu)$$

By setting $\mu_{32} = \mu_{21} = 0$ Case 1 reduces to the partial/catastrophic failure model of [3] whilst by setting $\mu_{32} = \lambda_{23} = 0$ it reduces to the failure to operate/failure to idle model of [4]. If instead $\lambda_{23} = \lambda_{12} = \mu_{31} = 0$, the model reduces to a 3-state reliability model which has been considered in connection with electrical systems (e.g. [5]).

Case 2 : Constant Upgrading Transition Rates, Sum of Weibull Degradation Rates

It is often the case (e.g. [6,7]) that in general the hazard function for a device has a 'bath tub' shape i.e. it is monotonically decreasing for small t (corresponding to burn in failures) fairly constant at medium time values and increasing for large t.

In order to model the whole of the bath-tub curve we use a sum of Weibull distributions to describe the degradation rates:

Additional Assumptions:- $\lambda_{ij}(t) = \theta_{ij}\, t^{\alpha_{ij}-1} + \phi_{ij}\, t^{\beta_{ij}-1}$

where $\beta_{ij} > 1$, $0 < \alpha_{ij} < 1$,

$\mu_{31}(t) = \mu_{21}(t) = \mu$ (constant)

$\mu_{32}(t) = \mu_{32}$ (constant)

In this case the differential equations are

$$\frac{d}{dt}\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} = \begin{bmatrix} -\lambda_{13}(t)-\lambda_{12}(t) & 0 & 0 \\ \lambda_{12}(t) & -\lambda_{23}(t) & 0 \\ \lambda_{13}(t) & \lambda_{23}(t) & 0 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} + \begin{bmatrix} 0 & \mu & \mu \\ 0 & -\mu & \mu_{32} \\ 0 & 0 & -\mu-\mu_{32} \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix}$$

heir solution is

$$p_1(t) = \frac{1}{G(t)}\left[\mu \int_0^t G(x)dx + 1\right]$$

$$p_2(t) = \frac{1}{H(t)}\left[\int_0^t H(x)\ \{\mu_{32} - \mu_{32}p_1(x) + \lambda_{12}(x)p_1(x)\}dx - 1\right]$$

$$p_3(t) = 1 - p_1(t) - p_2(t)$$

here

$$G(t) \equiv \exp\{\int_0^t [\mu + \lambda_{13}(x) + \lambda_{12}(x)]dx\}$$

$$H(t) \equiv \exp\{\int_0^t [\mu + \mu_{32} + \lambda_{23}(x)]dx\}$$

he steady-state availability of this model is

$$p_1(\infty) = 0$$

.e. constant rates for repair and replacements cannot keep up with increasing egradation and failure tendencies.

5-STATE MODEL

Notation:

$P_0(t)$ probability the device is good at time t

$P_{ij}(t)$ probability the device is in state (i,j) at time $t; i,j = 1, 2$

Equations:

The differential equations for this device, in matrix notation, are:-

$$\frac{d}{dt}\begin{bmatrix} P_0(t) \\ P_{11}(t) \\ P_{12}(t) \\ P_{21}(t) \\ P_{22}(t) \end{bmatrix} = \begin{bmatrix} -(\lambda_1+\lambda_2+\mu_1+\mu_2) & 0 & 0 & k_1 & k_2 \\ \lambda_1 & -\lambda_{12} & 0 & 0 & 0 \\ \mu_1 & 0 & -\mu_{12} & 0 & 0 \\ \lambda_2 & \lambda_{12} & 0 & -k_1 & 0 \\ \mu_2 & 0 & \mu_{12} & 0 & -k_2 \end{bmatrix} \begin{bmatrix} P_0(t) \\ P_{11}(t) \\ P_{12}(t) \\ P_{21}(t) \\ P_{22}(t) \end{bmatrix}$$

These equations can be solved by use of Laplace transforms or by classical integration methods to yield

$$P_0(t) = \sum_{i=1}^{5} D_i(\lambda_{12}-r_i)(\mu_{12}-r_i)(k_1-r_i)(k_2-r_i)\exp(-r_i t)$$

$$P_{11}(t) = \sum_{i=1}^{5} D_i(\mu_{12}-r_i)(k_1-r_i)(k_2-r_i)\lambda_1 \exp(-r_i t)$$

$$P_{12}(t) = \sum_{i=1}^{5} D_i(\lambda_{12}-r_i)(k_1-r_i)(k_2-r_i)\mu_1 \exp(-r_i t) \qquad (1)$$

$$P_{21}(t) = \sum_{i=1}^{5} D_i(\mu_{12}-r_i)(k_2-r_i)[\lambda_2(\lambda_{12}-r_i) + \lambda_1\lambda_{12}]\exp(-r_i t)$$

$$P_{22}(t) = \sum_{i=1}^{5} D_i(\lambda_{12}-r_i)(k_1-r_i)[\mu_2(\mu_{12}-r_i) + \lambda_1\lambda_{12}]\exp(-r_i t)$$

where

$$D_i \equiv \prod_{j\neq i=1}^{5} (r_j-r_i)^{-1}$$

and r_i, $i = 1,2,3,4,5$ are the five roots of the polynomial equation

$$p(r) \equiv (r+k_1)(r+k_2)(r+\lambda_{12})(r+\mu_{12})(r+\lambda_1+\lambda_2+\mu_1+\mu_2)$$
$$- (r+k_1)(r+\lambda_{12})k_2(\mu_1\mu_{12}+\mu_2\mu_{12}+r\mu_2)$$
$$- (r+k_2)(r+\mu_{12})k_1(\lambda_1\lambda_{12}+\lambda_2\lambda_{12}+r\lambda_2) = 0.$$

Note one of these roots r_1 (say) is zero.

The steady state availability of this device is

$$P_0(\infty) = \frac{\lambda_{12}\mu_{12}k_1k_2}{r_2r_3r_4r_5} = \frac{N}{D}$$

where

$$N \equiv \lambda_{12}\mu_{12}k_1k_2$$

$$D \equiv \lambda_{12}\mu_{12}k_1k_2 + \lambda_{12}k_1(\mu_1\mu_{12}+\mu_2\mu_{12}+\mu_1k_2) + \mu_{12}k_2(\lambda_1\lambda_{12}\ \lambda_2\lambda_{12}\ \lambda_1k_1)$$

Special cases:

1. If $k_2 = \mu_1 = \mu_2 = \mu_{12} = 0$ the model reduces to the model considered in [3]

2. If $\lambda_1 = \lambda_{12} = \mu_1 = \mu_{12} = 0$ the model reduces to the catastrophic failure model of [4]

3. In many situations the rates of replacement or repair of the device from state (2,1) and state (2,2) may be the same, i.e. $k_1 = k_2 \equiv k$. If also $\lambda_{12} = \mu_{12} \equiv \beta$ only three of the five exponential factors determining $P_0(t)$ and $P_{ij}(t)$ in (1) (say those corresponding to r_1, r_2 and r_3) will have non zero coefficients. As before $r_1 = 0$ and r_2, r_3 are the roots of the reduced polynomial equation

$$p_1(r) \equiv r^2 + r(\lambda_1+\lambda_2+\mu_1+\mu_2+\beta+k) + (\beta+k)(\lambda_1+\mu_1) + \beta(\lambda_2+\mu_2+k) = 0$$

The steady-state availability in this case is

$$P_0(\infty) = \frac{\beta k}{\beta(\lambda_1+\lambda_2+\mu_1+\mu_2) + k(\lambda_1+\mu_1+\beta)} .$$

If $\Delta^2 \equiv 4k(\lambda_1+\mu_1) - (\lambda_1+\lambda_2+\mu_1+\mu_2+k-\beta)^2 > 0$, r_2, r_3 are complex and $P_0(\infty)$ will be approached in a damped oscillatory manner with period $2\pi/\Delta$.

REFERENCES

[3] J.M.Kontoleon, N. Kontoleon, "Reliability analysis of a system subject to partial and catastrophic failures", IEEE Trans. Reliability, vol R-23, 1974 Oct, pp277-278.

[4] C.L. Proctor, B. Singh, "A reparable 3-state device", IEEE Trans. Reliability, vol r-23, 1976 Aug, pp210-211.

[5] R.N. Allan, M.F. De Oliveira, "Three-state reliability modelling of electrical networks", Proc 1976 Advances in Reliability Technology Symposium, Bradford, England. National Centre of Systems Reliability NCSR R12, 1977 Feb, ART/3/1-16. Available from the National Centre of Systems Reliability, Culcheth, Warrington, U.K.

[6] M.L. Shooman, Probabilistic Reliability: An Engineering Approach, New York: McGraw-Hill, 1968

[7] Z.A. Lomnicki, "Some aspects of the statistical approach to reliability", (with discussion), J R Statist.Soc.A, vol 136, 1973 Sep, pp395-417.

A3. Operating history and failure and degradation tendencies.

by

A. Bendell and S. Humble. IEEE Transactions on Reliability, Vol. R-27 (1978), 75-76.

Operating History and Failure and Degradation Tendencies

A. Bendell
S. Humble

Key Words—Competing failure modes, Degraded states, Random hazard

Reader Aids—
Purpose: Report of derivations
Special math needed for explanations: Probability
Special math needed for results: Probability, Integrals of special functions
Results useful to: Reliability theoreticians

Abstract—Models for the s-dependence of a unit's reliability and degraded states on its operating history are developed. The effect of a random environment is introduced into these models by the use of stochastic hazard functions.

1. INTRODUCTION

We consider failure and degradation models for a unit in which the failure and degradation behaviour depend on the following aspects of operating history:

1) total elapsed calendar-time, t_1
2) total accumulated on-time, t_2
3) length of current operating period, t_3
4) random environments.

The model of section 2 is concerned with the dependence of the failure tendency on aspects 1–3. Section 3 introduces a simple model for the partial degradation of a unit and its dependence on aspects 1–3. Section 4 adapts the models to deal with the effect of a random environment.

2. DEPENDENCE OF FAILURE TENDENCY ON OPERATING HISTORY

2.1 *Notation and General Model*

t_0	a suitable scaling factor (in appropriate time units)
τ_i	t_i/t_0, $i = 1,2,3$
h_i	constant hazard rate for mode-i failure
$H(t_1,t_2,t_3)$	cumulative hazard for unit by $t_3 \leq t_2 \leq t_1$
$\gamma(\psi, \alpha)$	$\int_0^\psi \tau^{\alpha-1}e^{-\tau}d\tau$, the incomplete gamma function

$$\text{pdf}\{\tau_2|\tau_1\} \quad \begin{cases} \tau_2^{\alpha-1}e^{-\tau_2}/\gamma(\tau_1,\alpha), & 0 < \tau_2 \leq \tau_1 \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

$$\text{pdf}\{\tau_3|\tau_2\} \quad \begin{cases} 1/\tau_2, & 0 < \tau_3 \leq \tau_2 \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

2.2 *Further Assumptions*

1. A unit has three s-independent and competing failure modes; so the unit can be regarded as composed of three s-independent 'components' in series (1-out-of 3:F).

2. s-Dependence between the modes is modeled by adding extra terms to the hazard rate for the unit. Here modes 1 and 2 are s-dependent in the following way—

$$H(t_1,t_2,t_3) \equiv h_1t_1 + h_2t_2 + h_3t_3 + h_4^*t_1t_2 \tag{2.3}$$

2.3 *Analysis*

After considerable manipulation, the Sf is obtained:

$$R(t_1) = \exp(-h_1t_1)\,[\theta^{1-\alpha}\gamma(\theta t_1, \alpha-1) - (h_3+\theta)^{1-\alpha}\,\gamma((h_3+\theta)t_1,\ \alpha-1)]/(t_0^\alpha h_3\gamma(\tau_1, \alpha))$$

$$\theta \equiv 1/t_0 + h_2 + h_4^*\tau_1 \tag{2.4}$$

It is shown in [1] that for this specialised model the reliability is asymptotically independent of the mode 3 failure mechanism.

3. DEPENDENCE OF DEGRADATION TENDENCY ON OPERATING HISTORY

3.1 *Basic Model*

We consider the degraded state s of a unit which has the following properties:

1. $0 \leq s \leq 1$; $s = 1$ indicates full operation and $s = 0$ indicates complete failure

2. Transfers are only possible to lower degraded states.

3. Transfer intensities at t_i are independent of degraded states experienced prior to t_i (the Markov property).

3.2 *Additional Notation*

$\phi_i(s,t_i)$	pdf for mode i ($i = 1,2,3$) for state s at time t_i
$h_i^*(\sigma,s,t_i)$	transfer intensity at time t_i for mode i, from state σ to state s.
$\Gamma(x)$	Gamma function
$\delta(s)$	Dirac delta function
$B(x, y)$	$\Gamma(x)\Gamma(y)/\Gamma(x+y)$ Beta function
$F(a,b; c; z)$	Hypergeometric function
$\exp f(\cdot)$, $\exp fc(\cdot)$	Cdf, Sf of exponential distribution

3.3 *Evaluation of* $\phi_i(s,t_i)$

$\phi_i(s,t_i)$ is related to $h_i^*(\sigma,s,t_i)$ by the differential equation

$$\partial\phi_i(s,t_i)/\partial t_i \equiv -\phi_i(s,t_i)\int_0^s h_i^*(s,\epsilon,t_i)\,d\epsilon + \int_s^1 \phi_i(\sigma,t_i)h^*(\sigma,s,t_i)\,d\sigma, \quad s > 0; \tag{3.1}$$

which is a special case of the Chapman-Kalmogorov equation [1,2]. Most solutions for $\phi_i(s,t_i)$ are complicated even when the transfer intensities are simple. However,

0018-9529/78/0400-0075 $00.75

if

$$h_i^*(\sigma,s,t_i) \equiv \lambda_i + \delta(s)\{\lambda_i(1-\sigma) - \lambda_i \log(1-\sigma) + \sigma/t_i\} \quad (3.2)$$

then provided $\phi_i(s,0) = 0$ for all $s \neq 1$

$$\phi_i(s,t_i) = \begin{cases} \lambda_i t_i (1-s)^{\lambda_i t_i - 1} \exp fc(\lambda_i t_i), & s \neq 0 \\ \exp f(\lambda_i t_i)\delta(s), & s = 0 \end{cases} \quad (3.3)$$

3.4 *Dependence on Operating History*

We assume further that

1. s depends upon aspects 1–3 in the Introduction.
2. The three degradation modes are s-independent
3. The degraded state of the unit is given by the product of the levels of degradation of the modes. This is discussed in [1].

Using (3.3) for each of the degradation modes we obtain

$$\text{pdf}\{s/t_1,t_2t_3\} = \left[\prod_{i=1}^{3} \lambda_i t_i (1-s)^{\lambda_i t_i} \exp fc(\lambda_i t_i)\right] X(s,t_1,t_2,t_3)/(1-s)$$

$$X(s,t_1,t_2,t_3) \equiv B(\lambda_1 t_1, \lambda_2 t_2) F(\lambda_1 t_1, \lambda_2 t_2 - 1; \lambda_1 t_1 + \lambda_2 t_2; (1-s)y) \int_0^1 dy (1-y)^{\lambda_3 t_3 - 1}(1-(1-s)y)^{1-\lambda_3 t_3} y^{\lambda_1 t_1 + \lambda_2 t_2 - 1} \quad (3.4)$$

Using (2.1), (2.2) it follows that near $s=1$ and $t_1 = 0$,

$$\text{pdf}\{s|t_1\} \approx (1-s)^{\lambda_1 t_1 - 1} \exp fc(\lambda_1 t_1)\{\lambda_1 + (\alpha/\alpha+1)(\lambda_2 + \lambda_3/2))t_1 \quad (3.5)$$

4. THE EFFECT OF A RANDOM ENVIRONMENT

4.1 *Failure Tendency Model*

1. Following [3] we model the effect of a random environment by making the hazard parameters h_i ($i = 1,2,3$) i.i.d. r.v.'s with uniform pdf's.

$$\text{pdf}\{h_i\} = \begin{cases} 1/((b_i-a_i)t_0) & 0 \leq a_i < h_i < b_i \\ 0, & \text{otherwise} \end{cases} \quad (4.1)$$

2. We assume that $h_4^* \equiv 0$.

It can be shown that the corresponding Sf is

$$R(t_1) = [\exp(-a_1 t_1) - \exp(-b_1 t_1)]Y(t_1) \Big/ \left[t_1 t_0^3 \prod_{i=1}^{3}(b_i-a_i)\cdot\gamma(\tau_1,\alpha)\right]$$

$$Y(t_1) \equiv \sum_{n=1}^{\alpha} (-1)^n (a_3^n - b_3^n)[\gamma((a_2+1/t_0)t_1, n+\alpha-2) \cdot (a_2+1/t_0)^{2-n-\alpha} - \gamma((b_2+1/t_0)t_1, n+\alpha-2) \cdot (b_2+1/t_0)^{2-n-\alpha}]/(n\cdot n!) \quad (4.2)$$

4.2 *Degradation Tendency Model*

We assume that λ_i in (3.3) is a r.v. with pdf (4.1). Then

$$\phi_i(s,t_i) = \begin{cases} t_i(z(b_i) - z(a_i))/\{(b_i-a_i)(t_0 \log(1-s) - t_i)^2\}, & s \neq 0 \\ 1 - [\exp(-a_i t_i) - \exp(-b_i t_i)]/[t_i(b_i-a_i)], & s = 0 \end{cases}$$

$$z(x) \equiv (1-s)^{xt_0-1}(xt_0 \log(1-s) - xt_i - 1)\exp(-xt_i) \quad (4.3)$$

In [1] a similar expression is obtained for the case where λ_i has a gamma pdf.

REFERENCES

[1] A. Bendell, S. Humble, "Some aspects of reliability theory", Sheffield City Polytechnic Report No. SCP/MS/12/1976. 17 pages—available from Department of Mathematics and Statistics; Sheffield City Polytechnic; Pond Street; Sheffield S1 1WB ENGLAND.

[2] W. Feller, *An Introduction to Probability Theory and Its Applications*, New York: Wiley, 1957.

[3] C. M. Harris, N. D. Singpurwalla, "Life distributions derived from stochastic hazard functions", *IEEE Trans. Reliability*, vol R-17, 1968 Jun, pp 70-79.

A. Bendell; Department of Mathematics and Statistics; Sheffield City Polytechnic; Pond Street; Sheffield S1 1WB ENGLAND.

Tony Bendell was born in London, England on 1949 Sep 20. He received a BSc(Econ) (1971) and a MSc in Statistics with Distinction (1972) from the London School of Economics and Political Science (London University). He is a Senior Lecturer in Statistics at Sheffield City Polytechnic, and a part-time author/tutor for the Rapid Results (correspondence) College, London.

S. Humble; Department of Mathematics and Statistics; Sheffield City Polytechnic; Pond Street; Sheffield S1 1WB ENGLAND.

Dr. Stephen Humble is a Senior Lecturer at Sheffield City Polytechnic; he was born in the North of England on 1943 Aug 8. He received a BSc Mathematics with first class honours (1964) and a PhD Theoretical Physics (1967) from Durham University. He has held research posts at Trinity College, Dublin 1966–67, Colorado University, 1967–69, SRC Laboratories, England 1969–1973 and was a visiting scientist at CERN, Geneva 1973–1975. He has published one book and over 40 papers in Particle Physics as well as other work in Number Theory, etc.

Manuscript received 1976 Nov 22; revised 1977 Jul 25, 1977 Oct 18.

□ □ □