On the existence of soliton and hairy black hole solutions of SU(N) Einstein–Yang–Mills theory with a negative cosmological constant

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On the existence of soliton and hairy black hole solutions of $\mathfrak{su}(N)$ Einstein-Yang-Mills theory with a negative cosmological constant

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Abstract. We study the existence of soliton and black hole solutions of four-dimensional $\mathfrak{su}(N)$ Einstein-Yang-Mills theory with a negative cosmological constant. We prove the existence of non-trivial solutions for any integer N, with $N-1$ gauge field degrees of freedom. In particular, we prove the existence of solutions in which all the gauge field functions have no zeros. For fixed values of the parameters (at the origin or event horizon, as applicable) defining the soliton or black hole solutions, if the magnitude of the cosmological constant is sufficiently large, then the gauge field functions all have no zeros. These latter solutions are of special interest because at least some of them will be linearly stable.

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1. Introduction

Since the discovery of non-trivial, spherically symmetric, soliton [1] and black hole [2] solutions of the $\mathfrak{su}(2)$ Einstein-Yang-Mills (EYM) equations in four-dimensional asymptotically flat space-time, there has been an explosion of interest in the properties of solutions of EYM theory and its variants (see, for example, [3] for a review). The original soliton solutions were a surprise discovery because there are no gravitational solitons [4] nor solitons in pure Yang-Mills theory in flat space-time [5]. The black hole solutions are counter-examples to the "no-hair" conjecture [6], in that they have no global charge, and are indistinguishable from a standard Schwarzschild black hole at infinity. However, while the "letter" of the no-hair conjecture is violated, its "spirit" remains intact because these black hole solutions, like the solitons, are linearly unstable [7].

Introducing a negative cosmological constant changes the picture completely (see [8] for a recent review). In asymptotically flat space, soliton and black hole solutions exist at discrete points in the parameter space, but in four-dimensional asymptotically anti-de Sitter (adS) space, $\mathfrak{su}(2)$ EYM solutions are found in continuous open sets of the parameter space [9, 10, 11]. Of particular interest are those solitons and black holes for which the single gauge field function ω has no zeros. In asymptotically flat space, all solutions are such that ω must have at least one zero, and the number of unstable perturbation modes of the solutions is $2k$, where k is the number of zeros of ω [12]. In asymptotically adS space, at least some (but, interestingly, not all [11]) solutions where $\omega(r)$ has no zeros are linearly stable under spherically symmetric [9, 10, 11] and non-spherically symmetric [13] perturbations, provided $|\Lambda|$ is sufficiently large. Stable black holes in this model violate the "no-hair" conjecture in that, at infinity, they are indistinguishable from magnetically-charged Reissner-Nordström-adS black holes. However, only one extra parameter (which can be taken to be the value of the gauge field function at the event horizon) is required to completely determine the geometry exterior to the event horizon. While this is not a global charge measurable at infinity, one might still argue that the "spirit" of the "no-hair" conjecture is still valid as only a small number of parameters are required to fully describe stable black holes.

Recently soliton and black hole solutions of four-dimensional $\mathfrak{su}(N)$ EYM theory with a negative cosmological constant have been found [8, 14, 15]. For purely magnetic solutions, the gauge field is described by $N-1$ gauge field functions ω_j. As in the $\mathfrak{su}(2)$ case, there are solutions for which all the gauge field functions have no zeros. For $|\Lambda|$ sufficiently large, at least some of these are stable, for any value of N [14, 16]. Stable black hole solutions in $\mathfrak{su}(N)$ EYM in adS therefore require an additional $N-1$ parameters to completely describe the configuration, even though the metric is identical to magnetically charged Reissner-Nordström-adS at infinity. Therefore an unbounded number of parameters is required to determine the structure of stable hairy black holes in adS.

Our purpose in the present paper is to prove analytically the existence of solitons and hairy black holes in four-dimensional $\mathfrak{su}(N)$ EYM in adS, confirming the numerical work presented in [15]. Existence theorems for the original soliton and black hole solutions of $\mathfrak{su}(2)$ EYM in asymptotically flat space were challenging to prove [17, 18], but it has been shown analytically in this case that there are an infinite number of soliton and black hole solutions of the field equations, parameterized by the radius of the event horizon (if there is one) and k, the number of zeros of the gauge field function ω. It has also been proven that $k > 0$. For $\mathfrak{su}(N)$ EYM in asymptotically flat space, rather less analytic work exists in the literature [19, 20]. Arguments for the existence of solutions for arbitrary N are presented in [20], but the only complete analytic work is for $\mathfrak{su}(3)$ [19]. For $\mathfrak{su}(2)$ EYM in adS, proving the existence of solutions [9] is considerably easier than in the asymptotically flat case, mostly due to the boundary conditions at infinity. It is therefore reasonable to imagine that the corresponding proof for $\mathfrak{su}(N)$ EYM in adS will also be significantly easier than in the asymptotically flat case.

The outline of this paper is as follows. We start, in Section 2, by outlining the field equations, gauge field ansatz and boundary conditions for soliton and hairy black hole solutions of $\mathfrak{su}(N)$ EYM theory with a negative cosmological constant in four space-time dimensions [15]. Some trivial solutions of the field equations are discussed in Section 2.3, the most important of which (for our later analysis) are embedded $\mathfrak{su}(2)$ solutions. We also review the salient features of the numerical solutions presented in [15]. The field equations, presented in Section 2.1, are singular at the origin, black hole event horizon (if there is one) and at infinity. The first part of the proof of the existence of solutions is therefore to prove local existence of solutions satisfying the appropriate boundary conditions (which are given in Section 2.2) at these singular points, and this is the subject of Section 3. It turns out that this is the most lengthy and technically intricate part of the whole existence proof, particularly for local solutions near the origin. Our method in Section 3 follows a well-established approach [17, 20, 21, 22] which involves casting the field equations in an appropriate form, so that a standard theorem (Theorem 1 [17]) can be applied. It is deriving a suitable form for the equations near the origin which is the most complicated part of the analysis. The standard theorem then gives us local existence of solutions in a neighbourhood of the singular point (i.e. the origin, event horizon or infinity), and an important consequence of this theorem is that the solutions are analytic in the parameters which determine them. This property of analyticity is central to the arguments, presented in Section 4, which prove the existence of genuinely $\mathfrak{su}(N)$ soliton and black hole solutions of the field equations.

Our existence theorems are presented in Sections 4.3 and 4.4. In Section 4.3 we show existence of $\mathfrak{su}(N)$ EYM solutions in a neighbourhood of any embedded $\mathfrak{su}(2)$ solution, and, as a corollary of this, the existence of $\mathfrak{su}(N)$ solutions for which all the gauge field functions have no zeros for any value of $|\Lambda|$. The behaviour of the solution space for large $|\Lambda|$ is discussed in Section 4.4, and, in particular, we show that for any fixed values of the "initial" parameters (defined in Section 4.4) determining the solution of the field equations, for sufficiently large $|\Lambda|$, the solution generated by these parameters will be such that all the gauge field functions have no zeros. Finally, our conclusions are presented in Section 5.

2. $\mathfrak{su}(N)$ Einstein-Yang-Mills theory

In this section we will describe in detail our field equations and boundary conditions for both soliton and black hole solutions. We will also describe some simple embedded solutions of the theory, as well as reviewing the key features of the numerical solutions studied in [15].

2.1. Ansatz and field equations

We consider four-dimensional $\mathfrak{su}(N)$ EYM theory with a negative cosmological constant, described by the following action, in suitable units:

$$S_{\rm EYM} = \frac{1}{2}\int d^4x\sqrt{-g}\left[R - 2\Lambda - {\rm Tr}\, F_{\mu\nu}F^{\mu\nu}\right], \tag{1}$$

where R is the Ricci scalar of the geometry and Λ the cosmological constant. Throughout this paper, the metric has signature $(-,+,+,+)$ and we use units in which $4\pi G = 1 = c$. In this article we focus on a negative cosmological constant, $\Lambda < 0$. Varying the action (1) gives the field equations

$$\begin{aligned} T_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}; \\ 0 &= D_\mu F_\nu{}^\mu = \nabla_\mu F_\nu{}^\mu + [A_\mu, F_\nu{}^\mu]; \end{aligned} \tag{2}$$

where the YM stress-energy tensor is

$$T_{\mu\nu} = {\rm Tr}\, F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}{\rm Tr}\, F_{\lambda\sigma}F^{\lambda\sigma}. \tag{3}$$

In equations (1–3) we have employed the usual Einstein summation convention, where a summation over repeated indices is understood. However, from now on all summations will be given explicitly, and this will be particularly important in Section 3.

In this paper we are interested in static, spherically symmetric soliton and black hole solutions of the field equations (2), and we write the metric in standard Schwarzschild-like co-ordinates as:

$$ds^2 = -\mu S^2\, dt^2 + \mu^{-1}\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2, \tag{4}$$

where the metric functions μ and S depend on the radial co-ordinate r only. In the presence of a negative cosmological constant $\Lambda < 0$, we write the metric function μ as

$$\mu(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}. \tag{5}$$

We emphasize that we are considering only spherically symmetric black holes and not topological black holes which have been found in the $\mathfrak{su}(2)$ case [23].

The most general, spherically symmetric, ansatz for the $\mathfrak{su}(N)$ gauge potential is [24]:

$$A = \mathcal{A}\, dt + \mathcal{B}\, dr + \frac{1}{2}\left(C - C^H\right)\, d\theta - \frac{i}{2}\left[\left(C + C^H\right)\sin\theta + D\cos\theta\right]\, d\phi, \tag{6}$$

where $\mathcal{A}$, $\mathcal{B}$, C and D are all $(N \times N)$ matrices and C^H is the Hermitian conjugate of C. The matrices $\mathcal{A}$ and $\mathcal{B}$ are purely imaginary, diagonal, traceless and depend only on the radial co-ordinate r. The matrix C is upper-triangular, with non-zero entries only immediately above the diagonal:

$$C_{j,j+1} = \omega_j(r) e^{i\gamma_j(r)}, \tag{7}$$

for $j = 1, \ldots, N-1$. In addition, D is a constant matrix:

$$D = \mathrm{Diag}\left(N-1, N-3, \ldots, -N+3, -N+1\right). \tag{8}$$

Here we are interested only in purely magnetic solutions, so we set $\mathcal{A} \equiv 0$. We may also take $\mathcal{B} \equiv 0$ by a choice of gauge [24]. From now on we will assume that all the $\omega_j(r)$ are non-zero (see, for example, [25] for the possibilities in asymptotically flat space if this assumption does not hold). In this case one of the Yang-Mills equations becomes [24]

$$\gamma_j = 0 \qquad \forall j = 1, \ldots, N-1. \tag{9}$$

Our ansatz for the Yang-Mills potential therefore reduces to

$$A = \frac{1}{2}\left(C - C^H\right) d\theta - \frac{i}{2}\left[\left(C + C^H\right)\sin\theta + D\cos\theta\right] d\phi, \tag{10}$$

where the only non-zero entries of the matrix C are

$$C_{j,j+1} = \omega_j(r). \tag{11}$$

The gauge field is therefore described by the $N-1$ functions $\omega_j(r)$. We comment that our ansatz (10) is by no means the only possible choice in $\mathfrak{su}(N)$ EYM. Techniques for finding *all* spherically symmetric $\mathfrak{su}(N)$ gauge potentials can be found in [26], where all irreducible models are explicitly listed for $N \leq 6$.

With the ansatz (10), there are $N-1$ non-trivial Yang-Mills equations for the $N-1$ gauge field functions ω_j:

$$r^2\mu\omega_j'' + \left(2m - 2r^3 p_\theta - \frac{2\Lambda r^3}{3}\right)\omega_j' + W_j\omega_j = 0 \tag{12}$$

for $j = 1, \ldots, N-1$, where a prime $'$ denotes d/dr,

$$p_\theta = \frac{1}{4r^4}\sum_{j=1}^{N}\left[\left(\omega_j^2 - \omega_{j-1}^2 - N - 1 + 2j\right)^2\right], \tag{13}$$

$$W_j = 1 - \omega_j^2 + \frac{1}{2}\left(\omega_{j-1}^2 + \omega_{j+1}^2\right), \tag{14}$$

and $\omega_0 = \omega_N = 0$. The Einstein equations take the form

$$m' = \mu G + r^2 p_\theta, \qquad \frac{S'}{S} = \frac{2G}{r}, \tag{15}$$

where

$$G = \sum_{j=1}^{N-1}\omega_j'^2. \tag{16}$$

Altogether, then, we have $N+1$ ordinary differential equations for the $N+1$ unknown functions $m(r)$, $S(r)$ and $\omega_j(r)$. The field equations (12,15) are invariant under the transformation

$$\omega_j(r) \to -\omega_j(r) \tag{17}$$

for each j independently, and also under the substitution:

$$j \to N - j. \tag{18}$$

2.2. Boundary conditions

We are interested in black hole and soliton solutions of the field equations (12,15). However, the field equations are singular at the origin $r = 0$ if we have a regular, soliton solution, at an event horizon $r = r_h$ if there is one, and at infinity $r \to \infty$. Since the cosmological constant $\Lambda < 0$, there is no cosmological horizon. We need to derive the boundary conditions at each of these singular points. For $\Lambda = 0$, local existence of solutions of the field equations near these singular points has been rigorously proved [21, 22]. The extension of these results to $\Lambda < 0$ will be the focus of Section 3.

2.2.1. Origin The boundary conditions at the origin are the most complicated of the three singular points. We postpone the detailed derivation of the form of the field variables near the origin to Section 3.1, and instead here briefly state the results of this derivation for reference.

We assume that the field variables $m(r)$, $S(r)$ and $\omega_j(r)$ have regular Taylor series expansions about $r = 0$:

$$\begin{aligned} m(r) &= m_0 + m_1 r + m_2 r^2 + O(r^3); \\ S(r) &= S_0 + S_1 r + S_2 r^2 + O(r^3); \\ \omega_j(r) &= \omega_{j,0} + \omega_{j,1} r + \omega_{j,2} r^2 + O(r^3); \end{aligned} \tag{19}$$

where the m_i, S_i and $\omega_{j,i}$ are constants. The constant S_0 is non-zero in order for the metric to be regular at the origin, but otherwise arbitrary since the field equations involve only derivatives of S.

Regularity of the metric and curvature at the origin immediately gives:

$$m_0 = m_1 = m_2 = 0, \qquad S_1 = 0, \qquad \omega_{j,1} = 0 \tag{20}$$

and

$$\omega_{j,0} = \pm\sqrt{j(N-j)}. \tag{21}$$

Without loss of generality (due to (17)), we take the positive square root in (21).

The expansions (19) are substituted into the field equations (12,15) to determine the values of the remaining constants in the expansions. The details of this analysis are presented in Section 3.1, but the upshot is as follows. We define a vector $\boldsymbol{\omega}(r) = (\omega_1(r), \omega_2(r), \ldots, \omega_{N-1}(r))^T$. It turns out that, in order to have $N-1$ independent parameters to describe the $N-1$ gauge field degrees of freedom $\omega_j(r)$, it is necessary to expand the $\omega_j(r)$ to order $O(r^N)$. Then the expansions (19) become

$$\begin{aligned} m(r) &= m_3 r^3 + O(r^4); \\ S(r) &= S_0 + O(r^2); \\ \boldsymbol{\omega}(r) &= \boldsymbol{\omega}_0 + \sum_{k=2}^{N} z_k(r)\boldsymbol{v}_k r^k + O(r^{N+1}), \end{aligned} \tag{22}$$

where

$$\boldsymbol{\omega}_0 = \left(\sqrt{N-1}, \sqrt{2(N-2)}, \ldots, \sqrt{(N-1)}\right)^T. \tag{23}$$

The functions $z_k(r)$, $k = 2, \ldots, N$ are determined completely by the arbitrary constants $\bar{z}_k = z_k(0)$ for $k = 2, \ldots, N$. Together with Λ, the constants $\bar{z}_2, \ldots, \bar{z}_N$ determine the constant m_3 (see equation (56)). The $N-1$ vectors $\boldsymbol{v}_k$ are constant and their components are fixed (see equation (62)). The expansions (22) therefore give the field variables in terms of the $N+1$ parameters $\bar{z}_2, \ldots, \bar{z}_N, \Lambda$ and S_0.

2.2.2. Event horizon For black hole solutions, we assume that there is a regular, non-extremal event horizon at $r = r_h$, where $\mu(r)$ has a single zero. This fixes the value of $m(r_h)$ to be

$$2m(r_h) = r_h - \frac{\Lambda r_h^3}{3}. \tag{24}$$

We assume that the field variables $\omega_j(r)$, $m(r)$ and $S(r)$ have regular Taylor series expansions about $r = r_h$:

$$\begin{aligned} m(r) &= m(r_h) + m'(r_h)\left(r - r_h\right) + O\left(r - r_h\right)^2; \\ \omega_j(r) &= \omega_j(r_h) + \omega_j'(r_h)\left(r - r_h\right) + O\left(r - r_h\right)^2; \\ S(r) &= S(r_h) + S'(r_h)\left(r - r_h\right) + O\left(r - r_h\right). \end{aligned} \tag{25}$$

Setting $\mu(r_h) = 0$ in the Yang-Mills equations (12) fixes the derivatives of the gauge field functions at the horizon:

$$\omega_j'(r_h) = -\frac{W_j(r_h)\omega_j(r_h)}{2m(r_h) - 2r_h^3 p_\theta(r_h) - \frac{2\Lambda r_h^3}{3}}. \tag{26}$$

Therefore the expansions (25) are determined by the $N+1$ quantities $\omega_j(r_h)$, r_h, $S(r_h)$ for fixed cosmological constant Λ. For the event horizon to be non-extremal, it must be the case that

$$2m'(r_h) = 2r_h^2 p_\theta(r_h) < 1 - \Lambda r_h^2, \tag{27}$$

which weakly constrains the possible values of the gauge field functions $\omega_j(r_h)$ at the event horizon. Since the field equations (12,15) are invariant under the transformation (17), we may consider $\omega_j(r_h) > 0$ without loss of generality.

2.2.3. Infinity At infinity, we require that the metric (4) approaches adS, and therefore the field variables $\omega_j(r)$, $m(r)$ and $S(r)$ converge to constant values as $r \to \infty$. We assume that the field variables have regular Taylor series expansions in r^{-1} near infinity:

$$\begin{aligned} m(r) &= M + O\left(r^{-1}\right); \qquad S(r) = 1 + O\left(r^{-1}\right); \\ \omega_j(r) &= \omega_{j,\infty} + O\left(r^{-1}\right). \end{aligned} \tag{28}$$

If the space-time is asymptotically flat, with $\Lambda = 0$, then the values of $\omega_{j,\infty}$ are constrained to be

$$\omega_{j,\infty} = \pm\sqrt{j(N-j)}. \tag{29}$$

This condition means that the asymptotically flat black holes have no magnetic charge at infinity, or, in other words, these solutions have no global magnetic charge. Therefore, at infinity, they are indistinguishable from Schwarzschild black holes. However, if the cosmological constant is negative, then there are no *a priori* constraints on the values of $\omega_{j,\infty}$. In general, therefore, the adS black holes will be magnetically charged. The fact that the boundary conditions at infinity in the $\Lambda < 0$ case are less restrictive than those in the $\Lambda = 0$ case leads to the expectation of many more solutions when $\Lambda < 0$ compared with $\Lambda = 0$.

2.3. Embedded solutions

The field equations (12,15) are non-linear and coupled, but they do have two analytic, trivial solutions.

Schwarzschild-adS Setting

$$\omega_j(r) \equiv \pm\sqrt{j(N-j)} \tag{30}$$

for all j gives the Schwarzschild-adS black hole with

$$m(r) = M = \text{constant} \tag{31}$$

We note that, by setting $M = 0$, pure adS space is also a solution.

Reissner-Nordström-adS Setting

$$\omega_j(r) \equiv 0 \tag{32}$$

for all j gives the Reissner-Nordström-adS black hole with metric function

$$\mu(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}, \tag{33}$$

where the magnetic charge Q is fixed by

$$Q^2 = \frac{1}{6}N\left(N+1\right)\left(N-1\right). \tag{34}$$

Only for this value of the magnetic charge is the Reissner-Nordström-adS black hole a solution of the field equations.

As well as these effectively abelian embedded solutions, there is also a class of embedded $\mathfrak{su}(2)$ non-abelian solutions. These will turn out to play an important role in our proof of the existence of genuinely $\mathfrak{su}(N)$ solutions in Section 4. The construction of the embedded $\mathfrak{su}(2)$ solutions in adS follows from that given in [22] in the asymptotically flat case. First, we write the $N-1$ gauge field functions $\omega_j(r)$ in terms of a single function $\omega(r)$ as follows:

$$\omega_j(r) = \pm\sqrt{j(N-j)}\,\omega(r) \qquad \forall j = 1, \ldots, N-1, \tag{35}$$

then define [22]

$$\lambda_N = \sqrt{\frac{1}{6}N\,(N-1)\,(N+1)}. \tag{36}$$

Next we rescale the field variables as follows:

$$\begin{aligned} R &= \lambda_N^{-1} r; \qquad \tilde{\Lambda} = \lambda_N^2 \Lambda; \qquad \tilde{m}(R) = \lambda_N^{-1} m(r); \\ \tilde{S}(R) &= S(r); \qquad \tilde{\omega}(R) = \omega(r). \end{aligned} \tag{37}$$

Note that we rescale the cosmological constant Λ (this is not necessary in [22] as there $\Lambda = 0$). The field equations satisfied by $\tilde{m}(R)$, $\tilde{S}(R)$ and $\tilde{\omega}(R)$ are then

$$\begin{aligned} \frac{d\tilde{m}}{dR} &= \mu\tilde{G} + R^2\tilde{p}_\theta; \\ \frac{1}{\tilde{S}}\frac{d\tilde{S}}{dR} &= -\frac{2\tilde{G}}{R}; \\ 0 &= R^2\mu\frac{d^2\tilde{\omega}}{dR^2} + \left[2\tilde{m} - 2R^3\tilde{p}_\theta - \frac{2\tilde{\Lambda}R^3}{3}\right]\frac{d\tilde{\omega}}{dR} + \left[1-\tilde{\omega}^2\right]\tilde{\omega}; \end{aligned} \tag{38}$$

where we now have

$$\mu = 1 - \frac{2\tilde{m}}{R} - \frac{\tilde{\Lambda}R^2}{3}, \tag{39}$$

and

$$\tilde{G} = \left(\frac{d\tilde{\omega}}{dR}\right)^2, \qquad \tilde{p}_\theta = \frac{1}{2R^4}\left(1-\tilde{\omega}^2\right)^2. \tag{40}$$

The equations (38) are precisely the $\mathfrak{su}(2)$ EYM field equations. Furthermore, the boundary conditions (22,25,28) also reduce to those for the $\mathfrak{su}(2)$ case. This is straightforward to see for the boundary conditions at the horizon (25) or at infinity (28). At the origin, the $\mathfrak{su}(2)$ embedded solutions are given by $\bar{z}_2 \neq 0$, with $\bar{z}_3 = \bar{z}_4 = \ldots = \bar{z}_N = 0$. Therefore any $\mathfrak{su}(2)$, asymptotically adS, EYM soliton or black hole solution can be embedded into $\mathfrak{su}(N)$ EYM to give an asymptotically adS soliton or black hole. Given that the solution space of $\mathfrak{su}(2)$ EYM solitons and black holes in adS has been extensively studied [9, 11, 15], we immediately have much useful information about the space of solutions of $\mathfrak{su}(N)$ EYM soltions and black holes. However, our purpose in this paper is to prove the existence of genuinely $\mathfrak{su}(N)$ (that is, not embedded) solutions.

2.4. Properties of the numerical $\mathfrak{su}(N)$ *solutions*

The numerical solution of the field equations (12,15) and the properties of the space of solutions are discussed in detail in [15]. In this section we briefly review the salient features of the solution space for our purposes here, and illustrate these features with some figures for the $\mathfrak{su}(3)$ case.

As in the $\mathfrak{su}(2)$ case [9, 10], we find solutions in open sets of the parameter space ($[\bar{z}_2, \ldots \bar{z}_N]$ (22) for solitons and $[\omega_1(r_h), \ldots \omega_{N-1}(r_h)]$ for black holes). For all negative values of the cosmological constant $\Lambda < 0$, we find solutions in which all the gauge field functions have no zeros, in a neighbourhood of the embedded pure adS (soliton) or Schwarzschild-adS (black hole) solution, although in practice the size of this neighbourhood decreases rapidly as $|\Lambda|$ decreases. The existence of such a region of solutions where all gauge field functions have no zeros is proven in Section 4.3, see Corollary 10. As $|\Lambda|$ decreases, the region of phase space where we have soliton or black hole solutions shrinks and breaks up, eventually becoming discrete points in the limit $|\Lambda| \to 0$.

The main interest in this paper is solutions for which the gauge field functions have no zeros, and particularly the behaviour of the space of solutions for large $|\Lambda|$. As $|\Lambda|$ increases, the region of phase space where solutions exist expands rapidly, as can be seen by comparing the size of the regions for $\Lambda = -2$ and $\Lambda = -3$ in Figures 1 and 2 (for black holes) and Figures 3 and 4 (for solitons). It can also be seen in these figures that the regions of solutions where at least one gauge field function has at least one zero shrink as $|\Lambda|$ increases. For black holes, Figure 10 in [15] shows that there are no black hole solutions for which at least one gauge field function has at least one zero when $\Lambda = -5$. As $|\Lambda|$ increases still further, all the solutions which exist are such that all the gauge field functions have no zeros. In addition, the region of phase space where there are solutions continues to expand without limit as $|\Lambda|$ increases. We will show in Section 4.4 that for any point in the phase space, for all sufficiently large $|\Lambda|$, this point leads to a black hole or soliton solution (as applicable) for which all the gauge field functions have no zeros.

We have $\mathfrak{su}(2)$ embedded solutions (35) along the lines $\omega_1(r_h) = \omega_2(r_h)$ for black hole solutions (indicated by a dashed line in Figures 1–2) and $\bar{z}_3 = 0$ for soliton solutions (not everywhere along the lines, only where the lines are contained within the region of solutions). The existence of genuinely $\mathfrak{su}(N)$ solutions in a neighbourhood of embedded $\mathfrak{su}(2)$ solutions, seen in Figures 1–4, is shown in Section 4.3.

3. Local existence of solutions

The first part of our proof of the existence of regular soliton and black hole solutions of the $\mathfrak{su}(N)$ EYM equations (12,15) is showing that regular solutions exist in neighbourhoods of the points $r = 0$, $r = r_h$ and $r = \infty$, where the field equations are singular.

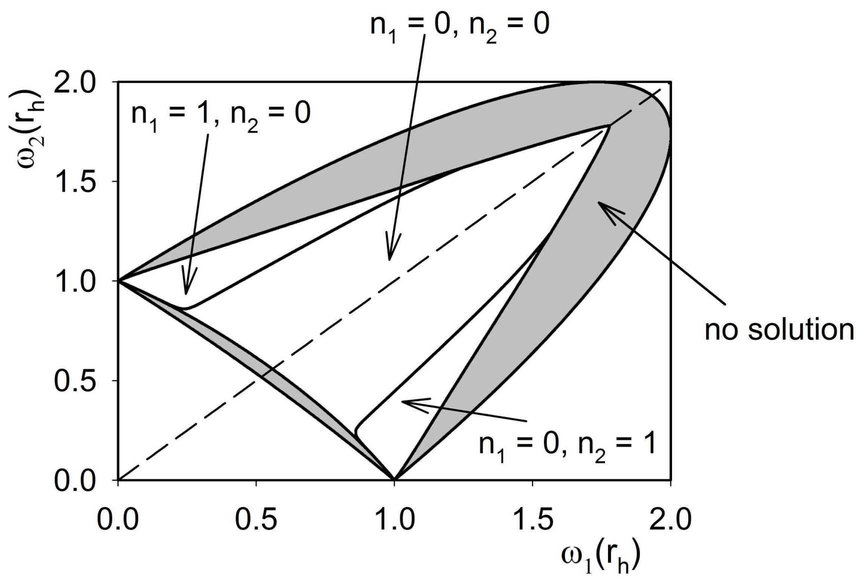

Figure 1. Solution space for $\mathfrak{su}(3)$ black holes with $r_h = 1$ and $\Lambda = -2$. The numbers of zeros of the gauge field functions, n_1 and n_2, are shown for the various regions of the solution space. The grey region denotes those areas of the $(\omega_1(r_h), \omega_2(r_h))$ plane for which the constraint (27) is satisfied, but for which we do not find solutions. Outside the shaded region the constraint (27) is not satisfied. The dashed line is $\omega_1(r_h) = \omega_2(r_h)$, along which lie the embedded $\mathfrak{su}(2)$ solutions.

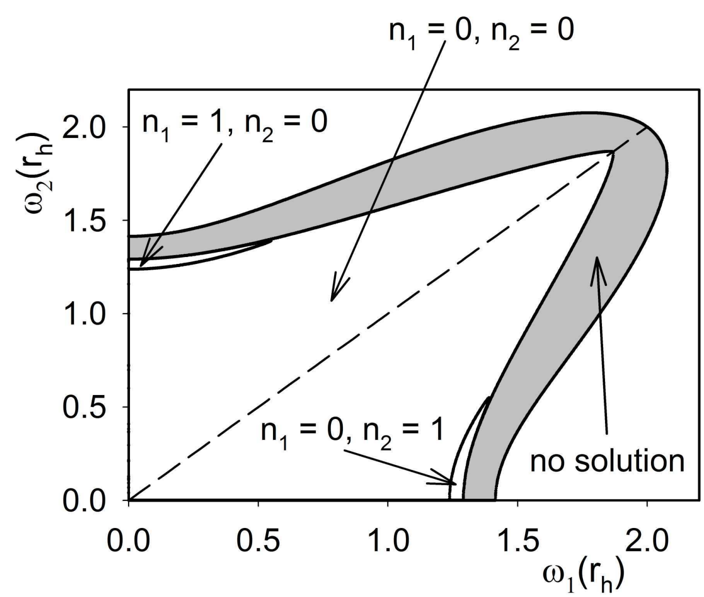

Figure 2. Solution space for $\mathfrak{su}(3)$ black holes with $r_h = 1$ and $\Lambda = -3$. The numbers of zeros of the gauge field functions, n_1 and n_2, are shown for the various regions of the solution space. The grey region denotes those areas of the $(\omega_1(r_h), \omega_2(r_h))$ plane for which the constraint (27) is satisfied, but for which we do not find solutions. Outside the shaded region the constraint (27) is not satisfied. The dashed line is $\omega_1(r_h) = \omega_2(r_h)$, along which lie the embedded $\mathfrak{su}(2)$ solutions.

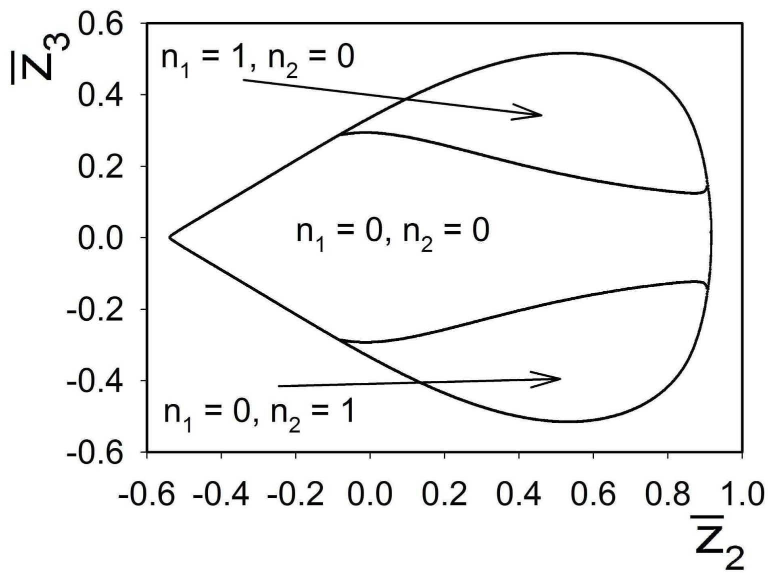

Figure 3. Solution space for $\mathfrak{su}(3)$ solitons with $\Lambda = -2$. The numbers of zeros of the gauge field functions, n_1 and n_2, are shown for the various regions of the solution space.

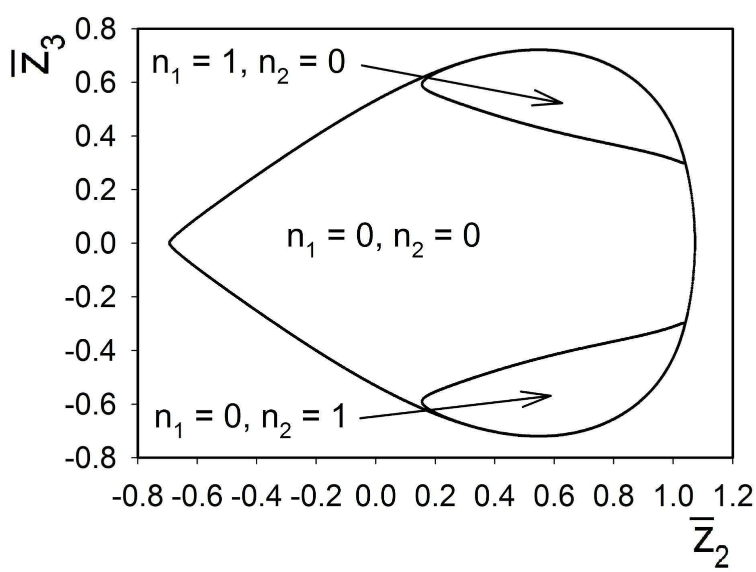

Figure 4. Solution space for $\mathfrak{su}(3)$ black holes with $\Lambda = -3$. The numbers of zeros of the gauge field functions, n_1 and n_2, are shown for the various regions of the solution space.

In each case, our method follows that employed in [17] for asymptotically flat $\mathfrak{su}(2)$ EYM, and subsequently generalized in asymptotically flat space to any compact gauge group in [21]. For $\mathfrak{su}(N)$ EYM in asymptotically flat space, the paper [22] used a different approach to prove local existence.

The method we employ here makes use of the following theorem, which is taken from [17], but which we restate for convenience:

Theorem 1 *[17] Consider a system of differential equations for* $n+m$ *functions* $\boldsymbol{u} = (u_1, u_2, \dots u_n)^T$ *and* $\boldsymbol{v} = (v_1, v_2, \dots v_m)^T$ *of the form*

$$
\begin{aligned}
x\frac{du_i}{dx} &= x^{\sigma_i} f_i\left(x, \boldsymbol{u}, \boldsymbol{v}\right); \\
x\frac{dv_i}{dx} &= -\tau_i v_i + x^{\varsigma_i} g_i\left(x, \boldsymbol{u}, \boldsymbol{v}\right);
\end{aligned}
\tag{41}
$$

with constants $\tau_i > 0$ *and integers* $\sigma_i, \varsigma_i \geq 1$*, and let* $\mathcal{C}$ *be an open subset of* $\mathbb{R}^n$ *such that the functions* f_i*,* $i = 1, \dots n$ *and* g_i*,* $i = 1, \dots m$ *are analytic in a neighbourhood of* $x = 0$*,* $\boldsymbol{u} = \boldsymbol{c}$*,* $\boldsymbol{v} = \boldsymbol{0}$ *for all* $\boldsymbol{c} \in \mathcal{C}$*. Then there exists an n-parameter family of solutions of the system (41) such that*

$$
u_i(x) = c_i + O\left(x^{\sigma_i}\right), \qquad v_i(x) = O\left(x^{\varsigma_i}\right),
\tag{42}
$$

where $u_i(x)$*,* $i = 1, \dots n$ *and* $v_i(x)$*,* $i = 1, \dots m$ *are defined for* $\boldsymbol{c} \in \mathcal{C}$*, for* $|x| < x_0(\boldsymbol{c})$ *(for some* $x_0(\boldsymbol{c}) > 0$*) and are analytic in* x *and* $\boldsymbol{c}$*.*

This theorem allows one to parameterize the family of solutions near a singular point of a set of ordinary differential equations. The key part of the method is putting the field equations into the form (41) with $x = 0$ at the singular point, which requires a change of variables. We need to consider the three singular points $r = 0$, $r = r_h$ and $r \to \infty$ separately. Once we have the field equations in the relevant form, it is then straightforward to prove local existence of solutions of the field equations with the required behaviour (22,25,28). The proof of the local existence of solutions near the origin is by far the most complicated of the three. The fact that the local solutions are analytic in the parameters c_i as well as the independent variable x will turn out to be extremely useful in Section 4.

3.1. Local existence of solutions near the origin

The local existence of solutions of the field equations near the origin is technically rather complicated. Local existence near the origin has been proved in the asymptotically flat case for $\mathfrak{su}(N)$ gauge fields in [22], using a method which is different from that employed here, and for general compact gauge groups in [21], whose method we follow. As might be expected, the inclusion of a negative cosmological constant Λ does not significantly change the analysis. However, the analysis is highly involved and in particular it is important to verify exactly how the cosmological constant appears in the derivation. For this reason, in this section we have included sufficient detail to make it clear where our analysis differs from that in [21, 22].

We begin by following [22] and define

$$\gamma_j = j\left(N-j\right), \qquad j=0,\ldots,N. \tag{43}$$

Then we define new variables $u_j(r)$ and $q_j(r)$ as follows [22]:

$$\begin{aligned}\omega_j(r) &= u_j(r)\gamma_j^{\frac{1}{2}}; \qquad j=1,\ldots,N-1;\\ q_j(r) &= \omega_j^2-\omega_{j-1}^2-N-1+2j; \qquad j=1,\ldots,N.\end{aligned} \tag{44}$$

Next define a new independent variable x by

$$x=\frac{r}{\lambda_N} \tag{45}$$

where λ_N is defined in (36). We then write the cosmological constant Λ and metric function $m(r)$ in terms of new quantities $\tilde{\Lambda}$ and $\tilde{m}$ as follows:

$$m(r)=\lambda_N\tilde{m}(x); \qquad \Lambda=\frac{\tilde{\Lambda}}{\lambda_N^2}; \tag{46}$$

and from now on consider $u_j(r)=u_j(\lambda_N x)$ and $S(r)=S(\lambda_N x)$ as functions of x instead of r. In terms of these variables the field equations become [22]

$$\frac{d\tilde{m}}{dx}=\frac{1}{\lambda_N^2}\left[\mu\tilde{G}+\tilde{P}\right]; \qquad \frac{dS}{dx}=\frac{2S\tilde{G}}{\lambda_N^2 x}; \tag{47}$$

$$0 = x^2\mu\frac{d^2u_j}{dx^2}+\left[2\tilde{m}-\frac{2\tilde{\Lambda}x^3}{3}-\frac{2x}{\lambda_N^2}\tilde{P}\right]\frac{du_j}{dx}+\frac{1}{2}\left[q_{j+1}-q_j\right]u_j, \tag{48}$$

where

$$\tilde{G}=\sum_{j=1}^{N-1}\gamma_j\left(\frac{du_j}{dx}\right)^2; \qquad \tilde{P}=\frac{1}{4x^2}\sum_{j=1}^{N}q_j^2. \tag{49}$$

The remainder of our analysis of the solutions of the field equations near the origin is split into two parts. Firstly, in Section 3.1.1, we derive the power series expansion (22) for the field variables near the origin. Secondly, in Section 3.1.2, we prove the local existence of solutions of the field equations having this form near the origin. In this section, we will not be using the Einstein summation convention, and all summations are written explicitly.

3.1.1. Derivation of the power series expansion In this section we seek a power series expansion for $\tilde{m}$ and u_j in a neighbourhood of the origin $x=0$, in the form

$$\tilde{m}(x)=\sum_{k=0}^{\infty}m_k x^k; \qquad u_j=\sum_{k=0}^{\infty}u_{j,k}x^k. \tag{50}$$

A straightforward analysis of the field equations shows that m_0, m_1, m_2 and $u_{j,1}$ are all zero and that $u_{j,0}=1$.

We seek to substitute these power series expansions into the scaled field equations (47,48). First we require the power series expansions for the quantities q_j, $\tilde{G}$ and $\tilde{P}$ in terms of the m_k and $u_{j,k}$. These do not depend on $\tilde{\Lambda}$ and so have the same form as in [22]:

$$q_j = \sum_{k=2}^{\infty} q_{j,k} x^k; \qquad \tilde{G} = \sum_{k=2}^{\infty} G_k x^k; \qquad \tilde{P} = \sum_{k=2}^{\infty} P_k x^k; \tag{51}$$

where

$$\begin{aligned} q_{j,k} &= \sum_{\ell=0}^{k} \gamma_j u_{j,\ell} u_{j,k-\ell} - \gamma_{j-1} u_{j-1,\ell} u_{j-1,k-\ell}; \\ G_k &= \sum_{j=1}^{N-1} \sum_{\ell=1}^{k-1} \gamma_j \left(\ell+1\right)\left(k-\ell+1\right) u_{j,\ell+1} u_{j,k-\ell+1}; \\ P_k &= \frac{1}{4} \sum_{j=1}^{N} \sum_{\ell=2}^{k} q_{j,\ell} q_{j,k-\ell+2}. \end{aligned} \tag{52}$$

It is also helpful to have a power series expansion for μ:

$$\mu = \sum_{k=0}^{\infty} \mu_k x^k, \tag{53}$$

where

$$\mu_0 = 1, \qquad \mu_1 = 0, \qquad \mu_2 = -2m_3 - \frac{\tilde{\Lambda}}{3}; \tag{54}$$

and, for $k \geq 3$,

$$\mu_k = -2m_{k+1}. \tag{55}$$

It is clear that there is a minor difference in this expansion compared with the asymptotically flat case, due to the dependence of μ_2 on $\tilde{\Lambda}$. It is this small difference of which we must carefully keep track if we are to verify that the analysis of [22] holds when $\tilde{\Lambda} \neq 0$.

Examining the coefficient of x^k in the first equation (47), we find, for $k \geq 2$:

$$m_{k+1} = \frac{1}{\lambda_N^2 \left(k+1\right)} \left[P_k + \sum_{\ell=0}^{k-2} \mu_\ell G_{k-\ell} \right]. \tag{56}$$

Here, P_k depends on $q_{j,2}, \ldots, q_{j,k}$ and hence on $u_{j,0}, \ldots, u_{j,k}$; the quantity μ_ℓ ($\ell = 0, \ldots, k-2$) depends on $m_3, \ldots, m_{k-1}$ and $\tilde{\Lambda}$, and G_ℓ ($\ell = 2, \ldots, k$) depends on $u_{j,2}, \ldots, u_{j,k}$. Therefore equation (56) determines m_{k+1} uniquely in terms of $m_3, \ldots, m_{k-1}, u_{j,0}, \ldots, u_{j,k}$ and $\tilde{\Lambda}$. Apart from the dependence on $\tilde{\Lambda}$, this is exactly the same situation as in [22].

Next we turn to the more complicated equation (48). After a great deal of algebra, we obtain, from the coefficient of x^k for $k \geq 2$, the following equation:

$$b_{j,k} = 2\gamma_j u_{j,k} - k\left(k-1\right) u_{j,k} + \gamma_{j+1} u_{j+1,k} - \gamma_{j-1} u_{j-1,k}\,; \tag{57}$$

for $j = 1, \ldots, N-1$, where $b_{j,k}$ is a complicated expression which we write as

$$b_{j,k} = c_{j,k} + \tilde{c}_{j,k}, \tag{58}$$

where

$$
\begin{aligned}
c_{j,k} = & -\sum_{\ell=2}^{k-2} u_{j,\ell} V_{k,\ell}; \\
\tilde{c}_{j,k} = & -\frac{1}{2}\sum_{\ell=2}^{k-2}\sum_{i=1}^{N-1} A_{i,j} u_{i,k} u_{i,k-\ell} - \sum_{\ell=2}^{k-2}\sum_{i=1}^{N-1} A_{i,j} u_{i,k} u_{j,k-\ell} \\
& - \sum_{\ell=2}^{k-2}\sum_{p=2}^{k-\ell-2}\sum_{i=1}^{N-1} A_{i,j} u_{j,p} u_{j,k-\ell-2-p};
\end{aligned}
\tag{59}
$$

and we have defined further quantities $V_{k,\ell}$ and $A_{i,j}$ as

$$
\begin{aligned}
V_{k,\ell} = & \frac{\tilde{\Lambda}}{3}\left(k-2\right)\left(k-1\right)\delta_{\ell,k-2} + \frac{2}{\lambda_N^2}\ell P_{k-\ell} + 2\ell\left(\ell-2\right) m_{k-\ell+1}; \\
A_{i,j} = & \gamma_j \left[2\delta_{i,j} - \delta_{i+1,j} - \delta_{i-1,j}\right];
\end{aligned}
\tag{60}
$$

and $\delta_{i,j}$ is the usual Kronecker δ. Using the $(N-1)\times(N-1)$ matrix A whose entries are $A_{i,j}$ we can write equation (57) as

$$
b_{j,k} = \sum_{i=1}^{N-1}\left[A_{i,j} - \delta_{i,j} k\left(k-1\right)\right] u_{i,k}. \tag{61}
$$

This is exactly the same as the equation for the corresponding quantities $b_{j,k}$ in the asymptotically flat case [22]. However, our $b_{j,k}$ is slightly different from that in [22], as $V_{j,k}$ contains $\tilde{\Lambda}$. We have examined the analysis in [22] very carefully, and found that $V_{j,k}$ arises only in Lemma 3 in [22], and, even in the proof of Lemma 3, the precise form of $V_{j,k}$ is not important. Therefore Theorem 1 in [22] holds also in our case, and we restate it here for convenience:

Theorem 2 *[22] The recurrence relations (56) and (61) determine uniquely all the coefficients* m_k *and* $u_{j,k}$ *for* $k > N$ *once* $N-1$ *arbitrary parameters have been chosen, one for each equation with* $k = 2, \ldots, N$.

We comment that in this section we have taken $k = 2, \ldots N$, whereas in [22], the parameter k has values $k = 1, \ldots N-1$, and the quantity $u_{j,k}$ multiplies x^{k+1} in [22] rather than x^k here. Some care is therefore needed in comparing results in [22] with those in this section.

The form of the coefficients $u_{j,k}$ for $j = 1, \ldots, N-1$, $k = 2, \ldots, N$ is derived in exactly the same way as in [22]. The $(N-1)\times(N-1)$ matrix A (60) has eigenvalues $k\left(k-1\right)$ for $k = 2, 3, \ldots, N$ (proved in [22]) with right-eigenvectors $\bar{\boldsymbol{v}}_k$ and left-eigenvectors $\bar{\boldsymbol{\sigma}}_k^T$, and we write the components of the vectors $\bar{\boldsymbol{v}}_k$ as $\bar{v}_{k,j}$ for $j = 1, \ldots, N-1$. The components $v_{k,j}$ are given in terms of Hahn polynomials [22, 27]:

$$
\bar{v}_{k,j} = \frac{N-1}{N-j}\,{}_3F_2\left(-k+1, -j+1, k; 2, -N+1; 1\right), \tag{62}
$$

where the ${}_3F_2$ are hypergeometric functions. We find [22]

$$
u_{j,2} = \bar{\beta}_2 \bar{v}_{2,j}, \qquad u_{j,3} = \bar{\beta}_3 \bar{v}_{3,j} \tag{63}
$$

and, for $k = 4, \ldots, N$,

$$u_{j,k} = u_{j,k}^* + \bar{\beta}_k \bar{v}_{k,j}. \tag{64}$$

Here the $\bar{\beta}_k$, for $k = 2, \ldots, N$, are arbitrary constant parameters, and $u_{j,k}^*$ is a special solution of (61) fixed by the requirement that

$$\bar{\boldsymbol{\sigma}}_k^T \boldsymbol{u}_{,k} = d_k \bar{\beta}_k \tag{65}$$

for $k = 2, \ldots, N$, where [22]

$$d_k = \bar{\boldsymbol{\sigma}}_k^T \bar{\boldsymbol{v}}_k = \frac{(N+k-1)!\,(N-k)!}{(N-1)!\,(N-2)!k\,(k-1)\,(2k-1)}, \tag{66}$$

and $\boldsymbol{u}_{,k}$ is the vector with components $u_{1,k}, \ldots, u_{N-1,k}$. The $u_{j,k}^*$ (64) are uniquely determined by Theorem 2, and the $N-1$ arbitrary constants $\bar{\beta}_k$ completely fix the coefficients $u_{j,k}$ (64).

It is convenient to expand the vector $\boldsymbol{u}_{,k}$ in terms of the eigenvectors of the matrix A, by expanding the particular solutions $u_{j,k}^*$ in terms of the $\bar{\boldsymbol{v}}_i$ and collecting terms:

$$\boldsymbol{u}_{,k} = \sum_{\ell=2}^{N} Y_k^\ell \bar{\boldsymbol{v}}_\ell, \qquad k = 2, \ldots, N. \tag{67}$$

The analysis of [22] shows that, independent of the form of $V_{j,k}$ (60),

$$Y_k^\ell = 0 \qquad \text{for } \ell > k, \tag{68}$$

so that $\boldsymbol{u}_{,k}$ depends only on $\bar{\boldsymbol{v}}_2, \ldots, \bar{\boldsymbol{v}}_k$. This result is not trivial because it depends on the properties of the particular solutions $u_{j,k}^*$. Furthermore, from [22], the particular solutions $u_{j,k}^*$ depend only on the components of the vectors $\bar{\boldsymbol{v}}_i$ for $i = 2, \ldots, k-1$, so that

$$Y_k^k = \bar{\beta}_k. \tag{69}$$

Therefore the first $N+1$ terms in the power series for the gauge field functions $\boldsymbol{u} = (u_1, \ldots, u_{N-1})^T$ can be written as

$$\boldsymbol{u} = \boldsymbol{u}_0 + \sum_{k=2}^{N} \sum_{\ell=2}^{k} Y_k^\ell \bar{\boldsymbol{v}}_\ell x^k = \boldsymbol{u}_0 + \sum_{k=2}^{N} \boldsymbol{\beta}_k(x) x^k; \tag{70}$$

where $\boldsymbol{u}_0 = (1, \ldots, 1)^T$; the Y_k^ℓ for $\ell < k$ are determined by $\bar{\beta}_i$ for $i = 2, \ldots, k-1$; and we have defined vector functions $\boldsymbol{\beta}_k(x)$, $k = 2, \ldots, N$, by

$$\boldsymbol{\beta}_k(x) = \sum_{\ell=k}^{N} Y_\ell^k x^{\ell-k} \bar{\boldsymbol{v}}_k = \bar{\boldsymbol{v}}_k \left[\bar{\beta}_k + Y_{k+1}^k x + \ldots + Y_{k+N}^k x^{N-k}\right]. \tag{71}$$

The vectors $\boldsymbol{\beta}_k(x)$ are proportional to $\bar{\boldsymbol{v}}_k$ for all values of x, and so are eigenvectors of the matrix A (60) for every x. The constants $\bar{\beta}_k$, $k = 2, \ldots, N$ in (71) are the same as those in (64), and are arbitrary, but the constants $Y_{k+1}^k, \ldots, Y_{k+N}^k$ are determined by the $\bar{\beta}_2, \ldots \bar{\beta}_N$ in a complicated way. Returning to the original variables, we find the power series expansion for the gauge field functions ω_j has the form (22), with the components of $\boldsymbol{\beta}_k(x)$ determining the functions $z_k(r)$ in (22) as follows:

$$\beta_{k,j}(x) = z_k(r)\bar{v}_{k,j}\lambda_N^k \gamma_j^{-\frac{1}{2}}, \tag{72}$$

where $\boldsymbol{\beta}_k(x) = (\beta_{k,1}, \ldots \beta_{k,N-1})^T$.

The last remaining piece in the power series expansion of the field variables is to find the expansion for S. First let $S = \exp\delta$, so that (47) becomes

$$\frac{d\delta}{dx} = \frac{2\tilde{G}}{\lambda_N^2 x}. \tag{73}$$

From this equation it is clear that δ has a power series expansion of the form

$$\delta = \delta_0 + \frac{2}{\lambda_N^2}\sum_{k=2}^{\infty}\frac{G_k}{k}x^k; \tag{74}$$

for some arbitrary constant δ_0. This then gives a power series expansion for S of the form (22).

3.1.2. Local existence To prove the local existence of solutions of the field equations (12,15) in a neighbourhood of the origin $r = 0$, with the power series expansions (22,70,71) derived in the previous sub-section, we follow the analysis of [21] in the asymptotically flat case, valid for any compact gauge group. As in the previous sub-section, the inclusion of a negative cosmological constant Λ does not significantly change the proof, but it is nonetheless important to carefully consider the details of each step.

Proposition 3 *There exists an $N+1$-parameter family of local solutions of the field equations (12,15) near $r = 0$, satisfying the boundary conditions (22), and analytic in Λ, $\bar{z}_k$, S_0 and r.*

Proof We begin with the field equations in the form (47,48). Defining a vector $\boldsymbol{u} = (u_1, \ldots, u_{N-1})^T$, we write the YM equation (48) as a vector equation:

$$x^2\mu\frac{d^2\boldsymbol{u}}{dx^2} + \left[2\tilde{m} - \frac{2\tilde{\Lambda}x^3}{3} - \frac{2x}{\lambda_N^2}\tilde{P}\right]\frac{d\boldsymbol{u}}{dx} + \frac{1}{2}\boldsymbol{\mathcal{W}} = 0, \tag{75}$$

where the vector $\boldsymbol{\mathcal{W}} = (\mathcal{W}_1, \ldots, \mathcal{W}_{N-1})$ has components

$$\mathcal{W}_j = (q_{j+1} - q_{j-1})\,u_j = 2u_j - \sum_{i=1}^{N-1} u_j A_{j,i} u_i^2 \tag{76}$$

and the $(N-1)\times(N-1)$ matrix A is defined in (60).

Our approach is to use the eigenvectors of the $(N-1)\times(N-1)$ matrix A as a basis for our $N-1$ gauge field functions, instead of using the functions $u_1, \ldots, u_{N-1}$. From the power series derivation in the previous subsection, we are motivated to write the vector $\boldsymbol{u}(x)$ as a sum over eigenvectors of the matrix A (cf. (70)):

$$\boldsymbol{u} = \boldsymbol{u}_0 + \sum_{k=2}^{N}\boldsymbol{\beta}_k(x)x^k; \tag{77}$$

where $\boldsymbol{u}_0 = (1, \ldots, 1)^T$ and the $\boldsymbol{\beta}_k(x)$ are vector functions of x which satisfy

$$A\boldsymbol{\beta}_k(x) = k\,(k-1)\,\boldsymbol{\beta}_k(x) \qquad \text{for all } x. \tag{78}$$

It should be emphasized that we are making no assumptions about the $\boldsymbol{\beta}_k(x)$ at this stage. In particular, we are not assuming that $\boldsymbol{\beta}_k(x)$ are regular at the origin, nor that they have the form (71). Since the $N-1$ vectors $\bar{\boldsymbol{v}}_k$ are eigenvectors of the matrix A corresponding to distinct eigenvalues, they constitute a basis of the space of $N-1$ vectors. If we write $\boldsymbol{\beta}_k(x) = \tilde{\zeta}_k(x)\bar{\boldsymbol{v}}_k$ for some scalar function $\tilde{\zeta}_k(x)$ (which will be a constant factor times the $\zeta_k(x)$ functions defined below), then we are using the $N-1$ independent functions $\tilde{\zeta}_k(x)$, $k = 2, \ldots, N$, instead of the $N-1$ independent functions $u_j(x)$, $j = 1, \ldots, N-1$, to describe our gauge field.

We now multiply both sides of (76) by the left-eigenvectors $\bar{\boldsymbol{\sigma}}_k^T$ of the matrix A:

$$x^2\mu\bar{\boldsymbol{\sigma}}_k^T\frac{d^2\boldsymbol{u}}{dx^2} + \left[2\tilde{m} - \frac{2\tilde{\Lambda}x^3}{3} - \frac{2x}{\lambda_N^2}\tilde{P}\right]\bar{\boldsymbol{\sigma}}_k^T\frac{d\boldsymbol{u}}{dx} + \frac{1}{2}\bar{\boldsymbol{\sigma}}_k^T\boldsymbol{\mathcal{W}} = 0. \tag{79}$$

Using the fact that the left- and right-eigenvectors of the matrix A are orthogonal [22] we have

$$\bar{\boldsymbol{\sigma}}_{k_1}^T\boldsymbol{\beta}_{k_2} = 0 \qquad \text{if } k_1 \neq k_2, \tag{80}$$

and defining new functions ζ_k by

$$\zeta_k(x) = \bar{\boldsymbol{\sigma}}_k^T\boldsymbol{\beta}_k, \tag{81}$$

we obtain

$$\begin{aligned} 0 = {} & x^2\mu\left[x^k\frac{d^2\zeta_k}{dx^2} + 2kx^{k-1}\frac{d\zeta_k}{dx} + k\,(k-1)\,x^{k-2}\zeta_k\right] \\ & + \left[2\tilde{m} - \frac{2\tilde{\Lambda}x^3}{3} - \frac{2x}{\lambda_N^2}\tilde{P}\right]\left[x^k\frac{d\zeta_k}{dx} + kx^{k-1}\zeta_k\right] + \frac{1}{2}\bar{\boldsymbol{\sigma}}_k^T\boldsymbol{\mathcal{W}}. \end{aligned} \tag{82}$$

The complicated part of the analysis lies in casting $\boldsymbol{\mathcal{W}}$ into a suitable form. However, the form of $\boldsymbol{\mathcal{W}}$ depends only on the structure of the $\mathfrak{su}(N)$ gauge field, without any reference to the form of the metric or the cosmological constant. Therefore we may appeal to the analysis of [21], which is valid for any compact semi-simple gauge group. The following general result was proved in [21] for any compact semi-simple gauge group and we have explicitly verified it for $\mathfrak{su}(N)$:

Lemma 4 *[21]*

$$\frac{1}{2}\boldsymbol{\mathcal{W}} = -\sum_{k=2}^{N} k\,(k-1)\,\boldsymbol{\beta}_k(x)x^k + \sum_{\ell=2}^{Z}\boldsymbol{\tau}_\ell x^\ell \tag{83}$$

for some $Z \in \mathbb{N}$ *and vectors* $\boldsymbol{\tau}_\ell(x)$ *which are regular at* $x = 0$.

The key part of this result is that the expansion of $\boldsymbol{\mathcal{W}}$ as a power series about $x = 0$ starts with terms of $O(x^2)$, i.e. there are no $\ell = 0$ or $\ell = 1$ terms in this expansion. The precise forms of the vectors $\boldsymbol{\tau}_\ell$ are not particularly illuminating so we do not give them here.

Using Lemma 4 we have, for $k = 2, \ldots, N$:

$$\frac{1}{2}\bar{\boldsymbol{\sigma}}_k^T \boldsymbol{\mathcal{W}} = -k\left(k-1\right)x^k \zeta_k + \sum_{\ell=2}^{Z} \bar{\boldsymbol{\sigma}}_k^T \boldsymbol{\tau}_\ell x^\ell. \tag{84}$$

To progress further, we need the following result on $\bar{\boldsymbol{\sigma}}_k^T \boldsymbol{\tau}_\ell$, which is proved for general gauge group in [21], and which we have verified directly for gauge group $\mathfrak{su}(N)$:

Lemma 5 *[21]*

$$\bar{\boldsymbol{\sigma}}_k^T \boldsymbol{\tau}_\ell = 0 \qquad \textit{if } \ell < k+1. \tag{85}$$

Using Lemma 5, equation (84) therefore simplifies to

$$\frac{1}{2}\bar{\boldsymbol{\sigma}}_k^T \boldsymbol{\mathcal{W}} = -k\left(k-1\right)x^k \zeta_{k+1} + \sum_{\ell=k+1}^{Z} \bar{\boldsymbol{\sigma}}_k^T \boldsymbol{\tau}_\ell x^\ell. \tag{86}$$

The importance of this result will become apparent on returning to (82). Defining a new variable $\chi_k(x)$ by $\chi_k(x) = \frac{d\zeta_k(x)}{dx}$, dividing throughout by $x^{k+1}\mu$ and rearranging, we obtain

$$\begin{aligned} x\frac{d\chi_k}{dx} = & -2k\chi_k - k\left(k-1\right)x^{-1}\zeta_k + k\left(k-1\right)x^{-1}\mu^{-1}\zeta_k \\ & -\frac{1}{x^2\mu}\left[2\tilde{m} - \frac{2\tilde{\Lambda}x^3}{3} - \frac{2x}{\lambda_N^2}\tilde{P}\right]\left[k\zeta_k + x\chi_k\right] \\ & -\sum_{\ell=0}^{Z-k-1} \bar{\boldsymbol{\sigma}}_k^T \boldsymbol{\tau}_{\ell+k+1} x^\ell. \end{aligned} \tag{87}$$

Next we turn to the forms of the functions $\tilde{G}$ and $\tilde{P}$ (49). We write $\boldsymbol{\beta}_2$ as

$$\boldsymbol{\beta}_2(x) = \left[\hat{\beta}_0 + x\hat{\beta}_1(x)\right]\bar{\boldsymbol{v}}_2 \tag{88}$$

for some constant $\hat{\beta}_0$ and scalar function $\hat{\beta}_1(x)$, where $\bar{\boldsymbol{v}}_2$ is a right-eigenvector of the matrix A, with eigenvalue 2. Once again, we make no assumption about the properties of $\hat{\beta}_1$. We can write $\hat{\beta}_1$ as an analytic function of ζ_2 using (81), but this precise expression is not important here. Using (88) and the properties of the right-eigenvector $\bar{\boldsymbol{v}}_2$, we are able to write $\tilde{G}$ and $\tilde{P}$ as follows:

$$\tilde{G} = \frac{2}{3}N\left(N^2-1\right)x^2\hat{\beta}_0^2 + x^3\hat{G}; \qquad \tilde{P} = \frac{1}{3}N\left(N^2-1\right)x^2\hat{\beta}_0^2 + x^3\hat{P}; \tag{89}$$

where $\hat{G}$ is an analytic function of x, ζ_k and χ_k, and $\hat{P}$ is an analytic function of x and the ζ_k. The differential equation satisfied by $\tilde{m}$ (47) then simplifies to

$$\frac{d\tilde{m}}{dx} = 2x^2\hat{\beta}_0^2\left[2\mu+1\right] + \frac{x^3}{\lambda_N^2}\left[\mu\hat{G} + \hat{P}\right]. \tag{90}$$

We define yet another new variable α by

$$\alpha = \frac{1}{x^3}\left[\tilde{m} - m_3x^3\right], \tag{91}$$

where $m_3 = 2\hat{\beta}_0^2$, and, after some algebra, we obtain the differential equation satisfied by α in the form

$$x\frac{d\alpha}{dx} = -3\alpha + x\mathcal{F}_\alpha, \tag{92}$$

where $\mathcal{F}_\alpha$ is an analytic function in α, x, ζ_k, χ_k, and $\tilde{\Lambda}$.

Returning now to the YM equation (87), we next write

$$\mu^{-1} = 1 + x^2\tilde{\mu}. \tag{93}$$

Again, we make no assumptions about $\tilde{\mu}$: it can be written as a function of the variables x and α, and the constants m_3 and $\tilde{\Lambda}$. When we apply Theorem 1 at the end of this section, we will obtain as a corollary that $\tilde{\mu}$ is a regular function of x, at least in a neighbourhood of $x = 0$. The YM equation (87) then takes the form

$$x\frac{d\chi_k}{dx} = -2k\chi_k + x\tilde{\mathcal{F}}_{\chi_k} + \sum_{\ell=0}^{Z-k-1} \bar{\boldsymbol{\sigma}}_k^T \boldsymbol{\tau}_{\ell+k+1} x^\ell, \tag{94}$$

where $\tilde{\mathcal{F}}_{\chi_k}$ is an analytic function of the variables. We make one last change of variables, defining [21]

$$\tilde{\chi}_k = \chi_k + \frac{1}{2k}\bar{\boldsymbol{\sigma}}_k^T \boldsymbol{\tau}_{k+1}. \tag{95}$$

The precise form of $\bar{\boldsymbol{\sigma}}_k^T \boldsymbol{\tau}_{k+1}$ is not important, suffice to say it is a polynomial in the ζ_ℓ and does not involve any powers of x.

Then, summarizing, the field equations can be written as:

$$\begin{aligned}
x\frac{d\alpha}{dx} &= -3\alpha + x\mathcal{F}_\alpha; \\
x\frac{d\zeta_k}{dx} &= x\mathcal{F}_{\zeta_k}; \\
x\frac{d\tilde{\chi}_k}{dx} &= -2k\tilde{\chi}_k + x\mathcal{F}_{\tilde{\chi}_k}; \\
x\frac{dS}{dx} &= x\mathcal{F}_S; \\
x\frac{d\tilde{\Lambda}}{dx} &= 0;
\end{aligned} \tag{96}$$

where all the $\mathcal{F}$ functions are analytic in x, $\tilde{\Lambda}$, α, $\tilde{\chi}_k$ and ζ_k, in a neighbourhood of $\alpha = 0$, $x = 0$. Applying Theorem 1 we then have, in a neighbourhood of the origin $x = 0$, solutions of the form

$$\begin{aligned}
&\alpha(x) = O(x); \qquad \zeta_k(x) = \zeta_{0,k} + O(x); \qquad \tilde{\chi}_k = O(x); \\
&S(x) = S_0 + O(x),
\end{aligned} \tag{97}$$

where $\zeta_{0,k}$ and S_0 are constants, and α, ζ_k, $\tilde{\chi}_k$ and S are analytic in x, $\tilde{\Lambda}$, S_0 and $\zeta_{0,k}$. Returning to the original variables, we have proven the existence of solutions of the form (22) in a neighbourhood of the origin, which are analytic in r and the parameters $\bar{z}_k$ and S_0.

3.2. Local existence of solutions near the event horizon

As might be anticipated, the inclusion of the cosmological constant Λ does not change significantly the proof of the local existence of solutions near the event horizon $r = r_h$. Our choice of variables follows those in [17, 20, 21]. For $\mathfrak{su}(2)$ EYM black holes in adS, the following result is stated in [9].

Proposition 6 *There exists an $N+2$-parameter family of local black hole solutions of the field equations (12,15) near $r = r_h > 0$, satisfying the boundary conditions (25), and analytic in r_h, Λ, $\omega_j(r_h)$ and r.*

Proof We define a new independent variable x by $x = r - r_h$, and new dependent variables as follows [17, 20, 21]:

$$\rho = r, \qquad \lambda = \frac{\mu}{x}, \qquad \psi_j = \omega_j, \qquad \xi_j = \frac{\mu\omega_j'}{x} = \lambda\omega_j'. \tag{98}$$

The field equations (12,15) then take the form

$$\begin{aligned}
x\frac{d\rho}{dx} &= x; \\
x\frac{d\lambda}{dx} &= -\lambda + xH_\lambda + F_\lambda; \\
x\frac{d\psi_j}{dx} &= \frac{x\xi_j}{\lambda}; \\
x\frac{d\xi_j}{dx} &= -\xi_j + xH_{\xi_j} + F_{\xi_j}; \\
x\frac{dS}{dx} &= \frac{2x}{\rho}GS; \\
x\frac{d\Lambda}{dx} &= 0,
\end{aligned} \tag{99}$$

where

$$\begin{aligned}
F_\lambda &= \frac{1}{\rho} - \Lambda\rho - \frac{2}{\rho^3}\mathcal{P}; \\
H_\lambda &= -\frac{\lambda}{\rho}\left(1+2G\right); \\
F_{\xi_j} &= -\frac{1}{\rho^2}W_j\psi_j; \\
H_{\xi_j} &= -\frac{2G}{\rho}\xi_j;
\end{aligned} \tag{100}$$

we have defined a quantity $\mathcal{P}$ which is a polynomial in the ψ_j:

$$\mathcal{P} = r^4 p_\theta = \frac{1}{4}\sum_{j=1}^{N}\left[\left(\psi_j^2 - \psi_{j-1}^2 - N - 1 + 2j\right)^2\right] \tag{101}$$

and p_θ, W_j and G are given in (13,14,16) respectively. In particular, G takes the form

$$G = \frac{1}{\lambda^2}\sum_{j=1}^{N-1}\xi_j^2. \tag{102}$$

The functions F_λ, F_{ξ_j}, H_λ and H_{ξ_j} are therefore polynomials in $1/\rho$, $1/\lambda$, ρ, λ, ψ_j, ξ_j and Λ.

The equations (99) are not yet in the required form (41). We require a further change of variables:

$$\tilde{\lambda} = \lambda - F_\lambda; \qquad \tilde{\xi}_j = \xi_j - F_{\xi_j}. \tag{103}$$

The differential equations for these new variables are:

$$\begin{aligned} x\frac{d\tilde{\lambda}}{dx} &= -\tilde{\lambda} + xG_\lambda; \\ x\frac{d\tilde{\xi}_j}{dx} &= -\tilde{\xi}_j + xG_{\xi_j}, \end{aligned} \tag{104}$$

where

$$G_\lambda = H_\lambda - \frac{dF_\lambda}{dx}; \qquad G_{\xi_j} = H_{\xi_j} - \frac{dF_{\xi_j}}{dx}. \tag{105}$$

We do not write out here the full forms of G_λ and G_{ξ_j}, which are rather lengthy. However, from the forms of F_λ and F_{ξ_j} (100), using the differential equations (99), it is clear that both G_λ and G_{ξ_j} are polynomials in $1/\rho$, $1/\lambda$, ρ, λ, $\tilde{\lambda}$, $\tilde{\xi}_j$ and Λ.

The equations (104) are now in the form (41) required for the application of Theorem 1. We have, in a neighbourhood of $x = 0$, solutions of the form

$$\begin{aligned} &\rho = r_h + O(x); \qquad \tilde{\lambda} = O(x); \qquad \psi_j = \omega_j(r_h) + O(x); \\ &\tilde{\xi}_j = O(x), \qquad S = S(r_h) + O(x), \end{aligned} \tag{106}$$

with ρ, $\tilde{\lambda}$, ψ_j and $\tilde{\xi}_j$ all analytic in x, r_h, $\omega_j(r_h)$, Λ and $S(r_h)$. Transforming back to our original variables, we have proven the existence of solutions of the field equations in a neighbourhood of a black hole event horizon $r = r_h$, satisfying the boundary conditions (25), and analytic in r, r_h, $\omega_j(r_h)$, Λ and $S(r_h)$.

3.3. Local existence of solutions near infinity

For asymptotically flat $\mathfrak{su}(N)$ solitons and black holes, the fact that the values of the gauge field functions as $r \to \infty$ are fixed by (29) makes the proof of local existence of solutions in a neighbourhood of infinity rather complicated. In fact, as shown in [21, 22], it is necessary to expand the gauge field functions $\omega_j(r)$ to order r^{-N+1} in a manner similar to the analysis in Section 3.1 near $r = 0$. In the asymptotically adS case, the fact that the values of the gauge field functions at infinity are not fixed *a priori* makes the proof of the local existence of solutions much easier. The result below is stated in the $\mathfrak{su}(2)$ case in [9].

Proposition 7 *There exists an* $2N$*-parameter family of local solutions of the field equations (12,15) near* $r = \infty$*, satisfying the boundary conditions (28), and analytic in* Λ*,* $\omega_{j,\infty}$*,* M *and* r^{-1}*.*

Proof In this case our new independent variable is $x = r^{-1}$, and we define new dependent variables as follows, following [17, 21]:

$$\lambda = 2m = r\left(1 - \mu - \frac{\Lambda r^2}{3}\right); \qquad \psi_j = \omega_j; \qquad \xi_j = r^2 \omega_j'. \tag{107}$$

Then the field equations (12,15) take the form

$$\begin{aligned}
x\frac{d\lambda}{dx} &= x f_\lambda; \\
x\frac{d\psi_j}{dx} &= -x\xi_j; \\
x\frac{d\xi_j}{dx} &= x f_{\xi_j}; \\
x\frac{dS}{dx} &= x^4 f_S; \\
x\frac{d\Lambda}{dx} &= 0.
\end{aligned} \tag{108}$$

The functions on the right-hand-side of (108) are given by

$$\begin{aligned}
f_\lambda &= -2\left(x^2 - x^3\lambda - \frac{\Lambda}{3}\right)\sum_{j=1}^{N-1}\xi_j^2 + \mathcal{P}; \\
f_{\xi_j} &= -\frac{3}{\Lambda}\left[1 - \frac{3x^2}{\Lambda} + \frac{3\lambda x^3}{\Lambda}\right]^{-1}\left[W_j\omega_j - 2x\xi_j + 3\lambda x^2\xi_j - 2x^3\xi_j\mathcal{P}\right]; \\
f_S &= -2S\sum_{j=1}^{N-1}\xi_j^2;
\end{aligned} \tag{109}$$

where $\mathcal{P} = x^{-4}p_\theta$ is a polynomial in the ψ_j, defined in equation (101). Immediately we have that the f's are analytic in a neighbourhood of $x = 0$, so applying Theorem 1 gives, in a neighbourhood of $x = 0$, solutions of the form

$$\begin{aligned}
&\lambda = 2M + O(x); \qquad \psi_j = \omega_{j,\infty} + O(x); \qquad \xi_j = c_j + O(x); \\
&S = S_\infty + O(x^4);
\end{aligned} \tag{110}$$

for constants M, $\omega_{j,\infty}$, c_j and S_∞, and the solutions are analytic in x, Λ, and the constants. If we fix $S_\infty = 1$ for the space-time to be asymptotically anti-de Sitter, then the boundary conditions (28) are satisfied, and the field variables are analytic in r^{-1}, Λ, M, $\omega_{j,\infty}$ and c_j. We note that we find additional free parameters in this case, corresponding to the coefficients of terms of $O(r^{-1})$ in the expansion of ω_j near infinity. These parameters will not be important in our subsequent analysis.

4. Existence of soliton and hairy black hole solutions

In this section we prove two existence theorems: firstly, the existence of $\mathfrak{su}(N)$ solutions in a neighbourhood of any embedded $\mathfrak{su}(2)$ solution; and secondly the existence of $\mathfrak{su}(N)$ solutions for which all the gauge field functions ω_j have no zeros provided $|\Lambda|$ is sufficiently large.

4.1. Regularity for $\mu > 0$

Our strategy in proving the existence of genuinely $\mathfrak{su}(N)$ soliton and black hole solutions of the field equations is to start with a local solution near the origin (in the soliton case) or black hole event horizon and then extend this solution out to infinity.

The focus of this section is to show that solutions can be extended for larger values of r provided that $\mu > 0$. This lemma was proved in [17] for $\mathfrak{su}(2)$ EYM with $\Lambda = 0$, and stated for $\Lambda < 0$ in [9]. For $\mathfrak{su}(N)$ EYM with $\Lambda = 0$, the corresponding lemma is stated in [20]. Our method of proof is exactly the same as in [17], so we only sketch the details.

Lemma 8 *As long as* $\mu > 0$ *all field variables are regular functions of* r*.*

Proof Consider an interval $\mathcal{I} = [r_1, r_2)$, where $r_1 < r_2$ and $r_1 > 0$ in the soliton case, or $r_1 > r_h$ in the black hole case. Assume that all field variables are regular on the interval $\mathcal{I}$ and that $\mu(r) > 0$ for all $r \in \bar{\mathcal{I}} = [r_1, r_2]$. We need to show that all other field variables (i.e. $\omega_j(r)$, $\omega_j'(r)$ and $S(r)$) remain regular at r_2.

From the fact that $\mu(r_2) > 0$ we obtain

$$r_2 - \frac{\Lambda r_2^3}{3} > 2m(r_2). \tag{111}$$

From (15), since $r^2 p_\theta$ is positive, we have

$$2m(r_2) \geq 2 \int_{r_1}^{r_2} \mu(r) \sum_{j=1}^{N-1} \omega_j'^2 \, dr. \tag{112}$$

Now $\mu(r)$ must have a minimum in the closed interval $\bar{\mathcal{I}}$, so defining

$$\mu_{\min} = \min \left\{ \mu(r) : r \in \bar{\mathcal{I}} \right\} > 0; \tag{113}$$

we find

$$\int_{r_1}^{r_2} \sum_{j=1}^{N-1} \omega_j'^2 \, dr < \frac{1}{2\mu_{\min}} \left(r_2 - \frac{\Lambda r_2^3}{3} \right). \tag{114}$$

Direct integration of the second Einstein equation (15) then gives that $\log S$ and therefore S is finite at r_2.

Using the Cauchy-Schwarz inequality, we have

$$\sum_{j=1}^{N-1} \left[\omega_j(r_1) - \omega_j(r_2) \right]^2 = \sum_{j=1}^{N-1} \left[\int_{r_1}^{r_2} \omega_j' \, dr \right]^2 \leq (r_2 - r_1) \int_{r_1}^{r_2} \sum_{j=1}^{N-1} \omega_j'^2 \, dr. \tag{115}$$

Since the left-hand-side of the above equation is a sum of positive terms, and the right-hand-side is bounded above, we deduce that each $\omega_j(r_2)$ is finite.

Finally, we need to show that each $\omega_j'(r_2)$ is finite. To do this, we write the Yang-Mills equation (12) in the form

$$\left(\mu\omega_j'\right)' + \frac{S'}{S}\mu\omega_j' = -\frac{W_j\omega_j}{r^2}. \tag{116}$$

Integrating this first order differential equation for $\mu\omega_j'$ gives

$$\omega_j'(r_2) = \frac{1}{S(r_2)\mu(r_2)}\left[S(r_1)\mu(r_1)\omega_j'(r_1) - \int_{r_1}^{r_2}\frac{SW_j\omega_j}{r^2}\,dr\right], \tag{117}$$

so that $\omega_j'(r_2)$ is finite and we have shown that all field variables are finite on the closed interval $\bar{\mathcal{I}}$.

4.2. Asymptotic behaviour as $r \to \infty$

One of the most important reasons for the abundance of solutions of $\mathfrak{su}(2)$ EYM with $\Lambda < 0$ compared with the asymptotically flat case is the difference in the behaviour of the Yang-Mills equations as $r \to \infty$. Therefore we next turn to the asymptotic behaviour of the $\mathfrak{su}(N)$ Yang-Mills equations (12).

As $r \to \infty$, these take the form

$$-\frac{\Lambda r^4}{3}\omega_j'' - \frac{2\Lambda r^3}{3}\omega_j' + W_j\omega_j = 0. \tag{118}$$

As in the $\mathfrak{su}(2)$ case [9], these can be made autonomous by the change of variable

$$\tau = \frac{1}{r}\sqrt{-\frac{\Lambda}{3}}, \tag{119}$$

the Yang-Mills equations becoming

$$\frac{d^2\omega_j}{d\tau^2} + W_j\omega_j = 0. \tag{120}$$

The autonomous equations (120) have critical points when

$$W_j\omega_j = \left(1 - \omega_j^2 + \frac{1}{2}\omega_{j+1}^2 + \frac{1}{2}\omega_{j-1}^2\right)\omega_j = 0. \tag{121}$$

If all the ω_j are non-zero at a critical point, their values are:

$$\omega_j = \pm\sqrt{j\,(N-j)}. \tag{122}$$

Linearization about the critical points reveals that the critical point at the origin is a centre, and all other critical points are saddles.

We are in exactly the same situation as in the $\mathfrak{su}(2)$ case: what is important is not so much the different critical point structure of the equation (120) compared with the asymptotically flat case [20], but rather the nature of the variable τ (119). In particular, when $\Lambda < 0$ the variable τ tends to zero as $r \to \infty$, so that if we consider the solutions of the equations (120) starting at some large value of $r = r_1$ and going all the way out to infinity, the corresponding values of τ will lie in some small interval $\tau \in [0, \tau_1]$. On the other hand, for asymptotically flat solutions, the variable $\tilde{\tau}$ which makes the Yang-Mills equations in flat space autonomous is $\tilde{\tau} = \log r$, which will tend to infinity as $r \to \infty$. Therefore, in the asymptotically flat case, any solution of the Yang-Mills equations must trace out the whole length of a phase path, whereas in the asymptotically adS case the solutions will only travel a small, finite distance along any one phase path.

Including a small gravitational perturbation into the Yang-Mills equations (118) will not change this key property of the variable τ nor the finiteness of the phase paths taken by the field variables as $r \to \infty$.

4.3. Existence of $\mathfrak{su}(N)$ *solutions in a neighbourhood of embedded* $\mathfrak{su}(2)$ *solutions*

We now have available all the ingredients we require to show the existence of genuinely $\mathfrak{su}(N)$ EYM solitons and black holes. In this subsection we prove our first existence theorem, namely the existence of $\mathfrak{su}(N)$ solutions in a neighbourhood of an embedded $\mathfrak{su}(2)$ solution. This result relies heavily on the analyticity of the local solutions, proved in Section 3.

Proposition 9 *Suppose there is an embedded* $\mathfrak{su}(2)$ *solution of the field equations, with the gauge field functions all having* k *zeros. Then, all initial parameters in a sufficiently small neighbourhood of the initial parameters giving the embedded* $\mathfrak{su}(2)$ *solution will give an* $\mathfrak{su}(N)$ *solution of the field equations in which all gauge field functions have* k *zeros.*

Proof Suppose we have an embedded, non-trivial, $\mathfrak{su}(2)$ solution of the field equations. If this is a soliton, the initial conditions at the origin will be given by (22) with $\bar{z}_2 \neq 0$ and $\bar{z}_3 = \bar{z}_4 = \ldots = \bar{z}_N = 0$. If this is a black hole, the initial conditions at the event horizon will be $\omega_1(r_h) = \omega_2(r_h) = \ldots = \omega_{N-1}(r_h) \neq 0$. From these initial conditions, we can integrate the field equations (12,15) all the way out to infinity to give a regular solution. Furthermore, the solution will satisfy the boundary conditions (28) at infinity. For the rest of this section, we will assume that the magnitude of the cosmological constant Λ is fixed, and, if we are considering black hole solutions, that the radius of the event horizon, r_h, is also fixed. We also suppose that the gauge field functions each have k zeros (they will all have the same number of zeros from (35), since this is an embedded $\mathfrak{su}(2)$ solution).

From the local existence theorems (Propositions 3 and 6 proved in Sections 3.1 and 3.2 respectively) we know that there are solutions of the field equations locally near the origin or event horizon as applicable, for any values of the initial parameters $\bar{z}_2, \ldots \bar{z}_N$ or $\omega_1(r_h), \ldots \omega_{N-1}(r_h)$, and that these solutions are analytic in these initial parameters. For the embedded $\mathfrak{su}(2)$ solution, it must be the case that $\mu(r) > 0$ for all $r \in [r_0, \infty)$ (where $r_0 = 0$ for solitons and $r_0 = r_h$ for black holes). Therefore, by analyticity, for initial parameters close to the initial parameters giving the embedded $\mathfrak{su}(2)$ solution, the corresponding local solutions will have $\mu(r) > 0$ for all $r \in [r_0, r_2]$ for some r_2. By Lemma 8, these local solutions will also be regular on the interval $[r_0, r_2]$.

Now fix some $r_1 \gg \max\{1, r_0\}$, such that $|\Lambda| r_1^2 \gg 1$ and such that, for the embedded $\mathfrak{su}(2)$ solution, we have $m(r_1)/r_1 \ll 1$. By the analyticity argument above, provided the initial parameters $\bar{z}_2, \ldots, \bar{z}_N$ or $\omega_1(r_h), \ldots \omega_{N-1}(r_h)$ (as applicable) are in a sufficiently small neighbourhood of the initial parameters giving the embedded $\mathfrak{su}(2)$ solution, the local $\mathfrak{su}(N)$ solutions whose existence is guaranteed by Propositions 3 and 6 will be regular on the whole interval $r \in [r_0, r_1]$. That is, in this interval $\mu > 0$ for all r and the gauge field functions ω_j will all have k zeros provided r_1 is larger than the largest zero of the gauge field function ω of the embedded $\mathfrak{su}(2)$ solution. Furthermore, at r_1, it will still be the case that $m(r_1)/r_1 \ll 1$ for these $\mathfrak{su}(N)$ solutions as well as the nearby embedded $\mathfrak{su}(2)$ solution.

Since $r_1 \gg \max\{1, r_0\}$ and $m(r_1)/r_1 \ll 1$ for these $\mathfrak{su}(N)$ solutions, we can use the asymptotically adS regime discussed in Section 4.2. Provided r_1 is sufficiently large, the solutions will not move very far along their phase path as r increases from r_1 to $r \to \infty$. In particular, the gauge field functions will have no further zeros, and both the gauge field functions and their derivatives will not vary very much from their values at $r = r_1$. This means that $m(r)/r$ will continue to be very small as the solutions move along their phase path, and the asymptotic adS regime will continue to be valid.

Therefore, we have shown the existence of genuinely $\mathfrak{su}(N)$ soliton and black hole solutions in a neighbourhood of any embedded $\mathfrak{su}(2)$ solution. The gauge field functions of these $\mathfrak{su}(N)$ solutions will have the same number of zeros as the gauge field functions of the embedded $\mathfrak{su}(2)$ solution. It should be emphasized that these are genuinely $\mathfrak{su}(N)$ solutions rather than embedded $\mathfrak{su}(2)$ solutions because the gauge field functions $\omega_j(r)$ will not satisfy (35).

This result can be applied to the embedded Schwarzschild-adS or pure adS solutions (with initial parameters, $\omega_j(r_h) = \sqrt{j\,(N-j)}$ and $\bar{z}_j = 0$ respectively) to give, for *any* negative value of the cosmological constant Λ, genuinely $\mathfrak{su}(N)$ solutions for which all the gauge field functions have no zeros.

Corollary 10 *For any* $\Lambda < 0$*, there exist* $\mathfrak{su}(N)$ *black holes and solitons for which all the gauge field functions have no zeros.*

In practice, the size of this neighbourhood about the embedded pure adS or Schwarzschild-adS solutions in which we have $\mathfrak{su}(N)$ soliton and black hole solutions where the gauge field functions have no zeros is negligibly small unless

$$|\Lambda| \gtrsim 10^{-1}\lambda_N^2 = \frac{10^{-1}}{6} N(N-1)(N+1). \tag{123}$$

For the $\mathfrak{su}(2)$ case, this can be seen in Figures 1 and 4 in [15]. The factor of λ_N^2 arises from the embedding of the $\mathfrak{su}(2)$ solutions in $\mathfrak{su}(N)$ (see Section 2.3), and means that the size of the neighbourhood decreases as N increases. This is illustrated in Figures 9 and 13 in [15], where the size of the region of parameter space where we have nodeless black hole solutions with $r_h = 1$ and $\Lambda = -1$ can be compared for $\mathfrak{su}(3)$ and $\mathfrak{su}(4)$ EYM, the latter region being significantly smaller than the former.

4.4. Existence of solutions for sufficiently large $|\Lambda|$

In the previous section we have proven the existence of black hole and soliton solutions of the $\mathfrak{su}(N)$ EYM field equations, for any negative value of the cosmological constant Λ, such that all the gauge field functions have no zeros. However, the size of the region of parameter space where we have these solutions shrinks as N increases, for fixed $|\Lambda|$. In this section we shall consider the behaviour of the solutions for large $|\Lambda|$. We shall prove that, given fixed initial parameters at the origin or black hole event horizon, as applicable, for sufficiently large $|\Lambda|$, the corresponding solutions of the field equations are such that all the gauge field functions have no zeros. These solutions are of particular interest because we expect [14] that at least some of them will be linearly stable.

Numerically (see Section 2.4 and [15]), we find that for $|\Lambda|$ sufficiently large, *all* soliton and black hole solutions of the field equations are such that all the gauge field functions have no zeros. This is not what we are able to prove in this section. Here we can show that, with fixed initial parameters, there are nodeless solutions for sufficiently large $|\Lambda|$, and it may well be that the magnitude of $|\Lambda|$ required for the existence of nodeless solutions varies depending on the values of the initial parameters. We observe this numerically, for example, comparing Figures 1 and 2 in the $\mathfrak{su}(3)$ case, we see that for $\omega_1(r_h) = 0.25$, $\omega_2(r_h) = 1$, the corresponding black hole solution is not nodeless when $\Lambda = -2$ but is nodeless when $\Lambda = -3$, whereas for $\omega_1(r_h) = 0.75$, $\omega_2(r_h) = 1$, the black hole is nodeless for both these values of Λ.

We need to consider black hole and soliton solutions separately.

4.4.1. Black holes In [9], for $\mathfrak{su}(2)$ EYM black holes, it is shown that for any fixed value of the event horizon radius r_h and any fixed value of the gauge field function on the horizon, $\omega(r_h)$, then for all Λ such that $|\Lambda|$ is sufficiently large, there is a black hole solution of the field equations such that the gauge field function $\omega(r)$ has no zeros.

The extension of this result to $\mathfrak{su}(N)$ EYM black holes, as might be expected, is not difficult so we just briefly sketch the derivation of the following result.

Proposition 11 *For fixed r_h and any fixed values of the gauge field functions at the event horizon, $\omega_j(r_h)$, for $|\Lambda|$ sufficiently large, there exists a black hole solution of the $\mathfrak{su}(N)$ EYM field equations such that all the gauge field functions $\omega_j(r)$ have no zeros.*

Proof Firstly, we note that for fixed r_h and $\omega_j(r_h)$, the constraint (27) for a regular event horizon is satisfied for all sufficiently large $|\Lambda|$. As in [9], it is helpful to define a length scale ℓ by $\ell^2 = -3/\Lambda$, and then new variables $\hat{m}$ and $\hat{\mu}$, which will be finite as $|\Lambda| \to \infty$, $\ell \to 0$ as follows:

$$\hat{m} = m\ell^2, \qquad \hat{\mu} = \mu\ell^2 = \ell^2 - \frac{2\hat{m}}{r} + r^2. \tag{124}$$

The field equations (12,15) then take the form

$$\hat{m}' = \left(\ell^2 - \frac{2\hat{m}}{r} + r^2\right) G + \ell^2 r^2 p_\theta; \qquad \frac{S'}{S} = \frac{2G}{r};$$

$$0 = r^2 \left(\ell^2 - \frac{2\hat{m}}{r} + r^2 \right) \omega_j'' + \left[2\hat{m} - 2\ell^2 r^3 p_\theta + 2r^3 \right] \omega_j' + \ell^2 W_j \omega_j. \quad (125)$$

In the limit $\ell \to 0$, these equations simplify considerably and have the unique solution

$$\hat{m}(r) = \hat{m}(r_h) = \frac{1}{2} r_h^3; \qquad S(r) = 0; \qquad \omega_j(r) = \omega_j(r_h). \quad (126)$$

We would like to extend these results to small, non-zero values of ℓ by analyticity, along the lines of the argument used in Section 4.3. The proof of Proposition 6, specifically contains Λ, however, we may use the straightforward change of variables (cf. (98)):

$$\hat{\lambda} = \ell^2 \lambda. \quad (127)$$

Then the equations (99) are unchanged except that we include $x\frac{d\ell}{dx} = 0$ instead of $x\frac{d\Lambda}{dx} = 0$ and have the following equation for $x\frac{d\hat{\lambda}}{dx}$:

$$x\frac{d\hat{\lambda}}{dx} = -\hat{\lambda} + xH_{\hat{\lambda}} + F_{\hat{\lambda}}; \quad (128)$$

where

$$\begin{aligned} H_{\hat{\lambda}} &= -\frac{\hat{\lambda}}{\rho}\left(1 + 2G\right); \\ F_{\hat{\lambda}} &= \frac{\ell^2}{\rho} + 3\rho - \frac{2\ell^2}{\rho^3}\mathcal{P}. \end{aligned} \quad (129)$$

The field equations are then all regular as $\ell \to 0$, and the local existence result Proposition 6 carries over to give regular solutions in a neighbourhood of the event horizon. Furthermore, these solutions will be analytic in r_h, $\omega_j(r_h)$ and ℓ.

Once we have the local existence of solutions near the event horizon, exactly the same argument as used in Section 4.3 shows that we have solutions for ℓ sufficiently small. In particular, fixing some $r_1 \gg r_h$, and fixing both the radius of the event horizon and the values of the gauge field functions on the event horizon, and varying just ℓ, for ℓ sufficiently small we will have local solutions near the event horizon which are regular for all $r \in [r_h, r_1]$ and for which all the gauge field functions have no zeros in this interval. Then, provided that r_1 is sufficiently large, we may use the asymptotic regime (see Section 4.2), and find that, as $r \to \infty$, these solutions remain regular and the gauge field functions will have no zeros.

The results of this section are illustrated in Figures 5 and 6, where we have plotted example black hole solutions with the same values of the event horizon radius r_h and values of the gauge field functions at the event horizon, $\omega_1(r_h)$ and $\omega_2(r_h)$, but with $\Lambda = -100$ and $\Lambda = -1000$ respectively.

The effect of increasing $|\Lambda|$ by an order of magnitude can be seen by comparing these two figures. In both cases, the gauge field functions hardly vary at all from their values at the event horizon, and the difference between their values at the event horizon and at infinity reduces as $|\Lambda|$ increases. In both figures $\log S$ is a small function, and it decreases as $|\Lambda|$ increases. One interesting point is that, although we are considering very large

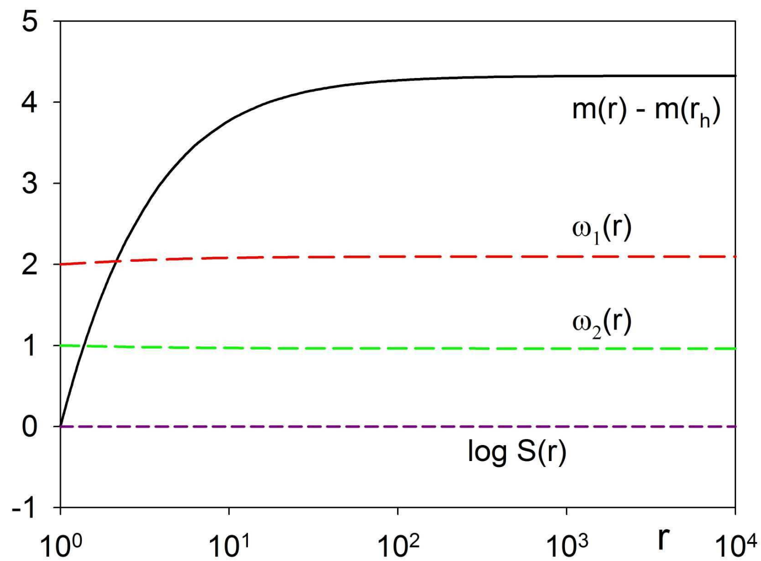

Figure 5. Example of an $\mathfrak{su}(3)$ black hole solution with $r_h = 1$, $\Lambda = -100$, $\omega_1(r_h) = 2$ and $\omega_2(r_h) = 1$. As $r \to \infty$, the gauge field functions tend to the following limits: $\omega_1 \to 2.0962$, $\omega_2 \to 0.9625$. The function $\log S$ is not identically zero: at the horizon $\log S = 2.6246 \times 10^{-3}$.

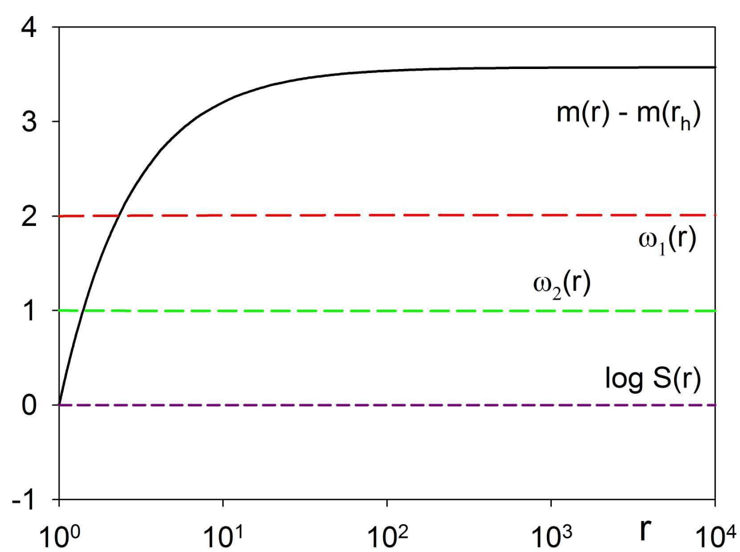

Figure 6. Example of an $\mathfrak{su}(3)$ black hole solution with $r_h = 1$, $\Lambda = -1000$, $\omega_1(r_h) = 2$ and $\omega_2(r_h) = 1$. As $r \to \infty$, the gauge field functions tend to the following limits: $\omega_1 \to 2.0091$, $\omega_2 \to 0.9964$. The function $\log S$ is not identically zero: at the horizon $\log S = 2.3968 \times 10^{-5}$.

values of $|\Lambda|$ in both these figures, the geometry is far from being Schwarzschild-adS. This is shown by the fact that the metric function $m(r)$ is far from being constant, although the difference between its values on the event horizon and at infinity is decreasing as $|\Lambda|$ increases, albeit slowly. Of course, Proposition 11 tells us that $m(r)$ will be approximately constant only for "sufficiently large" $|\Lambda|$, and does not tell us how large "sufficiently large" is. It is clear that $|\Lambda|$ will have to be very large indeed for $m(r)$ to be approximately constant, although the gauge field functions become approximately constant for much smaller values of $|\Lambda|$. However, our main interest in this paper is proving the existence of solutions, even if in an extreme situation.

4.4.2. Solitons Proving the existence of soliton solutions for *any* values of the initial parameters $(\bar{z}_2, \ldots, \bar{z}_N)$ (22), for sufficiently large $|\Lambda|$, is much more difficult than the corresponding result in the black hole case. The reason for this is that, in the black hole case, there are two length scales: the radius of the event horizon r_h and the length scale ℓ set by the negative cosmological constant ($\ell^2 = -3/\Lambda$), and we have $r \geq r_h$ for the region of space-time outside the event horizon. Therefore, with r_h fixed, we can take $\ell \to 0$ in a reasonably straightforward way because we know that r^{-1} is bounded. However, for soliton solutions, there is only one length scale, namely ℓ, and we need to consider the whole range of values of $r \in [0, \infty)$. We therefore have to be very much more careful in how we take the limit $\ell \to 0$.

For solitons, the result corresponding to Proposition 11 is:

Proposition 12 *For any fixed values of parameters* $\bar{z}_i$*,* $i = 2, \ldots, N$*, which parameterize the gauge field functions near the origin, for* $|\Lambda|$ *sufficiently large, there exists a soliton solution of the* $\mathfrak{su}(N)$ *EYM field equations such that all the gauge field functions* $\omega_j(r)$ *have no zeros.*

Proof Firstly, we rescale all the dimensionful quantities as follows:

$$r = \ell \hat{x}; \qquad m(r) = \ell \hat{m}(\hat{x}). \tag{130}$$

In the limit $\ell \to 0$, it is more convenient to write the Yang-Mills equations not in terms of the gauge field functions ω_j (12), but instead to write the gauge field functions in terms of new functions $\hat{\zeta}_j(r)$:

$$\boldsymbol{\omega}(r) = \boldsymbol{\omega}_0 + \sum_{k=2}^{N} \hat{\zeta}_k \boldsymbol{v}_k r^k = \boldsymbol{\omega}_0 + \sum_{k=2}^{N} \hat{\zeta}_k(\hat{x}) \ell^k \hat{x}^k \boldsymbol{v}_k. \tag{131}$$

These functions $\hat{\zeta}_k$ are essentially the same as the ζ_k in Section 3.1.2, modulo a factor of $\sqrt{j(N-j)}$ and the rescaling of the independent variable x (compare equations (45) and (130)). As shown in Section 3.1.2, the differential equations satisfied by the $\hat{\zeta}_k(x)$ (82) are rather complicated for general N. For $N = 3$, they are given explicitly in [15]. However, in the limit $\ell \to 0$, the field equations simplify considerably. The Einstein equations reduce to

$$\hat{m} \equiv 0, \qquad S \equiv 1, \tag{132}$$

so that $\mu = 1 + x^2$, and the differential equations for the $\hat{\zeta}_k(x)$ (82) decouple:

$$0 = x\left(1 + x^2\right)\frac{d^2\hat{\zeta}_k}{dx^2} + 2\left[k + (k+1)\,x^2\right]\frac{d\hat{\zeta}_k}{dx} + xk\,(k+1)\,\hat{\zeta}_k. \qquad (133)$$

This equation has a solution which is regular at $x = 0$:

$$\hat{\zeta}_k(x) \propto {}_2F_1\left(\frac{1}{2} + \frac{k}{2}, \frac{k}{2}; \frac{1}{2} + k; -x^2\right); \qquad (134)$$

where ${}_2F_1$ is a hypergeometric function. The constant of proportionality in (134) is simply $\bar{z}_k$ (22). It is straightforward to show, using the properties of hypergeometric functions, that the boundary conditions at infinity (28) are satisfied. From (131), when $\ell = 0$, all the gauge field functions are constant and given by $\omega_j \equiv \pm\sqrt{j(N-j)}$. However, it is important to find the forms of the $\hat{\zeta}_k$ functions in the limit $\ell \to 0$, even though they are multiplied by powers of ℓ and so do not affect the form of the gauge field functions in this limit. This is because they are important for small, non-zero ℓ.

The local existence result (Proposition 3) of Section 3.1.2 carries over, with trivial amendments, using the scaled radial co-ordinate $\hat{x}$ rather than x. Therefore in a neighbourhood of the origin, we have local solutions of the field equations which are analytic in the parameters $\bar{z}_k$ and ℓ. The same argument as used in Sections 4.3 and 4.4.1 can then easily be used to show that there are soliton solutions for the field equations for sufficiently small ℓ (that is, sufficiently large $|\Lambda|$) for which all the gauge field functions have no zeros.

As with the black hole solutions in the previous section, we illustrate this result with two example soliton solutions in Figures 7 and 8. Comparing these two figures, it can be seen how $m(r)$ and $\log S(r)$ tend to zero as $|\Lambda|$ increases, and the gauge field functions $\omega_{1,2}$ approach $\sqrt{2}$ everywhere. The metric function $m(r)$ is decreasing in size more quickly for solitons as $|\Lambda|$ increases, compared with the black hole case (cf. Figures 5 and 6). Our results in this section agree with [28], where it is shown that stable monopoles (in [28] both monopoles and dyons are considered, but here we have studied only purely magnetic monopole configurations) in $\mathfrak{su}(2)$ EYM with $\Lambda < 0$ are approximated by solutions on a pure adS background. There are non-trivial, nodeless, monopole solutions in pure adS (see Figure 1 in [28]) for the $\mathfrak{su}(2)$ case, and, as in Section 2.3, these can be embedded into $\mathfrak{su}(N)$. In Proposition 12, we have shown the existence of nodeless $\mathfrak{su}(N)$ solitons effectively approximated by solitons in pure adS (because $\hat{m} \approx 0$), but only in the limit of very large $|\Lambda|$, in which case the equations governing the YM degrees of freedom decouple (133).

5. Conclusions

In this paper we have studied the existence of four-dimensional, spherically symmetric, soliton and black hole solutions of $\mathfrak{su}(N)$ EYM with a negative cosmological constant.

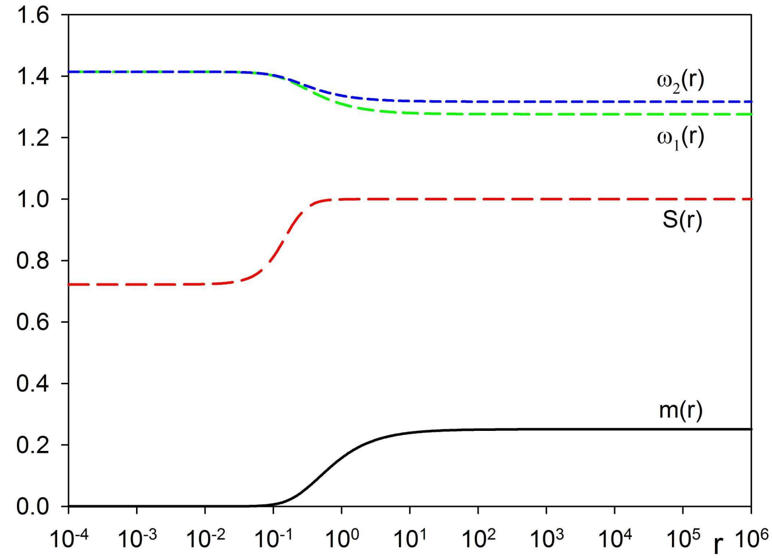

Figure 7. Example of an $\mathfrak{su}(3)$ soliton solution with $\Lambda = -100$, $\bar{z}_2 = -2$ and $\bar{z}_3 = -1$.

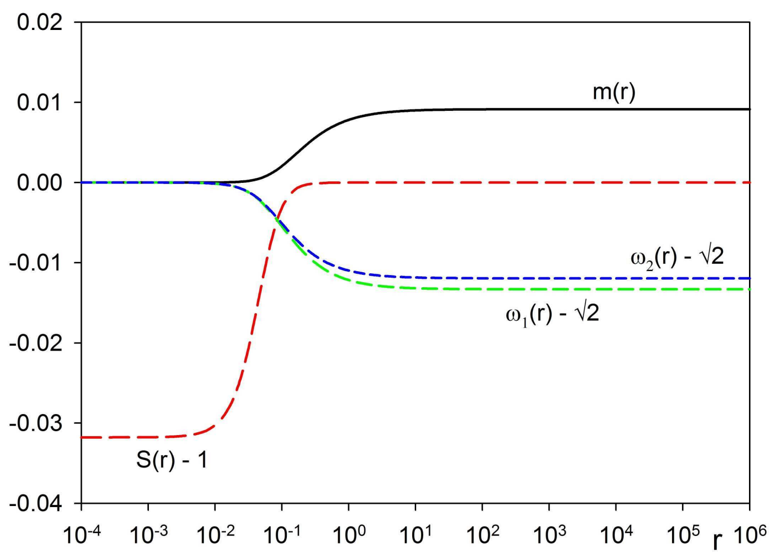

Figure 8. Example of an $\mathfrak{su}(3)$ soliton solution with $\Lambda = -1000$, $\bar{z}_2 = -2$ and $\bar{z}_3 = -1$.

Numerical solutions for $N = 3$ and $N = 4$ were presented in [15], but, of course, numerics can only find solutions for a small, finite number of values of N. The purpose in this paper was to prove the existence of four-dimensional, spherically symmetric, soliton and black hole solutions for any integer value of N. The general approach was briefly outlined in [14], but in this paper we have presented all the detailed results.

We began, in Section 2.1, by describing the field equations for $\mathfrak{su}(N)$ EYM with a negative cosmological constant, the ansatz for the metric and gauge potential and the boundary conditions at the origin, black hole event horizon (if there is one) and infinity. We considered only purely magnetic gauge fields, which are described by $N-1$ gauge field functions ω_j. We also described how any $\mathfrak{su}(2)$ EYM soliton or black hole solution can be embedded to give an $\mathfrak{su}(N)$ EYM solution for any value of N. The goal of the paper is to prove the existence of genuinely $\mathfrak{su}(N)$ solutions, that is, solutions of the $\mathfrak{su}(N)$ EYM field equations which are not embedded $\mathfrak{su}(2)$ solutions.

The field equations for $\mathfrak{su}(N)$ EYM are singular at the origin, black hole event horizon and at infinity. In Section 3 we proved local existence of solutions in neighbourhoods of these singular points, using an approach in [17, 21]. Our key existence results are in Section 4. Firstly, we are able to prove the existence of genuinely $\mathfrak{su}(N)$ solutions in a neighbourhood of an embedded $\mathfrak{su}(2)$ solution, and, as a corollary, the existence of $\mathfrak{su}(N)$ solutions for which all the gauge field functions have no zeros, for any negative value of the cosmological constant. Secondly, we show that, for any fixed values of the initial parameters at either the origin or the black hole event horizon, which fix the local solution in a neighbourhood of the relevant starting point, then for $|\Lambda|$ sufficiently large, this local solution can be extended to infinity to give a soliton or black hole solution for which all the gauge field functions have no zeros. These solutions when $|\Lambda|$ is large are of particular interest, because at least some of them are linearly stable under spherically symmetric perturbations. An outline of this result can be found in [14], and the details will be presented elsewhere [16].

Our main result is therefore the existence of four-dimensional, spherically symmetric, asymptotically adS, soliton and black hole solutions of the $\mathfrak{su}(N)$ EYM field equations with $N-1$ gauge field degrees of freedom. For black holes, this means that there is no limit on the number of gauge field degrees of freedom (or 'hair') with which the black hole may be endowed. Of course, it is the stability of at least some of these black holes which is central to their importance for the no-hair conjecture, an issue to which we shall return [16]. This result raises the interesting question of whether there are solitons or black holes with an infinite number of gauge field degrees of freedom. There is some evidence [29] for black hole solutions of $\mathfrak{su}(\infty)$ EYM, but more work is required in this area.

There are a number of generalizations of the model considered here which would merit further study. We have not considered topological black holes in adS, although solutions are known for the $\mathfrak{su}(2)$ gauge group [23]. For $\mathfrak{su}(2)$ EYM in adS, static, axially symmetric soliton [30] and black hole [31] solutions are known: it is likely that generalizations of these to $\mathfrak{su}(N)$ EYM exist. Rotating $\mathfrak{su}(2)$ EYM black holes in adS have also been found [32], and again one would expect generalizations to the larger gauge group to exist. However, the numerical challenges in finding such solutions cannot be underestimated. The stability of these solutions also remains an open question. In this paper we have restricted our attention to purely magnetic gauge fields. For $\mathfrak{su}(2)$ EYM in adS, dyonic regular and black hole solutions exist [10], even though non-trivial $\mathfrak{su}(2)$ EYM solutions in asymptotically flat space must be purely magnetic [33]. Since $\mathfrak{su}(N)$ solutions in asymptotically flat space may have electric as well as magnetic charge

[25], it would not be surprising to find dyonic $\mathfrak{su}(N)$ EYM solutions in adS, although including the electric part of the gauge field will only introduce a single additional gauge field degree of freedom. The stability of the dyonic solutions, even in the $\mathfrak{su}(2)$ case, remains an open question. Next, EYM solitons and black holes in more than four space-time dimensions have recently received much attention in the literature (see [34] for a review and references), and analogue higher dimensional solutions of $\mathfrak{su}(N)$ EYM would be expected. Finally, one would expect to be able to extend our results to arbitrary compact gauge group: in [21] local existence theorems are proved for compact gauge group in asymptotically flat space, and extending these to include a negative cosmological constant should be possible, leading to existence theorems along the lines of those proved in Section 4.

Having shown that there is no limit to the amount of hair a black hole in adS can be given, the natural question arises of the consequences of these black hole solutions for gravitational physics. Firstly, there is the impact on the status of the "no-hair" conjecture: these black holes require an unlimited number of parameters to fully describe their configurations, and therefore are contrary to the "spirit" of the "no-hair" conjecture, namely that black holes are fundamentally simple objects. The fact that these black holes with unlimited amounts of hair exist in anti-de Sitter space may be significant, particularly in view of the adS/CFT (conformal field theory) correspondence [35]. It has been conjectured [36] that there are observables in the dual (deformed) CFT which are sensitive to the presence of black hole hair, and an adS/CFT interpretation of some stable seven-dimensional black holes with $\mathfrak{so}(5)$ gauge fields is given in [37] (see also [38] for a discussion of non-abelian solutions in the context of the adS/CFT correspondence). Black holes with non-abelian gauge fields in supergravity have begun to attract attention recently [39] and understanding the supersymmetric analogues of the solutions we have found in this paper will be an important question to which we plan to return in the near future.

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