A professional development programme for teaching mathematics through problem solving

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Candidate Declaration

I hereby declare that:

1. I have not been enrolled for another award of the University, or other academic or professional organisation, whilst undertaking my research degree.
2. None of the material contained in the thesis has been used in any other submission for an academic award.
3. I am aware of and understand the University's policy on plagiarism and certify that this thesis is my own work. The use of all published or other sources of material consulted have been properly and fully acknowledged.
4. The work undertaken towards the thesis has been conducted in accordance with the SHU Principles of Integrity in Research and the SHU Research Ethics Policy.

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Abstract

This thesis documents the design, implementation and evaluation of a professional learning programme to support teachers in developing their ability to teach mathematics through problem solving. Teaching mathematics through problem solving is recognised as an effective way to teach mathematics. However, there are a number of different approaches with different descriptors such as problem posing and problem-based learning. In this study, teaching mathematics through problem solving is an approach where learners explore a problem or a task and their responses to the task are 'orchestrated' by the teacher to introduce and develop the mathematics to be learned. The development of this teaching approach involves the principles of task design and planning the sequencing of anticipated responses to be orchestrated.

The professional development programme was informed by research on task design and the features of effective professional learning for teachers. The programme was built around the professional learning programme Lesson Study and was modified with a 'pause' in the research lesson and the use of remote visualisers to observe and analyse the work of the pupils. This latter feature was an adaption of the original programme design to respond to conditions arising in schools during the Covid-19 pandemic. Also a number of additional modifications were made to accommodate the ambitions of the programme.

A qualitative approach was adopted using the methodology of design research and comprising two research cycles. The cohort in the first cycle were secondary school teachers and in the second cycle they were primary school teachers, all working in England. The aim of the study was to understand the effect of the programme on the participants' knowledge, skills and confidence in teaching mathematics through problem solving. Existing frameworks in the literature on the known characteristics of effective professional learning were synthesised to evaluate the effectiveness of the programme.

The main outcomes from this research confirm that the designed features of the PD programme were effective in supporting teachers in developing their ability to teach mathematics through problem solving. The use of task design principles in conjunction with the development of the teaching technique of orchestration resulted in gains in teacher confidence and subject and pedagogical knowledge. The pause in the research lesson and the use of visualisers to observe and analyse the pupils' work provided the participants with a focused learning opportunity to develop the teaching approach. The evaluation of the programme using the designed analytical framework for CPD identified [a range of] known characteristics of effective CPD in the programme as designed and implemented.

Three main implications emerged from the study, the first of which concerns the future development of teaching mathematics through problem solving. It was apparent that there were contextual barriers to the approach becoming an established part of teachers' practice. To address these, the design of mathematics teaching resources would need to reflect the ambition to teach mathematics through problem solving. This is an issue for curriculum designers. There would also need to be recognition of the problem-solving skills that pupils require to access the problems, which would need to be taught either as part of teaching through problem solving or as a separate approach known as teaching *for* problem solving.

The second implication identifies the challenges associated with introducing this type of programme into the current landscape of professional learning for teachers of mathematics in England, including the need for greater coherence between the professional learning of teachers of mathematics and their own workplace contexts.

Finally, the third implication relates to the use of Lesson Study. This study described how the use of design research can develop Lesson Study variants that can make a positive contribution to professional learning environments for teachers without compromising the key principles of Lesson Study.

For Clare – Thank you

Contents

List of Figures

List of Tables

Chapter 1 – The Study, its Origins and Ambition

Introduction

Accounts of qualitative studies, like many stories, involve a chronological interpretation of a sequence of events. This thesis tells the story of how I designed, implemented and evaluated a professional development (PD) programme for teachers of mathematics to support their development of teaching mathematics through problem solving. In this study, teaching mathematics through problem solving is defined as an approach where a learner explores a problem[1] or task, and the teacher uses the learners' responses and contributions in order to teach a mathematical concept or idea. The study originated from my personal and professional interest in teaching mathematics through problem solving – an interest that was informed by international research on problem solving and the current context for mathematics teacher development in the United Kingdom. The resulting PD programme was implemented with two groups of teachers from different phases of education and over two research cycles.

In this chapter I describe the three key areas that informed the rationale and shaped the design and content of the study. I provide an outline of the PD programme and the innovations that I devised to develop the areas of research. I summarise the research methodology and then address the contribution to knowledge by introducing the research questions. Finally, I set out the content and structure of the thesis and the relationship between the relevant chapters.

1.1 Rationale for the study

In this section I discuss the key areas that have shaped the rationale for the development of this study. Firstly, I draw on the findings from international research literature on the value of teaching mathematics through problem solving and how attention to this approach has evolved over the last four decades. Then I explain how my personal and professional experience in problem solving and teaching mathematics through problem solving has contributed to my understanding of the teaching approach that I describe in this thesis.

[1] For example, the exploration of why any six-digit number of the form 'abcabc' is always divisible by 13 can lead to the introduction of index notation and prime factorisation.

Finally, I comment on the current context of professional learning for teachers of mathematics in England.

1.1.1 Teaching mathematics through problem solving

In recent decades, one of the most significant developments in mathematics education worldwide, but particularly in the USA and the UK, is the preponderance of theories that put forward methods of effective teaching which have a focus on problem solving (Hembree & Marsh, 1993; Henningson & Stein, 1997; Hiebert & Wearne, 1997; Kroll, 1993; Stein et al., 1999).

In 1980, the USA's National Council for Teaching Mathematics[2] *Agenda for Action* stated that "problem solving must be the focus of school mathematics" (NCTM, 1980, p. 1). In the UK, Burkhardt and Bell's (2007) commentary on the development of problem solving over the last century describes how the mathematics curriculum shifted from a focus on rote learning to a recognition that mathematical reasoning is important. In 1988 the Education Reform Act gave further prominence to problem solving in mathematics. The resulting programme of study introduced a number of attainment targets, the most significant of which was Attainment Target 1 (AT1): Using and Applying Mathematics that contained eight levels[3] of performance designed to ensure that problem solving and mathematical reasoning was at the heart of the curriculum. Prior to this, the Shell Centre for Mathematical Education published *Problems with Patterns and Numbers*[4] (Board, 1984) a resource which became highly regarded by mathematics teachers in the teaching of AT1.

Disappointingly, despite these past endeavours to make problem solving central to the mathematics curriculum, there currently seems to be a diminishing focus on developing effective problem-solving approaches in the teaching of school mathematics in England. Despite the official recognition of the importance of problem solving in secondary GCSE examination syllabuses and many mathematical schemes and resources in primary education, the reality is that teaching practices using problem solving are underdeveloped. The approach is used infrequently by teachers who, mainly due to policy maker attention and their

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[2] Founded in 1920 in Cleveland, Ohio, in the USA, the NCTM is the world's largest mathematics organisation.
[3] An exceptional performance level is also included above level 8.
[4] This is a splendid resource that I have used not only to teach mathematics through problem solving but also as a highly effective CPD resource.

conflicting priorities, have not had the opportunity and time to explore and develop the use of problem solving into their teaching (Burkhart, 2014).

Often research that explores problem-solving pedagogies begins with the definition of the word 'problem'. Whilst there are many definitions in education literature,[5] the general consensus is that a problem is a task where, at the point of encounter, there is no known approach to solving it; therefore, a problem is only a problem until it is solved. By contrast, interpretations of the term 'problem solving' are much more diverse. Stanic and Kilpatrick (1989) describe problem solving as a cognitive enterprise, as something to be taught, and as something to teach through. The recent Ofsted review (2021) offers a different perspective, classifying mathematics curriculum content into declarative, procedural and conditional knowledge. This categorisation highlights that problem solving for young children is often about word problems and so the first barrier to overcome is language rather than the mathematics within the problem. This variance in the contributions from different professional bodies continues to prevent the development of consensus on a problem-solving curriculum.

Therefore, whilst the concepts of 'problem' and 'problem solving', and the relationships of both to teaching mathematics are debated, there is a lack of shared meanings. There are many different ways in which the term problem solving is used, and whilst the common feature to all is that problems or tasks are used to teach mathematics, problem-solving skills or both, they all incorporate different pedagogies and learning ecologies. For example, Enquiry Based Learning (EBL) is an approach to learning that is driven by a process of enquiry where the teacher establishes the task and supports or facilitates the process (Kahn & O'Rourke, 2005), whereas Problem Based learning (PBL) is a constructivist model of learning that focuses on the pupils' responses (Amalia et al., 2017). I discuss these in more detail in Chapter 2.

It is not my intention to categorise or evaluate these different approaches; however, it is important that at this stage I clearly define the method used in this study in order to underpin the assumptions on which the research was developed. Teaching mathematics through problem solving is an approach to teaching mathematics where the teacher uses a specially designed task in order to develop or introduce a new piece of mathematics. This is achieved by first allowing the pupils the opportunity to explore the task and to develop their thoughts, methods and ideas about how the problem might be solved. The teacher then 'orchestrates' these pupil responses in such a way (that is often pre-planned) so that the resulting analysis

[5] For example, Stephen Krulik and Jesse Rudnick (1980) in *Problem Solving: A Handbook for Teachers* define 'problem' as "a situation, quantitative or otherwise, that confronts an individual or group of individuals, that requires resolution, and for which the individual sees no apparent or obvious means or path to obtaining a solution" (p. 3).

and discussion of the different methods and solutions reveals the mathematics to be learned. Examples of the tasks and the approach are developed in Chapter 2.

1.1.2 My personal and professional journey with mathematics and problem solving

My own learning and interest in problem solving in mathematics began as a pupil four years after an unsuccessful experience with the 11 Plus[6] – an outcome that was a shock for me but not for my teachers. They knew of my capability in mathematics but they were also aware of my limitations in literacy which was an equal component of the test. During my final year at a secondary modern school, I was informed that I was to be entered for GCE mathematics in addition to the CSE examination, together with three other pupils. In preparation for this, additional tuition was provided by my mathematics teacher ‘Scratcher’. Scratcher was a dishevelled head of year who taught some mathematics. As far as I know he had no formal qualifications in the subject but his skill in devising imaginative mathematical problems, mainly concerning Euclidean geometry, inspires me to this day. His ability to frame mathematics in a way that attracted me to the subject beyond the desire to be successful is one of the seminal moments in my relationship with mathematics.

His teaching approach appeared to me to be a conversation about how to solve a single problem. That conversation usually extended for the duration of the lesson and was occasionally punctuated by a diversion into the beautiful game – cricket not football. The extra revision sessions also followed an unusual format. Taking place in the back room of a working men’s club, these final preparations were much more intense but still followed the same approach to problem solving. In these sessions we did few examples but discussed in serious detail the construct of each question and its relationship to other areas of mathematics. In my experience of learning mathematics, what was unique about Scratcher’s teaching was the way he utilised my ideas and approaches as possible building blocks towards a potential solution. Since then, I have considered that this method is an effective approach to teaching mathematics through problem solving. This longstanding belief has been affirmed by developments in my own professional learning as a teacher, teacher educator and researcher, as I go on to describe in the next section.

[6] A standardised examination (IQ test) for pupils in England and Northern Ireland in their last year of primary education, which governs admission to grammar schools and other secondary schools which use academic selection.

Alongside my personal journey, my belief that mathematics can be taught effectively through problem solving has been influenced by a wide range of professional experiences as a teacher educator in universities and working on national CPD multiplier cascade programmes such as the National Numeracy Strategy. More recently this belief has been affirmed through my involvement in the development of the professional learning programme known as Lesson Study.[7] During research visits to Japan as part of the International Math-teacher Professionalization Using Lesson Study (IMPULS) project, I observed many research lessons where the focus was the teaching of mathematics through problem solving.

Stigler and Hilbert (1999) referred to the approach as “structured problem solving”. It was clear from my observations in Japan that the teachers using this approach had a very good command of the subject and a robust understanding of the task and the mathematics that would develop from exploring the task. They were also highly competent in the technique of ‘orchestrating’ the work of the pupils. This refers to the way the teacher sequences the pupils’ responses in order to teach the mathematics. I will explain and discuss this technique later in this thesis.

As a result of these experiences, I was inspired to develop a Lesson Study programme in the UK. Through my role as Director of Schools in the Diocese of Hallam in Sheffield, England, I engaged a number of schools[8] in the diocese and local authority in the process of Japanese Lesson Study, all of whom agreed to focus their research lessons on the teaching of mathematics through problem solving. Three years and 22 Lesson Study cycles later, it is apparent that the effectiveness of this programme in supporting professional learning in the teaching of mathematics through problem solving was uncertain at best and ineffective at worst. Despite the significant amount of time, research and preparation invested in the Lesson Study process by the participating teachers, there was little evidence to suggest that the teaching of mathematics through problem solving had become a recognised part of their everyday teaching practice.

[7] Lesson Study is a form of teacher professional development which involves collaborative study of live classroom lessons; it has attracted growing international attention since the late 1990s and has been used widely in Japan for over 100 years (Fernandez & Yoshida, 2012; Lewis & Tsuchida, 1998; Stigler & Hiebert, 1999).

[8] Several of these schools continue to engage with Lesson Study through the Collaborative Lesson Research (CLR) community. CLR is a powerful form of teacher professional development focused on the design of lessons for student learning. https://www.collaborative-lesson-research.uk/

An evaluation of the Lesson Study cycles by the South Yorkshire Maths Hub Innovation Work Group on Developing Mathematical Reasoning in 2014 revealed two significant outcomes:

1. The orchestration element of the research lessons (known in Japanese problem-solving lessons as Neriage) was the most challenging for the teachers.

2. The teachers often made decisions that were not in the plan created by the Lesson Study research team. Frequently they deviated from the plan to such an extent that it was impossible to evaluate the effectiveness of the research lesson as originally planned.

1.1.3 The context for mathematics teacher professional development in England

Two issues have defined the current context for the professional learning of mathematics teachers in England. The first relates to concerns about standards in mathematics education in recent decades, prompted by evidence from numerous professional sources such as the OECD (2010; 2013) as well as government comparisons of international pupil performance data (Gibb, 2015; Gove, 2013), and ongoing concerns about the quality of mathematics teaching (Ofsted, 2008; Smith, 2004; Williams, 2008). The second issue concerns the marketplace that has emerged out of the policy context (Boylan & Adams, 2023) where a number of providers are currently competing to redistribute teaching ‘treasures’ from East Asia packaged in the form of the pedagogical approach ‘teaching for mastery’. Sitting uneasily alongside this provision is the Early Career Framework (DfE 2019a) for teachers in which subject-specific knowledge tends to be separated from pedagogy.

Over the last 40 years there has been no shortage of professional learning opportunities for teachers of mathematics and indeed no lack of focus in the pursuit of quality PD. The National Numeracy Strategy (NNS) was launched in 1998 and set out to raise standards in mathematics attainment in primary schools in England through the introduction of a Framework for Teaching Mathematics from Reception to Year 6 (National Numeracy Project 1998). This framework was supported by a range of resources such as Unit Plans and was delivered to teachers through national cascade programmes delivered by approximately 300 numeracy consultants.

In 2006, the National Centre for Excellence in the Teaching of Mathematics (NCETM) was formed to provide strategic leadership for mathematics-specific CPD, and since 2013 it has coordinated Maths Hubs in England to provide professional learning programmes to engender the 'teaching for mastery' pedagogy. In 2007 it commissioned a research project, Researching Effective CPD in Mathematics Education (NCTEM, 2009), to provide evidence-based advice and recommendations on effective CPD in this subject area. This large-scale project investigated 30 different PD initiatives for teachers of mathematics which had taken place in the academic year 2007-2008. The research involved collecting and analysing data about the PD structure and organisation and the responses of participating teachers. The report informed the characterisation of PD initiatives and drew on teachers' responses to the various characteristics – an approach that the authors suggested was one way of understanding the 'effectiveness' of PD programmes. The NCETM's principal goal is to ensure that all teachers of mathematics have easy access to high quality, evidence-based, maths-specific continuing professional development. However, in this statement, the terms 'high quality' and 'evidence-based' both precede the phrase 'continuing professional development' and therefore it is unclear whether CPD is of high quality *because* of the evidenced-based content or independently of that aspect. I understand I might be meddling with semantics but the reason for doing so is to highlight the fact that the design of current professional learning programmes continues to pay insufficient attention to the known characteristics of effective CPD.

As will be discussed further in Chapter 2, the research into the effective characteristics of professional learning for teachers is extensive and growing. Notwithstanding this growth, I have observed that current PD programmes, such as White Rose Maths, Ark Mathematics Mastery and even components of the Maths Hubs[9] programme, continue to employ PD approaches similar to those used in the NNS but have not acknowledged the effectiveness or limitations of these approaches. For example, the cascade training approaches categorised by Kennedy (2005) as transmission models are still very much in use.

Further, whilst there is abundant existing research exploring different PD delivery approaches and their effects (Harland & Kinder, 2014), this knowledge is rarely called upon in the design of current PD programmes. So, even though the characteristics of effective PD are defined

[9] However, I do acknowledge that the Maths Hubs among others are now using more transitional and transformative approaches such as the community of practice and action research models.

(Desimone, 2009; Harley & Valli, 1999; Simms et al., 2021), we still have much to learn about the alignment of these characteristics to PD programmes that are designed to support the professional learning of teachers of mathematics. In my experience, a lack of scrutiny of the necessary alignment is common in the development and delivery of educational reform. As will be discussed later, this weakness manifests itself as insufficient coherence between the PD programme and the professional conditions in which the participants operate. Nevertheless, this does not undermine the development of the programme described in this thesis nor the potential contribution to knowledge resulting from the design, implementation and evaluation of the programme.

Therefore, consistent with the arguments of other researchers such as Burkhardt and Bell (2007), I contend that insufficient attention has been given to the challenges of teaching mathematics through problem solving and that a number of important features need to be considered: the change in the role of the teacher; the design and selection of problems (with implications for teacher subject knowledge); and the integration of appropriate problems and tasks into the mathematics curriculum to enable the teacher to practise and learn how to teach mathematics through problem solving. The need for development of these features in current teacher professional learning programmes is a principal driver for the development of this research, alongside the insights from my own professional learning journey as described above. Below, I set out a number of positions that underpin this study some of which have evidence and others which are assumptions based on my professional knowledge and experience.

1. There is widespread agreement that teaching mathematics through problem solving holds the promise of fostering student learning and that it can be an effective approach to teaching mathematics (Liljedahl et al., 2016; Schroeder & Lester, 1989; Stanic & Kilpatrick, 1989). This resonates with my experience of observing this approach in Japanese classrooms through my participation in the IMPULS and CLR research programmes.
2. My experience as a teacher educator in a number of local and national contexts has led me to assume that professional development programmes on the teaching of mathematics through problem solving in the UK are uncommon and are not sufficiently focused on the technical skills required of teachers.

3. Whilst there is guidance for teachers on the teaching strategies for solving problems such as in the Education Endowment Foundation guidance report on improving mathematics in Key Stages 2 and 3 (Henderson et al., 2017) and the mathematics non statutory guidance (Department for Education, 2020), this guidance does not include frameworks for the teaching of mathematics through problem solving.
4. There are a number of sources for mathematical problems (NRICH etc.) and several published schemes (Ark Mathematics, Mastery Maths No Problem, White Rose Maths) that contain tasks that could be used to teach mathematics through problem solving. My examination of these sources and schemes indicates they that do not provide information on either the mathematics to be learned or the pedagogy to be used.

These assumptions and the reflections above led me to consider the design of a PD programme for teachers that would add to the existing body of practical knowledge on teaching mathematics through problem solving, as well as the methods and approaches for developing professional learning programmes for teachers of mathematics.

Whilst the PD programme did not seek to address directly the issues arising from the three stated assumptions, it was devised to examine the effects of engaging teachers in working collaboratively on task design and developing their experience in the teaching technique 'orchestrating the learning'. Also, it should be emphasised that it is not the study's objective to consider how best to teach mathematics through problem solving, and I do not contend that the use of the teaching technique 'orchestrating the learning' is the most effective way of doing so. However, I do suggest that the process of engaging teachers in professional learning – designed with the key components described above – can improve the teachers' subject knowledge and ability to teach mathematics through problem solving and also potentially influence their views on the value of teaching this way. As such, the learning from this research is intended to contribute to understanding about potential approaches that are effective in supporting the development of teaching mathematics through problem solving.

1.2 Outline of the PD programme

The design of the PD programme was based on two major strands. The first is the theoretical framework for an approach to teaching mathematics through problem solving which has been developed from the literature on task design and problem solving and also from my own experiences as a teacher and educator in mathematics education. This framework has three components:

- task design
- teaching about, for and through problem solving
- the teaching method 'orchestrating the learning'.

The second strand of the programme design related to the known characteristics of effective PD. The key components of the PD programme were:

- three PD sessions where participants collaboratively explored the principles of task design and teaching mathematics through problem solving
- a modified Lesson Study research cycle which contained two innovations within the research lesson
- a further PD session to review the orchestration sequence from the research lesson.

The programme used the PD approaches of Lesson Study and Teacher Design Teams (TDT) and also a number of facilitation strategies that I devised to use in the PD programme. These were:

- the use of documents to facilitate the participants' thinking
- use of breakout rooms as part of the remote learning
- no time limits on TDT tasks.

These strategies are explained and discussed in more detail in Chapters 4 and 5.

The programme was designed to be used with teachers from both the primary and secondary phases and with teams of teachers who worked together in the same school. The reason for working in different phases was to explore the effect of the programme on a range of teachers with differing levels of experience and approaches to teaching mathematics. The decision to work with teams of teachers from the same school was to maximise the opportunities afforded by a Teacher Design Team (TDT). TDTs have been shown to be more impactful on teacher learning and teacher practice where teachers from the same school or department participate collectively (Birman et al., 2000; Wayne et al., 2008).

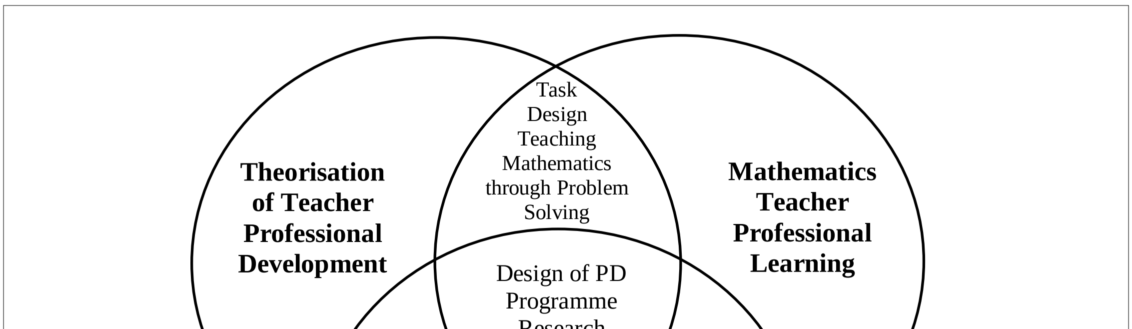
The inclusion of two research cycles (outlined further in the next section) enabled modifications to be made to the PD programme in Cycle 2. As the programme was delivered remotely using video-conferencing technology, each of the PD programme components could be easily reviewed. The analysis from this process together with formative evaluations by the participants informed the changes made in Cycle 2.

1.3 Outline of the research methodology and design process

After consideration of various research methodologies, I chose design research, which some experts refer to as design experimentation (Bakker, 2018). In this methodological approach, design and research are intertwined: the design is informed by research and the research is based on a design. In the present study, the research on task design and the teaching technique of orchestration has informed the design of the PD programme. Likewise, the design of the PD programme, including the ‘pause’ in the research lesson, has defined the research. An important goal was to explore how the participating teachers’ views on teaching mathematics through problem solving were affected by certain components of the PD programme: namely, the pause in the research lesson and the PD sessions on task design and orchestrating the learning.

To develop the PD programme, the research methodology and the research questions, I devised a model of three intersecting learning domains: mathematics teacher professional learning; theorisation of teacher PD; and design research principles and methodology. Figure 1.1 below depicts how the three domains interacted to develop PD programme components, the framework for presenting the programme and the associated research questions.

Figure 1.1 *Programme design and methods located in learning domains*



The purpose of the diagram is to make explicit which aspects the study is mainly concerned with and importantly which aspects it is not focused on. The PD programme was designed using the learning from the three intersecting domains and informed by my own personal experience and tacit knowledge of professional learning for mathematics teachers. Whilst the study draws on various aspects of each domain, it is not my intention to elaborate on these. Its purpose then is to simply illustrate in broad strokes, the framework for development of the different components of the PD programme. Therefore, the main narrative of the thesis is focused on the PD design and the evaluation of the innovations developed within it. Table 1.1 below summarises the four phases of the design research process and the timescale for each phase.

Table 1.1 *Summary of design research phases*

Design research process			
Phase	Design inputs	Research outputs	Timeline

Devise a theoretical framework for development of PD programme	Task Design Teaching mathematics through problem solving Lesson Study Teacher Design Teams Framework for effective PD	Establishment of PD programme within an effective PD domain	January 2018 to April 2019
Devise research methodology and methods	Design Research	Research questions Methods of analysis Ethics approval	April 2019 to January 2021
Implementation of research cycles	PD programme sessions Pause in the research cycle Use of visualisers Development of research lesson plan	Formative data Design and procedural iteration	Cycle 1 January 2021 to April 2021 Cycle 2 June 2021 to September 2021
Analysis and reporting	Conjecture mapping Coding PD analysis framework	Findings, conclusions and contribution to knowledge	September 2021 to April 2023

The development of the PD programme and an approach to teaching mathematics through problem solving drew from literature on the principles of task design and orchestrating pupil responses together with my own professional experience. The innovations were introduced through modifications to the Lesson Study process. A framework for analysis was developed to ensure that the design and implementation of the PD programme was consistent with the known characteristics of effective CPD.

The study involved two parallel and iterative research cycles. They were parallel in the sense that the same design principles were applied to both cycles. They were also iterative in that modifications were made between the two cycles but these adjustments, were mostly procedural and concerned with the delivery of the PD programme. The study took place during the Covid-19 pandemic and so the majority of the research took place online. As the study progressed, the pandemic restrictions changed and modifications to the study were made accordingly.

1.4 Contributions to knowledge

The design of the PD programme and the development of research questions were two interconnected and simultaneous processes. The PD programme was developed in response to the assumptions regarding the current knowledge on teaching mathematics through problem solving and was designed to incorporate the use of task design principles together with the teaching technique orchestrating the learning. At the same time, the premise that orchestrating the learning is an effective way of teaching mathematics through problem solving informed the design of the programme, such as the introduction of the pause in the research lesson. The expected contribution to knowledge therefore comes from the exploration of:

- the relationship between the principles of task design and the teaching technique orchestrating the learning
- the role of the pause in the research lesson in supporting the development of the teaching technique
- the evaluation of the PD programme in terms of its contribution to the professional learning of teachers.

The PD programme development led to the formulation of the three research questions below which will be revisited to inform the methods of analysis and the presentation of the findings.

1. How does the use of task design in conjunction with the teaching technique of 'orchestrating the learning' affect teachers' views on teaching mathematics through problem solving?
2. How does the introduction of a 'pause' in the research lesson support the development of 'orchestrating the learning' as a method of teaching mathematics through problem solving?
3. How do the designed features of the PD programme 'teaching mathematics through problem solving' contribute to knowledge about professional learning programmes and environments for teachers of mathematics?

The study also aimed to provide insights into the application of design research methods to the development of PD programmes by describing the affordances and limitations of the methods used and to contribute to the narrative on how teachers respond to and interact with professional development in the current contextual and political climate.

1.5 Thesis organisation and structure

In this thesis, in addition to a review of key empirical and theoretical literature on teaching mathematics through problem solving, task design and professional development, I integrate discussion of other relevant research evidence into the relevant chapters. The reviews of the literature on task design, the approaches used to teach mathematics through problem solving, and the effective features of professional learning for teachers are incorporated into Chapters 2 and 4.

In Chapter 2, I set out the theory underpinning the design of the PD programme by discussing the principles of task design and introducing a framework for teaching *about*, *for* and *through* problem solving. The concept of contingency is introduced to explain the teaching technique 'orchestrating the learning'. I then discuss the issues around effective PD and present a framework used in the study to ensure that the PD programme aligned with the known characteristics of effective professional learning. Finally, I describe the PD approaches of Lesson Study and Teacher Design Teams that were integral to the delivery of the programme.

Chapter 3 details the research methodology and the rationale for the data collection methods used. I introduce the method of 'conjecture mapping' to connect the research questions to the data. I explain how 'argumentative grammar' was achieved using conjectures and describe the process of coding and the process of abduction that led to the themes for analysis. In this chapter, I also discuss the recruitment of the participants and the ethical considerations emerging from the nature of my relationship with their schools.

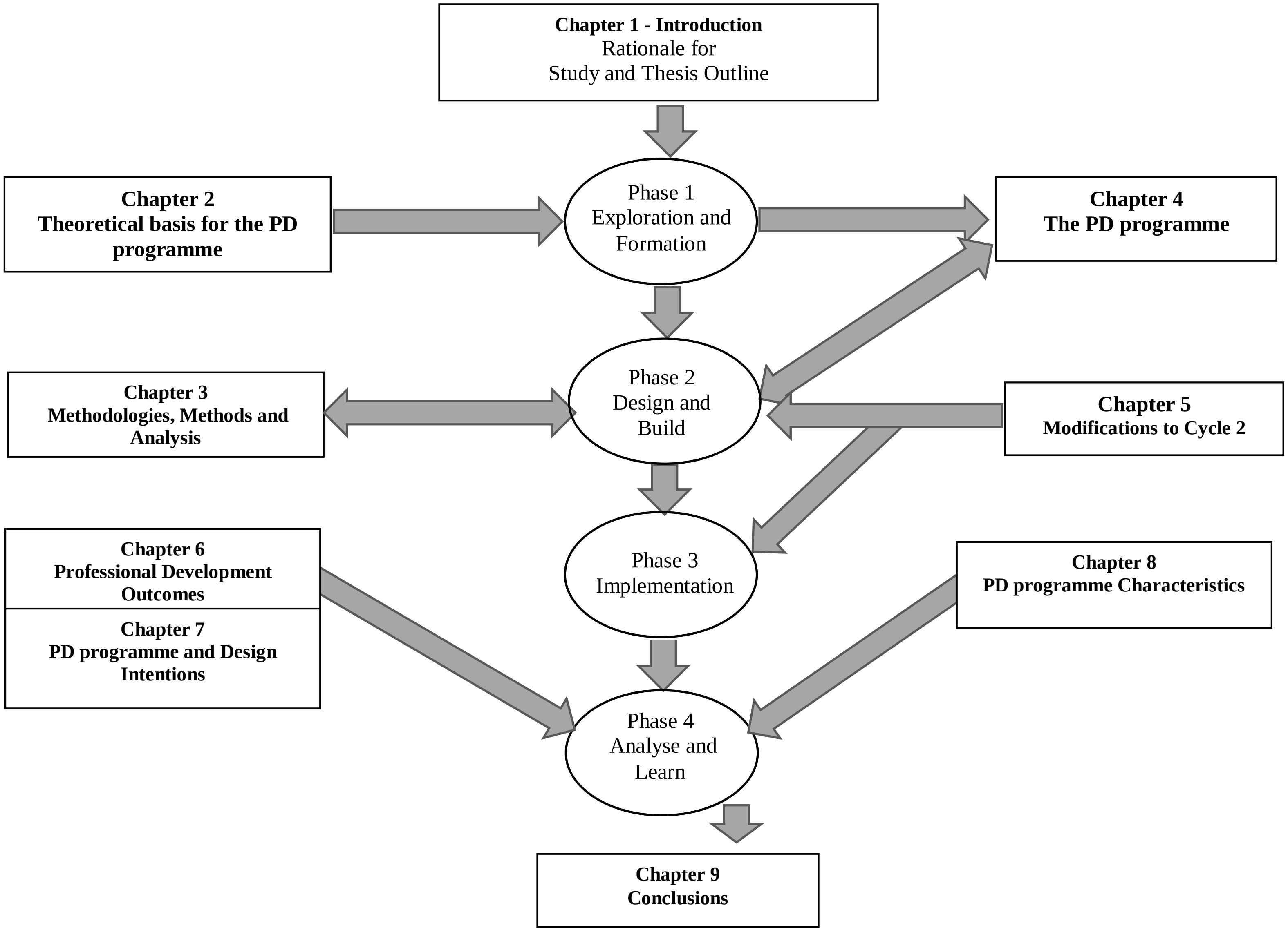
In Chapter 4, I describe the detail of PD programme components and discuss the facilitation approaches used to present the programme. I explain how formative evaluations were used to reflect on the effectiveness of these approaches. Finally, in this chapter, I further develop the CPD framework for analysis that was introduced in Chapter 2. In Chapter 5 I discuss the modifications made to Cycle 2 of the PD programme.

Chapter 6 documents the professional development outcomes for the participants in the study in relation to developments in: their views about teaching mathematics through problem solving, their subject and pedagogical knowledge and potential changes in practice. In Chapter 7 I consider the extent to which the professional development outcomes met the design intentions and in Chapter 8 I revisit the CPD framework for analysis to describe the positive characteristics of the PD programme.

In Chapter 9, I return to the research questions to consider the outcomes of the study and the contribution to knowledge. Finally, I set out the implications of the study, its limitations and potential future research possibilities.

Figure 1.2 below shows the relationship between the chapters and the design and implementation phases of the study.

Figure 1.2 *Chapter connectivity map*



Chapter 2 – Research and Practice Informing the PD Design

Introduction

In this chapter I discuss the three components of the theoretical framework that I used to set out the principles of teaching mathematics through problem solving as defined in this study. The first component describes the differences between teaching *about*, *for* and *through* problem solving. The second component considers the key features of task design that are important in the teaching of mathematics through problem solving. The third component, the technique of orchestrating the learning, is discussed in relation to the concept of contingency.

I then introduce a PD framework for analysis that was devised to support both the design and evaluation of the PD programme with regard to the characteristics of effective CPD. Finally, I discuss the PD approaches of Lesson Study and Teacher Design Teams that were incorporated into the PD programme design.

2.1 Teaching about, for and through problem solving

> The sheik addressed the three of them: "Here are my three friends. They are sheep rearers from Damascus. They are facing one of the strangest problems I have come across. It is this: as payment for a small flock of sheep, they received, here in Baghdad, a quantity of excellent wine, in 21 identical casks: 7 full, 7 half full, 7 empty. They now want to divide these casks so that each receives the same number of casks and the same quantity of wine. Dividing up the casks is easy – each would receive 7. The difficulty, as I understand it, is in dividing the wine without opening them, leaving them just as they are. *Now, calculator, is it possible to find a satisfactory answer to this problem?*
>
> From the man who counted Tahan (1993)

What a delightful problem!

The definition of problem solving has been located for some time within the work of Pólya. In their highly influential book that was first written in 1925 and translated in 1972, Pólya and Szegŏ (1972) expertly set out two pedagogical themes:

- the use of problems to systematically discover mathematics concepts and ideas
- the systematic development of a method (often now called problem-solving strategies) to solve the problem.

Burkhardt (2014, p. 3) defines the word 'problem' in mathematics as a task that is:

- Non-routine: A substantial part of the challenge is working out how to tackle the task. (If the student is expected to remember a well-defined method from prior teaching, the task is routine – an exercise not a problem).
- Mathematically rich: Substantial chains of reasoning, involving more than a few steps, are normally needed to solve a task that is worth calling a problem.
- Well-posed: Both the problem context and the kind of solution required are clearly specified. (In an 'investigation' the problem context is defined but the student is expected to pose questions as well as to answer them; investigations are implicit in the following discussion).
- Reasoning-focused: Answers are not enough; in problem solving, students are also expected to explain the reasoning that led to their solutions and why the result is true.

Burkhardt's definition of a 'problem' in mathematics and Pólya's articulation of the two pedagogical themes of 'problem solving' complement each other. Burkhardt defines the key characteristics of a problem whilst Pólya explains the purposes of solving such problems. Together they encapsulate why pupils should engage in the task: to help them discover mathematics concepts and ideas and so learn specified mathematical content, realising of course that as part of the process they will necessarily utilise and develop their problem-solving 'skills'.

In this study, a problem is any task presented to a learner where the solution is not known in advance (Liljedahl, 2016)[10]. Also, the ways in which the problem is presented to the learner and subsequently approached by them can illuminate new mathematical knowledge or develop problem-solving methods, or both. For example, in the wine problem above, the presentation of the problem using labelled 'paper' barrels could lead to a problem-solving 'sorting' approach where the 21 barrels are divided equally. Therefore, such problems require a creative process that some researchers refer to as 'flow' (Csíkszentmihalyi, 1997). In

[10] Liljedahl goes on to state that any problem in which you can see how to attack it by deliberate effort, is a routine problem, and cannot be an important discover.

problem solving it is often the trait of novelty with its companions of creativity and imagination that lead to success.

2.1.1 Teaching about and for problem solving

To explain the difference between teaching *about* problem solving and teaching *for* problem solving, in this study I make a distinction between the terms 'problem-solving strategies' and 'problem-solving skills'[11]. As will be explained below using a number of tasks, teaching *for* problem solving is where the task and the associated teacher pedagogy results in the teaching and learning of a problem-solving skill or skills, whereas teaching *about* problem solving is concerned with the pupil's own exploration of problem solving and the learning is largely the result of the pupil's own endeavours.

Founded in 1967, the Shell Centre for Mathematical Education is known internationally for its contribution to mathematics education. In 1984, *Problems with Patterns and Numbers* (Board, 1984) was published and became widely influential in the teaching of 'investigations' which was the term used at the time for the teaching of problem solving. As introduced in Chapter 1, the Shell resources supported teachers' efforts to meet the new National Curriculum requirements, particularly AT1 (Using and Applying Mathematics). The 'blue box', as the resources were known, comprised a range of non-routine problems, together with a number of associated problem-solving strategies. These included the following:

- try some simple cases
- find a helpful diagram
- organise systematically
- make a table
- spot patterns
- find a general rule
- explain why the rule works
- check regularly.

In the 1980s when coursework was part of the GCSE examination in England, these strategies were taught to pupils so that they could use them to carry out independent

[11] In one sense they can be regarded as the same, however I identify a key difference so as to be able to illustrate the different teaching approaches that can be associated with teaching about and for problem solving.

investigations into mathematical problems. Often, the teaching approach was to model the problem-solving strategy with a similar problem. Following this demonstration by the teacher, the pupils would explore a different task with little intervention by the teacher. This approach could be described as teaching *about* problem solving where the ambition of the teacher would be for the pupil to explore and develop such strategies in order to solve the problem**.**

Teaching *for* problem solving involves the teaching of a specific problem-solving skill which I argue is different from a problem-solving strategy. This difference is made clear by the classification in Figure 2.1 below, where problem-solving skills are grouped under four problem-solving strategies. This classification was developed as part of training materials for NNS numeracy consultants.

Figure 2.1 *Classification of problem-solving skills: consultant training materials (NNS, 2000)*

1. Generating data and listing

Pupils should be taught to:

- Generate data from a given rule or a set of conditions
- Derive a set of numbers or shapes that meet a list of criteria
- Find the largest and/or the smallest cases or values for given circumstances and conditions
- Systematically list and record all the possibilities in a set given a number of conditions

2. Sorting and classifying

Pupils should be taught to:

- Sort objects, numbers or shapes by deciding whether they meet a given criterion
- Classify a set of objects numbers, or shapes using a number of criteria or properties
- Identify criteria to describe sets of numbers, objects or shapes that have been sorted or classified
- Use sorts and/or classifications to complete sets with missing items

3. Identifying patterns and relationships

Pupils should be taught to:

- Organise data to generate and complete patterns
- Use symmetric properties in shapes, sets of numbers and calculations to establish relationships and enumerate lists
- Organise information into tables, charts and diagrams in order to recognise and discover patterns and relationships
- Describe relationships and patterns Manipulate these to generate new ones

4. Explaining and reasoning

Pupils should be taught to:

- Use calculations to support explanation and argument
- Look for a counter example to define the conditions and limits of a rule
- Use a relationship or pattern to justify or confirm others
- Use properties and relationships to reason and deduce
- Use and manipulate diagrams to support an explanation
- Generalise in order to prove

Using the classification above, the problem-solving strategy of sorting and classifying can be developed by teaching the ability (skill) to "identify criteria to describe sets of numbers, objects or shapes that have been sorted or classified". Therefore, in this model a problem-solving strategy is defined as any method that supports the solution of a problem, whereas a problem-solving skill is identified as a particular action that supports the development of the problem-solving strategy.

This classification of problem-solving skills is different to other taxonomies such as the Revised Blooms Taxonomy, SOLO Taxonomy, and Webb's Depth of Knowledge (Weay et al., 2016) in that this classification identifies specific strategies for solving problems rather than the hierarchical representation of general learning skills (outcomes). Whilst this classification could be described as hierarchical in that sorting and classifying could be a prerequisite for spotting patterns, I suggest that the problem-solving skill of sorting and classifying can be at least as complex as the skill of reasoning and proof, depending on the context and demands of the problem being tackled.

I do not contend that this list of strategies and skills is either complete or definitive but that it is a useful tool to demonstrate the difference between teaching *about* and *for* problem solving as defined in this study.

Teaching *for* problem solving in this study therefore is defined as a teaching approach in which teachers engage pupils in a problem-solving task in a particular way with the aim of drawing out and developing a specific problem-solving skill or skills. An example of such a teaching approach is set out below. Consider the following problem:

If the full-time score in a game of hockey was 3 – 4, what could the half-time score have been?

An obvious approach to solving this problem would be to start making a list of the possible half-time scores either randomly or systematically. The random approach could lead to duplicates and omissions, therefore an appropriate skill to teach pupils would be to "systematically list and record all the possibilities in a set given a number of conditions" (from the classification section Generating data). However, this problem could also be used to teach the problem-solving skill "use sorts and/or classifications to complete sets with missing items", by presenting the problem with an additional instruction as follows:

The full-time score in a game of hockey was 3 – 4. With a partner write down as many different possible half-time scores on the pieces of card provided.

By asking the pupils to record their half-time scores on small rectangular pieces of card, the pupils could then be asked to sort their cards according to their own criteria or the criteria defined by the teacher.

According to Piaget (Lavatelli, 1970), simple sorting is viewed as a beginners' type of grouping task in which the way objects are to be sorted is shown or told to the child. For example, if a child is given a set of plain shapes, they could be asked to sort the shapes into sets of different colours. Here, the children are given or told the grouping pattern for the objects. A classifying task, on the other hand, requires children to decide how a given set of objects might be grouped according to some criteria. For example, the children could be asked to put the shapes into a number of groups and then describe the criteria by which that group has been formed. In the half-time scores problem above, the children could be asked to sort their cards under the headings given to them as shown in Figure 2.2 (incomplete diagram) below.

Figure 2.2 *Incomplete sort for half-time scores problem*

Sort Card Headings

Draw	**0 – ?**	**1 – ?**	**2 – ?**	**3 – ?**
0 – 0	0 – 1	1 – 0	2 – 0	3 – 0
1 – 1		1 – 2	2 – 1	
2 – 2	0 – 3	1 – 3	2 – 3	
3 – 3	0 – 4	1 – 4	2 – 4	3 – 4

This 'sort' introduces the children to a problem-solving strategy that can be used to solve 'combinatorix' type problems. For example, by sorting the cards as in Figure 2.2 above, the missing half-time scores can now be more easily be identified:

0 – 2   3 – 1   3 – 2

This example illustrates how a task can be developed for the purpose of teaching a specific problem-solving skill that is used to work towards a solution. Asking the pupils to sort according to some given criteria is different to asking the pupils to sort their cards into a grouping or pattern of their choosing. Therefore, the decision on which approach to use is central to the task design process. Figures 2.3 and 2.4 show other possible classifications: ‘sum of the scores’ and ‘score difference’.

Figure 2.3 *Sum sort for half-time scores*

Sum of Scores

0 – 0	0 – 1	0 – 2	0 – 3	0 – 4	1 – 4	2 – 4	3 – 4
	1 – 0	2 – 0	3 – 0	3 – 1	3 – 2	3 – 3	
		1 – 1	2 – 1	2 – 2	2 – 3		
			1 – 2	1 – 3			

Figure 2.4 *Score difference sort for half-time scores*

Score Difference

0 Diff	**1 Diff**	**2 Diff**	**3 Diff**	**4 Diff**
0 – 0	0 – 1	0 – 2	0 – 3	4 – 0
1 – 1	1 – 2	1 – 3	1 – 4	
2 – 2	2 – 1	2 – 0	3 – 0	
3 – 3	3 – 2	3 – 1		
	2 – 3	2 – 4		
	1 – 0			
	3 – 4			

It is important to appreciate that the task design process can also contain the teaching approach that indicates how the task will be used with the pupils. The above task could have been given to pupils without access to pieces of card. This then changes how the pupils will engage in the task and naturally inhibits the use of a sorting strategy. Both teaching *about* problem solving and teaching *for* problem solving require the identification of a set of problem-solving strategies and problem-solving skills that can be used to the solve the problem.

Schoenfeld (1980) posed two interesting questions in relation to the teaching of problem solving:

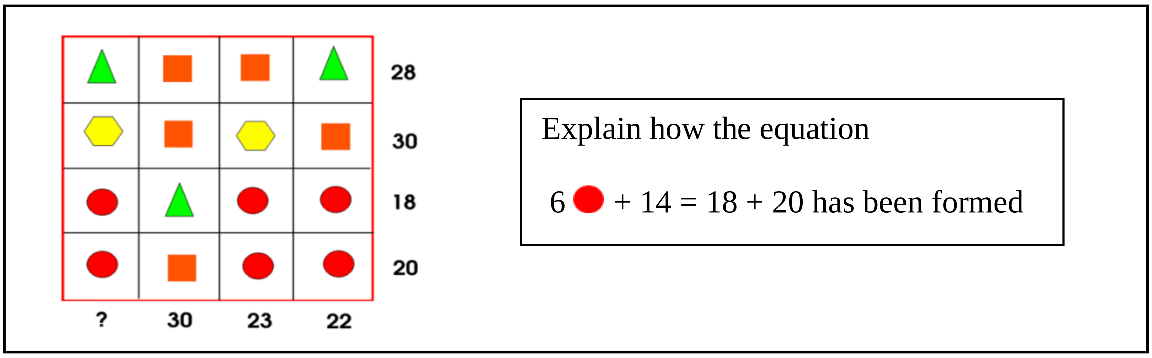
1. Can we accurately describe the strategies used by ‘expert’ mathematicians to solve problems?
2. Can we teach students to use those strategies?

The answer to question 1 is dependent on the interpretation of the word ‘expert’ while question 2 presents a related concern regarding the teaching of mathematics through problem solving. Irrespective of the absolute answers to these two questions, in order to teach mathematics through problem solving it is important to appreciate that children will attempt to solve mathematical problems using a range of strategies and only some of these may relate to those used by an ‘expert’ mathematician.

2.1.2 Teaching mathematics through problem solving

Teaching mathematics *through* problem solving means engaging pupils in problems that have been designed to introduce the pupils to new mathematics. For example, the task below was designed to help pupils to develop a conceptual understanding of the formation and manipulation of linear equations, leading to the algebraic technique of ‘substitution’.

Figure 2.5 *Example of problem used to introduce linear equations*

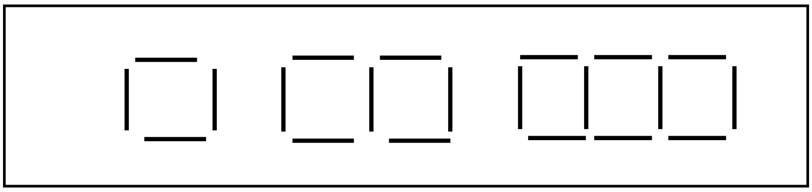


The explanation that 2  + 2  = 28 could lead to the introduction of the linear equations 2S + 2T = 14 and S + T =14.

I include this example not only to show the design of a task to teach mathematics through problem solving but also to illuminate the level of challenge in the design process.
Whilst this may look like a suitable problem to introduce the concepts of algebraic expressions and linear equations, it also potentially raises an issue concerning the use of the multiplicand and the multiplier. In the term ‘2T’, 2 is the multiplier and T is the multiplicand (here the multiplier is written before the multiplicand[12]). However, if the term 2T is read as ‘2 multiplied by T’ this means that the term 2T is equal to T ‘lots of’ 2, i.e. 2 + 2 +2 + 2 + … (T times!) and therefore the positions of the multiplier and multiplicand are reversed. It should then follow that the algebraic equation from the top row in the diagram should be S2 + T2 = 28. Of course, this is not how we structure algebraic equations and many would argue that this is irrelevant due to the commutative nature of multiplication.

Whilst the position of the multiplier may well be unimportant for the solution to this problem, there is a potential issue with the problem design and it might lead to unnecessary confusion later on, when in the expression 3x it is the ‘x’ term that is the variable (Usiskin, 1988). In the problem above this would mean that T is the variable; whilst the number of triangles can vary, the properties of the triangle cannot. Also, understanding this issue is important in the concept of division where problems can occur in understanding the difference between partitive and quotative division. In other words, ‘3n’ may be equal to ‘n3’ but is certainly not identical to it! As a possible alternative, consider the task below adapted from the work of John Mason (Mason, 1996).

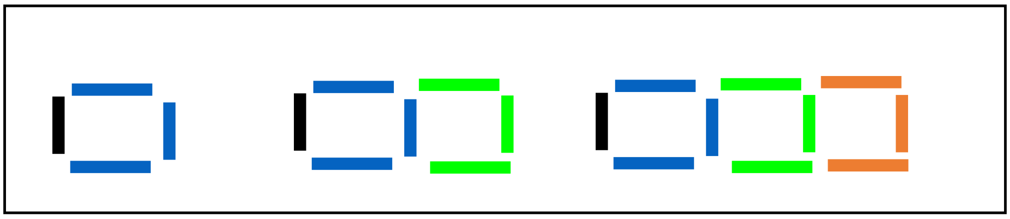
Figure 2.6 *Matchstick sequence*



[12] For an interesting discussion on whether the multiplier should precede the multiplicand, read the article by Saradakanta Ganguli (Ganguli, 1932).

In the diagram above we can see the growing sequence 4, 7, 10 … leading to the n^{th} term 3n +1. By indicating how the pattern grows, as in Figure 2.7 below.

Figure 2.7 *Matchstick sequence - growing*



We can show that the pattern grows as follows:

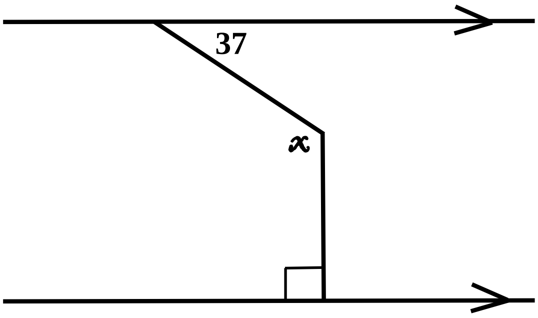
1 + 3, 1 + 3 + 3, 1 + 3 + 3 + 3, … leading to 1 + 3n.

This representation of the sequence models the term '3n' with n being the multiplier and not the multiplicand as in the previous example.

The last example in this section uses a task to connect the process of task design with teaching of mathematics *for* problem solving and *through* problem solving. Consider the problem below:

Figure 2.8 *Parallel lines angle problem*

Find the value of the angle marked x



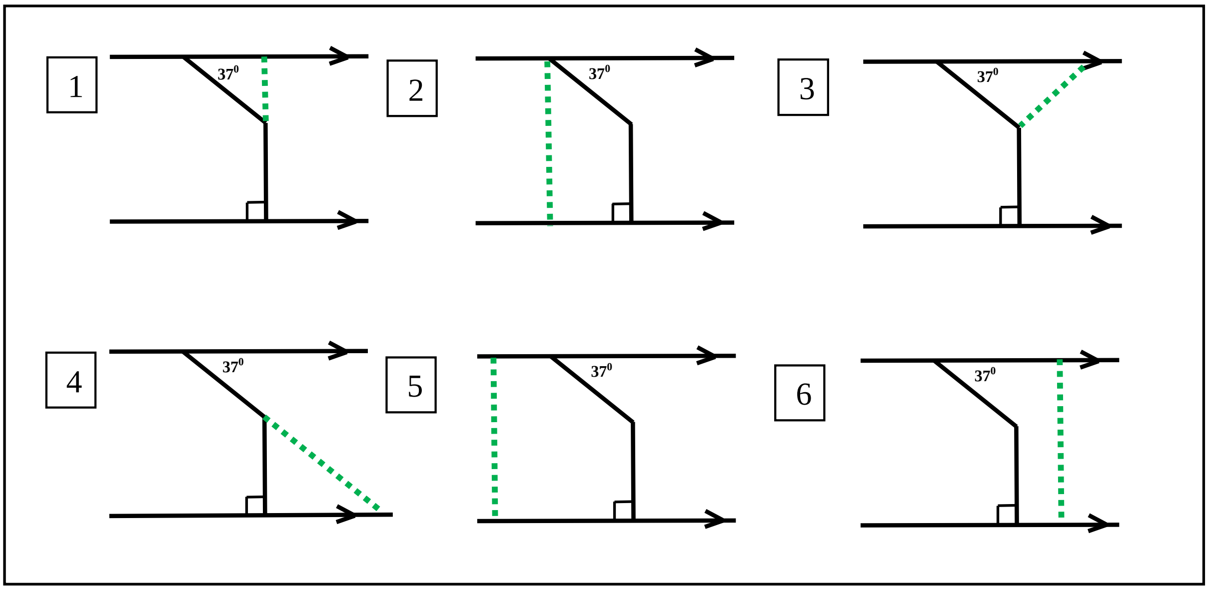
The way the instruction is posed, there are a number of different methods to solving this problem, many of which involve imagining or constructing additional lines. However, consider the task if the question is augmented with an additional instruction:

'By drawing just one additional straight line, justify that $x = 127^0$'

Now pupils are required to pursue the solution to this problem by first adding an additional line to the figure thereby encouraging the use and development of a specific problem-solving skill: ‘Use and manipulate diagrams to support an explanation’.

The pupils could also be required to provide an explanation and reason for the location of their line. The following set of diagrams in Figure 2.9 illustrate some of the potential responses. In each case the ‘one additional line’ is drawn in green (dotted line).

Figure 2.9 *Possible solutions by adding just one line*



Not only does this modification of the task enable the teaching of a specific skill and so supports the approach of teaching *for* problem solving, in this instance the problem can also be used to teach mathematics *through* problem solving. For example, by exploring the solutions 2, 5 and 6 in the diagram above, the teacher can introduce the sum of the angles in any polygon (from the assumed knowledge that the angle sum in a triangle is 180^0). Therefore, the task selected and the teaching approach used are both critical in the method of teaching mathematics through problem solving. Importantly, then, it is not only the task itself that determines whether it should be used to teach *about*, *for* or *through* problem solving, but is also about how the task is used by the teacher.

Teaching mathematics *through* problem solving should not be confused with Enquiry Based Learning (EBL). Although there are several similarities, EBL is an approach to learning that is driven by a process of enquiry where the teacher establishes the task and supports or facilitates the process, but the students pursue their own lines of enquiry, draw on their existing knowledge and identify the consequent learning needs. They seek evidence to support their ideas and take responsibility for analysing and presenting this appropriately, either as part of a group or as an individual supported by others. They are thus engaged as partners in the learning process (Kahn & O'Rourke, 2005).

The key difference between EBL and teaching mathematics through problem solving is that in the latter approach the task presented to pupils has been 'engineered' to enable the teacher to 'reveal' and develop a particular mathematical concept or idea through the exploration of the different responses and solutions anticipated by the teacher and used by them to introduce new learning.

However, both approaches require the pupils to spend time exploring a problem and developing their thoughts and ideas. This approach is often challenging for teachers who are not used to their pupils working in this way. Putnam and Borko (2000) argue that teachers who may be seeking to change their practice may not have useful images from their personal experience and may not have vision of what teaching mathematics through problem solving should look like. In fact, and as reflected by the teachers in this study, beginning the lesson with a problem that would be used for the whole lesson to teach the intended mathematics is rarely considered.

The relationship between teaching about, for, and through problem solving is also important. As defined above, in this study a problem is something that you do not know how to solve at the point of encounter. This could be because of a lack of experience of the type of problem but could also be because of not having the necessary skills at the point of encounter (Liljedahl, 2016). Therefore, in order to teach mathematics through problem solving, the learner should have a range of problem-solving skills sufficient to explore the mathematics and produce solutions or partial solutions so that the teacher can then use these to illuminate and develop the mathematics from the learner's work. The point here is that if the task designed requires particular problem-solving skills in order to access the problem then these need to be already held by the learner or be capable of being developed through the exploration of the problem itself.

The PD programme in this study has been designed to support teachers in the development of teaching mathematics *through* problem solving which I assert is different to teaching only *about* and *for* problem solving. However, as discussed below there is an important connectivity between these notions that needs to be recognised in the design of tasks for teaching mathematics through problem solving. I exemplify the differences in section 2.3. Teaching mathematics through problem solving in this study refers to a teaching approach where pupils learn mathematics while solving tasks that have been designed to introduce them to a new mathematical idea or concept *and* the teacher uses a pedagogy that orchestrates the pupils' responses to develop the learning. This approach is sometimes referred to as 'problem-based instruction' (King, 2019). Within both definitions, the pupils work through a problem and subsequently discuss different methods to build a procedural and conceptual understanding about the mathematics involved (Cai, 2003). However, the trouble with the term 'problem-based instruction' in this case is the word 'instruction' which in the UK evokes a particular image of the teaching approach which does not capture the pupils' full contribution or the teacher's particular pedagogy in working with the pupil responses. Further, it is important to distinguish this approach of teaching mathematics through problem solving from Problem Based Learning (PBL). There are similarities in that PBL is a constructivist model of learning that focuses on the pupils' responses (Yelland et al., 2008). Amalia et al. (2017) set out the steps of PBL:

a. defining the problem
b. self-learning
c. investigation
d. exchange knowledge
e. assessment.

Terms such as 'self-learning' and 'exchange knowledge' do not directly relate to the approach of teaching mathematics through problem solving as described above. In teaching mathematics through problem solving, as pupils first explore the problem they may learn or invent new approaches to solve it, but these on their own may not always lead to pupils understanding the new mathematics to be learned, as might be implied by the term 'self-learning'. As an example, see the area problem in section 2.4. Also, the term 'knowledge exchange' suggests that the new learning is present somewhere in the class and will be revealed simply by pupils exchanging ideas. Again, this does not reflect the crucial role of the teacher in orchestrating pupils' contributions to develop the new learning, nor does it recognise the importance of the design of the task in relation to the mathematics to be learned.

Finally in this section, the distinction between the words ‘technique’ and ‘pedagogy’ is important. A technique within the practice of the teacher refers to a specific action that the teacher uses to incorporate a particular pedagogy. For example, if the teacher believes that cooperative learning is an effective pedagogy, they may use techniques such as Kagan structures (Kagan, 1994) which require pupils to take turns to speak and listen. In a similar way, the orchestration of pupils’ responses to a problem is referred to as a teaching technique that is used within the pedagogy of teaching mathematics through problem solving.

2.2 Task Design

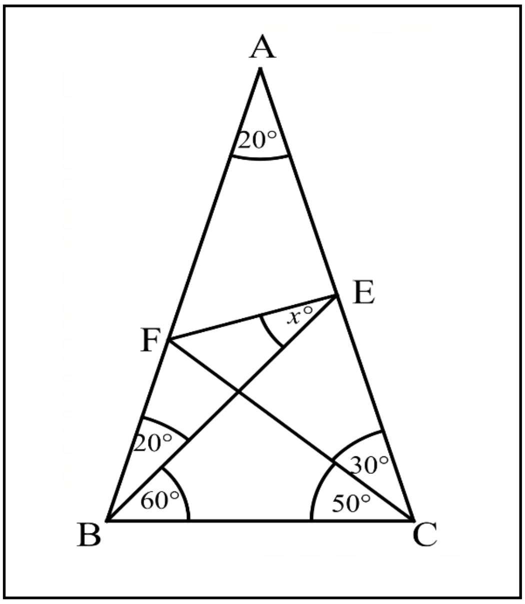
The use of task design was a key component in the PD programme and underpinned the development of the approach of teaching mathematics through problem solving. Here, the definition of a task is “anything that a teacher uses to demonstrate mathematics, to pursue interactively with students, or to ask students to do something” (Watson et al., 2013, p. 12). Hiebert and Wearne (1993) suggest that “what students learn is largely defined by the tasks they are given” (p. 395). It would be reasonable to assume, therefore, that the tasks chosen by teachers reflect their understanding of the mathematics to be learned by the pupils; so, as Sullivan and Mousley (2001) have argued, teacher professional development should focus on supporting teachers in understanding the complexity of decision-making about classroom tasks.

The use of task design processes within a range of effective PD methodologies can have a significant impact not only on teacher subject knowledge and pedagogy but also on teacher beliefs (Wilson & Cooney, 2002). The ICME Study 22 Task Design in Mathematics Education (Watson & Ohtani, 2015) is a compendium of wide-ranging articles that offer insights into the use of theoretical frameworks and their affordances in the development of tasks, as well as student perspectives in relation to task design, and issues related to textbook-based tasks. The development of tasks in this study has incorporated some of the principles of task design using the frameworks of Komatsu and Tsujiyama (2013) and which are summarised by Kieran et al. (2015). The three principles are:

1. Educators and teachers should select or develop certain kinds of proof problems with diagrams where students can find counterexamples or non-examples and engage in deductive guessing through changing the diagrams.
2. Teachers should encourage their students to change the diagrams while keeping the conditions of the statements, so that they find counterexamples or non-examples of the statements.
3. After students face the counterexamples or non-examples, teachers should plan their instructional guidance by which students can utilize their proofs of initial problems to invent more general statements that hold true for these examples.

For example, in the task in the previous section, the problem on finding an unknown angle was designed to encourage pupils to think about modifying the diagram, a strategy that is essential for solving problems such as the Adventitious Angles problem posed by Edward Mann Langley (1922, p. 173), as shown in the figure below.

Figure 2.10 *Edward Langley's original problem (1922)*



There are over 40 solutions to this problem that do not need the use of trigonometry but can be solved using elementary geometry and the introduction of additional lines to create isosceles and equilateral triangles.

In addition to the principles above, I assert that an important element of task design is the potential for the task to prompt the learning of the intended mathematical concepts (Sullivan et al., 2015). This is achieved not just through the careful selection of representations, context and choice of numbers but also in the associated pedagogy. Whilst there is no guarantee that a design developed for one particular setting will be effective in other settings, it is therefore important that the design of the tasks reflects the context in which they are to be presented. Also, the implementation of tasks can subvert the aims of the task designer, for example when teachers modify tasks to increase or decrease the level of challenge. Care should be taken to ensure that any such adaptations do not result in changes in the learning goals (Tzur et al., 2008).

For some time, the modification of tasks has been a focus in the professional development of teachers. For example, following the introduction of the NCTM Standards (NCTM, 1989), PD programmes for teachers incorporated the use of tasks which had been modified in order to encourage teachers to think differently about how to present mathematical content and knowledge. The rationale for this approach was based on the belief that teachers are strongly influenced by the ways that they themselves learned the subject matter (Kagan, 1992). Zaslavsky (1995, p. 15) describes how teachers responded to this type of PD activity in workshops where they were presented with a closed task (one solution) and a modified open task (multiple solutions), as set out in Figure 2.11 below.

Figure 2.11 *Open-ended tasks as a trigger for mathematics teachers' professional development (Zaslavsky, 1995)*

A standard task	A modified task
How many intersection points does the parabola $y = x^2 + 4x + 5$ have with the straight line $y = 2x + 5$?	Find an equation of a straight line that has two intersection points with the parabola $y = x^2 + 4x + 5$

After working on both tasks, the teachers reflected on their experiences as learners through professional discourse. Zaslavsky observed that although the modified task only required one solution, the teachers demonstrated a desire to continue to work on the task in a number of different ways. The teachers then began to design mathematical tasks, similar in nature to the above, and implement them in their own classrooms on a regular basis. As a result, Zaslavsky concluded that “through such activities teachers seemed to develop both aspirations to change and the confidence that they can teach differently” (1995, p. 19).

Indeed, according to Males (2018), most research on the use of curriculum materials supports the notion that there is some kind of interaction between the teacher and the materials. For example, Remillard (2005) describes this as a participatory interaction where the influence is bi-directional, meaning that the teacher influences the materials and the materials influence the teacher. Sullivan et al. (2015) state that "the role of the teacher is to select modify, design, redesign, implement and evaluate the tasks given to their pupils" (p. 83) and recognise that tasks and pedagogies are interrelated. In the present study, the relationship between the selected task, the anticipated responses and the resulting learning is central to the design of the lesson planning process. Hence the PD programme acknowledged the critical importance of teachers' knowledge to accurately present mathematical ideas and to examine and understand unusual methods and solutions to problems (Hill et al., 2008).

The decisions teachers make about how tasks are chosen, devised or augmented are closely related to their beliefs (Wilson & Cooney, 2002). For example, Drageset (2010) argues that teachers who believe that reasoning and proof are the most important aspects of mathematics will strengthen their own learning and understanding of reasoning and proof because they spend more time on these aspects, and that this will be evident in the type of tasks they present to pupils. Other teachers, however, may believe that procedural fluency is the most important aspect of the students' mathematical knowledge. These teachers will tend to choose tasks that require procedural knowledge, thus promoting the repeated practice needed to acquire procedural fluency. In both cases, the belief systems are strengthened through their practice but at the same time other aspects of their practice are weakened. The first teacher will become more skilled at teaching through reasoning and proof and the second teacher more skilled in procedural fluency. Subsequently, they may each have underdeveloped expertise in the complementary aspect which further strengthens their beliefs about their preferred practice.

Being able to link particular beliefs to barriers in learning is a starting point for those beliefs to be challenged. However, as Drageset (2010) explains, whether or not beliefs are amenable to change also depends on the nature of the belief. According to Wheatley (2002), studies of teacher efficacy have identified two types of beliefs. The first type is general teacher efficacy where the teachers' beliefs relate to the ability of teachers in general to influence student outcomes. Personal teacher efficacy refers to the teachers' beliefs about their own ability to affect student outcomes. Clearly some beliefs pertain to both types of teacher efficacy.

Understanding how teacher efficacy can be influenced is clearly an important dimension of the development of PD programmes for teachers.

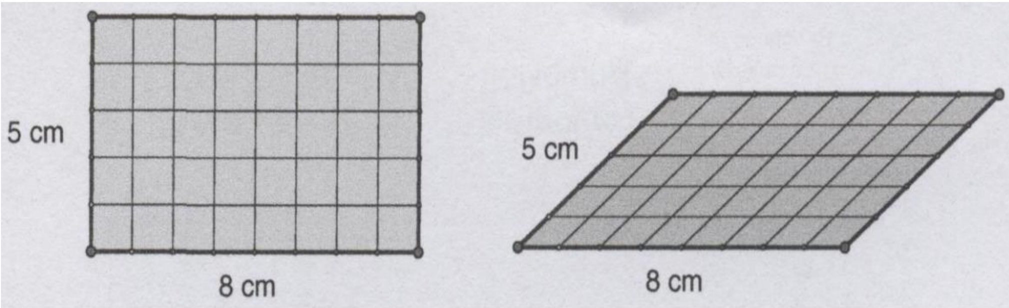
The challenge of identifying beliefs that are associated with affordances and barriers in learning and which may be amenable to change is compounded by the fact that we know little about how teachers learn to use mathematical tasks and problems in the classroom, despite the fact that the majority of teachers use some type of resource (Males et al., 2018).

Whilst engaging in task design methodology can impinge on attitudes and beliefs it is not clear whether any resulting changes will lead to positive changes in outcomes for learners, except where teachers' own attitudes towards mathematics impact on the attitudes of learners (Cheng & Lo, 2013). In fact, Loewenberg Ball and Forzani (2009) argue for making practice the core of teachers' professional preparation. They suggest that a PD focus on the development of beliefs and knowledge on orientations and commitments,[13] rather than on detailed professional training on the tasks and activities of teaching, impedes the improvement of teachers' preparation for the work of teaching. However, I consider that it is still important to understand teachers' enacted beliefs, as it is reasonable to assume that what teachers do in the classroom is connected to their understanding of how mathematics should be taught and that this is largely based on their previous experience. In this study, the teachers' enacted beliefs were explored by observing changes to the participants' views and attitudes to teaching mathematics through problem solving as a result of engaging in the PD programme, and also by identifying those beliefs or views that appeared not to be open to change.

Whilst the use of curriculum materials is integral to teaching, Wasserman (2015) argues that the complexity involved in decompressing (unpacking), trimming and bridging mathematics tasks and activities requires teachers to have a depth and breadth of mathematical understanding which extends far beyond the specific content they are teaching. For example, consider the problem in Figure 2.12 (Wasserman, 2015, p. 82) below.

[13] For example, the current Early Career Framework for teachers in England (Department for Education, 2019) sets out what early career teachers are entitled to learn about and learn how to do when they start their careers. The framework underpins a new entitlement to two years of professional development designed to help early career teachers develop their practice, knowledge and working habits, but does not include subject-specific training on teaching the subject.

Figure 2.12 *Unpacking teachers' moves in the classroom (Wasserman, 2015)*

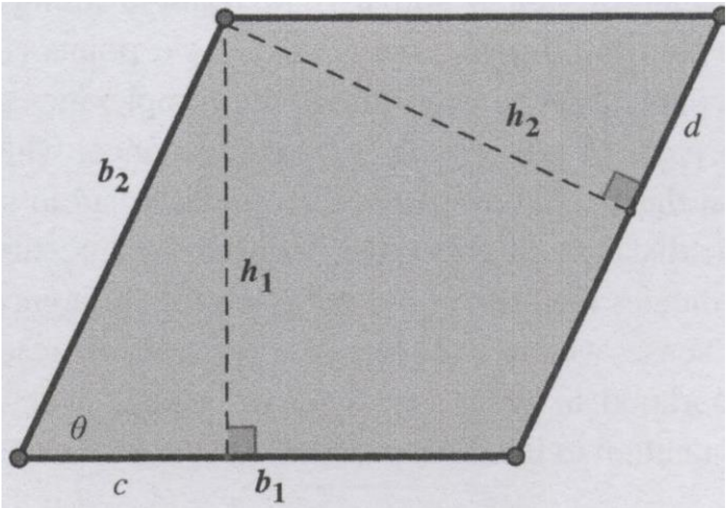


Calculate the area of both shapes in the diagram above.

As Wasserman explains, understanding that squares are used as the unit of measurement for area is conceptually important for developing a 'sense' of area especially in shapes where squares do not seem to fit well, such as parallelograms, circles and polygons. The challenge for teachers in this example is that whilst there are clearly 40 squares in the rectangle, the parallelogram also contains 40 of something but they are not squares. How does the teacher in this instance 'unpack' their own subject knowledge in order to explain to the pupils that the areas of these two shapes are not the same?

In many cases it would be easier for the teacher to avoid this situation and simply introduce the method of turning the parallelogram into a rectangle by removing and relocating a triangle. However, if used alone, this method does not address the fact that parallelograms are more complex than rectangles because of the relationship between the side lengths and heights – often parallelograms have irrational lengths due to the Pythagorean relationships within the right-angled triangles. Tackling these conceptual issues cannot be done simply by introducing a formula for the parallelogram (Area = base x vertical height) which is justified by modelling the transformation of the parallelogram into a rectangle. Further, if we want pupils to understand that the area of the parallelogram is the enumeration of square units, then it is important to show that this can be achieved by using either dimension of the parallelogram as the base. This requires the identification of a parallelogram that has four integer measurements (h_1, h_2, b_1 and b_2) as shown in Figure 2.13 below.

Figure 2.13 *Unpacking teachers' moves in the classroom (Wasserman, 2015)*



The process of designing a parallelogram with the above characteristics is an example of task trimming. Wasserman (2015) explains that the purpose of trimming[14] is to remove complexity. Using a parallelogram with the above properties enables the calculation of the area to be enumerated as a number of squares so as to obtain an area of an integer number of squares units, thus removing complexity for the learner (depending on the accompanying teacher instruction).

However, for the teacher this process of task design involves some crucial mathematical thinking. The two bases b_1 and b_2 will have a common divisor and the common divisor will at least include the hypotenuse of a primitive Pythagorean triple. In addition, the angle θ must be such that $\sin(\theta) = a/c$ and $\sin(\theta) = b/c$, meaning that $h_1/b_2 = h_2/b_1$. This is an example of the features of good task design where careful consideration of the numbers is aligned with the key intentions of the task.

In consideration of the discussion above, certain design choices were made in relation to the tasks used in the PD programme. The first relates to the work of Zaslavsky who demonstrated that the activity of working with modified tasks can affect the way in which teachers think about their teaching. This strategy was used in the PD programme to explain how tasks could be modified to support the teaching of problem-solving skills and teaching mathematics through problem solving. The second design choice relates to the selection of tasks within the

[14] It is important to state that trimming should not be confused with scaffolding. Whilst these are similar, trimming removes the complexity from the problem whereas scaffolding manages complexity within the problem.

PD programme which were chosen to exemplify the importance of task design in relation to teaching problem solving and teaching mathematics through problem solving. Also, the trimming technique in task design was identified as important (albeit not discussed in the detail given by Wasserman above) with regard to the purpose of the task and specific design features such as the context of the problem and the choice of numbers to be used to develop a mathematical concept or idea. Examples of the problems incorporating these design features are detailed in the sections below.

2.3 Orchestrating the learning

Orchestrating the learning is a teaching technique used in this study in the approach to teaching mathematics through problem solving. It involves a teacher-led discussion which the teacher shapes using a sequence of the pupils' responses that is normally pre-planned.

Rowland and Carson (2001) state that "constructivism's popularity seems largely due to the consensus that the learner is not a passive recipient of knowledge but that knowledge is 'constructed' by the learner in some way" (p. 1). Constructivism also assumes that learning is a social process where individuals learn through interacting with other people (Pritchard & Woollard, 2010). Thus, a constructivist approach requires the teacher to create a learning environment in which the learner can construct their experience and knowledge. By contrast, Johnson (2005) refers to 'instructionism' as educational practices that are teacher-focused, skill-based, product-oriented, non-interactive and highly prescribed, but argues that these two apparently contradictory orientations of instructionism and constructivism can in fact be highly compatible. In this study I assert that 'orchestrating the learning' is an example of a technique where instructional teaching is interwoven with constructivist approaches.

In their study of trainee teachers, Rowland et al. (2005) introduced a framework through which the mathematics-related knowledge of these beginning teachers could be observed in practice. The framework is known as the Knowledge Quartet (KQ) and comprises four dimensions established from 18 observational codes (teacher practices and responses). These four dimensions are:

- foundation
- transformation
- connection

- contingency.

The last of these, contingency, is described by the authors as ‘knowledge-in-interaction’ and is revealed by the ability of a teacher to ‘think on their feet’ and respond effectively to a contribution made by a pupil. The codes used to form this dimension were:

- responding to children’s ideas
- use of opportunities and deviation from agenda.

These are skills that a teacher can develop over time and through experience. However, I would proffer that almost every teacher as part of that experience would have found themselves, at some point, being unable to respond to a pupil’s contribution because they did not comprehend the idea, method or question being presented by the pupil. I recall from my own experience having to ‘deflect’ a response by using the phrase “that is really interesting, ‘Alicia’, we will come back to this later”.

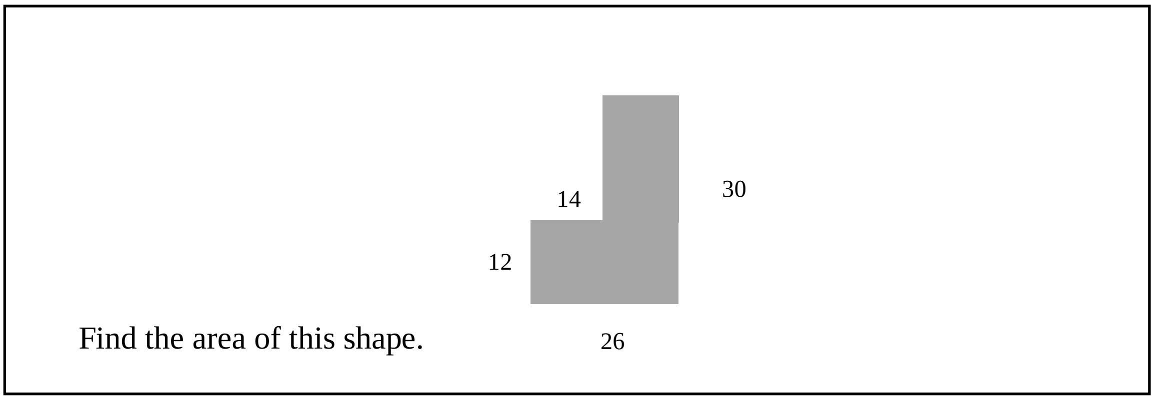
In other words, the teacher is so surprised and unprepared in that moment by the contribution from the pupil that they are unable to offer a suitable response. Coles and Scott (2015) state that contingency and the unexpected are intimately connected and that the teacher demonstrates contingency if they are able to respond in a manner they had not planned and which accommodates the student’s contribution, to a greater or lesser extent, in the subsequent events of the lesson. The majority of Japanese problem-solving lessons (‘structured problem solving’) follow the same format described by Fujii (2015) as a lesson focusing on a single task and containing four phases:

> Phase 1. Presenting the problem for the day (5 to 10 minutes)
> Phase 2. Problem solving by the students (10 to 20 minutes)
> Phase 3. Comparing and discussing (neriage) (10 to 20 minutes)
> Phase 4. Summing up by the teacher (matome) (5 minutes).

Phase 3 of the lesson requires the teacher to plan for a range of pupil responses so that during the neriage the teacher is better equipped to demonstrate contingency. The planning for these lessons also contains information about how the teacher will sequence the expected or anticipated responses. The purpose of this teaching approach is to maximise the learning during the neriage part of the lesson.

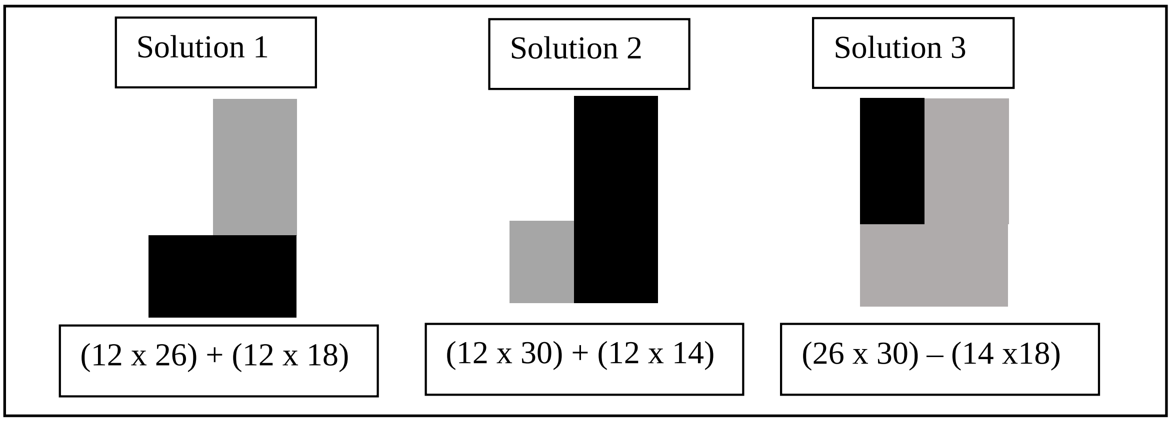
In this study, this sequencing of pupil responses is called 'orchestrating the learning' and is the approach that is developed in the PD programme. Whilst there is no assertion that this is the most effective way to teach mathematics through problem solving, it is identified as an approach to be used in conjunction with task design to develop teachers' ability to teach mathematics through problem solving. The approach is exemplified by the task shown in Figure 2.14 below and the explanation that follows.

Figure 2.14 *Area problem*



There are several ways in which a solution[15] to this problem could be approached, three of which are set out in Figure 2.15 below.

Figure 2.15 *Three solutions to the area problem*



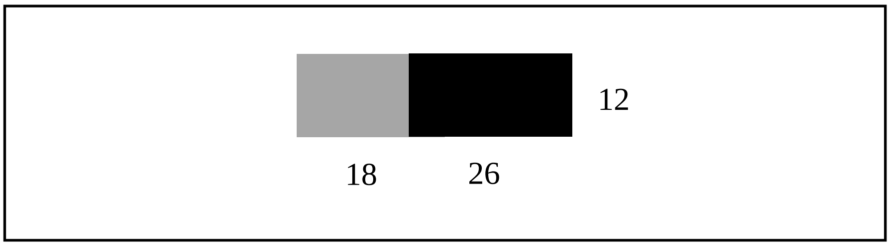
[15] Note that in this problem no units have been attached to the numbers; this is something that would be uncommon in Japanese problem-solving lessons involving area.

In a lesson that produced these three different methods (importantly there are others – see below), the order in which these solutions are discussed by the whole class could affect the way in which the understanding of the mathematics is developed.

For example, if solutions 1 and 2 were considered first and the calculations for each were examined, it could be proposed that both calculations can be expressed as equivalent to the calculation 12 x 44 (note that the calculations for solutions 1 and 2 have been written consistently so that the '12' in each case is the first term (which in this context could be the multiplier or the multiplicand).

However, with the appropriate choice of numbers (as in this example), two further approaches could be considered that connect the algebraic manipulation of the calculation to a physical representation. Consider solution 1 in Figure 2.15 above. By separating the grey rectangle from the black rectangle and rotating this grey rectangle through 90^0 (best shown as an animation), this rectangle can now be re-joined with the black rectangle to form a new rectangle with dimensions as shown in Figure 2.16.

Figure 2.16 *Transformation solution to area problem*



The area of this shape can now be expressed by the calculation 12(26 + 18). If this method was one that appeared in the lesson, then the comparison of this method with solution 1 above could be used to introduce some new mathematics. Through the principle of conservation of area, this calculation and the calculation from solution 1 are equal. We can now write (12 x 26) + (12 x 18) = 12(26 + 18) which could lead to the introduction of factorisation[16]. Thus, the concept of factorisation has been *revealed* to the pupils through the careful design of a task and a planned sequence of orchestration.

[16] Again note the challenge with this form as the two multipliers in the expression (12 x 26) + (12 x 18) are different.

Note that by partitioning solution 2 in a similar way the following equation can be developed: (12 x 30) + (12 x 14) = 12(30 + 14).

If we want the method of separation and reorientation to occur in the lesson, the task has to be designed using numbers that enable the 'L' shape to be transformed into a rectangle as described above. However, I have frequently been challenged when sharing this task with teachers. Often they assert "*my pupils would never think of doing that!"*. My courteous reply has always been to suggest that whilst they might not have before, with this task the pupils now have the opportunity to think of doing the reorientation, whereas if the task used numbers that do not allow this method, the pupils never will. The inference in my retort is that in order for children to be imaginative we must give them the playground in which to do so!

It is important to state that in such planning for orchestration there may be a number of different but equally effective sequences. For example, if the purpose of the lesson was to introduce factorisation, one should consider how the calculation for solution 3 would be used with solutions 1 and 2. How would the discussion progress to show in a similar way that the calculation (26 x 30) – (14 x18) was equivalent to 12(26 + 18) and 12(30 + 14)?

As an activity that might be used with teachers, the purpose of the exercise is not to try and establish the 'correct' sequence but to show that exploring and experimenting with the different possible sequences is an effective professional learning activity in the study of teaching mathematics through problem solving.

Earlier in this section, I stated that this orchestration technique was an example of how constructivism and instructionism could be compatible bedfellows. The constructivist approach is evident in the fact that the pupils will first explore the problem independently[17] and the teacher will subsequently facilitate the learning by assimilating and discussing the pupil responses. However, there is also the need to direct and instruct the pupils to compare the solutions that the teacher has chosen. For me, therefore, the line between constructivism and instructionism becomes naturally blurred and in this situation I consider this a welcome affordance.

[17] 'Independently' refers to being independent of the teacher but not necessarily on their own.

2.4 Framework for developing the professional development programme

In this section, I discuss the literature on the frameworks to evaluate the quality of Continuing Professional Development (CPD) that were used to inform the design of the PD programme and to establish the extent to which the delivery mechanisms and attributes of the programme align with the known features of effective CPD. I also include a discussion of the CPD approaches[18] that were used in the delivery of the programme.

Over recent years, many PD programmes have been criticised for their duration, content or purpose, in light of evaluations which indicate failures to produce meaningful improvements in teaching quality or student outcomes. Cordingley et al. (2015) express concerns about the effectiveness of short courses. Hendrick (2017) considers that the content of many CPD programmes has never been requested or identified by teachers. Kennedy (2005) defines the ‘deficit model’ for CPD the purpose of which is to contribute to the performance management of individual teachers and which is designed to overcome their identified needs relative to an often abstract set of externally defined goals.

McCormick (2010) in his review entitled ‘The state of the nation in CPD’ states that “the CPD literature has not served the field well in terms of a paucity of literature on what happens in ordinary schools and under-theorised work, particularly in terms of teacher learning” (p. 395). The report indicates that there is a problem with the theoretical basis of the CPD literature: whilst the empirical evidence has been established for many of the common features of CPD, there are insufficient theoretical frameworks for the effectiveness of particular programmes.

My personal experience resonates strongly with McCormick’s findings and includes programmes such as the nation-wide training of numeracy consultants in the roll-out of the National Numeracy Strategy (NNS) where PD materials devised by ‘strategy experts’ were presented to consultants who then were responsible for disseminating these materials to lead teachers in all primary schools in England.

During the last 20 years, many frameworks and criteria have been developed to improve our understanding of teacher CPD (Darling-Hammond., 2017; Desimone, 2009; Harris et al.,

[18] In this study CPD approaches are defined as the structures or delivery mechanisms that support the effective implementation of the PD programme, namely Lesson Study and Teacher Design Teams.

2006; Kennedy, 2014; Lewin & Stuart, 2003). The majority of frameworks now consider both the content and the delivery mechanisms of CPD simultaneously. For example, Desimone (2009) argues that appropriate content focus, active learning, coherence, duration and collective participation are all essential for effective professional development. (These aspects will be addressed in more detail later in this section.) The convergence of criteria that define effective professional development is clearly beneficial as CPD designers can utilise the criteria in the further development of programmes.

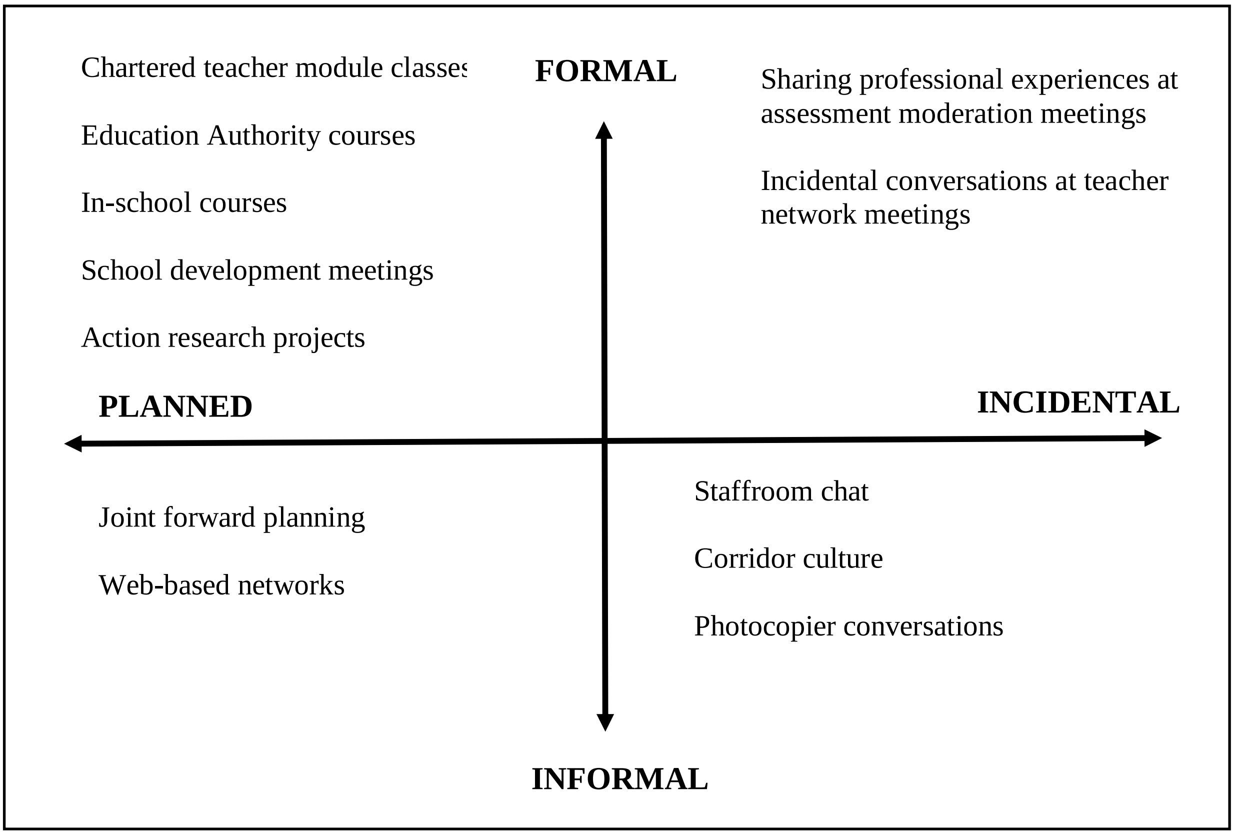
Whilst many of these frameworks form a broad consensus of the key components of effective CPD (Desimone, 2009), they also illuminate the complexities of professional development and professional learning (Fraser et al., 2007). Middlewood et al. (2005) consider that a key difference between development and learning can be identified by its purpose; professional development often points to activity that meets departmental and corporate needs but does not always lead to a process of self-development leading to personal growth as well as development of skills and knowledge that facilitates the education of young people (professional learning). These complexities are further demonstrated by Sancar et al. (2021) who are critical of past frameworks for failing to capture all aspects of teacher CPD. For example, they consider that Desimone's (2009) model on teacher CPD focuses on a limited set of teacher characteristics and supposes only one motive for teachers to engage in CPD. Similarly, while the study by Hildebrandt and Eom (2011) investigated different motives for teacher CPD – and the teachers most likely to be led by specific motives – their study did not relate the motives to the content that teachers learn or the learning methods they use.

It is not the intention of this study to focus on the challenges of developing theoretical frameworks for the evaluation of CPD programmes, but it was important to ensure that the PD programme acknowledged and incorporated the known characteristics of effective CPD. Accordingly, I decided to use a model devised by Frazer (2007) involving a triple-lensed framework. The first lens is Kennedy's framework (2005) describing nine models of CPD ranging from familiar types such as training, coaching and mentoring, to those such as action research and communities of practice. The characteristics of each of these models illuminate both the delivery mechanism and the nature of the content. The nine models sit on a continuum from transmissive to transitional to transformative. Within this framework it is possible to locate different types of CPD on the continuum and within the associated definitions, and to evaluate the effectiveness of each type according to a range of criteria.

The second lens is the application of Bell and Gilbert's (1996) framework. Its use of social, personal and occupational dimensions suggests that the impetus for change is located within the personal aspect of professional learning and that this is contingent on motivation and interest. In addition, the links between theory and practice need to be strong, particularly with regard to the development of the occupational aspect.

The third lens uses Reid's quadrants of teacher learning (Frazer et al., 2007), a model which enables the CPD to be categorised in relation to the 'sphere of action' (where the learning takes place). By using an array with two dimensions – formal to informal, and planned to incidental – the location in which the professional learning takes place helps to provide insight into the opportunities and limitations of the different delivery mechanisms experienced by teachers. The quadrants are shown in Figure 2.17 below.

Figure 2.17 *Reid's quadrants of teacher learning*



In developing the framework for this study, I was also interested in the conceptual framework proposed by Desimone (2009) who suggests that for CPD to be effective in improving teaching practice (and student learning), then at least five features need to be built in appropriately. These are:

- duration
- content focus
- active learning
- coherence
- collective participation.

In particular, I was drawn to the feature of coherence which Desimone describes in part as the extent to which the content and aims of the PD programme are consistent with the school's curriculum and goals, as well as teachers' knowledge and beliefs. The completed framework for analysing the PD programme in this study incorporates Desimone's five features, as shown in Table 2.1 below.

Table 2.1 *Framework for analysing the PD programme*

CPD Analysis Framework	
PD component	• the PD sessions 1, 2 and 3 • the research lesson including the pause and post-lesson discussion • the TDT analysis of the orchestration teaching sequence (Cycle 2 only)
Lenses	**Design Features**
Kennedy	Aligned to one or more of the nine features
Bell and Gilbert	***Personal:*** ***Social:*** ***Occupational:***
Reid's Domain of Practice	Aligned to one or more of the four quadrants
Relevant Desimone Features	***Duration:*** ***Content focus:*** ***Active learning:*** ***Coherence:*** ***Collective participation:***

In Chapter 5, I develop this framework to identify the features that were crucial in the PD programme and which could be used to discuss the outcomes of the programme that are detailed in Chapter 8.

2.5 CPD approaches for mathematics teacher development

I use the term 'CPD approach' to describe the mechanism[19] that configures the programme specifically in relation to the content and the method of sharing the content with the participants. For example, the training model which can take place within school or off-site (and more recently, remotely) is usually delivered by an 'expert' who presents some new knowledge[20] that will subsequently be used by the teacher. In this instance the CPD mechanism might be a workshop or a lecture interspersed with individual or group activities. By contrast, in the award-bearing model such as the Chartered Status for teachers (through the Chartered College for Teaching), the CPD approach would be delivery of the four Chartered Teaching assessment units, the successful completion of which leads to the 'CTeach' award. This approach utilises an environment of professional study, self-evaluation and evidence collation using given criteria.

Not all models have a clearly identifiable CPD approach within them. For example, the deficit model, designed to address deficiencies in teacher performance, might involve performance management as the tool to identify the perceived weaknesses and to suggest targets for improvement. Clearly this tool addresses the identified need but in itself might not necessarily define the specific nature of the professional learning programme required to tackle the perceived deficiency. This study has identified two CPD approaches: Lesson Study (modified) and Teacher Design Teams. The reasons for using these approaches is set out in the sections below.

2.5.1 Lesson Study

At the beginning of this study my intention was to design a professional learning programme for teachers of mathematics and that a component of the programme would be drawn from aspects of Lesson Study. In other words, the PD programme would be more than just Lesson Study and which would not include all of the components of Lesson Study. However, as I

[19] The term mechanism here has a different meaning to the one used in the EEF review (Simms et al., 2021).
[20] For example, a new teaching scheme, an evidenced-based pedagogy, or the use of an ICT device or platform, all designed to improve pupil outcomes or to respond to some nationally identified area for improvement.

explain below, this may not be the case – and it is arguable that the whole programme could be characterised as yet another variant of Japanese Lesson Study.

In their systematic review of the effects of Lesson Study, Cheung and Wong (2014) concluded that Lesson Study is a powerful tool to help teachers examine their practices and enhance student learning. One of the major strengths of Lesson Study is that it places teachers at the centre of the learning process (Murata, 2011). The Teacher Development Trust describes Lesson Study as:

> a Japanese model of teacher-led research in which… teachers work together to target an identified area for development in their students' learning. Using existing evidence, participants collaboratively research, plan, teach and observe… lessons, using ongoing discussion, reflection and expert input to track and refine their interventions. (Teacher Development Trust, 2018)

This rich definition and others, however, do not detail the Lesson Study components, some of which are highly complex and have a sophisticated rationale that is deeply embedded in Japanese culture. Therefore, it is not surprising that as different countries have implemented Japanese Lesson Study over the last 20 years, many variants have emerged. Takahashi and McDougal (2016) describe this wide variation in how Lesson Study is used outside Japan. For example, the research lesson in Japan is rarely repeated whereas in some countries the research lesson is repeated up to six times.

It is important to clarify here that Lesson Study is not confined to teaching mathematics through problem solving[21] and that it is not the only way of developing the teaching of mathematics through problem solving. Also, it should be noted that Lesson Study is not without its critics (Murphy et al., 2017, p. 4). However, more recent reviews such as the Education Endowment Foundation's meta-analysis on effective teacher professional development (Simms et al., 2021) suggest that Lesson Study, along with instructional coaching and communities of practice, has a positive effect on pupil attainment.

Buchard and Martin (2017, p. 2) identified three types of Lesson Study that have developed through international partnerships. Lesson Study from Japan was exported directly to the

[21] In Japan and more recently in the UK, Lesson Study is used to evaluate and improve many different areas of the school curriculum.

USA. The development of 'Learning Studies' was the result of an adaptation of Lesson Study by Sweden and Hong Kong.

Figure 2.18 *Models of Lesson Study and component steps (adapted from Buchard and Martin, 2017)*

Learning Study Sweden / Hong Kong	Lesson Study Japan / USA	UK Lesson Study UK
Specific criteria ←	1.Recruit members	
	2.Specify a research theme	
	3.Choose a suitable lesson	
Very important ←	4.Pre-test	→ No Pre-test
	5.Information gathering	
Focus: taught concept ←	6.Goal setting	
	7.Lesson planning	
	8.Research lesson	
Post test ←		→ Interview and or observations with case pupils
	9.Discussion analysis	
Possible simultaneous lessons ←	10.Cycle 7 - 9 (optional)	
	11.Final Discussion	
	12.Sharing	

Figure 2.18 sets out the component steps for the Japan/USA model. The arrows show the variance between the Japan/USA model and the other two models. Importantly, the figure also points to how the original model of Japanese *jugyo kenkyuu* (Lesson Study) has been interpreted. Also, the language used to describe the components does not align well with other descriptors such as the component known as *kyozaikenkyu*, a crucial part of authentic Japanese Lesson Study which is explained below. In this study the key components of Japanese Lesson Study were the lesson planning, the research lesson and the post-lesson discussion.

The above observations have been recorded here not to criticise the modifications made to the original Japanese Lesson Study, nor to cast doubt on the representations of these variations. In fact, I currently hold the view that it is important to consider modifications to Lesson Study because of the particular context and culture in which the original model was developed. However, it is important that any modifications made do not result in the removal of the key components of Lesson Study and that the coherence between them is retained.

In this PD programme, I made five modifications: two that were prompted by differences in culture and context, and three that were devised to meet the design intentions of the programme. The first modification was to restrict the observation of the research lesson to the participants in the study and the researcher. In Japan, the number of observers in the research lesson can be many and regularly extends beyond the research team. In my last visit to Japan, I took part in research lessons where more than 50 observers were present. Indeed, in regional Lesson Study events, the number of observers can reach three figures.

Those who consider Lesson Study to be powerful do so because of the absence of any element of judgement about the performance of the teacher who taught the research lesson. Lesson Study critiques lessons, not teachers, which is one of its central strengths. The culture in Japan is to perfect the delivery of content and the development of skills. Lesson Study approaches this challenge through a ‘learned collaborative’ that delves deeply into the understanding of concepts, knowledge and skills and uses research and experience to anticipate and resolve barriers to learning presented by the children.

Establishing this philosophy can be difficult within the English accountability framework where the focus is on improving the teacher through methods such as monitoring and feedback, coaching and training. The system for monitoring the quality of teaching in England uses a range of evaluation mechanisms such as learning walks (observation of learning and teaching), book scrutinies and pupil interviews, together with analysis of attainment data. What is surprising and perhaps remarkable is the frequency with which this monitoring is done with all teachers. More importantly, for many mathematics teachers, this system of monitoring contributes very little to their professional growth and their understanding of how children learn key concepts in mathematics (Hill & Grossman, 2013). What it does do however is create an accountability system and culture in which the performance of the teacher is of great importance and any identified deficiencies are unfavourably received. Within the Lesson Study approach, even though in a research lesson it is the planning team who are responsible for the design of the lesson, and the outcomes for pupils can remain unknown, teachers in the UK sometimes find it hard to discuss openly the aspects of a lesson they have delivered without taking any negative aspects personally in the learning process.

From my previous experience of using Lesson Study, I was aware of the potential of this phenomenon to adversely influence the effectiveness of the PD programme. The participants in this study were committed to developing their expertise in the teaching of mathematics through problem solving, so it was important to ensure that the evaluation of their endeavours was not negatively affected by external observations that were not informed by experience of the programme. Therefore, I decided that the research lesson in this study would only be observed by the researcher and the study participants.

The second modification was in relation to the component of Lesson Study known in Japan as *kyozaikenkyu* which is the study of the curriculum materials before the development of the research lesson. This detailed examination of the existing curriculum materials supports the development of the research. The Japanese mathematics curriculum is much more coherent and established than in England, and improvements are achieved through a continuous iterative process. In this study, an adapted small-scale version of *kyozaikenkyu* was incorporated into the programme through the discussion of a paper that provided additional information about the mathematics that could be explored by the task to be used in the research lesson.

The three other modifications made related to the design and objectives of the PD programme. These were:

1. The research lesson contained a pause during which a PD session took place.
2. The post-lesson discussion focused predominantly on the second half of the research lesson.
3. The final commentary from a ‘knowledgeable other’ was replaced with an analysis of the orchestration sequence.

At the beginning of this section, I suggested that this PD programme might actually be characterised as a modified Lesson Study programme. The PD sessions on task design and orchestrating the learning could be regarded as a type of *kyozaikenkyu.* The fact that the headteachers of the schools involved agreed to the professional development of their teachers in this way could be seen as a warrant for a research proposal. As discussed below, the corpus of a TDT can be the same as a research team in Lesson Study despite their different ambitions. Also, the observation of the orchestration sequence by members of the TDT could be regarded as an effective professional learning experience similar to that in a post lesson discussion. Equally though, the programme has not been developed to respond to a particular school improvement issue; TDTs were formed to engage in the professional learning programme and so were bound by their common personal interests and not by a research finding or proposal from the school. Also, there was no final commentary provided by an external expert. Nevertheless, whether or not this PD programme is in fact a Lesson Study does not affect the design intentions of the programme.

2.5.2 Teacher Design Teams

The theoretical basis for the formation of a Teacher Design Team (TDT) centres on the need to provide creative spaces where teachers can work and learn together (Lieberman & Miller, 2005). TDTs were incorporated into the PD programme design because their characteristics supported the design intentions of the programme. A TDT is a group of two or more teachers who work collaboratively to design or redesign curriculum materials (Handelzalts, 2019). They can focus on any area of professional learning ranging from curriculum content and sequencing to developments in teacher practice and pedagogy. This focus for the TDT connects to the principles of task design and therefore is a suitable structure for the participants to work on common tasks to explore how they would be used in the teaching of mathematics through problem solving.

Vogt et al. (2015) investigated how TDTs contributed to teacher professional learning in the context of curriculum reform. They identified potential benefits of TDTs for the professional development of teachers in that they can create environments where they can solve problems and make decisions and importantly where they can articulate their tacit and practical knowledge (Verloop et al., 2001). Further, Voogt et al. (2011) have shown that the interaction of TDTs with the external expertise provided by a facilitator positively contributes to the quality of the curriculum design and to teachers' learning. Importantly, the expertise can also be in the form of curriculum materials. In this study, expertise resides in the design of the PD materials based on the findings in literature.

In many ways the research team within Lesson Study exhibits several characteristics of a TDT. The teachers in a research team work together on a research proposal that can typically take between 8 and 10 weeks to produce and involves a review of a part of the school's mathematics curriculum and related scheme of work. As such these teachers are often involved in the process of task design and focus on improving their teaching through collaborative professional development.

However, there are differences between the TDTs in this study and a research team in Lesson Study. The participants in the TDT are principally working together for the betterment of their own professional development whereas a research team in Lesson Study is generally working to solve a problem or improve on an issue identified by the school or department which is used to develop a research question for the team to explore.

Also, Handelzalts (2009) states that a TDT works together to develop practice which they then enact. As such they recognise the need for each other's contributions in order to succeed in their own work. This is a subtle but important difference. In both situations the inputs from the team members contribute to the outcomes of the research or development; in other words, everyone (including the organisation) learns. In the case of the research team in Lesson Study, the learning can be formed from a range of different and sometimes opposing views and beliefs. Whilst this can also be the case in TDTs, they are established on the premise that the success of any individual is enhanced by the success of another. Accepting this as a condition for a TDT impacts on the relationship between the participants and therefore the participants need to feel some degree of 'fit' between naturally occurring teacher relationships and the artificially constructed links that are introduced (or imposed) in the service of improvement initiatives (Hargreaves, 2003). It was therefore crucial to establish this purpose with the participants at the beginning of the research programme, defining the group as a professional learning community whose main purpose was development of their own practice *and* that of their colleagues within the TDT.

2.6 Summary

In this chapter I have described the theoretical basis on which the PD programme was devised, namely the principles of task design and orchestrating the learning, and their contribution to teaching mathematics through problem solving. The summary of the literature on task design identified the core features of good design and emphasised the importance of aligning designs with pedagogy and instruction. The summary of the research on problem solving described the teaching of mathematics through problem solving, and the framework of problem-solving skills was used to explain the key differences between teaching about, for and through problem solving.

I also explained how the review of the literature on effective CPD led to the building of a framework for developing and reviewing the PD programme components used in this study. The framework adopts the triple lens approach proposed by Frazer (2007) which incorporates the models of Kennedy (2005), Bell and Gilbert (1996) and Reid's quadrants of teacher learning (Frazer et al., 2007). These three lenses in conjunction with the five key characteristics of effective CPD identified by Desimone (2009) have shaped the framework for analysis that I used to ensure the PD programme incorporates the known features of effective CPD.

Finally, I explained the rationale for the use of two PD approaches in this study. With regard to Lesson Study, I described the modifications made to the original Lesson Study protocol and the reasons for doing so. I have also explained how the establishment of a Teacher Design Team contributed to the objectives of the PD programme.

As an extended footnote to this chapter, this study took place during the Covid-19 pandemic and so a review of the literature on remote learning in the context of the PD programme was carried out (Appendix 1). Whilst it was not an objective of this research to evaluate the effects of remote learning used in this study, several affordances were noted. The most notable of these was the use of visualisers to observe the pupils working in the research lesson which was an innovation in response to the Covid-19 restrictions regarding the number of teachers in a classroom and their permitted proximity to the pupils.

Chapter 3 – Methodologies, Methods and Analysis

Introduction

In this chapter, I consider the features of the methodologies that were potentially suitable for this study and explain my rationale for the research methodology I have chosen. The selection of methodologies and methods was influenced by a number of logistical and ethical challenges including my existing professional relationship with the participants. As will be discussed below, I used the principles of Participatory Action Research (PAR) to ameliorate the possibility of my professional relationships affecting the outcomes of this study.

I begin by explaining the rationale for choosing design research and discuss the key characteristics of this methodology. I then describe the methods used to collect the data, the participants involved in the study, and the ethical considerations for this research. Finally, I describe the methods of analysis used and explain the need for 'argumentative grammar' in design research. I show how the research questions were used in the development of conjecture mappings and how the method of coding the data informed the themes for analysis.

3.1 Methodology and choices

Analogically, a methodology can be envisioned as a domain or a map, while a method refers to a set of steps to travel between two places on the map (Jonker & Pennink, 2010). Initially, I considered the qualitative methodologies of case studies, action research and grounded theory as potentially suitable for this project. However, the nature of the design components within this study ultimately informed my decision to move away from a case study approach. There are two innovations within the PD programme: the pause in the research lesson and the use of visualisers[22] to observe the work of the pupils. Whilst these innovations are bounded by space and time, they currently do not have a natural context (Algozzine & Hancock, 2017), given that the pause in a lesson and the use of visualisers would currently only happen in a professional learning situation and not in the course of the participants' everyday practice.

[22] These were incorporated into the study as a means of observing pupils' work during the Covid-19 pandemic where teachers and observers were unable to move around the classroom and interact with the pupils.

Similarly, action research with its focus on "learning in and through action and reflection" (McNiff, 2013, p. 15) also initially appeared to be an appropriate methodology. However, because the central feature of this study was the development, design and evaluation of a PD programme, I concluded that I would be doing more than just exploring a phenomenon or a solution to a problem.

I therefore decided that the best methodology for this study was design research. The principles of Participatory Action Research (Reason & Bradbury, 2008) were also incorporated to respond to the ethical issue relating to my particular relationship with the participants, as explained further in Section 3.5.

3.2 Design research methodology

In his classic work, *The Sciences of the Artificial*, Herbert Simon (1969) argues that science develops knowledge about what already is, whereas design involves human beings using knowledge to create what could be, that is, things that do not yet exist. Mintrop (2020) described the term 'design' as "an intervention that consists of a sequence of activities that together or in combination intervene in existing knowledge, beliefs, dispositions or routines in order to prompt new learning that leads to new practices" (p. 156).

A central question for educational research is how to design interventions[23] that move beyond describing what is, or confirming what works, to designing a strategy or intervention that might work better. Bakker (2018) defines design research in education as research in which the design of new educational materials (including professional development programmes) is a crucial part of the research. Bereiter (2002) suggests that design research is defined by the objectives of the researcher and the characteristics of the research:

> Design research is not defined by its methods but by the goals of those who pursue it. Design research is constituted within communities of practice that have certain characteristics of innovativeness, responsiveness to evidence, connectivity to basic science, and dedication to continual improvement. (p. 321)

[23] In design research the terms intervention, innovations and design experiments are sometimes used interchangeably.

In addition to the definitions and descriptions above, design research is sometimes used to test or validate theories as a result of the study of educational interventions, such as learning processes or learning environments (Bakker, 2018). However, my purpose was to develop research-based solutions for complex problems in educational practice, a field known as development studies (Gravemeijer & Cobb, 2006). A significant difference between design research methodology and other methodologies is the type of knowledge created. My aim was to design and evaluate a PD programme for teaching mathematics through problem solving to generate new knowledge in relation to the key characteristics and innovations within the programme.

Van den Akker et al. (2006) list three motivations for engaging in design research, the third of which drew me to the field of design research:

- the desire to increase the relevance of educational research for educational policy and practice
- the ability to develop empirically grounded theories
- the opportunity to make more explicit the learning that can be applied to further designs.

In the research design, I considered the five characteristics of design research as identified by Cobb et al. (2003). Whilst these are regarded as a family of characteristics, not all of them necessarily had to be brought into the design as they do not all need to be incorporated together. The five characteristics are summarised as follows. Design research:

- develops theories about learning which can support the learning
- is interventionist
- contains both prospective and reflective components
- is cyclic in nature
- should ensure that the theory does real work.

Cobb and colleagues refer to these five characteristics as cross-cutting themes which exemplify that design experiments are both pragmatic and theoretical in orientation. The second theme is the highly interventionist nature of the methodology. In this research the intention is to study new learning by engineering a situation through the design and implementation of a PD programme. With regard to the cyclic nature of design research, two cycles were used. However, their iterative character was confined mainly to the facilitation of the PD programme and did not apply to the specific content which had already been augmented to cater for the different characteristics of the two TDTs.

Also design research describes an approach that is committed to:

- the production of innovative learning environments
- the acquisition of knowledge about how such environments work in the settings for which they are designed
- a fundamental contribution to knowledge about learning or teaching.

(Cobb et al., 2003).

Whilst it is easy to see how different research methodologies can achieve one or more of these three commitments, it is the unique claim of design research that it can simultaneously embrace all of them. Educational design research programmes can lead to the production of innovative learning environments that subsequently lay claim to a fundamental contribution to knowledge about teaching and learning.

3.3 Methods of data collection

The data collection methods were informed by the research questions. The first two research questions in this study enquire about possible changes in teachers' views about the teaching of problem solving and the effect of the design innovation known as the pause in the research lesson in supporting the teaching technique 'orchestrating the learning'. I have therefore chosen a combination of interviews, observations and questionnaires, with the main emphasis on the first two methods. A summary of the data collection procedures for both research cycles is shown in the Table 3.1 below.

Table 3.1 *Data collection methods and purposes*

Data Collection Summary			
Data	When	Detail	Purpose
Questionnaire	Before PD Programme	1 questionnaire Cycle 1 – 5 participants Cycle 2 – 4 participants	To establish views on teaching and teaching mathematics through problem solving to support development of PD programme
Pre PD programme interview	Before PD programme	1 interview for each participant Cycle 1 – 5 participants Cycle 2 – 4 participants	To establish views on teaching and teaching mathematics through problem solving to support development of PD programme
Formative questionnaire	During PD programme	2 questionnaires for each participant in Cycle 1	To inform/augment Cycle 2
Johari Window[24]	During PD programme	Johari window completed as a group in Cycle 2	To contribute to research findings data
Observation of the 'Pause' PD session	During PD programme	Transcript of discussion obtained from each cycle	To contribute to research findings data
Observation of the post-lesson discussion	During PD programme	Transcript of discussion obtained from each cycle	To contribute to research findings data
Post PD programme interview	After PD programme	1 interview for each participant Cycle 1 – 4 participants Cycle 2 – 4 participants	To contribute to research findings data
Analysis of recordings of PD sessions	After Cycle 1	Analysis of each PD session	To augment Cycle 2
Observation of participants' review of orchestration sequence	After research Cycle 2	Transcript of discussion involving two teachers one of whom taught the research lesson	To contribute to research findings data

[24] The Johari Window is a device that can be used to collect data on beliefs and feelings. A description of how it was used in this study is given in Chapter 4.

3.3.1 Questionnaires

In this research, both pre-study and formative questionnaires were used to:

- establish key beliefs held by individuals which should be accommodated and/or challenged within the PD programme;
- obtain a profile of the TDT to support the facilitator to effectively facilitate the PD session and discussions and choices made by the participants;
- obtain the participants' perceptions and views on a number of specific PD facilitation techniques used in first two PD sessions.

The pre-study questionnaire (Appendix 2) was used to capture data with regard to the first two purposes. It was adapted from the questionnaire used in the international study Teacher Education Development Study - Mathematics (TEDS-M) (Tatto, 2020) and using components of the Teaching and Learning International Survey (TALIS) (OECD, 2013). The pre-study questionnaire consists of two parts. In the first part there is a series of 13 statements and participants indicate the extent to which they agree or disagree with each statement (4-point Likert scale).

The second part of the questionnaire consists of four sets of four statements. For each set, the participants are asked to rank the four statements in order of importance. For example, the set of statements in Figure 3.1 below were chosen to establish the participants' views on their approach to teaching problem solving. Giving the rank of 1 to the first statement might suggest that the teacher would introduce a problem to pupils by first allowing them to explore the problem on their own. This interpretation is then considered with responses from the other sets of statements and the information used to inform the facilitation of the PD sessions and the planning of the research lesson, including the pause in the research lesson and the post-lesson discussion.

Figure 3.1 *Example of statements in ranking task used in pre-study questionnaire*

S3	
Statement	Rank
Pupils must be able to decide on their own procedures or methods	
Pupils must explore alternative methods for solutions	
I should teach the most efficient way to solve a particular kind of problem	
I should direct pupils away from non-standard or inefficient methods	

Formative questionnaires were created to establish participants' views and feedback on the particular facilitation strategies that were used in PD sessions 1 and 2. These approaches and strategies (which are discussed in Chapter 4) are based on my experience as a facilitator and which I have developed over time. Some of the strategies became more accessible due to the remote delivery of the PD sessions. The participants were asked to give their views on the following strategies used in each session:

PD session 1

- Participants being presented with a classification of problem-solving skills after they had worked as a TDT to devise their own framework of skills.

PD session 2

- The effectiveness of the Johari Window as a tool to reflect on their own beliefs, and the extent to which it impacted on the intended learning from the session.
- Being given unlimited time in the breakout room to work on the research lesson.
- The effect of being given a paper with additional information (Diophantine Equations) about the problem to be used in the research lesson after working on the lesson in the breakout room.

Appendix 4 shows the statements that were presented to the participants and the set of response options from which they were asked to select one. The questions were presented on PowerPoint at the end of each session and the participants submitted their replies by email.

3.3.2 Interviews

Two sets of interviews took place in this study: one at the pre-PD stage and one post-PD programme. The decision to carry out pre-PD interviews in addition to the questionnaires and post-PD interviews was made for two reasons. Firstly, it was anticipated that the data obtained from the pre-study interviews would usefully complement the questionnaire data to inform the design and facilitation of the PD programme. Secondly, to gauge the effect of the programme on the participants, it was important to understand their teaching approaches and views on teaching problem solving prior to the programme.

Wahyuni (2012) states that “the main feature of an interview is to facilitate the interviewees to share their perspectives, stories and experience regarding a particular social phenomenon being observed by the interviewer” (p. 73). According to Brinkmann and Kvale (2015), interview questions can be evaluated with respect to both a thematic and a dynamic dimension. Good qualitative interview questions should invite a process of exploration and discovery and contribute thematically to knowledge production and dynamically to good interactions between the researcher and interviewee. Additionally, in order to prepare data for analysis, researchers must align the theoretical assumptions about interviewing with the kind of research design and interview methods used to generate data (Roulston, 2016).
Of the many interview approaches available, I chose the narrative[25] approach for the pre-PD interviews and semi-structured interviews for the post-PD stage. Each approach will be discussed in turn, including its advantages and disadvantages relative to the other, and the rationale for its selection. Each subsection also describes the development of the interview topics and questions.

Narrative interviews: Pre-PD programme

The narrative approach enables ‘flow’ in the accounts by the interviewee and they are not as potentially susceptible to undue influence or being ‘led’ by the researcher as could be the case in semi-structured situations. Giving the interviewee a forum to narrate freely has its advantages for the data collection, as articulated by Christel Hopf (cited in Flick, 2004):

[25] This form of interviewing was developed in the 1970s by Fritz Schütze, originally in connection with a research project on municipal merging where community politicians gave an account of the "chains of incidences" (Schütze, 1982).

> Narratives are strongly oriented to concrete action sequences and less to the ideologies and rationalizations of the interviewees. Interviewees who narrate freely may, in particular instances, reveal thoughts and memories that they would not and could not express in response to direct questioning (p. 207).

I decided that this approach would be suitable for the pre-PD programme interviews as allowing participants to talk for extended periods without prompts or interruptions by the researcher would result in a more accurate evaluation of their current views and beliefs.

The pre-PD interview questions were developed by first considering the potential effects of the programme, informed by the relevant research questions, to generate areas for exploration with the participants. Table 3.2 below sets out these points of departure for the pre-PD questions.

Table 3.2 *Points of departure for pre-PD programme interview questions*

Research Question	Potential effects of the PD programme	Areas for exploration
How does the use of task design in conjunction with the teaching technique of ‘orchestrating the learning’ affect teachers’ views on teaching mathematics through problem solving?	Changes in views about: • learning mathematics • problem solving Changes in teacher subject knowledge	How do teachers plan to teach different topics? How do teachers choose the tasks that pupils do for teaching problem solving?
How does the introduction of a ‘pause’ in the research lesson support the development of ‘orchestrating the learning’ as a method of teaching mathematics through problem solving?	Changes in: • beliefs on teaching • views on developing pupils’ learning The establishment of a view on the effectiveness of the teaching technique	What is the role of the teacher in the classroom? How do teachers know what their pupils are learning? How do teachers develop the learning in the lesson?

The development of the pre-PD interview questions was further informed by the work of Luft and Roehrig (2007) who developed a framework, the Teacher Belief Interview (TBI), to capture the beliefs of beginning secondary science teachers. The TBI comprised eight questions that had been designed following research on teacher beliefs and building on the work of Berg (2001) and Patton (1990). These questions were trialled with pre-service mathematics teachers in order to determine the generalisability of the TBI, and to test the reliability of the questions following concerns that teachers from different disciplines would all give similar answers, thus suggesting that the questions were not reliable enough to capture beliefs about teaching subject-specific content. Reassuringly, in their answers to the TBI questions, the pre-service mathematics teachers clearly drew upon their content knowledge and their understanding of the nature of knowledge construction in mathematics. The answers provided by mathematics teachers differed from those of the science teachers, thus supporting the reliability of the questions.

I considered that the TBI questions were suitable for the purposes of my study. They aligned well with the broad areas for exploration initially identified (Table 3.2) and would contribute to my understanding of the participants' actions and responses during the programme as well as the information provided by them in the post-study interviews. Ultimately, my interview schedule included five questions from the TBI plus a sixth question specifically to explore the participants' approaches to the teaching of mathematics through problem solving. The six questions are set out in Figure 3.2 below.

Figure 3.2 *Pre-PD programme interview questions*

Pre-PD programme interview questions

1. How do you describe your role as a teacher?
2. How do you know when your students understand?
3. How do you decide when to move on to a new topic in your classroom?
4. How do you know when learning is taking place in your classroom?
5. How do you maximize student learning in your classroom?
6. How do plan to teach the topic of problem solving?

Luft and Roehrig's (2007) framework also includes five category descriptors which support coding of participants' responses to the questions, using a continuum that ranges from 'traditional' to 'transformative'. These category descriptors were used to produce a profile of each TDT, as discussed further in Chapter 6.

Semi-structured interviews: Post-PD programme

For the post-PD programme interviews I used a semi-structured approach. As the term implies, the semi-structured interview is a format which has structured and unstructured components and gives the interviewer the opportunity to explore and clarify, and the participant to clarify and exemplify. These affordances were important given that the participants' views on the PD programme might still be forming and contain elements of uncertainty.

Whilst the researcher in a semi-structured interview can refine the direction of the conversation in order to 'extract' data more readily than in narrative interviews, the format potentially leaves the participant more open to the researcher's influence depending on the nature of their relationship. However, an enduring prerequisite of any productive interview is the presence of trust: irrespective of the technique used, the participants must feel able to engage in the discussions freely and without fear of judgement. Therefore, in the semi-structured post-PD interviews, I was aware that the prompts used should avoid conveying judgements, particularly given my relationship with the participants which is explained in more detail in Section 3.5.5.

In the post-PD interviews I asked the participants two sets of questions. The first set was common to all participants and was designed to generate answers to the research questions pertaining to the expected outcomes in the conjecture mappings (detailed in Section 3.5). In the first cycle the set of questions asked of all participants is shown in Figure 3.3 below.

Figure 3.3 *Post-PD programme interview questions*

Cycle 1 Post-PD programme interview questions

1. In PD session 1 we considered the design of tasks that had been developed to teach mathematics through problem solving. What are your reflections on this?
2. Since then have you considered any other problems that could be used in the same way?
3. In PD session 2 we discussed how pupils might respond to a specific problem and considered different sequences of coordinating these responses. How did this process affect the way in which you think about planning problem-solving lessons?
4. How useful was it to think about the anticipated responses and then to sequence those responses in your teaching?
5. Do you think this is a difficult process?
6. What were the advantages of being able to observe the work of the children in this way?
7. In the research lesson we modified the second part of the lesson as a result of observing the work of pupils. Do you have any comments about that part of the process?
8. In thinking about the PD programme and in relation to previous professional learning programmes could you talk about the similarities and differences of this programme to other programmes?
9. The PD programme comprised the following components:
    - taught sessions on problem solving, comprising ideas on task design and orchestrating the learning
    - planning the research lesson
    - the research lesson with a pause to review the lesson plan
    - a post-lesson discussion.

   Can you reflect on this programme and talk about the value of each component?
10. Is there anything about the programme that you would have liked to be different?

The second set of questions was specific to each participant and was devised in response to the analysis of the data in the questionnaires and the pre-study interviews. These questions were posed in cases where the pre-PD interviews and/or questionnaires revealed information that warranted further exploration following the participant's engagement in the programme. For example, in Cycle 1, one of the participants in response to Q1 in the pre-PD interview talked about maximising learning by “being able to create an environment for discussion so as to be able to highlight misconceptions”. This led to the development of the following two bespoke questions.

1. In the first interview you talked about maximising learning by being able to create an environment for discussion so as to be able to highlight misconceptions. Does the approach used in the PD support the way in which you would like to teach as you describe?
2. What are the barriers to achieving this on a more regular basis?

Question 1 was used to verify the assumption that the PD programme, through the development of the research lesson, supported the establishment of an environment for discussion that highlighted and addressed misconceptions. Question 2 was devised in order to ascertain potential barriers that might have been present before the programme or became evident as a result of the programme.

Finally, specific questions were developed for the teacher who taught the research lesson, to establish their views on the orchestration episode of the teaching.

3.3.3 Observations

Williamson (2000) categorises observation as a data collection technique because it can be used in a variety of research methods, whereas Baker (2006) muses that the use of observation as a research method is unclear and complex because it often requires the researcher to play a number of roles and use a number of techniques. Gold (1958) builds on Buford Junker’s typology of the four roles researchers can play in their efforts to study and develop relationships with insiders (insiders is the term given to those who are being studied). These four roles are:

- complete observer
- observer-as-participant
- participant-as-observer
- complete participant.

I adopted this useful framework in this study, acknowledging the model as “a range of flexible positions in a continuum of participatory involvement” (Gorman & Clayton, 2005, p. 106).

Initially, I took up the position of complete observer, with my identity as a researcher evident to the insiders (participants) throughout the formal data collection processes such as the pre-PD programme interviews and questionnaires. In this role, the relationship between the observer and the insiders has advantages and disadvantages. The main advantage is that the observer does not intentionally influence the insiders (and this chimes with the decision to use a narrative approach in the pre-PD programme interviews). The disadvantage of course is that there is no opportunity to seek clarification or explore interesting observations. With regard to observing interviews, Arvey and Campion (1982) assert that non-verbal cues (visual cues) should be taken into account and that the best way to make use of them is by video recording. However, I recognise that particular competencies are required to interpret these cues and that without sufficient expertise there is a danger of ‘assumptional’ data being introduced into the study. Having said that, the analysis of videos alongside the transcripts was used to note any non-verbal aspects that were easily identified, such as uncertainty of response or strength of feeling, and to clarify ambiguity or missing words.

The second role that I took was that of participant-as-observer during the delivery of the PD programme. As Adler and Adler (1994) state, in this role the researcher becomes more involved with the insiders’ central activities but does not fully commit to the insiders’ values and goals. The danger here is that the researcher’s active participation in the study can operate to change relationships and this may then influence the views of the participants and the interpretation of the data collected. However, because I was able to analyse the recordings of the PD sessions and the post-study interviews, it was possible to look out for instances where my facilitation of the PD programme and the developing relationships with the participants may have influenced their views of the programme. Also, these recordings together with the post-PD session feedback questionnaires provided valuable information for potential iterations between cycles.

Finally, in the pause in the research lesson and in the post-lesson discussion (and in Cycle 2 the analysis of the orchestration sequence) I adopted the role of observer-as-participant. Here, I mostly observed the interactions of the insiders and as such remained strongly research focused. The main advantage of adopting the observer-as-participant role in these situations was that it ensured that I did not ‘go native’ (Pearsall, 1970). This role was the most challenging one to take up because during the pause discussion there was a temptation to adopt the role of a complete participant and contribute to the discussion so as to influence the design of the second part of the research lesson. As Spradley (1980) cautioned, “the more you know about a situation . . . the more difficult it is to study it as an ethnographer” (p. 61).

3.4 Participants

I was interested in working with teachers who had experience of Lesson Study and who had either experience or interest in research lessons that had a focus on teaching mathematics through problem solving. Of particular interest were those teachers whose teaching appeared to have not been influenced in any way by their engagement in the previous learning experiences from the Lesson Study cycles they had participated in. I was also interested in the effect of the programme on teachers working in different education phases: due to the study comprising two research cycles, I took the opportunity to recruit participants from the primary and secondary phases[26]. Also, as discussed in Chapter 2, the design of the PD programme required the participants to work as part of a Teacher Design Team (TDT), so it was important for the teachers to have a professional working relationship and work in the same school. Therefore, it was desirable for the participants to have:

- previous experience and engagement in Lesson Study cycles that contained a research focus on problem solving
- an interest in teaching mathematics through problem solving
- common interests and strong professional relationships.

Using the above criteria to recruit participants who were not known to me would have meant that the process of recruiting would be challenging and significantly elongated. I therefore decided to recruit teachers from schools in the Diocese of Hallam[27] (in which I worked as Director of Schools) who had engaged in Lesson Study cycles with the research theme of teaching mathematics through problem solving. In the next section on ethical considerations,

[26] The primary phase is for pupils aged 5 to 11. The secondary phase is for pupils aged 11 to 16.
[27] Diocesan schools are schools grouped by faith. There are 22 Catholic Dioceses in England and Wales and 42 Anglican Dioceses. The schools in this study are part of a Diocese comprising 40 primary schools, six high schools and one 3-16 through school.

I discuss the implications of working with participants with whom I had this existing relationship.

The school selected for the first cycle was a 3-16 through school with the participants all teaching in the secondary phase. The school used in the second cycle was a 5-11 primary school. In both cycles the TDT contained one member who had previously taught a research lesson as part of the South Yorkshire Maths Hub Lesson Study project described in Chapter 1. As already explained, the two cycles were parallel in that the same principles of teaching mathematics through problem solving were used and the problem used in the research lesson was appropriate for both Year 6 and Year 7 pupils. In designing the PD sessions on task design and orchestrating the learning, I was confident that all the teachers in the programme could access the mathematics being presented and the pedagogies being proposed.

My relationship with the two schools was similar in that I was known to them as the lead for the Lesson Study Project and as Director of Schools for Secondary. However, in my director role, the secondary school also knew me as the evaluator of the school's performance to whom the school leadership was accountable. Whilst there were several diocesan schools with which I had this relationship, I chose these two because both had previously:

- developed their teaching approaches in mathematics and had recently introduced the Ark Mathematics Mastery scheme[28] as a resource to develop the teaching of mathematics;
- explored the teaching of mathematics through problem solving as a result of being involved in previous Lesson Study cycles through their engagement in the Maths Hub Lesson Study Project and the Diocesan Lesson Study programme.

The details of the participants in each cycle are summarised in Tables 3.3 and 3.4 below.

[28] Ark Mathematics Mastery https://www.arkcurriculumplus.org.uk/?gclid=Cj0KCQjwr4eYBhDrARIsANPywCgqrfg3GqB2o_1u2uq5JKziRvkUPgeY83RLxO2cTlRLWb7jRuOjaJ4aAskcEALw_wcB

Table 3.3 *Cycle 1 Participants*

Cycle 1 Participants – Secondary Phase			
Teacher	Role	Experience	Participation in previous LS Cycles
Joe	Leadership	12 years	3 research cycles Delivered 1 research lesson
Adam	Key Stage responsibility	20 years	3 research cycles
Marie	Class teacher	RQT	2 research cycles
Dave	Class teacher	RQT	2 research cycles
Ruth	Key Stage responsibility	15 years	3 research cycles

Table 3.4 *Cycle 2 Participants*

Cycle 2 Participants – Primary Phase			
Teacher	Role	Experience	Participation in previous LS Cycles
Tom	Class teacher Key Stage Responsibility	7 years	2 research cycles
Paul	Class Teacher	9 years	1 research cycle
Clare	Class Teacher	4 years	1 research cycle
Charlotte	Class Teacher	5 years	None

It is important to recap that the schools and the teachers who took part in the study are not typical in that they had previous experience in professional learning on teaching mathematics through problem solving and had participated in previous Lesson Study projects. This is a potential limitation of the study and will be discussed further in the final chapter.

3.5 Ethical considerations

The study was carried out in accordance with the British Educational Research Association's (BERA) Ethical Guidelines, fourth edition (BERA, 2018) and I obtained ethical approval from the university (ref no: ER8926768). Data was stored and used in accordance with the relevant sections of the Data Protection Act and as set out in my data management plan which formed part of the ethics application approved by the university. Prior to any communication with potential participants, I obtained permission from the headteacher of each selected school to approach the teachers, collect data, video-record the research lesson, and collate and therefore potentially publish the pupils’ work from the research lesson. Participant information sheets, explaining the key details of the programme and the required time commitment, were provided to all participants.

All participants gave their written informed consent to participating in the research. The terms of consent included permission for the PD sessions and interviews to be video-recorded, as face-to-face events at the time were subject to Covid-19 restrictions. It was agreed with the participants that pseudonyms would be used to present the findings from the study. Permission was also obtained from the parents of the pupils to be filmed during the research lesson and for their images to be used for the purposes of the research.

I acknowledged that ethical issues relating to power, consent and transparency could emerge in the study as a result of working with teachers who were known to me and with whom I have previously worked in my role as Director of Schools within the Schools Department of the Diocese of Hallam. In this role, I report to the Diocesan Trustees on the attainment and progress of the schools in the Diocese. As such I am required to work with the headteachers to establish their achievements and priorities for development but I do not have any authority relating to individual teachers within the school. Whilst my director role is broadly supportive and consultative, the authority of the Bishop flows through this position, which raises two significant ethical issues concerning the perceived imbalance of power that could affect the participants and the outcomes of the study. Firstly, my status and position of authority could result in the school and participants feeling that they were obliged to take part in the study. Secondly, my position and working relationship with the participants could lead them to believe that they are required to give the researcher positive reviews of the PD programme and to indicate more readily that the technique of orchestrating the learning in the teaching of problem solving would become part of their practice as a result of their participation in the programme.

Fujii (2012) comments that such imbalances of power and authority are a major source of ethical dilemmas and can impinge on the freedom of participation. Therefore, great care was taken to ensure that the participants understood their role in the study and that they could see the potential personal benefits. This was achieved by heeding the principles of Participatory Action Research (PAR). Reason and Bradbury (2008) define PAR as:

> a participatory process concerned with developing practical knowing in the pursuit of worthwhile human purposes. It seeks to bring together action and reflection, theory and practice, in participation with others, in the pursuit of practical solutions to issues of pressing concern to people, and more generally the flourishing of individual persons and their communities. (p. 4).

The participants were informed of this approach and that their reflections and views could potentially influence the future design of PD programmes of this nature. As such the participants were treated as being 'on par' with the researcher, to give them greater agency in, and ownership of, the innovations within the PD programme. This approach sought to foster their sense of investment in the study and provided reassurance that the outcomes of this research were more dependent on the design of the PD rather than the quality of their participation. This sense of equality and collective participation was sustained by continually informing the participants of my design process and presenting the TDT with choices in the PD programme and research lesson. The use of Lesson Study and TDT also reflected a commitment to the principles of PAR. Whilst I recognise the potential of PAR, I also acknowledge its limitations given the nature of my existing role within the Diocese.

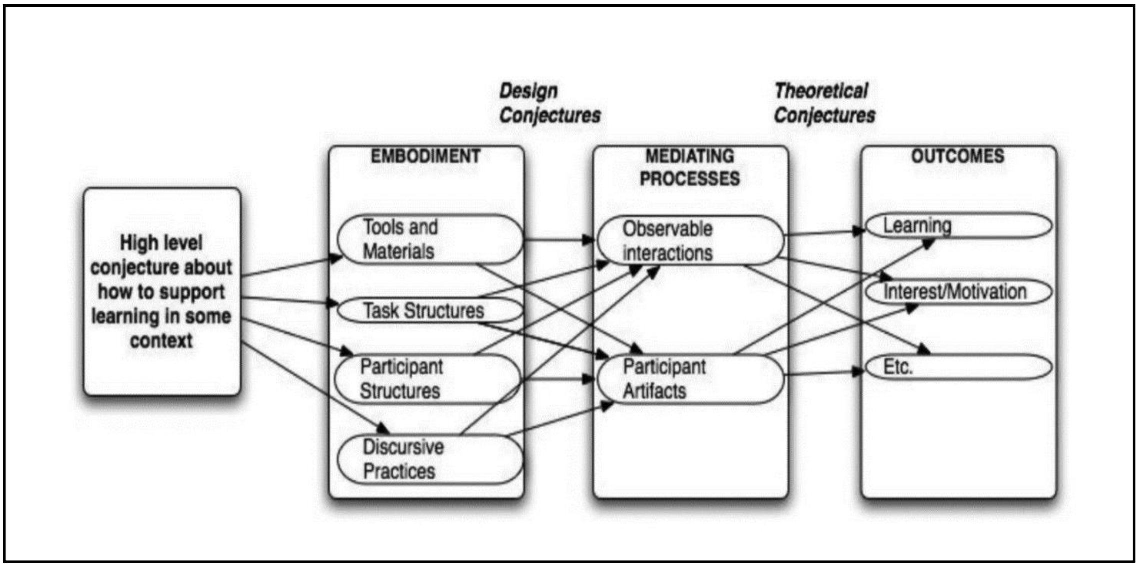
3.6 Analysis

In this section I explain how conjecture mappings were used to link the research questions to the expected outcomes. I go on to describe how the data was analysed. The process of coding collated the data into themes that were analysed to assess the claims relating to the expected outcomes. I also discuss the need for 'argumentative grammar' in the analysis and explain how the use of conjecture mappings support argumentation.

3.6.1 Conjecture mapping

Conjecture mapping is based on the presumption that the design of a learning environment is a theoretical activity and that learning environments intrinsically embody hypotheses about how learning happens in some context (Cobb et al., 2015; Sandoval, 2004). As such it should be evident in the design process that the design work is informed by ideas of how learning might happen or be made to happen. Sandoval (2014) describes the use of conjecture maps as a way of connecting key conjectures with the design components. Figure 3.4 shows a generalised form of a conjecture map; read from left to right.

Figure 3.4 *Generalised from of conjecture mapping (Sandoval, 2004)*



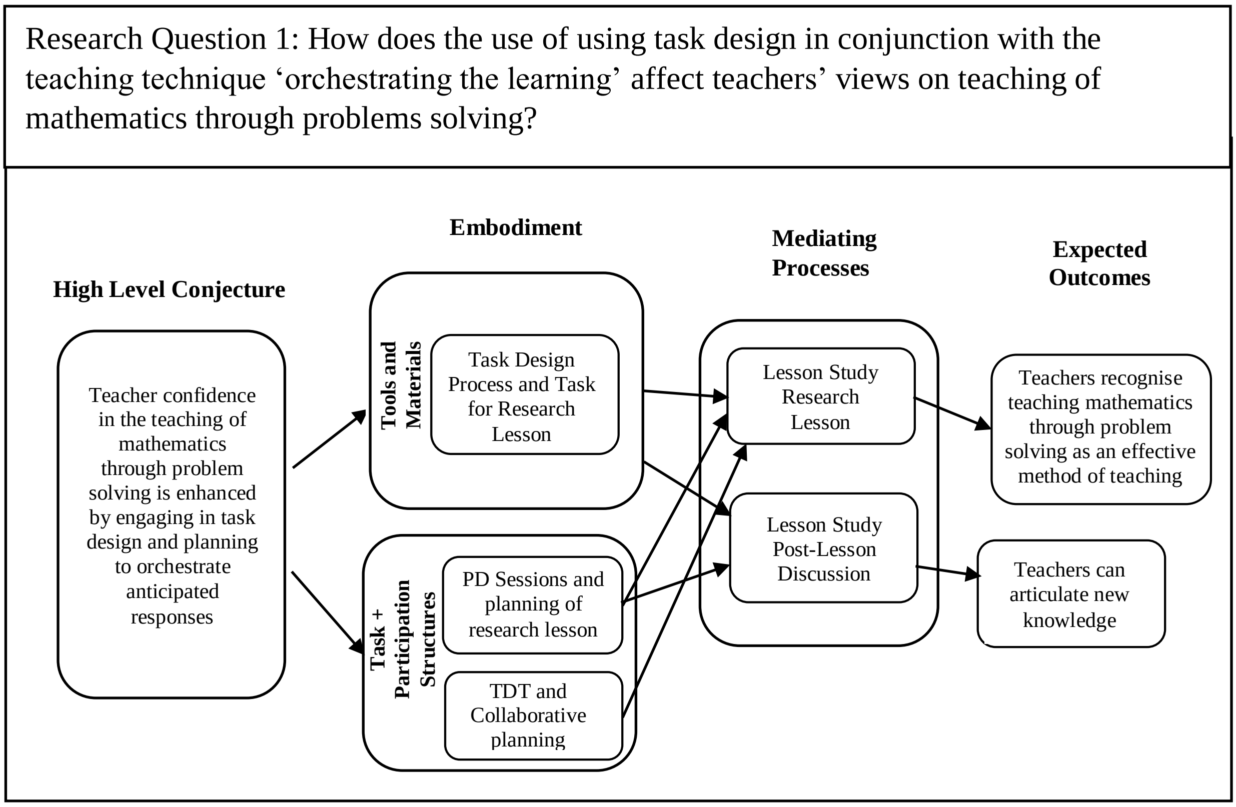
The purpose of conjecture mapping in this way is to model how a high-level conjecture (about supporting learning), which could be formulated from a research question, is to be translated into specific activity within a particular context. As introduced in Chapter 1, the research questions for this study are:

1. How does the use of task design in conjunction with the teaching technique of ‘orchestrating the learning’ affect teachers’ views on teaching mathematics through problem solving?
2. How does the introduction of a ‘pause’ in the research lesson support the development of ‘orchestrating the learning’ as a method of teaching mathematics through problem solving?
3. How do the designed features of the PD programme ‘teaching mathematics through problem solving’ contribute to knowledge about professional learning programmes and environments for teachers of mathematics?

In design research, the formulation of research questions generally falls into two categories (Bakker, 2018). The first of these are theoretical questions, the answers to which can often be found in the literature depending on the extent to which the concept has been developed. For example, consider the third research question. This is a theoretical question, the answers to which may be found in the literature, and which the outcomes of this study may add to. By contrast, the first and second questions are examples of what Bakker called the “researcher’s questions”. Whilst some answers may be found in the literature, these questions have been posed to address specific design features in a particular context. This type of question is one that the researcher formulates not just to research the phenomenon being studied but also to support the design of the study. This distinction between the different types of research question is discussed further in the account of the conjecture mapping for the first research question.

The three figures below (Figures 3.5 to 3.7) set out the high-level conjectures that were devised from each of the three research questions and how they have been embodied in the tools and materials, and the tasks and participation structures. The conjecture mappings therefore connect the research questions to the expected outcomes.

Figure 3.5 *Conjecture mapping for research question 1*



In the mapping above, the first research question *How does the use of using task design in conjunction with the teaching technique ‘orchestrating the learning’ affect teachers’ views on* *teaching of mathematics through problems solving?* was used to generate a high-level conjecture. This is a 'design conjecture' as opposed to a theoretical conjecture. The difference is that a design conjecture establishes the premise that 'if the learner does this in this context (the embodiment) then this will happen', whereas a theoretical conjecture takes the form 'if this mediating process occurs it will lead to this outcome'.

The mapping outlines the actions to be taken and the context in which they will operate. The mediating processes are the conduits through which the outcomes might be observed. In line with this mapping, evidence from the research lesson and the post-study interviews will be used to affirm the conjecture. What this mapping does not do is provide a model for refuting the conjecture. For example, the absence of the expected outcomes would not extrapolate to an erroneous conjecture but would prompt modifications and iterations for the design with respect to the tools, participation structures and mediating processes. In other words, the mapping only works one way. It is also important to note that this process does not investigate the three assumptions set out in Chapter 1 that led to the design of the research questions. The conjecture mappings for research questions 2 and 3 are shown in the Figures 3.6 and 3.7 below.

Figure 3.6 *Conjecture mapping for research question 2*

Research Question 2: How does the introduction of a pause in the research lesson, support the development of 'orchestrating the learning' as a method of teaching mathematics through problem solving?

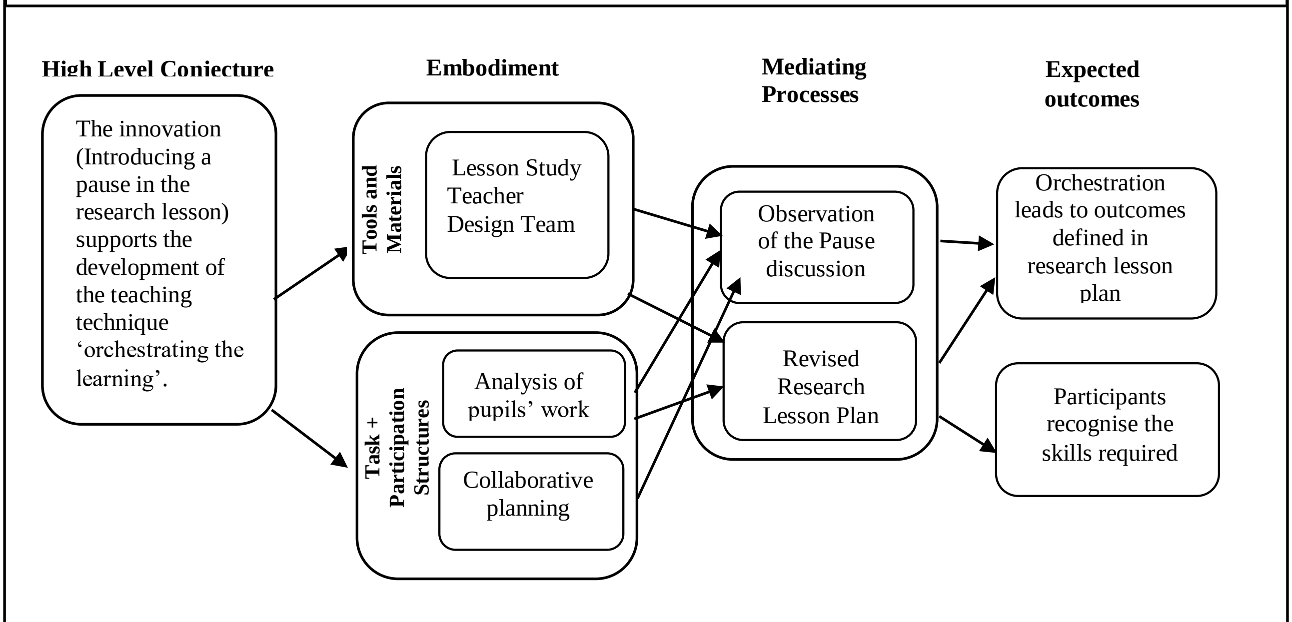
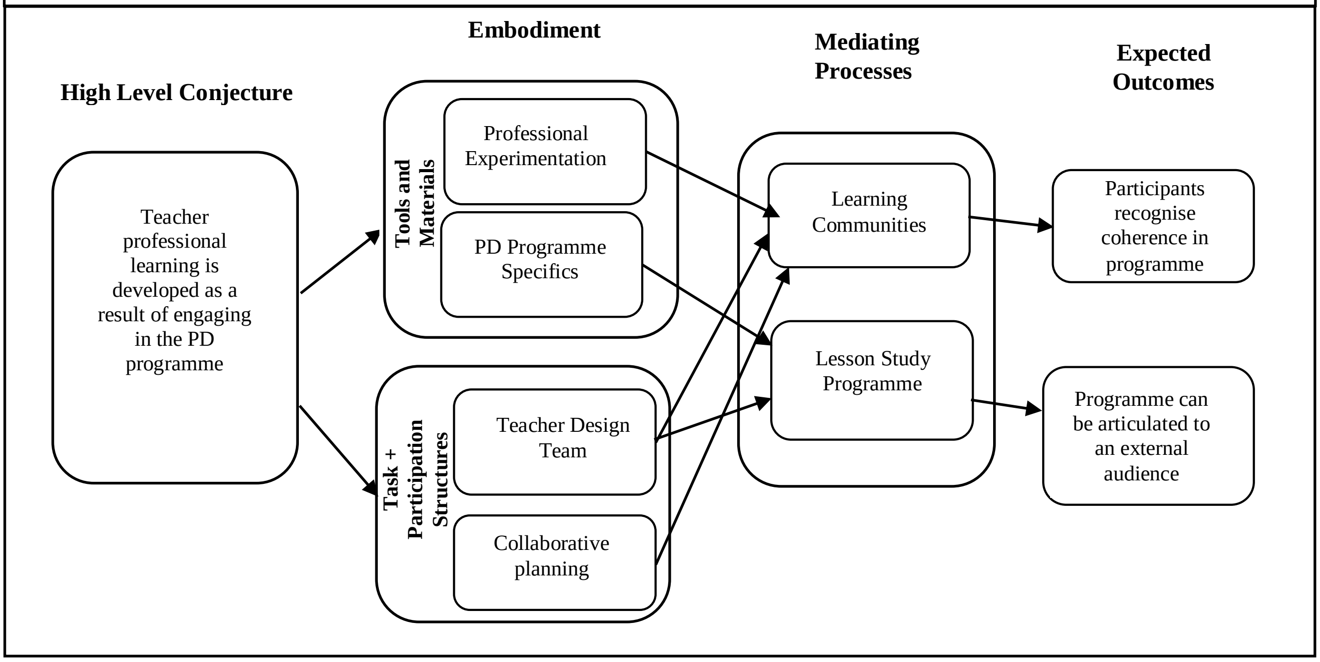


Figure 3.7 *Conjecture mapping for research question 3*

Research Question 3: How do the designed features of the PD programme 'teaching mathematics through problem solving' contribute to knowledge about professional learning programmes and environments for teachers of mathematics?



3.6.2 Developing the themes

In this section, I show how the conjecture mappings were used with inductive and deductive codes to develop themes that were used to collate and present the findings from each of the research cycles. Flick (2018) states that the analysis of data, either existing data or produced for the research, is carried out using two basic methods, namely: (1) coding and categorisation and (2) investigating data in context. Coding may be described as the deciphering or interpretation of data and includes the naming of concepts and explaining and discussing them in more detail (Böhm, 2004). A "good code" is one that captures the qualitative richness of the phenomenon (Boyatzis, 1998). How codes are developed depends on the relationship between the specified code and the data. Inductive codes are formed directly from the data set and therefore have the advantage of 'staying close' to the data. Working systematically with inductive codes allows the researcher to achieve transparency in the analysis and thus offer credible interpretations of the empirical material (Gioia et al., 2013).

However, using inductive codes exclusively can become complicated especially if many codes are generated as this can lead to a lack of focus. In such cases it can be more productive to consider the use of deductive codes. These codes are pre-generated and link to the relevant literature and to the research questions; the codes tend to be smaller in number and for large data sets are typically located in a coding frame (Schreier, 2012). Often in practice (and as in this study), a combination of inductive and deductive codes is developed. This approach is known as the process of abduction (Alvesson & Kärreman, 2007). Using this systematic approach, my first stage in the coding process was to develop a large number of inductive codes from the words or phrases taken directly from the participants' transcripts, items that I judged could be associated to the expected outcomes in the conjecture mappings. For example, in the first conjecture mapping, the expected outcome 'Teachers can demonstrate new knowledge' could be evidenced by the participants in many ways such as describing their views on the process of task design, demonstrating understanding of the teaching technique, or expressing a feeling or emotion in connection to task design or the teaching technique. In the analysis of the transcripts, it was evident that many codes could be identified using these criteria but that not all of the codes would have utility in responding to the research questions. To reduce the number of inductive codes, five deductive codes were established:

- Teacher beliefs
- Teacher subject knowledge
- Ambition to change
- Effective PD design
- Effective CPD design and teacher development.

These codes were established by defining, for each research question, a key focus which could be framed and defined in the relevant literature. For example, the first research question was concerned with the effect of using task design processes in conjunction with the technique of orchestrating the learning. In the literature the use of task design is associated with developments in subject knowledge, so the participants' views on their own subject knowledge could be captured under the deductive code 'teacher subject knowledge'.

During the matching process it became clear that some of the inductive codes could be assigned to more than one of the identified deductive codes. Finally, by returning to the data and the research questions, the deductive codes and the associated inductive codes were arranged into themes. These themes are the headings under which the data is presented in Chapters 6, 7 and 8. The relationships between the codes and the themes are shown in the table below.

Table 3.5 *Coding analysis and thematic summary*

<table>
<tr><th colspan="4">Coding analysis and thematic summary</th></tr>
<tr><td>Research questions</td><td colspan="3">1. How does the use of task design in conjunction with the teaching technique of ‘orchestrating the learning’ affect teachers’ views on teaching mathematics through problem solving?
2. How does the introduction of a ‘pause’ in the research lesson support the development of ‘orchestrating the learning’ as a method of teaching mathematics through problem solving?
3. How do the designed features of the PD programme ‘teaching mathematics through problem solving’ contribute to the knowledge of professional learning programmes and environments for teachers of mathematics?</td></tr>
<tr><td>Inductive codes
Established directly from the data using the outcomes from the conjecture mappings</td><td>Deductive codes
Linked to the literature and/or research design</td><td>Resulting themes for analysis of data</td><td>Associated research questions</td></tr>
<tr><td>Uncertainty
Pupil attitudes</td><td>Teacher beliefs</td><td rowspan="3">Professional development outcomes from the PD programme (Chapter 6)</td><td rowspan="3">1</td></tr>
<tr><td>Teacher confidence
Task Design
Misconceptions
Task analysis</td><td>Teacher subject knowledge</td></tr>
<tr><td>Personal barriers to changes
Teacher context
Teacher confidence</td><td>Teacher ambition to change</td></tr>
<tr><td>Observing pupils’ work
Anticipating responses
Orchestrating the learning</td><td>Teacher subject knowledge
Teacher beliefs</td><td rowspan="2">PD programme design and design intentions (Chapter 7)</td><td rowspan="2">2</td></tr>
<tr><td>The pause
The visualisers
Teacher Design Teams
Collaborative working
Interactivity</td><td>Effective PD design</td></tr>
</table>

Coherence System barriers to change Suitability of PD programme for different teachers	Effective CPD design and teacher development	CPD Characteristics (Chapter 8)	3

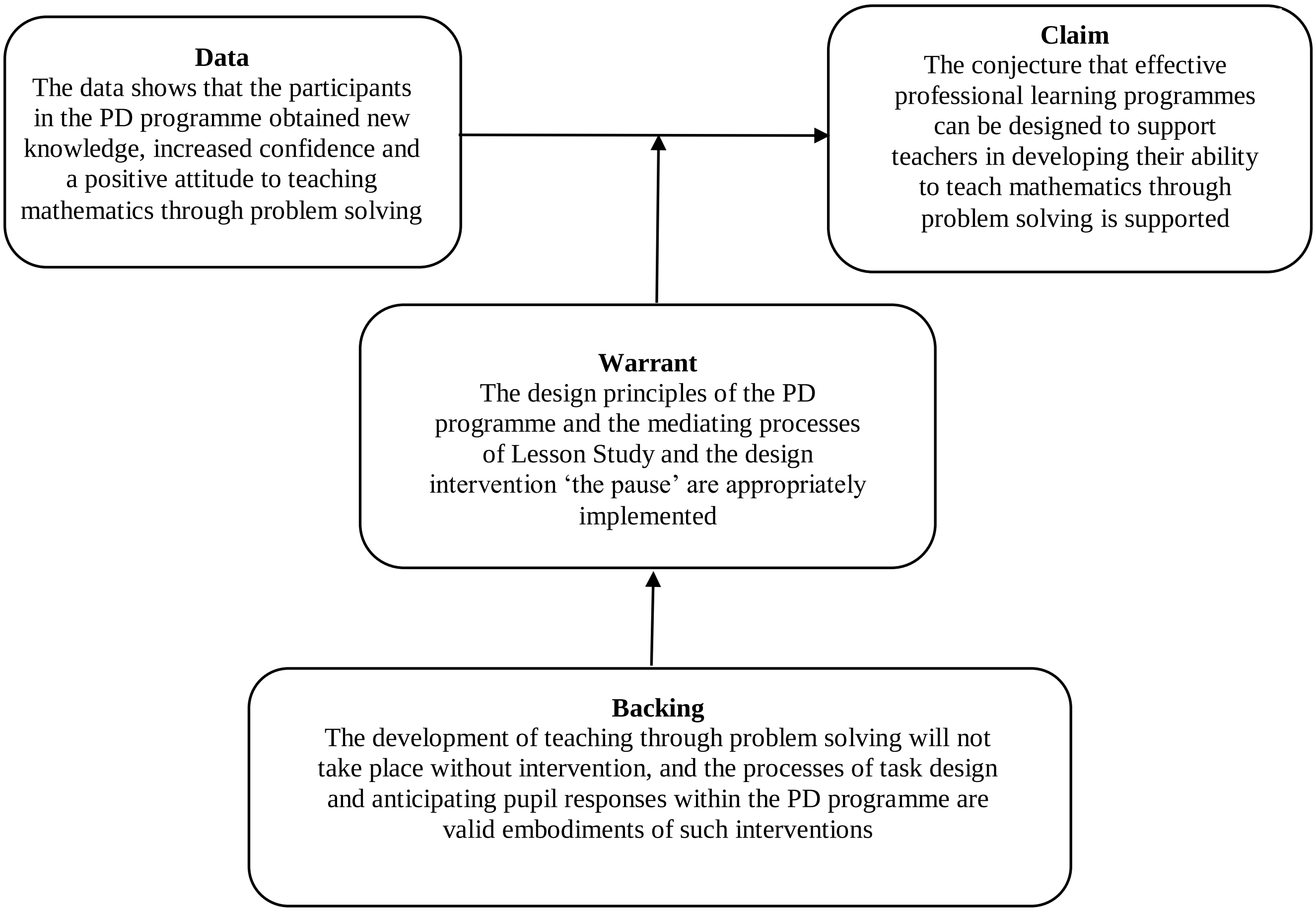
3.6.3 Analysis in design research

The need for 'argumentative grammar' in design research analysis

For design research to become a methodology, Kelly (2004) argues that an 'argumentative grammar' is required. He defines this as "the logic that guides the use of a method and that supports reasoning about its data" (p. 118). The key function of such grammars is to "link research questions to data, data to analysis, and analysis to final claims and assertions" (Cobb et al., 2015, p. 489). Insufficient attention to the conjectures and hypotheses relating to the designed learning environments can lead to a lack of argumentative grammar. This can be the case even when the design research makes clear the characteristics of the design and the resulting affordances.

In this study, the conjecture mapping provides guidelines for how the embodied design components lead to the intended observable effects or outcomes and thus supports the argumentation that the 'how and why' of the intended outcomes is valid. Figure 3.8 below shows how argumentation for this methodology has been developed using the principles within the generalised conjecture mapping and the key ambitions of the research design.

Figure 3.8 *Argumentation grammar for the methodology using conjecture mapping*



3.7 Summary

In this chapter, I have explained the rationale for using design research and have discussed the key principles of this methodology. I have described the research methods of questionnaires, interviews and observation used to collect the data and have explained how the participants were identified and recruited. I also discussed the ethical considerations and the measures taken to safeguard the study from the potential risks arising from the existing relationships between my role as Director of Schools and the workplace of the participants.

I have explained how conjecture mapping and coding were used to establish the themes for analysis and how the research questions related and contributed to the design of the conjecture mappings. Finally, I have discussed the need for argumentative grammar and have shown how the process of conjecture mapping can support the development of argumentation.

Chapter 4 – The PD Programme Cycle 1

Introduction

This chapter details the PD programme content which was developed using the theoretical basis for the programme as discussed in Chapter 2. The PD approaches of Lesson Study and Teacher Design Teams are regarded as effective mechanisms[29] to support the professional learning of teachers and as such feature throughout the PD programme. Although the programme incorporates several of the main features of Lesson Study, it was not designed wholly as a Lesson Study programme. As discussed in Chapter 3 the participants worked as part of a Teacher Design Team (TDT). For each of the programme components below I describe the content and the approach used.

- PD Sessions 1, 2 and 3
- The research lesson
- The pause in the research lesson
- The post-lesson discussion
- The TDT review of the orchestration teaching sequence (Cycle 2 only).

The facilitation approaches used in Cycle 1 are introduced in the section on the relevant PD component. I reviewed these strategies at the start of the subsequent PD session. For example, I used a formative questionnaire with the participants to review the strategy of presenting a classification of problem-solving skills after the participants had explored their own thoughts on the problem-solving skills. The participants' views on these approaches are discussed in the next chapter where I describe the modifications that were made to Cycle 2.

Also in this chapter I continue to explain the development of my framework for the analysis of the PD programme.

[29] It is important not to confuse this term with the term used by the EEF (Sims et al., 2021) in their rapid review of effective CPD.

4.1 The PD programme

In summary, the full programme was delivered to five teachers[30] over a period of five weeks in Cycle 1, and to four teachers over a longer period in Cycle 2 due to the inclusion of an additional PD session where the TDT and the researcher reviewed the orchestration sequence from the research lesson. All of the programme components were modified from their original design in order for the study to proceed during the Covid-19 pandemic. A summary of the programme for each cycle is shown in Tables 4.1 and 4.2 below.

Table 4.1 *Cycle 1 Programme*

PD Programme Timeline for Cycle 1		
Date	Activity	Who
1 February 2021	PD Session 1: Task design and teaching mathematics through problem solving	TDT/researcher
1–8 February 2021	Gap Task[31] Analysis of problem	TDT collaboratively
8 February 2021	PD Session 2 Orchestrating the learning	TDT/researcher
8–15 February 2021	Gap Task Exploration of problem to be used in research lesson	TDT individually
1 March 2021	PD session 3 Planning the research lesson	TDT/researcher
9 March 2021	The research lesson The pause The post-research lesson discussion	TDT/researcher

[30] One of the teachers in Cycle 1 was unable to take part in the evaluation of the programme beyond the post-lesson discussion.
[31] A gap task is known in UK professional learning as an activity that is undertaken by the participants between PD programme sessions. The task can be completed collectively or individually depending on the nature of the task.

Table 4.2 *Cycle 2 Programme*

PD Programme Timeline for Cycle 2		
Date	Activity	Who
14 June 2021	PD Session 1 Teaching mathematics through problem solving using task design and orchestrating the learning (Part 1)	TDT/researcher
14–21 June 2021	Gap Task Analysis of problem	TDT collaboratively
21 June 2021	PD Session 2 Teaching mathematics through problem solving using task design and orchestrating the learning (Part 2)	TDT/researcher
21–28 June 2021	Gap Task Exploration of problem to be used in research lesson	TDT individually
5 July 2021	PD Session 3 Planning the research lesson	TDT/researcher
16 July 2021	The research lesson The pause The post-research lesson discussion	TDT collaboratively
8 September 2021	TDT analysis of orchestration sequence	TDT/researcher

In both cycles, the three PD taught sessions were delivered remotely and all sessions were recorded using the video conferencing technology known as Zoom. In Cycle 1, the participants observed the research lesson from outside the classroom. They were able to watch the lesson remotely and also to observe the pupils working using remote visualisers that were located on the pupils' desks. The technology allowed the participants to select any pair of pupils (in both cycles, the pupils worked in pairs) and observe their work and then switch to another pair of pupils when they wished. In Cycle 2, the participants were able to be present in the classroom and so were able to observe the lesson directly. However, they were not allowed to move around the classroom and so again they observed the work of the pupils through the remote visualisers. Also in Cycle 2, due to the limitation in technology it was not possible for the participants to access every visualiser remotely and so they could not observe the pupils working in the same way as in Cycle 1. Therefore, in Cycle 2, each participant was required to select just two pairs of pupils to observe for the duration of the lesson.

All of these modifications significantly changed the delivery of the programme, but they brought benefits. Firstly, it was now possible to analyse the recordings of the delivery of the taught PD sessions to observe the participants' responses and reactions to both the content and the facilitation strategies. Secondly, the use of the visualisers contributed to teacher learning as the participants were able to examine in detail the pupils' methods and approaches as they were constructed in response to the problem presented. This provided additional information that could be used in the pause in the research lesson, the post-lesson discussion and the analysis of the orchestration teaching episode.

Formative evaluations were built into the programme and used to inform and revise the delivery of the content in Cycle 2 and also to note any differences in views between participants from different phases of education. As explained in Chapter 2, the participants in Cycle 1 were all from the secondary phase of education and in Cycle 2 the participants were from the primary phase.

At the start of the programme, it was important to establish the expectation that the participants would work as a TDT and would develop the characteristics of such a team in relation to the development of the research lesson and their own personal learning. At the same time, it was also important to ensure that the individual learning and reflections of the participants were not influenced by knowledge of designated roles in the team. To achieve this, it was agreed by the team that the member of the TDT who would teach the research lesson would not be identified until the near completion of the planning for the research lesson.

4.1.1 PD Session 1 – Task design and teaching mathematics through problem solving

This session introduced the participants to the principles of task design and the framework for teaching *about*, *for* and *through* problem solving, as explained in Chapter 2. The session comprised three parts.

Part 1

The session began by engaging the participants in the number cells problem that was introduced to explain how a task could be used to teach a mathematical concept or idea. The task is shown in Figure 4.1 below.

Figure 4.1 *Number cells problem*

This is a number cell. Each number, after the first two, is generated by adding the preceding two numbers together.

4	7	11	18	29	37	66

In the number cell below two of the numbers are missing. Find the two missing numbers.

3			93

The participants were asked to explore the task individually by:

- solving the problem
- considering the problem-solving strategies that could be taught using this problem
- identifying mathematical content that could be taught using this problem.

In discussing the principles of task design, the participants were asked to reflect on the size of the number cell (in the problem above, a 4-cell task was used) and the two numbers chosen, being 3 and 93 which are both odd numbers. They were also asked to consider the different methods that may be used by the pupils. Following this individual activity, the participants then shared their work as a TDT. Appendix 12 shows how the problem can be developed to introduce linear equations and then on to simultaneous equations. It also shows how a diagram with symbols can be used to demonstrate different strategies, for example, finding the number in the second cell by adding the contents of the first cell to the last cell and then dividing the total by 2 to get 45.

Part 2

To facilitate a discussion on the differences between:

- teaching *about* problem solving, teaching *for* problem solving and teaching mathematics *through* problem solving
- a strategy and a skill

the participants were asked, as a group, to consider the problem-solving skills that children should be taught. The aim of this activity was to introduce the participants to the classification of problem-solving skills that was discussed in Chapter 2. The skills were grouped under four problem-solving strategies:

- generating data
- sorting and classifying
- patterns and relationships
- reasoning and proof.

I considered two facilitation approaches for introducing the classification of skills. The first is a tactic well used in PD sessions where participants are given the area or focus first and then asked to contribute to or develop that area or focus. In this situation the participants would work as a group to identify the key problem-solving skills that should be taught to children. The facilitator would then build on the contribution from the participants by introducing a framework comprising a set of problem-solving skills. Alternatively, the framework can be shared at the outset and then the participants are asked to consider the framework through a process of critical reflection and potential refinement of the classification.

Both approaches have built-in constraints that can impact on the learning of the participants. A danger with the first approach is that the introduction of the classification of problem-solving skills after the participants have spent time developing their own classification can sometimes result in the participants feeling that their contribution is being 'marked' and/or is not as valued. With the second approach, the participants might feel uneasy about being too critical of a classification that has been presented by a person with knowledge of this area. The extent to which either approach impacts on the effectiveness of the activity is dependent on the relationship between the facilitator and the participants. In Cycle 1, I decided to adopt the first approach and did so because of my previous working relationship with the participants. This approach was reviewed using a formative questionnaire at the beginning of the next PD session. The participants' views on this approach and resulting modifications to Cycle 2 are discussed in Chapter 5.

In order to exemplify this classification further, the participants were asked to explore a problem and to identify the problem-solving skills that could be taught or would be required as a result of engaging in the problem. In Cycle 1, the half-time scores problem was used and in Cycle 2 the area problem was discussed. These problems were discussed in Chapter 2.

Part 3

The last activity in this PD session was to set up a gap task for the TDT in preparation for PD Session 2 – Orchestrating the learning. The participants were introduced to a problem.
The problem given is shown in Figure 4.2 below.

Figure 4.2 *The Three Cards Problem*

Three Cards Problem

1 2 3 4 5 6 7 8 9

Anne picks up 3 cards from the set above and notes that the sum of her numbers is a square number. David takes another 3 cards and notices that the product of his numbers is 63. Freda picks up the last 3 cards. Which cards did Freda pick up?

Again, the participants were asked to consider the problem individually and to reflect on the problem-solving skills that could be taught (or learned) and the mathematics that could be introduced (for example, factors and primes).

After carrying out the problem themselves, the participants were asked to anticipate the methods and responses that the pupils might produce. At the end of the session, the TDT was presented with a gap task: to begin to consider the possible ways in which the different approaches and solutions might be discussed so as to introduce the mathematics to be learned. It was agreed that the gap task would be carried out collaboratively.

4.1.2 PD Session 2 – Orchestrating the learning

This session began with the participants reflecting on the facilitation approach that was used to develop the classification of problem-solving skills. They were asked to read the statement in Figure 4.3 below and then email a response of either A or B.

Figure 4.3 *Question 1 from the formative questionnaire*

> In the last PD session, a classification of problem-solving skills (categorised into four strategies: generating data, sorting and classifying, patterns and relationships, reasoning and proof) was shared after you had considered the problem-solving strategies pupils should be taught and learn.
>
> A. I would have preferred to see the classification first to provide a stimulus to my thinking.
>
> B. It was useful for us to think about problem-solving skills before the classification was shared.

The participants' responses are discussed in Chapter 5.

The session continued by capturing the participants' perceptions on their personal learning following their engagement in the first PD session on task design and teaching mathematics through problem solving. This was achieved by using a model known as the Johari Window (Luft, 1961) which can support the development of self-awareness. It was named after its creators, Joseph Luft and Harrington Ingram. The original four-paned window depicted in Figure 4.4 below was used as a heuristic device to speculate about human relations.

Figure 4.4 *The original Johari Window*

Open Self	Blind Self
Hidden Self	Mystery Self

An explanation of the different types of 'self' was given. For example, when a participant is learning about teaching mathematics through problem solving, what could be happening in the 'mystery self' box is that the learning is rejected by the participant but they are not conscious that they have done so. Furthermore, the people with whom the participant has been discussing these ideas also do not know that the participant has rejected them. An interpretation of the window is set out in Figure 4.5 below.

Figure 4.5 *Interpretation of the Johari Window*

Open Known to self and known to others	**Blind** Unknown to self but known to others
Hidden Known to self but unknown to others	**Mystery** Unknown to self and unknown to others

The participants were asked individually to complete their own Johari Window by inserting into the applicable 'self' quadrants any references to knowledge, beliefs or feelings that they felt were now part of their knowledge as a result of participating in PD Session 1. The purpose of asking the participants to complete the window was to give agency to the participants and to obtain data that could subsequently be used to analyse the effects of the PD programme.

The participants were then asked to consider the gap task from PD Session 1 (exploration of the three-cards problem) and discuss how the range of anticipated responses that pupils might produce could be sequenced to develop the learning. Next, they were shown a video of a teacher who demonstrates the orchestration technique by selecting and discussing, in a pre-planned sequence, the different methods used by the pupils in solving the three-cards problem. This approach of orchestrating the pupils' responses was then explored further by the TDT by returning to the half-time scores problem (in Cycle 2, the area problem) to explore what mathematics could be taught using the task and how the pupils' responses might be orchestrated to develop the learning.

Finally in this session, the task for the research lesson was introduced. The task is known as the Tripods and Bipods problem. The problem was presented as follows:

Figure 4.6 *The problem for the research lesson*

> Some Tripods and Bipods flew from Planet Zero. There were at least two of each of them. Tripods have 3 legs; Bipods have 2 legs. There were 23 legs altogether. How many Tripods and Bipods were there?

This problem is one of a family of problems that involves combining multiples of given numbers, in this problem 3 and 2, to make a given total, in this case 23. Essentially there are two variables to manipulate in order to make the total: the number of Tripods' legs and the number of Bipods' legs. There are many other problems that belong to this family of problems and they occur in a range of different contexts, including in real life. The participants were asked first to work on the task individually to consider the mathematics that could be learned from exploring this task. They then worked on the problem as a TDT to share their thoughts about the mathematics that could be learned from exploring/solving the problem.

The TDT worked on the research lesson plan in an online breakout room (made possible by the use of the remote platform Zoom) without the presence of the facilitator. It was agreed that this would not be a time-limited activity and it was for the team to decide when they would return to the PD session.

Upon their return from the breakout room, the participants were presented with a document entitled 'A discussion on Tripods and Bipods' (Appendix 3) that was developed to show how the task could be used to teach sorting and listing strategies and to introduce algebraic terms through the development of Diophantine linear equations. The document was discussed alongside the participants' initial thinking. The participants then agreed to continue to work collaboratively (as a gap task) on the research lesson plan which would be discussed and developed further at the next PD session.

4.1.3 PD Session 3 – Planning the research lesson

This session began by obtaining the participants' views on the three CPD facilitation strategies that were used in PD Session 2. The strategies were:

1. the use of the Johari Window to reflect on the participants' learning

2. giving unlimited time for the breakout activity and without the presence of the facilitator
3. presenting a paper on how the Tripods and Bipods task could be used to introduce the concept of Diophantine equations.

As at the start of PD Session 3, the participants were presented with the statements S2 to S4 from the formative questionnaire (Appendix 4) and then asked to email a response to each statement. The participants' responses to the statements are discussed in Chapter 5.

The purpose of this session was to continue to develop the detail in the research lesson plan and to establish where the pause in the lesson would be located. In general, the activities and approaches in this session were familiar to the participants due to their previous participation in Lesson Study cycles. My role was to facilitate the planning of the research lesson and I used the following questions (Figure 4.7) to stimulate the planning work and to ensure that the research lesson plan focused on the development of an orchestration sequence within the lesson.

Figure 4.7 *Facilitation questions to support the development of the research lesson plan*

1. Is this a lesson about teaching problem-solving skills, or introducing some new mathematical content or both?
2. How will the problem be introduced?
3. What anticipated responses will we plan for?
4. At what point in the lesson will we begin the orchestration?
5. How will the pupils' work be collated and shared?
6. What will be the order in which the responses will be shared with the class?
7. What questions/statements will be used to connect the learning between the responses?
8. What will the pupils learn from working on the problem and from exploring the different methods?
9. What will the pupils be expected to record at the end of the lesson?

The TDT then worked collaboratively to complete and refine the plan for the research lesson to be taught by a member of the team and who was identified at this point in the programme.

In both cycles the teachers had previous but differing experience of writing and contributing to Lesson Study research proposals. As a result, the lesson plans varied considerably between cycles. In Cycle 1, the lesson plan (Appendix 6) broadly followed the format previously used by the participants and contained a section on their review of the literature on problem solving. Surprisingly, the lesson plan produced by the TDT in Cycle 2 was very brief in relation to both the plan produced by the TDT in Cycle 1 and their own previous research lesson plans (Appendices 7 and 8). An explanation for this might be the fact that the participants in Cycle 1 all had experience of working in the same research team in previous Lesson Study cycles, whereas the participants in Cycle 2 had less experience overall and had not worked together as a team before this study.

4.1.4 The research lesson

In Cycle 1, the research lesson was taught to a group of 14 Year 7 pupils and in Cycle 2 to a group of 12 Year 6 pupils. As explained earlier, the visualisers were used to observe the pupils working on the problem during the lesson. This technology also enabled their work to be recorded which meant that it could be easily reviewed. The TDT in each cycle agreed the point at which the lesson would be paused. The pupils were told in advance that this would happen. The pause was one hour in duration. In Cycle 1, Covid-19 restrictions meant that the pause discussion took place remotely. However, in Cycle 2, due to an easing of restrictions, it was possible to carry out the discussion together in the same room and also to examine the pupils' work directly.

4.1.5 The pause in the research lesson

The pause in the research lesson was a major component in the study and was devised following my previous experience of Lesson Study cycles that had incorporated the orchestrating the learning technique. In conversations with the teachers of previous research lessons who had used this technique for teaching mathematics through problem solving, they indicated that orchestration was a challenging process and they did not feel they had sufficient experience to use it effectively.

However, the principles of this 'innovation' are not new. As part of my continuing research into this area, after the PD programme I became aware of an approach by Gibbons et al. (2017) who carried out an in-depth study of a professional learning programme called

Teacher Time Out (TTO) where a lesson designed by teachers and school leaders is regularly paused in order for the teachers to think aloud, share decision-making, and/or determine where to steer instruction. A protocol within the TTO routine is that all members of the team have permission to pause the lesson by taking a TTO. This TTO routine is what Grossman et al. (2009) call a *pedagogy of enactment*. However, there are important differences between the TTO routine and the pause in the research lesson. In the present study the pupils are not involved in the discussion between the teachers. Also, the remit of the teachers during the pause is focused on a single activity – to decide on the sequencing of the pupils' responses – and therefore does not involve potential ongoing changes to the lesson.

In Chapter 2, I discussed the framework known as the Knowledge Quartet (Rowland et al., 2005). This framework has four dimensions: foundation, transformation, connection and contingency. Contingency refers to the teacher responding effectively to the pupils' responses and contributions arising from their encounter with the mathematical task or problem. I consider that the teaching technique of orchestrating the learning is not only an effective way to develop contingency but also that it enables the teacher to interact with the pupils' responses in order to introduce, develop and explain the mathematics to be learned. Therefore, to orchestrate pupils' learning, the teacher needs to plan for the anticipated responses but must also be able to accommodate unexpected opportunities that may arise.

The design principle here is that the pause introduced to the lesson provides a training opportunity for the teacher which will increase their confidence making 'in the moment' decisions about how the work in the class should be orchestrated.

Essentially then the pause was a one-hour break[32] in the middle of the lesson in which the TDT and the teacher reviewed the work of the pupils thus far and then agreed how the second half of the lesson should proceed in the light of this review. In revising the research lesson plan, the TDT documented the range of anticipated responses observed and agreed a sequence in which some or all of these responses would be discussed. To prompt the discussion during the pause, I used the questions in Figure 4.8 below.

[32] During the break the pupils had refreshments and exercise. In Cycle 1, they were the only pupils in school due to Covid-19 restrictions. In Cycle 2, the pupils had lunch with the rest of the pupils in school.

Figure 4.8 *Questions used in the PD pause session*

1. Did all the anticipated responses planned for appear in the work of the pupils?
2. Did we observe any responses that we did not plan for and were unexpected?
3. Are there any misconceptions to be addressed during the orchestration?
4. How should the orchestration sequence change in the light of our observations?

4.1.6 Post-lesson discussion

In most Lesson Study programmes, the post-lesson discussion begins with the teacher and/or the research team reflecting on how the lesson went in relation to the plan. In the present study, the post-lesson discussion focused mostly on the second part of the research lesson. It should also be noted that in Japanese Lesson Study the post-research lesson discussion normally includes a final commentary from someone outside of the planning team, known as a ‘knowledgeable other’, who has deep expertise in the relevant content (Takahashi & McDougal, 2016). Often the knowledgeable other will provide a different perspective on how the content might be delivered and some critical reflection on the lesson plan (which they would have received several weeks prior to the research lesson). In the PD programme in the present study, my role was not that of a ‘knowledgeable other’ but as a facilitator of the post-lesson discussion which focused on:

- reflections on the success of the lesson in relation to the changes made during the PD pause session
- the effectiveness of the pause in developing the second half of the research lesson
- initial reflections from the TDT on the value of the PD programme specifically with regard to the teaching of mathematics through problem solving.

Facilitation of the discussion was achieved by displaying a set of questions that the TDT could consider as part of their deliberations (see Figure 4.9 below). The participants were not asked the questions in sequence but were asked to read them all before the start of the discussion.

Figure 4.9 *Post-research lesson facilitation questions*

1. Did the orchestration lead to the achievement of the desired learning outcomes?
2. Could the order of sequencing have been different?
3. How helpful was the pause in supporting the planning for the second half of the research lesson?
4. What are your views about teaching mathematics through problem solving?

4.1.7 TDT review of the 'orchestration'

The purpose of the review was to establish the teachers' views on the effectiveness of the orchestration in terms of the sequencing of the pupils' responses to the problem. In Cycle 1, this review was done as part of the post-lesson discussion. However, the video analysis and audio transcript of this activity indicated that the participants were suffering from fatigue. As a result, in Cycle 2, the TDT review of the orchestration was introduced as a separate event. The review was supported by the use of facilitation questions that followed the lines of enquiry below:

- How did the class discussion and orchestration develop as a result of choosing the first method?
- How was this method built on by the addition of other responses and teacher questions?
- To what extent did the orchestration contribute to the learning experienced by the pupils?

4.2 Design expectations

As part of the development of the PD programme and prior to implementation in Cycle 1, I used the CPD framework (introduced in Chapter 2) to establish the extent to which the PD programme, as designed, aligned with the characteristics of effective CPD identified by the framework. As discussed in Chapter 2, the framework was based on the triple lens model devised by Frazer et al. (2007) and which also acknowledged the effective characteristics described by Desimone (2009). Each model referenced in the framework represents a lens through which the PD programme components can be viewed. Originally, for the purposes of the analysis framework, the PD components were grouped into two categories: one for PD sessions 1, 2 and 3 (Table 4.1) and one for the research lesson, the pause in the research lesson and the post-lesson discussion (Table 4.2). However, as already indicated, the analysis of the orchestration sequence was taken out of the post-lesson discussion and became an additional PD session. Therefore, Table 4.3 pertains to the PD session in which the participants reviewed the orchestration episode within the research lesson.

The completion of the analysis frameworks in these tables was undertaken from a design perspective, in other words before the start of the PD programme. I analysed each of the design features of the PD programme in this study through all four lenses (the triple-lens model and Desimone's approach). I then decided whether they aligned to the effective characteristics 'by design' or incidentally ('potentially could'). For example, in Table 4.3, I indicate that the use of Teacher Design Teams aligns with Bell and Gilbert's framework, specifically the social aspect of professional learning that could lead to active participation. Also, the features identified by Desimone, such as duration, active learning, collective learning and coherence, were evident in the design of the three taught PD sessions. Therefore, each of these features are recorded in the 'by design' column in the table.

However, in the course of this analysis, I also noted that there were potentially other effective characteristics that could be attributed to the PD programme but were not part of the intended design. For example, in Table 4.3, I noted that the content of the PD sessions had the potential to help participants forge strong links between theory and practice. However, I did not deliberately design the PD programme to educate the participants about the links between the theory of task design and teacher learning, nor was it my intention to promote their interest in researching educational theories on teaching mathematics through problem solving. Therefore, I have recorded this potential characteristic in the column headed 'potentially could'. These tabled associations between the programme design features and the known characteristics of effective CPD are based purely on my personal reflections. However, they do serve as a useful tool for evaluating the design of the PD programme, which will be discussed further in Chapter 8.

Table 4.3 *PD Programme: Analysis framework Part 1*

PD Programme - Analysis Framework Part 1		
PD component	PD Session 1: Task design and teaching through problem solving PD Session 2: Orchestrating the learning PD session 3: Planning the research lesson	
Lens	**Design Features**	
	By Design	**Potentially Could**
Kennedy	***Community of practice:*** Collaborative development of research lesson.	***Training:*** Effective remote learning platform.
Bell and Gilbert	***Personal:*** Self-selection to PD programme. All participants demonstrated a desire to be part of the study and PD programme. ***Social:*** Learning focused through Teacher Design Team to develop a community of practice. ***Occupational:*** PD programme includes opportunities for personal development of mathematical subject knowledge.	***Occupational:*** Content of the PD programme sessions can provide strong links between theory and practice.
Reid's Domain of Practice	***Formal and planned***: Using Design Research methodology.	
Relevant Desimone features	***Duration:*** Pre-planned 5-week programme mainly resourced during school time. ***Active learning:*** Engagement in task design. Activities designed to ensure active participation. ***Collective participation:*** TDT works collaboratively to produce research lesson plan and contribute to PD programme design. ***Coherence:*** Learning from PD sessions directly relevant to planning and delivery of research lesson.	***Content focus:*** Examples of tasks and associated teaching approaches provide a focus for the consideration of teaching mathematics through problem solving.

Table 4.4 *PD Programme: Analysis framework Part 2*

PD Programme - Analysis Framework part 2		
PD components	The research lesson The pause Post-lesson discussion	
Lens	**Design Features**	
	By Design	**Potentially Could**
Kennedy	**Community of practice:** Facilitation of development of research lesson and analysis of pupils' work in the 'pause' session. Designed to ensure mutual engagement. **Action research**: Participants involved as researchers in the analysis of pupils' work in order to understand and contribute to re-planning of research lesson. The post-lesson discussion enables the TDT to ask 'critical questions' of their collective practice.	
Bell and Gilbert	***Occupational:*** Analysis of pupils' work in the research lesson adds to teacher subject knowledge. ***Personal***: Interest and motivation sustained by PD events being time-limited and outcome-driven, i.e. development of research lesson.	***Social:*** Relationships strengthened in the pause; peer support generated for the teacher who is teaching the research lesson.
Reid's Domain of Practice	**Formal and planned:** Using design research methodology.	
Relevant Desimone features	***Duration:*** 4 weeks' planning for research lesson. ***Active learning:*** Engagement in task design and lesson planning.	***Collective participation:*** Shared responsibility for analysis of pupils' work and subsequent input into revisions of lesson plan.

Table 4.5 *PD Programme: Analysis framework Part 3*

PD Programme - Analysis framework Part 3		
PD components	TDT Analysis of the 'orchestration'	
Lens	**Design Features**	
	By Design	**Potentially Could**
Kennedy	**Action Research:** The post-lesson discussion enables the TDT to ask 'critical questions' of their collective practice.	**Transformative:** TDT take ownership of PD programme outputs. They establish the strength of the endeavour and agree (or not) to locate learning into practice.
Bell and Gilbert	***Occupational:*** Analysis of orchestration event leads to acquisition of new knowledge that is derived from the work of the TDT.	***Personal:*** Interest and motivation sustained by PD events being time-limited and outcome-driven, i.e. development of research lesson. ***Social:*** Development of trust and professional commitment to improve provision with regard to development of teaching resources.
Reid's Domain of Practice	**Formal and incidental:** The facilitation device is the orchestration event. As such the learning is defined as incidental and solely within the domain of the TDT.	
Relevant Desimone features	***Active learning:*** Teachers engaged as researchers. ***Coherence:*** TDT members understand where learning has come from and where it should be now located.	

4.3 Summary

In this chapter, I described the content of the PD programme components and the methods of delivery and facilitation. The sections which detailed each of the PD sessions included information on the formative questionnaires developed to review the delivered components and inform the development of Cycle 2. Each of the PD components reflects the intentions and purposes specified in the design. I have explained how the PD approaches of Lesson Study and Teacher Design Teams were used in the programme. The resulting modifications to Cycle 2 of the PD programme and the rationale for so doing are explained in the next chapter.

I presented an analysis of the design of the PD programme using the framework developed to establish the extent to which the known features of effective CPD were incorporated into the programme design. The development of this framework is revisited in Chapter 8 which describes its use retrospectively to identify the effective characteristics of professional learning that were observed during the PD programme and which emerged from the analysis of the participants' reflections.

Chapter 5 – PD programme Cycle 2 modifications

Introduction

In this chapter I discuss the modifications made to Cycle 2 of the PD programme as a result of the evaluations through the formative questionnaires used in PD Sessions 2 and 3, and also following my reflections on the video recordings of each of the PD components in Cycle 1. Modifications were made to both the programme design and the programme facilitation approaches. The participants in Cycle 2 were all from the primary education phase and therefore some of the modifications acknowledge this difference.

I begin by discussing the design changes I made to PD Sessions 1 and 2. I then discuss the modifications to the facilitation approaches that were used in these sessions. The approaches are shown in Table 5.1 below.

Table 5.1 *Facilitation approaches used in PD sessions*

PD Session	Facilitation approach
PD Session 1	Presentation of classification of problem-solving skills document after participant discussion on problem-solving skills
PD Session 2	The use of the breakout room as a virtual environment for the TDT to work and no time limit placed on the planning the research lesson activity
	Presentation of the document 'Discussion about Tripods and Bipods' after the planning of the research lesson

Finally, I discuss the modifications I made to the facilitation of the pause and the post-lesson discussion as a result of my review of the videos of these PD components.

5.1 PD programme design modifications

In this section I discuss the modifications that were made to the design of the PD programme. Design research methodology includes a cyclic component the purpose of which is to 'feed' a new cycle. Of course, other types of research also build on previous studies or experiments, but in design research the changes can take place during the same intervention or a series of interventions. Also, as Bakker (2018) remarks, another difference between design research and other forms of research is the perspective on variation.

> Where experimentally oriented researchers mostly try to control or plan variation, design researchers welcome unexpected variation to see how robust their ideas and design are (irrespective of the question to what extent control is really possible in naturalistic settings). (p. 18)

Importantly, the cyclic features of design research do not confine themselves to just modifications and changes. In design research it is also important to 'lock in' to subsequent cycles those aspects of the design that have shown to be beneficial or necessary. Therefore, a number of design features in Cycle 1 were maintained in Cycle 2. It is also important to state that evaluations of activities in Cycle 1 did not always lead to modifications of the programme and delivery of those activities in Cycle 2.

5.1.1 Modifications to PD programme Sessions 1 and 2

Following evaluation of the Cycle 1 PD programme, it was apparent that the participants saw the processes of task design and orchestration as separate concepts. For example, in the pause discussion, some of the participants focused on completing the table of solutions without sequencing the responses they had planned and observed. This suggested that not enough emphasis was given to the interconnections between the process of task design and the teaching technique orchestrating the learning in the context of teaching mathematics through problem solving. A possible explanation is that the principles of these two processes were presented in separate PD sessions. PD Session 1 focused on task design and PD Session 2 introduced the technique of orchestrating anticipated pupil responses to develop the mathematics.

As a result, I redesigned PD Sessions 1 and 2 by bringing in the process of task design alongside the teaching technique of orchestrating the learning. This was achieved by using just one task in each PD session to develop a planning and teaching sequence. For example, PD Session 1 was modified as follows:

- introduce the task – number cells
- explore the mathematics that could be taught from the task
- discuss and collate a range of anticipated pupil methods and approaches

- identify those responses that can be used to develop the intended learning
- begin to think about the ordering of the responses as part of a teaching sequence that would best illuminate the mathematics to be learned.

The above model was repeated in PD Session 2 using the area problem described in Chapter 2. The analysis of problem-solving skills that originally took place in part 2 of PD Session 1 was introduced at the beginning of PD Session 2.

It is important to note that the participants' lack of awareness of the connectivity between task design and orchestrating the learning could have been due in part to their own contexts and experiences. The participants in Cycle 2 taught in a different phase (primary) and so their experiences might not have produced the same outcome. However, it was not the purpose of this study to explore this particular difference between the two sets of participants. The modification described above was made on the basis of an evaluation of the PD session and therefore it would have been made irrespective of the nature of the participants in Cycle 2.

5.1.2 The use of the Johari Window

In the last chapter, I explained that the Johari Window was used to draw out the participants' perceptions on their personal learning from PD Session 1. Statement S2 (Figure 5.1 below) from the formative questionnaire was used to establish the participants' views on this approach.

Figure 5.1 *Statement S2 from the formative questionnaire*

The Johari Window was introduced to examine how we, in our role as a research team, were responding to the PD programme and to consider how our own beliefs were impacting on the intended learning from the session. How helpful was this in identifying issues in the PD programme?

A. Did not help with reflecting on the PD programme.

B. Did not help but nor did it impact negatively on the PD programme.

C. Was useful as a tool to think about how the PD programme was impacting on my own beliefs in relation to the objectives of the PD programme.

Whilst three out of five of the TDT members responded that the Johari Window was a useful tool for their own reflection, the analysis of the recording of the PD session indicated that there was an uncomfortableness with the task that seemed to be borne out of difficulty either with accepting the model or with the request to document thoughts about themselves, particularly those in the Johari Window quadrant 'hidden' self (known to self but unknown to others). For example, when describing the contents of each of the 'self' boxes, in response to the explanation for the 'mystery self' box, one member of the TDT said:

> I think that if we did reject the ideas and principles around the teaching of problem solving then we would know that we had rejected them. I cannot understand how we would not know that we had done so… I think surely consciously we would know.
>
> [Dave, secondary teacher. Cycle 1, Extract from transcript of PD session]

The subsequent group discussion was useful in illuminating some of the reasons for the differences between espoused and enacted beliefs (for example, day-to-day pressures of teaching, and pupil attitudes), but it was clear that the participants felt 'exposed' by being asked to provide their reflections using this framework.

It is also evident that my explanation of the different self-boxes in the original model was clearly unsatisfactory as the example I gave did not adequately explain the characteristics of the mystery-self box and therefore it was understandable that Dave was confused by the activity.

As the use of the Johari Window (and my presentation of it) had adversely affected the participants' engagement, I decided to modify the approach in Cycle 2. I also decided not to use the data from the Johari Window in Cycle 1. Due to the anxiety the participants seemed to experience, I felt that the data did not accurately reflect the participants' views. In Cycle 2, the modification to the Johari Window reflected the work of Oliver and Duncan (2019) who adapted the model (Figure 5.2 below) to refer to knowledge about groups of people rather than individuals.

Figure 5.2 *Johari Window (adapted by Oliver and Duncan, 2019)*

Common knowledge What we and others know about our world	**Others' knowledge** What others know about our world that we do not
Our own knowledge What we know about our world that others do not	**The unknown** What none of us know about our world

Oliver and Duncan (2019) argue that this model offers a way of growing all four domains so that more knowledge is shared. Their example of how this is achieved involved a medical research team working with their patients by sharing all available research in order to improve the treatment of Lyme's disease. When using this model, care must be taken to properly define the term 'our world', which in this study was defined as the knowledge of problem solving and teaching mathematics through problem solving. In Cycle 2, the participants completed the Johari Window as a group. They did this by discussing and agreeing what they all knew – and presumed that others also knew – and then recording this information in the 'common knowledge' box.

5.1.3 Modifications to the review of the orchestration teaching sequence

In Cycle 1, the review of the orchestration sequence was carried out during the post-lesson discussion. During the interaction, I noticed that the quality of discussion dropped and I inferred that the TDT members were showing signs of fatigue. The teachers had been looking at a computer screen for nearly half a day and although a number of comfort breaks had been built into the sessions, the teachers were finding it hard to maintain focus and concentration. In light of the above, I decided that in Cycle 2 the orchestration sequence would be recorded to provide the option of reviewing this teaching sequence collectively at a later date.

This resulted in an additional PD session for Cycle 2. The TDT met several weeks after the research lesson and after the post-study interviews with the participants. They met to review the orchestration part of the research lesson by observing the video which showed how the work of the pupils was orchestrated to introduce the Diophantine equation 3T + 2B = 23 and the algebraic terms 3T and 2B.

5.2 PD programme facilitation modifications

In this section I discuss the participants' views on the facilitation approaches that were used in the PD sessions. The discussion is informed by the participants' responses to the formative questionnaire statements that were given following PD Sessions 2 and 3 and the data from the post-PD programme interviews. I also reflect on the facilitation of the pause and post-lesson discussions by reviewing the videos of those sessions. The facilitation approaches used were:

- presenting the participants with a classification of problem-solving skills after they had worked as a TDT to devise their own list of skills
- the use of the breakout room and having no time limit on the research lesson planning activity
- presenting the participants with a paper on Tripods and Bipods after they had worked as a TDT on exploring the problem.

5.2.1 The classification of problem-solving skills

In the first PD programme session, the participants were asked to consider what problem-solving skills children should be taught, and subsequently the participants were presented with a classification of problem-solving skills (Appendix 5). As mentioned in Chapter 4, an alternative approach would have been to introduce the classification first in order to stimulate discussion about the range of problem-solving skills that children should be taught and then to discuss additions or augmentation to the classification. To explore the participants' views on these two approaches, they were asked to consider statement S1 (Figure 5.3 below) and then to select the response that best reflected their view on how the classification of problem-solving skills document was used.

Figure 5.3 *Statement S1 from the formative questionnaire*

S1. In PD Session 1, the classification of problem-solving skills (generating data, sorting and classifying, patterns and relationships, reasoning and proof) was shared after you had considered the problem-solving strategies pupils should be taught and learn. A. I would have preferred to see the classification first to provide a stimulus to my thinking. B. It was useful for us to think about problem-solving skills before the classification was shared.

All of the participants selected response B. In the post-study interviews, the participants were asked whether they preferred having sight of the classification prior to the discussion. The view that the document should be shared after the discussion was explained by Paul.

> I think it was helpful to come up with our own ideas first, we could see that what we got and it linked quite well with yours, […] And I felt that it is important that we kind of realise that our ideas were linking with the kind of thing you were looking for as well. I think that this way this let us think more about the problem-solving skills to be taught. I don't know if it would have been the same if we'd have not had chance to explore the skills first.
>
> [Paul, secondary teacher. Cycle 1 Post-PD programme interview]

In the remote delivery of the PD programme, in order to avoid the complication of trying to view too many documents (and participants) simultaneously, the classification document was sent in advance of the session. Whether the participants in Cycle 1 studied the document prior to the PD session is unknown. As a result, in Cycle 2 the session was planned so that the participants would receive the document during the session by email. The participants were notified in advance that this would be happening so that they could make appropriate arrangements for viewing the document, such as on another screen or by printing off a copy at the appropriate point during the session.

5.2.2 The use of the breakout room

In the second PD session, all the participants spent time in a virtual breakout room exploring the task for the research lesson before returning to the main PD session to discuss their ideas with the facilitator. In the post-study interviews, the participants were asked for their views on the use of the breakout room to work on the development of the research lesson with no input or observation from the facilitator. The following responses were obtained.

> This approach put you (the facilitator) in the role of judging what we have done and that straight away makes us feel cautious and uncomfortable… that feeling of here we go we are being judged on something again without you being part of the process not knowing how we have made this decision or that decision. For example, in the choice of numbers you will only see the numbers we have chosen and will not know about the numbers that we considered and then rejected and why.
>
> [Marie, secondary teacher. Cycle 1 Post-PD programme interview]

> I think doing the way you said allows us to give our slant on it rather than be guided I suppose, which is then beneficial to you as you can see our train of thought and where that might match with where you are wanting to go with it.
>
> [Adam, secondary teacher. Cycle 1 Post-PD programme interview]

> I think we need both. We need an expert input but at the same time we need time on our own and at our own pace to think and develop ideas.
>
> [Dave, secondary teacher. Cycle 1 Post-PD programme interview]

These different reactions reflect the complexity of the dynamics that exist in any group. For PD programme designers, the reactions illuminate the importance of being aware of possible unintended interpretations of the facilitation approaches being used. In this situation, the approach was designed to provide the participants with the space to develop ‘a community of practice’ that enabled ‘active learning’ – which are features from the Bell and Gilbert framework and Reid’s Domain of Practice. However, the participants’ comments above also indicate that some saw the process as an assessment activity where their outputs would be judged by an ‘expert’. As a result of this reflection, I decided that in Cycle 2 I would explain the rationale for the use of the breakout room and ask the TDT to choose whether or not to work independently of the facilitator. The participants expressed a preference for me to be present with them in the breakout room, which could suggest that they required ongoing reassurance. I agreed to be present in the breakout room but it was interesting to note that the only questions they asked me were procedural ones such as "should we record this on the flipchart or the whiteboard?".

5.2.3 No time limit on the research lesson planning activity

It was agreed that the decision to return from the breakout room would be made by the TDT. In response to the statement below (Figure 5.4) from the formative questionnaire, all of the participants answered B.

Figure 5.4 *Statement S3 from the formative questionnaire*

S3.	When we were exploring the task for the research lesson in the breakout room, we were given no time limit on how long we could spend working on the plan.
A.	I would have preferred a time limit being given to the activity.
B.	I was happy that we could choose when to return to the PD session.

Also, in the post-study interviews, all members of the team indicated that they appreciated having control of the amount of time spent on the problem. They reflected that this was the first time in their collective CPD experience that this approach had been used. When asked about the facilitator joining them in the room to ascertain the level of progress, as an indirect way of allocating time to an activity, one participant said:

> If someone comes to join you midway through when you are discussing… you might feel uncomfortable. I would feel uncomfortable if that happened because I am thinking that they are expecting you to get so far in a certain amount of time.
>
> [Ruth, secondary teacher. Cycle 1 Post-PD programme interview]

From this information, I decided to use the same approach in Cycle 2 and to let the team decide when to return from the breakout room.

5.2.4 The use of the discussion paper on Tripods and Bipods

Following their work on the research lesson in PD Session 2, on returning from the breakout room the participants were presented with the discussion paper on Tripods and Bipods. An alternative strategy would have been to share the paper before exploring the task. A statement in the formative questionnaire was used to gauge the participants' views on the approach used (see Figure 5.5).

Figure 5.5 *Statement S4 from the formative questionnaire*

> S4. The Tripods and Bipods problem was presented and we were asked to explore the problem as group. After this, a paper on the problem setting out possible solutions and linking it to an aspect of linear equations (Diophantine) was presented to the team. How helpful was the paper?
>
> A. The paper on the problem was helpful and I would have liked to have seen it before we started working on the problem.
>
> B. The paper was helpful and I was happy to see it after we had worked on the problem.
>
> C. The paper was not helpful in planning the research lesson.

Four of the participants gave response B and one participant responded C. However, in subsequent discussion about the initial plans for the research lesson, they recognised that they had not given sufficient consideration to the mathematics to be taught and had focused more on the strategies that the pupils would use to approach the problem. As a result, the team then decided to explore how the problem could be used to introduce algebraic terms and the subsequent development of a linear equation comprising such terms.

This observation aligns with the empirical evidence in reports of other Lesson Studies in the UK and the USA that often the design of the research lesson does not contain enough information about where the intended mathematics content fits into the mathematics curriculum sequence and its contribution to mathematical knowledge (Seleznyov, 2018). This picture contrasts with practice in Japan where this part of the Lesson Study process is crucial and is known as *Kyosai Kenkyu*.[33] In Cycle 2, I provided the discussion paper to the TDT before they began their planning of the research lesson. I took this decision because I was unsure if the same outcome would happen as in Cycle 1 and thought it might be more likely because of the participants' lack of experience in the process of Kyosai Kenkyu.

[33]Kyosai Kenkyu is the component towards the beginning of the Lesson Study process where a detailed review is carried out on the current programme of study.

5.2.5 Changes to the pause

From the analysis of the video of the pause PD session in Cycle 1, I judged that I made too many interventions between the various inputs provided by the participants. As discussed above, this was probably due to the participants appearing to move away from orchestration of the pupils' work to a more conventional approach where the teacher focused on completing a table of the solutions without using the responses of the pupils. In Cycle 2, I endeavoured to keep my interventions to a minimum but I did remind the participants that the objective was to work with the responses provided by the pupils.

5.2.6 Changes to the facilitation questions in the post-lesson discussion

During a review of the video recording of the post-lesson discussion, I observed that the participants touched on all four of the facilitation questions (which were displayed for the participants to consider and were not posed directly) but it was question 1 that took most of the discussion time. I was interested to see if this question was the most important to the TDT or whether the time spent was just because it was the first question in the list.

The flow of the questions was originally designed to lead to the main discussion point (for the research) which was establishing the participants' views on how helpful the pause was in developing the second half of the research lesson in relation to achieving the planned outcomes. I assumed that the answers to the first two questions (as shown for Cycle 1) would naturally inform the thinking around questions 3 and 4. In Cycle 2, I interchanged questions 1 and 2 (as shown in Figure 5.6 below) to see if this would make any difference to the time spent on each question.

Figure 5.6 *Modification to presentation of facilitation questions in the post-lesson discussion*

Cycle 1 Question Order	**Cycle 2 Question Order**
1. Did the orchestration lead to the achievement of the desired learning outcomes? 2. Could the order of sequencing have been different? 3. How helpful was the pause in supporting the planning for the second half of the research lesson? 4. What are your views about teaching mathematics through problem solving?	1. Could the order of sequencing have been different? 2. Did the orchestration lead to the achievement of the desired learning outcomes? 3. How helpful was the pause in supporting the planning for the second half of the research lesson? 4. What are your views about teaching mathematics through problem solving?

The result was that the participants spent less time on the first question and again spent the most time on question 2 (the first question in Cycle 1). Whilst I recognise that this was an experiment that was not formally grounded in a research methodology, it was interesting to explore which of the questions the participants appeared to find most the most valuable to discuss, which in this case was the extent to which the orchestration led to the desired outcomes.

5.3 Summary

In this chapter, I have described the changes that were made to the PD programme in Cycle 2. I explained how the formative questionnaires, post-PD programme interviews and data from the analyses of the videos of the PD components informed the changes that were made to both the design of the PD programme and the facilitation approaches used. A summary of the changes that were made and adopted in Cycle 2 are set out in Table 5.2 below.

Table 5.2 *Summary of key modifications made between Cycle 1 and Cycle 2*

Modification Type	Programme component	Modification	Rationale
Design	PD Sessions 1 and 2	Redesign of both sessions to include the principles of task design and orchestrating the learning in each session.	In Cycle 1, participants saw the principles of task design and orchestrating the learning as separate.
Design and Facilitation	PD Session 2	Modified the Johari Window based on the work of Oliver and Duncan (2019). The participants complete the Johari Window as a group.	To reduce anxiety and obtain greater clarity on the participants' views on the process of task design in conjunction with the teaching technique 'orchestrating the learning'.
Design	Post-lesson discussion	Removed the analysis of the orchestration sequence from the post-lesson discussion and added an additional PD session to focus just on the orchestration sequence.	In Cycle 1, participants showed fatigue in post-lesson discussion and I considered that the orchestration sequence in Cycle 2 was an important outcome of the research lesson.
Facilitation	PD Session 1	Email document on classification of problem-solving skills during PD session at a pre-agreed time.	To overcome logistical challenge of showing a complex document on a split screen.
Facilitation	PD Session 3	Share the problem-solving document on Tripods and Bipods with the TDT before the planning of the research lesson in PD Session 3.	Cycle 1 TDT recognised that an important area of mathematics could be taught using this problem that they had not previously considered. Cycle 2 participants' limited experience of Kyosai Kenkyu.
Facilitation	Research lesson	Logistical changes concerning observation of pupils' work.	Experience from Cycle 1 and changes to Covid-19 regulations.
Facilitation	The pause	Changes to facilitation of TDT planning of second half of research lesson.	Judgement made from analysis of Cycle 1 – observation of recording of PD session.
Facilitation	Post-lesson discussion	Changes to ordering of facilitation questions.	Enquiry into relative importance of the facilitation questions.

Chapter 6 – Professional Development Outcomes

Introduction

In this chapter, I describe the professional development outcomes of the programme for Cycle 1 and Cycle 2. The outcomes identified were:

- the participants' views about teaching mathematics through problem solving
- developments in their professional learning with regard to orchestrating the learning
- developments in their subject and pedagogical knowledge
- potential changes in the participants' practice and barriers to change.

The outcomes reported below are drawn from data gathered in interviews with the participants and from observations of the participants' responses to the PD activities. However, in this chapter, I do not seek to describe participants' outcomes specifically for each individual PD programme component. Chapter 7 will address the findings for each of the designed components.

I then categorise these outcomes using the Harland and Kinder (2014) typology of In-Service Education and Training (INSET) outcomes. I align the observed outcomes with the relevant category descriptors from the typology.

As reported in Chapter 4, the five participants in Cycle 1 were all secondary mathematics teachers who taught pupils aged 11 to 16. Due to personal reasons, one of the participants was unable to take part in the post-PD programme interviews. In Cycle 2, the four participants were all teachers from the primary phase (children aged 5 to 11). They were not mathematics specialists.

6.1 Views about teaching mathematics through problem solving

In their post-PD programme interviews, the participants were asked for their views on teaching mathematics through problem solving. All of the participants indicated that the approach presented in the PD programme had merit. For example, in Cycle 1, Ruth explained why she thought teaching mathematics through problem solving is a better method than ‘traditional teaching’ by highlighting the opportunities for discussion, the exploration of different pupils’ ideas, and the opportunities for revealing and using misconceptions.

> …with more traditional teaching, […] it doesn't feel like there’s an opening for a discussion point when you're telling them how to do something, and they're practising. Whereas with this, because the problem itself should create chances for students to do it in different ways, there's always going to be discussion points about how someone has done it compared to someone else. And if you're picking the right things, you can create really good discussion points out of maybe a misconception that someone's had, and then move on to someone that understood it a little bit better, and why their methods, taking them closer to the finish or something like that. I think problem solving is a much better way of doing that, than the more traditional kind of methods.
>
> [Ruth, secondary teacher. Cycle 1 post-PD programme interview]

She also indicated an understanding of the principles of task design by explaining that the task should provide opportunities for pupils to approach the problem in different ways. She went on to explain that there were advantages to this approach in relation to the teacher being able to understand the problems the children will have.

> Yes, I can see lots of advantages of using problems […] and also to think about anticipated responses and that thinking about these can help with understanding the problems the children will have.
>
> [Ruth, secondary teacher. Cycle 1 post-PD programme interview]

In Cycle 2, Clare explained that she liked teaching using a ‘problem-solving investigation’ and that observing the pupils’ different approaches can be valuable and powerful.

> I really like doing some sort of problem-solving investigation, particularly when there are different ways to approach a task, which is always good to see, knowing, how the different children are going to do it […] can be really valuable. I think it can be really powerful for the pupils and teachers.
>
> [Clare, primary teacher. Cycle 2 post-PD programme interview]

These comments by Ruth and Clare, which were typical among all of the participants, highlighted positive features of teaching mathematics through problem solving and that this approach could be beneficial for the teacher as well as the pupils.

Tom suggested that teaching mathematics through problem solving was an approach that could be used with any problem, but he also recognised the importance of analysing the task in order to establish the mathematics that pupils could learn from engaging with the problem.

> I would say that any problem can be used to teach mathematics through problem solving as long as you take the time to work out the mathematics you want the children to learn from it […] and that you think carefully about the context and the numbers being used.
>
> [Tom, primary teacher. Cycle 2 post-PD programme interview]

This is an interesting perspective on task design: Tom was suggesting that the process involves exploring the task to find the mathematics rather than designing the task to produce the mathematics.

6.1.1 Views on limiting factors

Whilst the participants' views on teaching mathematics through problem solving were overall positive, their views appeared to be somewhat conditional in that they saw the effectiveness of the approach as dependent on the attitudes and skills of the pupils. For example, Dave was uncertain about whether teaching mathematics through problem solving would work with all pupils.

> Yeah - My opinions have changed throughout the whole time I have been with you, constantly since I have been working with you… I have thought it's been a great idea, I have thought right it does not work with some students and it is a really bad idea. I think it just… If I am honest it depends on the students themselves whether or not this approach works.
>
> [Dave, secondary teacher. Cycle 1 post-PD programme interview]

Dave explained that if the pupils had what he described as "a positive attitude to learning" then they could learn from the ideas of other pupils, and these types of lessons might work. He also went on to explain that in some of his other classes, his pupils did not like to learn in this way and were much happier when presented with a number of questions to work through by themselves in a traditional manner.

Marie also thought that the success of teaching mathematics through problem solving was dependent on the skills that were required of the pupils. She explained that for many of her classes this approach would lead to discipline problems:

> I do not think many of the classes I teach would have the skills to be able to sit and listen to the answers from the other pupils. Some of the pupils would lose interest and this would cause discipline problems.
>
> [Marie, secondary teacher. Cycle 1 post-PD programme interview]

Adam described the challenge of planning and teaching this way, not just because of pupil attitudes, but also because of their previous experiences and expectations of what normally happens in a mathematics lesson.

> I think you've got to be quite clear as to how to build up the problem and introduce it, that that's critical. I think I've particularly found the pupils at the school very reluctant to try new approaches, or just to get started on a problem. It's about it's like trying to crack a nut. I think once they get into it, they've cracked the shell then we can start to explore the problem more, it's sort of trying to get that over that initial inertia. And I think that's where the planning for that activity really comes in, that you've got to be clear in what you share with the class so you don't over share. Because although there's a tendency, I think, just to fill in the gap, fill in the awkward silences, to give them more information rather than wait to see if someone does actually come up with a suggestion or an approach. Again, it's all about that reluctance to do anything that's out of the comfort zone and it’s as much for the teachers as for the students themselves.
>
> [Adam, secondary teacher. Cycle 1 post-PD programme interview]

Here, Adam presented some insightful comments on the implications of this approach for the pupils as well as the teachers, particularly when the pupils are new to this approach. He highlighted the need for teachers to think carefully about scaffolding the problem to give pupils the confidence to ‘get started’, and also for teachers to be wary of the tendency to provide too much input by filling the gaps during moments of pupil silence. These are important reflections which flag an issue for the PD itself: ensuring that participants recognise that pupils as well as the teachers may need to develop skills in order for the teaching approach to work effectively. However, Adam's use of the metaphor 'to crack a nut' suggested that once the pupils become used to this approach then they will engage in the problem and further develop their confidence to talk about their methods. Tellingly, he also showed insight into the challenge that any practice change presents for both the pupil and the teacher.

Charlotte indicated that the approach would work with some children but may not be the most effective approach for other children. However, she also went on to reflect on the benefits of active discovery by pupils rather than passive learning, and this enables the pupils to think more 'mathematically'.

> I think it's really, really helpful for those children […] if they've got some basic mathematical skills I think, for those children that perhaps are a bit more math phobic […] and for those children that find things quite difficult I think those children find it particularly hard to learn through problem solving. But for the children who can, I think the learning perhaps sticks more for those children because they've discovered it from themselves, rather than just having to watch someone else do it. They have that realization for themselves. And they probably think more mathematically in the long run because they've got that that thought process of how am I going to approach this? What strategies can have use for this, and how am I actually going to do it? What's the problem asking me to do? Whereas children that are more reliant on the process? I think they find it particularly hard and need that support and guidance and a structure to follow perhaps.
>
> [Charlotte, primary teacher. Cycle 2 post-PD programme interview]

6.1.2 Perceptions of potential barriers

These reflections on the limitations (and merits) of teaching mathematics through problem solving point to potential barriers to teachers' practice change. The teachers' perceptions of these barriers may have been part of their views before starting the PD programme. In Chapter 3, I explained how the pre-study interview questions were taken from the Teacher

Belief Interview framework developed by Luft and Roehrig (2007). To recap: this framework includes five category descriptors which can be used to code teachers' responses on a continuum that ranges from traditional to transformative. In this study, the participants' responses to six questions were analysed and coded against the most relevant descriptor. A response to a question could be recorded against one or mode codes. Appendix 11 presents a set of coded responses for one anonymised participant. A summary of all participants' coded responses, by cycle, is shown in Table 6.1 below.

Table 6.1 *Summary of participants' responses to pre-study questionnaire, coded by category (Luft & Roehrig, 2007)*

Cycle	Traditional	Instructive	Transitional	Responsive	transformative
1	******** (8)	************ (12)	****** (6)	***** (5)	* (1)
2	** (2)	*** (3)	*********** (11)	********** (10)	**** (4)

The coding summary suggests a difference in teaching orientation/mind set between the group of participants in Cycle 1 and those in Cycle 2, with the important caveat that participant numbers in this study are small. It appears that the responses in Cycle 1 locate the group (all secondary mathematics teachers) more towards the traditional end of the continuum than the group in Cycle 2 (all primary teachers). This may shed light on the majority view of the Cycle 1 participants who all said the approach had merit but that it would not work with certain types of pupils, implying that existing traditional/instructive approaches were better suited to those pupils.

As an alternative framework, Ernest (2018) argued that a mathematics teacher's belief system has three components that Askew et al. (1997) characterised as 'transmission', 'discovery' and 'connectionist'. Swan (2006) described these three categories as follows:

- transmission-oriented teachers are those who believe mathematics is a set of factual information that must be conveyed or presented to students, and typically enact didactic, teacher-centred methods;

- discovery-oriented teachers view mathematics as a set of knowledge best learned through student-guided exploration, and frequently tend to focus on designing effective classroom experiences that are appropriately sequenced;
- connectionist-oriented teachers view mathematics as an intertwined set of concepts, and they rely heavily on experiences to help students learn about the connections between mathematical topics.

Teachers who believe that teaching mathematics through problem solving is an effective method could be described as having discovery-oriented and/or connectionist-oriented attitudes and beliefs. It would seem reasonable to assume that teachers with these orientations would recognise the connection between the process of task design and the teaching technique of orchestration. As for transmission-oriented teachers, there is no reason to automatically assume that they would not appreciate the principles of task design in conjunction with the technique of orchestration. However, it could be that participants in this study with transmission-oriented beliefs might have difficulty espousing the general idea of teaching mathematics through problem solving and therefore might not accept the importance of task design and orchestrating the learning.

Another difference between the participants' perceptions in the two cycles was that the Cycle 2 primary teachers thought that the effectiveness of the approach would be dependent not only on the pupils' attitudes and skills but also on their age and experience, and that this factor was related to the nature of the curriculum for younger pupils. For example, Paul explained that whether teachers could successfully adopt teaching mathematics through problem solving was dependent on the curriculum within the Key Stage[34].

> I think sometimes it's fairly alien for primary school teachers, particularly those working in Key Stage 1. A lot of the time we're just teaching them skills. Where I'm working in the moment, it's not teaching mathematics through problem solving kind of, it's almost as if teaching the skill is the end point, a bit like when you described teaching for problem solving… working with these tasks has made me see the different uses for problems.
>
> [Paul, primary teacher. Cycle 2 post-PD programme interview]

[34] In England the compulsory school curriculum is split into four Key Stages. Key Stages 1 and 2 are in the primary phase.

His comments suggest that teachers working in Key Stage 1 focus more on the teaching of problem-solving skills (teaching *for* problem solving) rather than teaching mathematics *through* problem solving.

6.2 Professional learning – Orchestrating the learning

Before reporting the participants' views on the orchestration technique, I revisit what the technique requires. Orchestrating the learning, as explained earlier, involves the sequencing of pupil responses and is supported by planning a sequence of anticipated responses. It has been described by researchers in a number of different ways, but a feature common to all the descriptions is the fact that the teacher must make on-the-spot decisions about what mathematics to pursue and how to pursue it. Heaton (2000) characterises this succinctly as 'improvisation' while Chazan and Ball (1999) explain that "teacher moves are selected and invented in response to the situation at hand, to the particulars of the child, and to the needs of the mathematics" (p. 7). The ability to make 'correct' on-the-spot decisions is clearly linked to the teacher's expertise and experience. It is also important that teachers understand that decisions which may lead to a deviation from the lesson plan are a natural part of orchestration. In the following subsections I discuss the participant's views about this teaching approach and the challenges of developing the technique of orchestration.

6.2.1 Views about the technique of orchestration

In Cycle 1, Adam described his current understanding of the process of orchestration, highlighting its dynamic nature and making an analogy with reading interactive books.

> …one way of thinking about it is like the kid reading these interactive books where you choose what happens next in the story, you get to the end of the page and it asks, 'What do you want to do next?', if you decide this then go to page such and such, if you decide this go to another page.
>
> [Adam, secondary teacher. Cycle 1 post-PD programme interview]

In the post-research lesson discussion in Cycle 1, both Marie and Ruth considered that the planned sequencing of the responses and the resulting orchestration was achieved as planned and was successful.

> I really liked the way in which the table we developed was used to draw out the equation… that worked really well and I think that the ordering of the pupils' work could not have been better!
>
> [Marie, secondary teacher. Cycle 1 post-research lesson discussion]

> I think the combination of using the table with selecting the particular pupils' work in the order we planned was really powerful.
>
> [Ruth, secondary teacher. Cycle 1 post-research lesson discussion]

Ruth indicated that the process was useful because of the way it made her think about the mathematics. However, it appears that she was thinking about her own sequencing of the steps to solve the problem rather than the sequencing of the anticipated responses from the pupils – which may not be the same.

> Thinking about the way you orchestrate their feedback it makes me think about the process […] what I like about it is even if you're only at step 1 in the problem we can see how to get the next level and where to go next and they can see how to take their answer and get to the next step, and even the last one who has got the right answer it might be that they have not done it in the most efficient way [...] so this process makes me think a little bit more about how we get to answers and what's the best way and what's the most efficient way.
>
> [Ruth, secondary teacher. Cycle 1 post-PD programme interview]

This reveals another interpretation of the orchestration technique. Orchestrating the learning in this study is defined as the sequencing of the pupil responses in a way that leads to the introduction or development of a mathematical idea or concept. Ruth has developed her understanding of the technique as one where the pupils' responses are sequenced in order to solve the problem or get the best solution, which is not necessarily the same as sequencing responses to reveal the mathematics to be taught. This particular interpretation of orchestrating the learning was also evident in the design of the second half of the research lesson in Cycle 1.

In Cycle 2, Clare indicated that she now thinks more about the responses she may get from her pupils and the way in which she will interact with those responses.

> In the past I have just talked about a good problem. But now I am thinking about how to structure it and what responses you can expect and how you're going to then interact with those responses. This has been challenging but very useful.
>
> [Clare, primary teacher. Cycle 2 post-PD programme interview]

The orchestration sequence in Cycle 2 was delivered as planned and led to the desired end point in the generation of the equation 3T + 2B = 23. In the post-lesson discussion, Clare agreed that this had been a very successful teaching episode.

> At first I was unsure about the way in which we had planned to sequence the responses however even if there was another way I think the outcome was really good as the children clearly understood where the equation had come from.
>
> [Clare, primary teacher. Cycle 2 post-research lesson discussion]

Charlotte indicated that exploration of the task was important and that she understood that probing the mathematics in the problem and orchestrating the learning should be developed together.

> I found the exploration of the tasks very helpful in terms of the thinking about how we present the tasks to children […]. I think you have to think about the maths in the problem and the sequencing together and this was something that we perhaps wouldn't have had maybe the time to do previously.
>
> [Charlotte, primary teacher. Cycle 2 post-PD programme interview]

For Tom, whether or not you analyse a task in order to plan for anticipated responses is dependent on the nature of the particular problem and whether you are also seeking to develop a problem-solving skill.

> I think it depends on the problem really, for example where there's a lot of data generation like in the video example, which might need to take place first. And then they need to work with that data, to then hypothesise I think it is quite important to show them that there is a sequence to doing that. If that's the skill, we need to teach that skill but I'm not sure if it would be essential, in every lesson.
>
> [Tom, primary teacher. Cycle 2 post-PD programme interview]

Here Tom was referring to the framework for problem-solving skills discussed in PD Session 1 and also to the video example of orchestrating the learning in PD Session 2.

Also, Charlotte indicated that how you plan for anticipated responses can often depend on the nature of the class and the knowledge you have about the pupils' approaches to solving problems.

> I think, I suppose, it depends on your group of children very much. I think knowing your class would help you think about the anticipated responses. So it's not just anticipated responses, in general, it's anticipating responses from your children, and knowing them well enough, and knowing what they need and what they are likely to come up with. So I know the class that I've got this year would probably come up with a very different approach to the class that I had last year, so perhaps, I'd have thought they're going to come up with something else first, and go for more of the trial and error. Whereas my last year's lot would have thought more – oh, there must be a process for this and use the process of elimination. It's just knowing the children.
>
> [Charlotte, primary teacher. Cycle 2 post-PD programme interview]

These examples from Charlotte and Tom suggest further dimensions in the process of task design and task analysis. Firstly, they believed that the task should be analysed in order to identify the problem-solving skills that may need to be taught in order to access the problem. Secondly, in the analysis the teacher should consider the characteristics of the class of pupils for whom the task has been chosen. This focus on planning the orchestration sequence in light of the characteristics of the class was not included in the design of the PD sessions and it relates to earlier comments by participants about potential barriers to this approach for some pupils. I therefore presume that these views have emerged from Charlotte's and Tom's analysis of the tasks alongside their reflections on their own context.

6.2.2 Challenges of orchestration

In both cycles of this PD programme, the participants expressed that it was a challenge to become proficient in the technique of orchestration and that it can be hard to think about and plan for the anticipated responses but it does get easier with practice. They also indicated that the level of challenge was linked to the nature of the task and the knowledge of the class. For

Dave, orchestrating pupil responses was the hardest part of teaching mathematics through problem solving. Nevertheless, his remarks show that notwithstanding the difficulty he valued the strategy of anticipating responses and was using it in his teaching preparation and practice.

> I think I have mentioned this before – the orchestration is the bit that I found the hardest part to learn. Yeah [...] I now try to anticipate responses from my class […] I look at my lesson beforehand to think what students might say… what I might want to draw out of that to guide them in the right direction to make them understand things.
>
> [Dave, secondary teacher. Cycle 1 post-PD programme interview]

Here it was unclear if the main challenge lay in identifying possible responses or the sequencing of the responses. When asked for clarification, Dave responded:

> Yeah that's a difficult one […] I do not know the effect of different sequences in different lessons. I've never had the opportunity to try and sequence anything differently […] But I think depending on what you are doing, there might be something specific that you are looking for within their answer that you might want to save until last and get other methods out of the way to then focus on one that might be the most efficient or that might lead you onto a different area of mathematics.
>
> [Dave, secondary teacher. Cycle 1 post-PD programme interview]

Here, Dave expressed uncertainty about the value of sequencing different responses and he interpreted the purpose of doing so to remove "other methods" in order to get to the most efficient solution. This suggests that he did not see the orchestration sequence as a process by which the mathematical concept or idea is developed from the contributions of the pupils. He also acknowledged that different sequences might lead to different outcomes but that he has not had the opportunity to explore this. This could be due to the significant pedagogical demands that are involved in orchestrating pupils' thinking (e.g., Ball, 2014; Brown & Campione, 1994; Lampert, 2001; Schoenfeld, 1998). Therefore, the technique of orchestration as presented could have been too complex for teachers who have limited experience of connecting mathematical ideas in this way.

Marie spoke of the importance of planning for anticipated responses and the implications of not doing during the orchestration.

> I think one of the reasons why people find it so hard to teach like that is because the pupils can throw up questions that you've never even thought of or planned for. With more traditional teaching, you've often taught it so many times before, you kind of know what's going to come out of it, […] it's more streamlined, it's kind of very bottlenecked to where you want it to go. Whereas with problems, I think it can go all over the place that if you don't think about what the responses are going to be, then it's really difficult to manage it all. So I think yeah, without doing that, teaching through problem solving, or using problem solving as a way to teach is really difficult.
>
> [Marie, secondary teacher. Cycle 1 post-PD programme interview]

Her use of terms like "bottlenecked" and "streamlined" to describe traditional teaching convey a sense of *narrowness* which makes it easier to manage the direction of learning. She implied that in "traditional"[35] teaching there are fewer surprises from the pupils. For Marie, it appears that a key difference between her teaching and teaching through problem solving is that the traditional approach results in fewer enquiries from the pupils, and repeated experience of traditional mathematics teaching means that you know what responses are going to come up.

In Cycle 2, Paul emphasised the challenge of thinking about the pupils' responses but said that it becomes easier with experience of the process.

> I think to begin with, it's quite hard. But then when you've gone through the process, and thought at a child's level, how they might experience a particular problem, and looked at it through their eyes, it does get easier, but it is definitely time-consuming and is something that I still find quite challenging.
>
> [Paul, primary teacher. Cycle 2 post-PD programme interview]

[35] In this comment and her earlier one in the previous section, Marie frequently referred to the term 'traditional teaching' which I confirmed with her is how she currently refers to her own practice.

Paul went on to explain his understanding of the purpose of task analysis, i.e. to plan for the anticipated responses, and again indicated that it gets easier with practice. He also remarked on his enjoyment of teaching in this way.

> I'd say that probably the main reason for doing it is to obtain the anticipated responses, once you've done it a couple of times is not too difficult, but then the level of thinking required to sequence them is the challenging part. Because you need to know where you want the lesson to go. That's challenging, but I think that's probably the part I enjoyed most about our lesson, because you can begin with quite a basic problem that turns into some quite challenging maths.
>
> [Paul, primary teacher. Cycle 2 post-PD programme interview]

I asked Paul to say more about his comment that "the level of thinking required to sequence is the challenging part". In response, he explained:

> Yes, I sometimes find it difficult to keep the focus on the mathematics and the pupil responses together. I think you have to know the task really well and the mathematics that will come out of it. […] This takes time to learn.
>
> [Paul, primary teacher. Cycle 2 post-PD programme interview]

Paul drew attention to the time it takes to develop the required knowledge and skill and the importance of knowing the task "really well" and the mathematics it can be used to teach.

6.3 Professional learning – Subject and pedagogical knowledge

In 1986, Lee Shulman introduced a taxonomy of teacher knowledge that distinguished pedagogical content knowledge (PCK) from subject-matter knowledge (SMK). The conceptual distinction is that PCK is specific to the subject matter being taught. According to Shulman (1986), SMK is "the amount and organization of the knowledge per se in the mind of the teacher" whereas PCK consists of "the ways of representing the subject which makes it comprehensible to others ... [it] also includes an understanding of what makes the learning of specific topics easy or difficult ..." (p. 9). In this study, the participants reflected on the effect of the programme on both types of knowledge, as illustrated in the observations below.

When asked about the exploration of tasks to develop the teaching of mathematics through problem solving, all of the participants indicated that they would now think more carefully about the design of the task. For example, Ruth said that she was now more aware of the importance of choosing the 'right' numbers in a task.

> Previously I had not considered how important choosing the right numbers is when we are designing a task. I would say that before this session, I did not pay as much attention to what actual numbers we used and which could lead to misconceptions for example not thinking about the importance of prime numbers when simplifying fractions.
>
> [Ruth, secondary teacher. Cycle 1 post-PD programme interview]

Ruth conveyed her understanding that the numbers used could lead to misconceptions and went on to explain how the selection of numbers could influence how the learning develops.

> Yeah quite often we always have lots of problems in our head but we don't always stop and think about the numbers that we use and what numbers will cause misconceptions and missed learning […] sometimes the numbers look really good. Say you are doing pie charts using 180, 360, 720. These are lovely numbers to use in a pie chart or representing data in a pie chart, but what if we were to use 500… what learning comes from that …
>
> [Ruth, secondary teacher. Cycle 1 post-PD programme interview]

After Ruth made this comment, I followed up by asking what she meant about "missed learning".

> So missed learning is something where […] maths is interwoven all the time erm what we do sometimes as teachers is we look at what learning so we have got to teach … so when I have got to teach pie charts in the past I have got so focused [on] the numbers always adding up to 360 or multiple of so that it is a nice integer for the students and so it's an easy angle to draw but there is an opportunity there ... but if I use 500 which isn't an easy number the students then have to consider rounding ... do we round up or round down ... we consider the point five, but if we round up all the time we can end up with 361 ... so that brings in new learning about rounding … does that make sense?

[Ruth, secondary teacher. Cycle 1 post-PD programme interview]

For Ruth, the mathematics curriculum is an interwoven set of concepts and ideas. Her response above suggested that the PD programme has strengthened this view (which she also expressed in the pre-PD programme questionnaire) and improved her subject knowledge in relation to the process of task design and her grasp of the potential of this process to help pupils make stronger mathematical connections.

In contrast, Marie explained that the rigid way she was taught mathematics has hindered her ability to see the links between areas of mathematics, making it more difficult for her to teach mathematics through problem solving.

> I think part of it is the intricacies of the problem and because of how much is interlinked, because I was taught very prescriptively, this is how you do it. Here's how you do it, practise it. I never made those links when I was learning, which I think probably makes it harder for me to teach that way even though I do enjoy teaching like that, if that makes sense.
>
> [Marie, secondary teacher. Cycle 1 post-PD programme interview]

She went on to explain that her own mathematical subject knowledge is not sufficient to teach "those lessons" effectively.

> So yeah, I try and do as much as I can, within my teaching. Whether, you know, we're pushed still in that direction, or not anymore, I don't know. I still enjoy those lessons. But I find them very difficult to teach. Because I think my own understanding of maths is not as good as it needs to be to be able to teach those lessons really, really well.
>
> [Marie, secondary teacher. Cycle 1 post-PD programme interview]

When asked what she meant by "not as good as it needs to be", she said:

> Well I can do the problems but because I was just taught to get the answer rather than explore different ways or solutions, I find it hard to think about the mathematics that can be taught from doing a problem.
>
> [Marie, secondary teacher. Cycle 1 post-PD programme interview]

Here, Marie provided a qualitative assessment of her subject knowledge, acknowledging that it needs to be greater in order to teach mathematics through problem solving effectively. Her comment about “being pushed in that direction” also suggests that the decision to teach in this way was not just her own choice and may have been subject to direction or pressure from elsewhere. These two issues could indicate that teaching mathematics through problem solving is unlikely to form a significant part of her future practice without further professional development to support her own mathematical knowledge.

In Cycle 2, Tom’s reflections on the area problem clearly indicated that he understood how the task had been designed in order to teach the mathematics through problem solving. From Tom’s repeated reference to “light bulb” moments it can be inferred that he has developed new knowledge and insights into the relationships between an algebraic expression and a physical representation.

> The area model was interesting […] but when you showed how the shape could be split up and then put together and that there was a completely different way to one which I thought about […] that was an interesting light bulb moment, I could just see that from the perspective of a child as well. We were all coming at it from the same point of view, you could split it into two rectangles and find the area of each, you could have a larger one and then almost have a little bit cut out. But then when you showed where you could split it up differently and put it together differently. And I'd never had thought about how that one could be used to introduce the algebraic structure of number. The area model in particular, was a really, really effective light bulb moment about teaching some mathematics through problem solving.
>
> [Tom, primary teacher. Cycle 2 post-PD programme interview]

In response to an explicit follow-up question on whether this impacted on his subject knowledge**,** Tom said:

> Definitely. For the area model one in particular, it made me think in a way that I hadn't thought before. And then the discussion around the cells problem to reflect the number of cells, which ones you'd leave out, how many you'd leave. That made me think deeply about the way the children would approach the task,

> particularly depending on which ones here you take out, how well they would be able to access it.
>
> [Tom, primary teacher. Cycle 2 post-PD programme interview]

Clare indicated that working with others to learn and talk about mathematics had advanced her own subject knowledge and that she had been "tested".

> I suppose what was nice about this programme was that there were lots of opportunities to actually do things together. Having some time to learn and talk about mathematics was really useful. This programme has developed my own subject knowledge. It has certainly been tested.
>
> [Clare, primary teacher. Cycle 2 post-PD programme interview]

Paul highlighted the value in his new understanding of the difference between teaching mathematics through problem solving and teaching for problem solving. He recognised now that teaching mathematics through problem solving requires an understanding of the mathematics that is 'in' the problem.

> Where I'm working in the moment, I had not appreciated that I was teaching for problem solving which I now understand is different to teaching through problem solving […] it is not just about the skills that you've been learning along the way but the mathematics that is in the problem as well. And I think I really, really like it, it's kind of brought a fresh perspective to my thinking about where teaching through problem solving fits in my planning.
>
> [Paul, primary teacher. Cycle 2 post-PD programme interview]

Charlotte described her new insights into a specific area of subject knowledge – multiplication. In the extract below she reported that her new understanding of this concept in relation to the number of groups and number in the group was now informing her teaching.

> I understand multiplicand and multiplier much more … once we get that equation and we had 2B rather than B2, I never thought about that before. Now with that in my mind when I'm teaching multiplication, I am now aware of the importance of identifying the number of groups and the number in each group.
>
> [Charlotte, primary teacher. Cycle 2 post-PD programme interview]

This new learning relates to reflections by Charlotte on the orchestration sequence from the research lesson, reported in Section 7.6 in the next chapter.

6.3.1 Insights from the Johari Window

As explained in Chapter 5, in Cycle 2 the participants were asked to complete the Johari Window as a group activity. The completed window in shown in Figure 6.1 below.

Figure 6.1 *Cycle 2 Group Johari Window (reflecting learning from PD Sessions 1 and 2)*

Common knowledge – what we and others know	**Others' knowledge – what others know and we do not**
• Teaching mathematics through problem solving is an effective method • Planning for anticipated responses takes time and requires a level of expertise • Understanding how pupils approach problems is an essential part of mathematics teaching	• How to design mathematical tasks that can be used to teach different areas of mathematics • Knowing how to sequence the pupils' responses to maximise pupil learning.
Our own knowledge – what we know and others do not	**The unknown – what none of us know**
	• Teaching mathematics through problem solving will improve pupil outcomes

The participants began populating the 'common knowledge' window by first discussing what they all agreed they 'knew'. They agreed that teaching mathematics through problem solving is an effective method of teaching mathematics and that planning for anticipated responses takes time and requires a level of expertise. They also agreed that understanding how pupils approach problems is an essential part of mathematics teaching and that this requires a high level of subject knowledge. Although the team demonstrated confidence in this knowledge, when they were asked why they had located this item in the 'common knowledge' quadrant rather than the 'own knowledge' quadrant, the participants explained that they presumed the process of task design and teaching mathematics through problem solving had been developed by educational experts – which meant that this was knowledge that many others already had. This was the principal reason why they had decided not to populate the ‘own knowledge’ quadrant.

While studies show that successful engagement in CPD programmes regularly leads to improved confidence (Furengetti, 2007), the group’s decision to locate the items in the ‘others’ knowledge’ quadrant points to nuanced outcomes in terms of confidence. The participants had increased their confidence in understanding the process of task design and sequencing pupil responses, but this confidence did not extend to a belief that they could now independently design tasks and plan the sequencing of pupils’ responses.

With regard to the quadrant for ‘the unknown’, the TDT as a whole demonstrated uncertainty about the impact of teaching mathematics through problem solving on pupil outcomes. This is significant because it could suggest that, irrespective of the participants’ positive gains from the PD programme, their future commitment to this approach to teaching mathematics might be contingent on the conjecture that teaching mathematics through problem solving will improve pupil outcomes. It was particularly interesting that not only did the participants not know if this teaching approach would lead to better outcomes, but in their view nobody else knew either. This was surprising because I had assumed that the participants would recognise that a reason for engaging in the programme would be to improve their teaching which in turn should lead to better outcomes, even though I had not *explicitly* claimed that teaching mathematics through problem solving would improve pupils’ outcomes. Nevertheless, this provides a point of learning in that it is clearly be beneficial to be explicit in describing the full rationale of the PD programme.

6.3.2 Developments in questioning

In the post-lesson discussion, the participants commented on the type of questions used during the orchestration sequence. In Cycle 1, they noted that the questions became more specific. For example, during the post-lesson discussions all of the participants agreed with the observation that the teacher’s questions to pupils had shifted from general ‘what did you do?’ type questions to more specific enquiries such as *‘Why did you start with the number 15?’*. There was also agreement that the reason for this was that the teacher already had knowledge of what the pupils had done, as a result of analysing their work, and so was able to ask more targeted ‘why or how’ questions.

In Cycle 2, participants noted that the teacher's questions often had a rhetorical component. For example, Charlotte asked the pupils:

> So you ruled out the 1 because it did not follow the rule, 9 because it was too many and 7 because that does not follow the rule, is that correct?
>
> [Charlotte, primary teacher. Cycle 2 research lesson transcript]

Teachers often ask very general questions to avoid 'asking by telling'. In fact, many teachers hold the view that in order for discussions to be focused on the students' thinking, they must avoid providing any substantive guidance at all (Lobato et al., 2005). It was interesting to note Charlotte's reflections when she was reminded in the post PD interview that she had done this.

> I had not really noticed that I did this a lot but I felt that at various points in this lesson, it was important to summarise each contribution so that I could then build on this with the next method.
>
> [Charlotte, primary teacher. Cycle 2 post-PD programme interview]

The key point here is that there is a complex relationship between the teacher commentary during orchestration and the general view that little or no guidance should be provided by the teacher.

The observations above indicate that the participants could demonstrate and articulate new knowledge as a result of engaging in task design within a specified planning process. Five out of the eight participants indicated that the programme had impacted on either their subject knowledge and/or pedagogical knowledge and recognised the 'skill' associated with the technique of anticipating and sequencing pupil responses and that this takes time and practice. Furthermore, their interactions with the tasks appeared to be a participatory two-way process: the teachers were influenced by the resources and the design of resources was influenced by the teachers. This finding supports the view of Remillard (2005), discussed in Chapter 2, and the results of other studies in relation to the use of task design. When teachers work with mathematical tasks by adapting and appropriating them, the teachers enhance their mathematical knowledge and their mathematics-didactical design capacity (Pepin, 2015).

6.4 Potential changes in participants' practice

In both cycles of the PD programme, all of the participants talked in their interviews about the components of teaching mathematics through problem solving and said that they had considered incorporating some of them into their practice or were already doing so. For example, in Cycle 1, Ruth envisaged the task design process possibly extending into lessons in addition to problem-solving lessons.

> I am now trying to think about how I introduce and use the tasks more carefully […] and not just in problem-solving lessons. I think this will take time to get good at as each lesson takes a great deal of time […] and time to be honest that is currently difficult to find.
>
> [Ruth, secondary teacher. Cycle 1 post-PD programme interview]

When asked about how this might look in other lessons, Ruth explained that she would think more carefully about the choice of numbers in problems and the questions used to develop the pupils' learning. She would also plan more carefully the way in which the task was modelled. She added that since the PD programme she had given thought to changing the format of the six-part mastery lesson so that the talk task became an orchestration sequence. However, Ruth's comments included a 'get-out clause' in the form of the caveat that lack of time may hinder her plans.

Marie spoke of her intention to introduce more problems into her teaching as a way of improving pupil engagement.

> Yes, I feel that giving a problem to the pupils will improve engagement and so I am thinking about where and when I can use these problems. If I can get the students to work on their own solutions so that we can have a discussion, I think that will help those students who often do not say anything.
>
> [Marie, secondary teacher. Cycle 1 post-PD programme interview]

However, it was unclear if this would become part of her regular practice given her uncertainty about where and when she would make the changes and also her remarks (reported earlier) that the success of the approach would depend upon the response from the pupils.

All the participants in Cycle 1 commented on the importance of choosing “good” tasks for teaching mathematics through problem solving and on how it could be problematic to identify the right task and find the time to do so. However, Marie suggested that once a good task is established it can then be built into the curriculum for repeated use.

> I think, yeah, just knowing where to look for problems, and even having some examples of really good problems […] would be useful too, to allow teachers to learn specifically how the problem works, and to then have this as part of your curriculum that you then use every time and you learn from your experience of teaching it more and more and more.
>
> [Marie, secondary teacher. Cycle 1 post-PD programme interview]

In Cycle 2, some of the participants expressed their intention to consider introducing the teaching of mathematics through problem solving by exploring structural changes to the scheme of work. For example, Tom explained that he was looking into replacing the “develop learning” part of the lesson with a designed task that would be used to teach the mathematics for this part of the lesson through problem solving.

> I was thinking really deeply about anticipating responses and how much I have learned about building the lesson around these. In our scheme of work, we are tied to our flip charts but in the develop learning section of the teaching sequence I was thinking we could leave this section blank, and then design a problem that would connect to the taught task and independent work. Since we did the programme I've been going into lessons thinking how we're going to address misconceptions by designing tasks in the develop learning section and to develop more understanding. So I think that this approach would enable us to do this even though it would take some time.
>
> [Tom, primary teacher. Cycle 2 post-PD programme interview]

He went on to explain that since the PD programme (which, interestingly, he referred to as “training”), he has used problems with his pupils that are a more “open”. In addition to exploring this approach in the ‘develop learning’ part of the six-part mastery lessons, Tom also has considered more open problems in a current maths initiative for Year 5 to Year 8[36].

[36] Here, Tom was referring to the Continuity project for Years 5-8, a fully funded Maths Hubs programme.

> Since the training I been giving them something a little bit more open than I might have given them previously, and taken the opportunity to explore and then share those ideas more than I used to. And I'd say, it's probably not been something I've been doing every day because we use the maths mastery scheme. But I've done it few times in the develop learning part of the lesson and in part of the year five to eight maths group project.
>
> [Tom, primary teacher. Cycle 2 post-PD programme interview]

6.4.1 Barriers to change

Whilst there were some indicators of potential change, five of the eight participants identified a number of barriers that would prevent the approach of teaching mathematics through problem solving (or aspects of it) being incorporated into their regular practice. Some of these barriers have already been touched on in Section 6.2 in relation to reported perceptions about the attitudes and skills of the pupils. In addition to these factors, the barriers of time, the cognitive demands on the teacher, the constraints within existing schemes of work, and post-pandemic remediation pressures were also raised by participants.

In Cycle 1, Dave explained his interest in exploring problems but his comment about doing so "alongside everything else" suggests that he does not see this work replacing any elements of his current teaching programme. However, Dave did recognise that the development of a task and the planning of the teacher's questions are important.

> I have definitely looked into more problems [...] and tried actually to develop my own … but […] I don't think I have done it as much this year as I would have liked to. Planning these types of lessons takes much more time because you have to think more about how to develop the task and then think about the questions and alongside everything else it is difficult to find the time.
>
> [Dave, secondary teacher. Cycle 1 post-PD programme interview]

Similarly, Marie described the effort and time involved in teaching mathematics through problem-solving and highlighted the "pressure" to get through the work.

> Yes. Yeah. I think the ability to manage it all is difficult and it takes time … to be able to get better at it […] know which responses you want to put first and what questions you're looking for and what questions are integral to answer, to make sure that the problem doesn't fall apart when you when you're trying to teach it. There is a lot to think about […] all at the same time and this with the pressure to get through the work makes me think that it is difficult to learn.
>
> [Marie, secondary teacher. Cycle 1 post-PD programme interview]

However, Adam alluded to this time/work barrier being overcome to an extent once the right task has been found – it then makes teaching the mathematics "seem easy". Interestingly, this remark is slightly at odds with his earlier comment about the attitudes and experiences of the pupils being a barrier.

> I think the time is probably the main issue, picking the task that's suitable for this line of work. That's the hard part […] is finding the time for doing the research trying to find the task that's suitable or that can be amended to lend itself to children looking at it through problem solving approach. Once you've got the task, it seems easy.
>
> [Adam, secondary teacher. Cycle 1 post-PD programme interview]

However, at the time of interview, Adam did not think that teaching mathematics through problem solving would align with the agenda of 'catching up' lost learning after the Covid-19 pandemic.

> I think currently, it's more about catching up. So it's probably on the back burner with everything that's happened with two lockdowns and the remote teaching … I think the focus now is more on trying to get the children up to speed with the basics ... if I use the phrase quick fix, it probably conjures up the wrong connotation. But with the approach through problem solving, we don't get instant results. Yes, you're working to build up something that will eventually reach fruition, much further down the line. But at the moment the thrust is more about trying to get results quickly. And I think with the GCSE as well you have got to be fairly confident that this is going to pay off. Otherwise, it's going to affect the school's results.
>
> [Adam, secondary teacher. Cycle 1 post-PD programme interview]

Adam's view that the approach would not contribute to helping pupils get up-to-speed with the basics is perhaps illustrative of the challenge of introducing new teaching methods within learning environments that require 'quick fixes'. His comment that teaching mathematics through problem solving does not lead to instant results suggests that he does not regard this method as comparable to other approaches where the pupils learn and then practise a piece of mathematics. He was not confident that this approach would support pupils' academic recovery and attainment rapidly enough and therefore it could affect the school's results.

In Cycle 2, Paul explained that teaching through problem solving takes a lot of time and required careful and detailed thinking in preparation and delivery. He also recognised that new skills were required and said he would like to develop these skills but did not have sufficient time to do so at present.

> So it's the thing of anticipating responses, really thinking in fine detail where the lesson could go where the misconceptions could fit in. I've really enjoyed that. And obviously, it takes a lot of time, it takes a lot of skill, and it is skills that you don't have to begin with, you kind of have to experience it a number of times to know, right, if I use this problem, the children could take it down this direction or this direction. So I need to be prepared for that to happen. Something that I don't feel like we have enough time to do all the time. But to kind of give a certain problem a go like that is definitely really useful.
>
> [Paul, primary teacher. Cycle 2 post-PD programme interview]

Paul's comments reflected a common concern among the participants that adequate time needs to be given to planning for the different pupil responses. There also appeared to be uncertainty about how to develop this teaching approach. Alongside the issue of time for planning, participants recognised that developing the skills and experience of teaching in this way is also important. There seems to be a cyclic argument emerging here: a teacher cannot develop the skills without the experience and they cannot obtain the experience without developing the skills.

Clare reported that she has tried to teach using problems but that there were constraints due to the way the school's scheme of work is set out and also because there were few problems in the scheme. She also mentioned that sometimes these problems (known as independent tasks in the scheme) are omitted in lessons due to the pressure of time. However, her use of the term “problem-solving investigation” in the extract below may suggest that Clare associated the method of teaching mathematics through problem solving with a different methodology that does not necessarily incorporate the features of task design and orchestrating the learning.

> I mean, yeah I have tried. Obviously it’s not something we do consistently because of the way that Maths Mastery plans are set out, but I really like doing some sort of problem-solving investigation […]. It’s hard, because there’s not a great deal of problems in the scheme and then I think everyone sort of panics so much because they have got so much to get through within the scheme so they do not use these problems. I’m still struggling to get through everything in the year anyway.
>
> [Clare, primary teacher. Cycle 2 post-PD programme interview]

In essence, Clare’s view was that it is difficult to teach using problems because of the mathematics scheme used by the school. She also indicated that teaching problem solving (which she enjoys) is difficult because of the pressure to get through the work in the scheme.

The observations above point to pre-existing barriers that inhibit the participants from fully incorporating the teaching of mathematics through problem solving into their practice. In both cycles, the most significant barrier cited was time but there were also reported constraints associated with schemes of work, teacher skill development, and the pressure to justify the approach in the context of pandemic recovery in schools. Of the possible explanations as to why these barriers operate against the PD programme’s full potential, I suggest that a lack of coherence between the PD programme intentions and the participants’ current classroom practice is the most significant (Desimone et al., 2002; Pedder et al., 2008). I return to this issue in Chapter 8.

6.5 Categorising the professional development outcomes

In this section I refer to the Harland and Kinder (2014) typology of In-Service Education and Training (INSET) outcomes to probe and classify the outcomes observed. The typology comprises nine broad categories which are shown in Figure 6.2 below.

Figure 6.2 *Typology of INSET outcomes (Harland & Kinder, 2014)*

1. **Material and provisionary** – development of physical resources e.g. worksheets or activities
2. **Informational outcomes** – fact-based information, e.g. about new policies or schemes
3. **New awareness** – a perceptual or conceptual shift, teachers becoming aware of new ideas and values
4. **Value congruence** – the extent to which teachers' own values and attitudes fit in with those which the CPD is trying to promote
5. **Affective outcomes** – how teachers feel emotionally after the CPD, may be negative (e.g. demoralised) or positive (e.g. confidence)
6. **Motivation and attitude** – such as enthusiasm and determination to implement changes e.g. may feel inspired
7. **Knowledge and skills** – both curricular and pedagogical, combined with awareness, flexibility and critical thought
8. **Institutional outcomes** – on groups of teachers, such as consensus, collaboration and support

I chose this framework because four of the nine category descriptors resonated with the design intentions of the programme and with the observations from the participants and my interpretations of them. The four categories identified in this study were:

- affective outcomes
- motivation and attitude outcomes
- knowledge and skills outcomes
- new awareness outcomes.

In the subsections below I explain each of the identified categories from the Harland and Kinder typology and provide examples from the observed PD outcomes.

6.5.1 Affective outcomes

Affective outcomes are those that indicate an emotional experience arising in or from a learning situation. These outcomes can be positive feelings (such as confidence) or negative states (such as feeling demoralised). Positive affective outcomes (such as enjoyment or enthusiasm) can be short-lived unless there are additional outcomes, such as knowledge and skills, that can maintain the participant's interest and motivation. Nevertheless, the immediate affective outcomes can be useful precursors for changing practice.

The observations of the pause and post-lesson discussions indicated that participants in both cycles demonstrated growing confidence in the use of the teaching technique orchestrating the learning. In Cycle 1, the orchestration sequence was judged a success, particularly by Marie and Ruth. This led to an enthusiastic discussion about the next lesson and the potential sequencing of responses that would come from introducing the modification to the problem which would produce the equation $3N + 6M = 41$ to introduce algebraic proof by demonstrating that no integer solutions could be found.

Hargreaves and Fullan's (2013) account of teacher professional capital emphasises the importance of the professional agency of teachers (i.e. decisional capital). Achieving agency requires teachers to have the confidence to take risks and to try out new ideas and strategies in their pedagogic work. Therefore, confidence along with commitment are essential elements of teacher professional practice (Eraut, 2011). These 'affective outcomes' reflect the emotional experience in this learning situation; whilst the observed changes may not be permanent they do provide a useful, and even necessary, precursor for changing practice.

6.5.2 Motivation and attitude outcomes

I intended the programme to lead to increased desire by the participants to develop their practice but recognised that this might not be evidenced by actual changes in practice by the time of the post-PD interviews. However, an *intention* to develop practice is a positive outcome in itself. It can present as enhanced enthusiasm and motivation to implement the ideas and approaches presented during the PD programme. For example, as mentioned earlier in this chapter, in the first cycle Ruth indicated that she was thinking about how the talk task within the Mathematics Mastery six-part lesson could be developed to incorporate orchestration of anticipated responses. In the second cycle, Tom explained that he was considering replacing the 'develop learning' part of the lesson with a designed task that would be used to teach the mathematics for this segment through problem solving. These examples indicate the actions of "an active professional agent [which] implies perceiving oneself as an active learner who is able to act intentionally, make decisions and reflect thoroughly on the impact of one's actions". (Toom et al., 2015, p. 616). They also illustrate the participants' motivation to implement and develop their learning and expertise. These observations are examples of Harland and Kinder's category of motivation and attitude outcomes and point to the effects of the PD programme. Like affective outcomes, this kind can be short-lived especially if they are not bolstered and sustained by other professional learning to develop the required skills – which in this case would be the opportunity to develop and refine the orchestration of anticipated responses.

6.5.3 Knowledge and skills outcomes

Knowledge and skills outcomes pertain to the development of deeper levels of understanding and practical experience in a technique. Much of what teachers know can be thought of as tacit knowledge (Eraut, 2011).

In Cycle 1 the participants talked about how a task could be designed or modified to develop the mathematics being taught. The examples they gave were in relation to the selection of numbers in a task, such as for the topic of pie charts, and they indicated that this was 'new knowledge' acquired as a result of being involved in the programme. The participants also made new connections between different aspects of mathematics. Dave showed how the area of a trapezium could be used to illustrate the formula for the number of half-time scores (displayed in Chapter 7.1). Similarly, in Cycle 2, the participants indicated that the analysis of the task in the research lesson had developed their own subject knowledge regarding the 'multiplicand' and the 'multiplier' in relation to algebraic terms such as '3T' and '2B'. This evidence suggests that the participants were able to demonstrate and articulate new knowledge as a result of engaging in task design together with the orchestration of anticipated responses.

It was interesting to probe the types of knowledge that were obtained by the participants as a result of their engagement in task design. Wilson (1987) identified seven aspects of teacher knowledge which in the context of mathematics teaching were defined as:

- mathematical content knowledge
- general pedagogical knowledge
- mathematics curriculum knowledge
- mathematical pedagogical content knowledge
- knowledge of mathematics learners
- knowledge of education contexts

- knowledge of the ends, purposes and values of education related to mathematics.

Clearly the knowledge gained by the participants, as identified above, could be designated in one or more of these categories, which I suggest illuminates the value of teachers engaging in the process of task design. Moreover, I contend that with the correct resources and tasks, task design is an activity that can be accommodated in regular teacher planning.

6.5.4 New awareness outcomes

The outcomes discussed earlier suggest that the participants became more aware of the potential of teaching mathematics through problem solving and recognised that the approach can support and develop their existing practice. The participants in Cycle 1 understood how the approach can support discussion in the classroom and help the teacher identify pupils' misconceptions. They also considered that whether this approach could be used in their classrooms would depend on the attitudes, skills and prior experiences of the pupils. In Cycle 2, the participants became more aware of the principles of teaching mathematics through problem solving and the links between them, including the relationship between task design and orchestrating pupil responses. For example, Charlotte conveyed her understanding that the processes of exploring the mathematics in the problem and orchestrating the learning should be developed together.

The participants also grew to understand that new skills were required of them to use the teaching technique orchestrating the learning. Questioning technique was the most prominent of these new skills and it was discussed by the TDT in both cycles. There was new awareness that the questioning used in the orchestration sequence was different from the general 'what did you do?' type questions, moving to more specific enquiry question such as 'Why did you start with the number 15?'. In Cycle 2, the analysis of the orchestration sequence identified that the teacher was using particular phrases which I define as 'teacher connectives'. These were used to clarify the approaches used by the pupils and to show how the different contributions from each method were building blocks towards a particular solution. Often these connectives by the teacher took the form of rhetorical questions. For example, in the research lesson during the orchestration sequence, the teacher used a question to confirm with the pupils why they had selected some numbers and not others.

These questioning strategies are valuable for guiding whole-class discussions towards important mathematics (Cobb & Wood, 1993). They are similar to some of the approaches suggested by Chazan and Ball (1999) who describe "expanded telling actions" which include actions to remind pupils of a conclusion they have already agreed and to rephrase pupils' comments for the whole class. Whether or not these strategies are part of the participants' regular teaching practice, it is clear that the analysis of the orchestration sequence brought them new awareness of the development of questioning in the research lesson and its pivotal role when the teacher is building on the previous contributions from the pupils.

6.6 Summary

In this chapter I have described the effect of the programme on the participants' views about teaching mathematics through problem solving and on their understanding of the teaching technique of orchestrating the learning. I have also identified developments in the participants' subject knowledge and potential indicators of changes in practice. Whether these outcomes are transitory or permanent is unknown, but the findings do suggest that in both cycles the participants had:

- established positive views about teaching mathematics through problem solving, and their emerging understanding of the approach was broadly in line with the definitions and descriptions presented in the PD programme;
- developed their own subject knowledge and pedagogical knowledge which had contributed to their professional growth as teachers of mathematics;
- considered incorporating some aspects of teaching mathematics through problem solving into their practice.

I identified a number of barriers that might prevent the participants from fully adopting this approach to teaching mathematics in their practice and which could relate to issues of coherence between the PD programme and the participants' classroom contexts. The coherence question is discussed in more detail in Chapter 8. Finally, in this chapter I used a typology of INSET outcomes to describe and exemplify the categories of outcomes observed in the PD programme.

Chapter 7 – The PD Programme – Design Intentions and Outcomes

Introduction

In the previous chapter I identified four areas of professional development outcomes for the participants as a result of their participation in the PD programme. In this chapter I consider the relationship between the programme's designed components and the CPD approaches, and the professional development outcomes.

The designed components of the PD programme were:

- the PD Sessions 1, 2 and 3
- the use of visualisers in the research lesson
- the pause in the research lesson
- the TDT analysis of the orchestration teaching sequence (Cycle 2 only).

The CPD approaches used were Lesson Study and Teacher Design Teams.

In Cycle 2, a number of changes were made to the designed components, as explained in Chapter 5. The results of these modifications in Cycle 2 are documented in the relevant sections below.

7.1 PD programme Sessions 1, 2 and 3

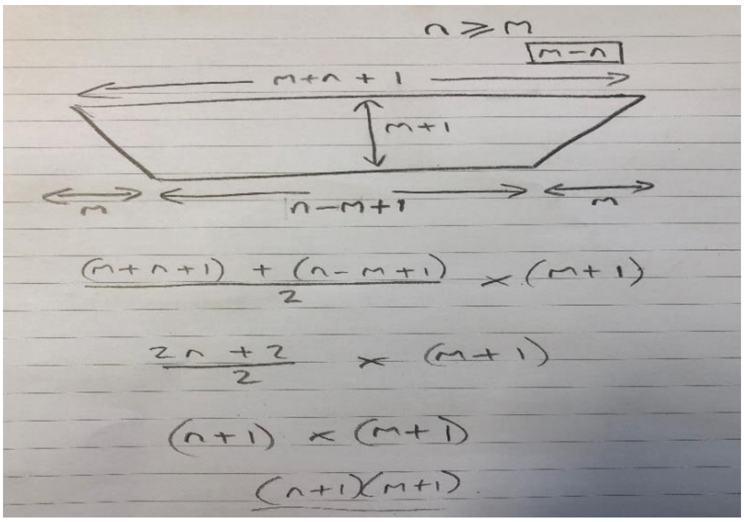
The first three PD sessions were designed to introduce the participants to teaching mathematics through problem solving and to develop a research lesson as part of a Lesson Study programme. As discussed in Chapter 2, the literature in this area confirms that the process of task design can influence teacher planning, teacher actions and student activity (Coles & Brown, 2016). Therefore, task design and the teaching technique orchestrating the learning were central features of the PD sessions. The design intentions of these sessions are summarised in the table below.

Table 7.1 *Summary of design intentions of PD sessions 1, 2 and 3*

Overview of PD sessions 1, 2 and 3		
PD Session	Focus	Intended outcomes
Teaching mathematics through problem solving	Task design and task analysis	An understanding of how teaching mathematics through problem solving can be used to teach pupils to learn mathematics
Orchestrating the learning	Anticipating and sequencing pupil responses	Develop expertise in task analysis and planning the sequencing of pupil responses
Planning for the research lesson	Task analysis and planning for sequencing pupil responses	Collaborative learning on task analysis and planning for the sequencing of pupil responses

The participants in both cycles indicated the value of engaging in the process of task design and planning to sequence anticipated responses and expressed that this professional learning had improved their subject knowledge. They also valued the time in the sessions to develop their expertise in analysing tasks to develop the mathematics to be learned and were enthused to explore how problems could be used to teach different areas of mathematics. For example, in Cycle 1, following the PD session on task design, Dave explored the half-time scores problem (described in Chapter 2) and shared his thinking about how to derive an expression for the number of half-time scores from a game ending with a score **m – n**. He showed how the general formula for the number of half-time scores (m+1)(n+1) could be proved using the formula for the area of a trapezium (Figure 7.1 below).

Figure 7.1 *Dave's proof using the sort displayed in Chapter 2*



The teachers from the primary phase (Cycle 2) were especially interested in the use of problems that could provide physical or diagrammatical models to introduce or develop a mathematical concept. Tom, for example, had a “light bulb” moment when he was shown how the area task could be used to compare equivalent algebraic expressions. These examples illustrate the insights that teachers can gain from working with mathematical tasks (Pepin, 2015).

In Chapter 2, I discussed the literature showing that engaging teachers in the process of task design can support developments in subject knowledge (Watson & Ohtani, 2015; Wilson & Cooney, 2002; Zaslavsy, 1995). This effect was observed in this study and also it appeared that inclusion of the teaching approach of orchestration alongside this process had not reduced the positive effects of task design. In both cycles, the participants talked about how their pedagogical and subject knowledge had developed as a result of being involved in the programme. The Cycle 1 participants talked about how the choice of numbers in a task was important, and the Cycle 2 participants described how the analysis of the task in the research lesson had enhanced their understanding of the ‘multiplicand’ and the ‘multiplier’ in terms of the development of algebraic notation. This greater understanding was apparent in the way the teacher used language in the research lesson, for example “the number of legs is multiplied by the number of tripods” which the participants agreed had improved the pupils’ understanding of the expressions ‘2B’ and ‘3T’.

However, the participants expressed different views about the principles of teaching mathematics through problem solving. It appeared that the PD sessions in Cycle 1 had not fully established in the participants’ minds the relationship between the process of task design and orchestrating the learning, as intended[37]. The purpose of these connected concepts was to support the development of teaching mathematics through problem solving by using the pupils’ responses to develop the mathematics to be learned. The data appeared to suggest that the participants saw the two concepts as separate and therefore they made decisions about the value of each, which was then reflected in their views and in the development of the research lesson. For example, in Cycle 1, Ruth explained that teaching mathematics through problem solving was useful in promoting discussion, exploring different approaches to solving the problem, and revealing misconceptions. However, she did not refer to the main objective of

[37] This resonates with the views of Swan and Swain (2007) who state that teachers interpret designs in ways that were not intended by the researcher.

the approach which is to teach mathematics by exploring the pupils' responses to the problem. She also indicated that the purpose of orchestrating the learning was to get to the best solution rather than developing the mathematics to be learned.

Also, in Cycle 1 when the participants re-planned the second half of the research lesson during the pause, they initially focused on the completion of a table that showed all four integer solutions for the Diophantine equation $3T + 2B = 23$, without reference to the pupils' own methods. This action appeared to move away from the intended teaching method which was to develop an understanding of the construction of the Diophantine equation using the responses from the pupils. This phenomenon did not occur in Cycle 2 where the participants planned the second half of the lesson using only the pupils' responses and there appeared to be a greater recognition of the relationship between the task design process and orchestrating the learning.

The planning during the pause in Cycle 2 was more in line with the design intentions of the programme and could have been due to the modifications made to the PD sessions in Cycle 2 to present the principles of task design and orchestrating the learning as a single, more coherent teaching approach rather than two separate strategies. Therefore, the PD programme was more effective in developing participants' understanding of the overall teaching approach. The issue of coherence between the two designed features of the programme is discussed further in Chapter 8.

7.2 The use of visualisers in the research lesson

The use of visualisers to observe pupils' work during the research lesson was initially a logistical solution to some of the constraints of Covid-19 rather than an original design feature. However, as described below, the use of visualisers had several positive effects so this feature was retained in Cycle 2 and has contributed to the overall design intentions of the PD programme.

In Cycle 1, the visualisers made it possible for participants to observe the work of every pupil (not simultaneously). This was equivalent to virtually walking the classroom without interference and it allowed an observer to return to a particular pupil at any point in the lesson. In Cycle 2, the use of the visualisers was more restricted due to bandwidth limitations, so the observers were limited to watching just two pairs of pupils for the duration of the lesson. The initial concern that having a large piece of hardware on the corner of every pupil's desk could be a distraction quickly diminished, in the same way that observers looking over a pupil's shoulder are eventually ignored.

Ngang and Sam (2015) explain that in a research lesson the observers should "observe closely the way pupils react, how excellently they learn and make improvement and how well the design of the lesson meets pupils' needs and engages them in learning" (p. 135). This observation activity can be quite challenging for a number of reasons.

- With multiple observers in the classroom, it can be difficult to observe several different pupils' work.
- Understanding what the pupils have done, or the particular approach they have used, can often take time. Too little time spent may lead to incorrect assumptions about their methods and solutions.
- To capture the progression in learning it may be necessary to return to a particular pupil or pupils – which may not always be possible.
- Taking notes[38] of observations can be challenging particularly if the observer is making comparisons and connections between different pupils.

In both cycles, all members of the TDT indicated the value of being able to analyse the pupils' work, to inform any changes to the lesson plan that were discussed in the pause in the research lesson. A common reflection was that the visualisers allowed the teacher to watch the pupils develop their work in real time without standing over the pupils, and so provided the teacher with more information than would be obtained by just looking at their books. For example, Ruth explained that she was able to observe the build-up of their methods, their jottings and scribbles. She also valued listening to the words they used when discussing the problem. The visualisers had microphones and so the participants could hear as well as see the pupils working, again without standing over them. Ruth elaborated on the value of being able to observe the pupils' work in this way:

> Watching them when they are actually doing the work themselves […] you could see their thought processes, you could see them pause. […] If you were to get their work at the end of the lesson you don't know how long they have

[38] Software called 'LessonNote' has been developed for this purpose. https://apps.apple.com/us/app/lessonnote/id507466065

> actually spent doing that work… how long has it taken them to get to that point? What conversations? What thought processes have taken place, here we see them pause and can watch what have they actually done to get them to the point where they think they have got their answer… that is huge, just listening to some of the words that the students use, some of the jottings, some of the work that is scribbled out… quite often the bit that they have scribbled out is the most important part because it shows how they have worked from the question to get an actual answer.
>
> [Ruth, secondary teacher. Cycle 1 post-PD programme interview]

Ruth was emphasising the critical difference between being able to see the development of the pupils' work in real time as opposed to a mere snapshot of the work during lesson or the completed work at the end: "here we see them pause and can watch what have they actually done to get them to the point where they think they have got their answer… that is huge!".

Similarly, Adam valued the visualisers for the opportunity they gave him to "stand back" and look at the mathematics to see how the pupils were developing the solutions and to spot any misconceptions. He also recognised it as a useful CPD opportunity.

> I think it is very good for CPD, because it's not something that you you'd normally have access to in real time, even when you're teaching because there's so much that you're dealing with on a minute by… well, second-by-second basis. In the lesson you can't stand back and really look at the maths. Here we were able to move from one group of pupils to another so you could see how they were tackling the problem. So it gives you more of an insight on how they're building up their own solutions and any misconceptions.
>
> [Adam, secondary teacher. Cycle 1 post-PD programme interview]

In Cycle 2, the participants commented on the value of using the visualisers to focus on just two children. For example, Paul provided this thoughtful account of the experience:

> Yes, it was really useful because we had just the two children, and you could just watch what they were doing the whole lesson. So I didn't really focus on what the teacher was doing… which was kind of a background noise. And what I was really focused on was making notes on what the children's response to

> what the teacher had said, and so I was able to see a map of their learning journey. And that was really powerful, because there are certain times when you ask a child a question, you just get their answer, but in this case, you saw they're working out as it was happening in real time. So it was really a really good experience to kind of be part of.
>
> [Paul, primary teacher. Cycle 2 post-PD programme interview]

Paul's use of the term "background noise" to describe the role of the teacher is interesting and raises an important issue regarding the observation of learning in a classroom. Often in lesson observations, we observe the work of the pupils in order to make judgements about the teaching as much as the learning. The dual purpose of observation can dilute the effectiveness of each aspect of the activity. In a research lesson that is planned in detail, the members of the planning team already know the specific actions of the teacher and the sequence of activities that the pupils will engage in (if they stick to the plan!). Also, Paul's comment that he was "really focused" on the pupils' learning to obtain a "map of their learning journey" conveys a sense of the intensity of the observation made possible by the use of visualisers.

Clare also appreciated being able to observe one or two pupils' work in detail throughout a learning sequence and she contrasted this "luxury" with the normal approach to monitoring the work of pupils as they independently engage in problems. Her use of the word "float" may allude to making only general or cursory observations of pupils' work in typical classroom conditions.

> I think it was really useful particularly because we were really focused on one or two children or two groups of children. It was interesting just to see. Because, you know, you rarely get to see this sort of detail […] normally you float around the room, and you might stay with one child for a bit. So then you move on to someone that you know, but it was nice just to spend the whole time just seeing how one or two pupils approach the work. And you get to see their thought process of what they're doing in detail. Obviously, it's kind of a luxury really, because we don't get that opportunity to do that very often.
>
> [Clare, primary teacher. Cycle 2 post-PD programme interview]

Clare also noted the difference between looking at the pupils' completed work in books and observing pupils in the actual process of doing their problem solving, especially when pupils are working as a pair. She highlighted the usefulness of being able to hear the conversations between two pupils.

> …you can often see their working out in books, but when we were observing a pair constantly, you're hearing more, you get to know what they're talking about, and just being able to hear them talk in pairs was really useful.
>
> [Clare, primary teacher. Cycle 2 post-PD programme interview]

The fact that the visualisers afforded *detailed* observations was a continuing theme for the participants in their interviews. Tom indicated the level of detail obtained from observing the pupils' workings in this way and he highlighted the value of being able to scrutinise unexpected responses.

> I found it very interesting to see the way they worked. And particularly with some of them that did trial and error or what seemed to be trial and error. However, when they discussed it […] It seemed like there was a bit more to it than that. I noticed the one who had said there had to be at least two of each so that would be a minimum of 10 legs. So they went 10, and then 20. And I thought there might have been a misconception here […] and that was a method we haven't really thought about and that's why I suppose it was important [to have] that break in the middle so that we could talk this method to see if we wanted to explore this misconception…
>
> [Tom, primary teacher. Cycle 2 post-PD programme interview]

Here Tom acknowledged that teachers' observations are sometimes not what they seem, by giving the example of the pupils he initially thought were using trial and error but on closer inspection were in fact using a systematic approach to find possible solutions.

It was also interesting to note the comments by Charlotte who taught the research lesson in Cycle 2. When asked about her thoughts on the observations from her colleagues, in relation to her own observations of the pupils, she said:

> So then hearing what had happened between the little snippets that I got was really helpful to kind of join the pieces of the puzzle together. And because I'd kind of seen all of the children connect those ideas up from one person to

> another because they'd seen different things perhaps. So then it was helpful to hear the more detailed version of it from the little snippets that I got.
>
> [Charlotte, primary teacher. Cycle 2 post-PD programme interview]

Charlotte's use of the word "snippets" suggests the depth of her observations and could reflect the way she normally gathers information from the children's work. It echoes other participants' use of expressions such as "floating around" the classroom.

The participants' comments above indicated that the knowledge gained via the use of the visualiser was not new knowledge as such; rather, they were saying that being able to observe the pupils 'live' was a different and generally more informative means of identifying methods, diagnosing errors and revealing misconceptions. Whilst the participants demonstrated new awareness of the different ways of analysing the pupils' work, the comments also revealed 'tacit association' – where teachers who are introduced to a new approach often believe that it is already part of their practice (Swan & Swain, 2010).

All of the participants recognised the advantages of being able to observe the pupils in this way and how it differed from their normal approaches in class and looking at the work in their books. In these observation sessions, the most significant difference between the two cycles related to the number of pupils that the participants were able to observe. Clearly, in Cycle 1 the affordance of being able to see the work of all of the pupils gave participants an overview of all the methods being used in the class. Equally though, in Cycle 2, having the opportunity to focus on the work of just two pupils meant that much more detailed information could be obtained and that particular approaches and/or misconceptions could be subsequently explored in more detail to develop a deep understanding of the pupils' methods. In particular, this close focus on two pupils enabled the participants to observe moments of hesitancy and uncertainty.

The observations in this section have indicated that the use of visualisers in the research lesson made a significant contribution to the design intentions of the programme and as such was an effective design component. Also, as discussed in the next section, I consider that this innovation supported the design intentions of the pause in the research lesson. It was clear from the quality of the discussion during the lesson pause that the participants had obtained a great deal of information about how the pupils had approached the problem. This would not have been the case without the enhanced, in-depth access to the pupils' work made possible by the use of visualisers.

7.3 The pause in the research lesson

The pause in the research lesson was an innovation to provide a learning opportunity for participants to develop the orchestration of a sequence of anticipated responses. It can be thought of as a simulation or rehearsal where it is possible to break down a situation in real time into a sequence of expanded events that can be analysed and responded to individually and then collectively. Its purpose therefore was to provide the teacher with an opportunity to 'practise', just as one might do in a sport or a medical procedure using virtual reality, and a subsequent opportunity to analyse the effect of the jointly planned orchestration sequence.

All the participants in both cycles valued the pause as a professional learning activity. Their views have been collated into the four themes described in the subsections below.

7.3.1 Enjoyment and working collaboratively

In Cycle 1, Marie enjoyed the interactivity of the process, as conveyed in her comment below. The reference to "changing tack" indicates that the discussion led to changes from the original lesson plan. She also recognised that although significant time was allocated to the process there was still more that could be learned from it.

> I enjoyed it because you're actually discussing something that was live so you could bounce ideas off one another. And I think from my recollection of the discussions, we changed tack I think as we were talking about it, and it sort of fed into what to do next. Even though it was quite a long process, I still felt like there was so much more we could have done.
>
> [Marie, secondary teacher. Cycle 1 post-PD programme interview]

Ruth also expressed her enjoyment of the PD and identified the pause in the research lesson as her favourite part of the programme. She explained that the pause was an important part of her learning and highlighted how much she enjoyed the opportunity to work as a team to analyse the pupils' work.

> I enjoyed the whole programme […] one of my favourite things that you introduced was the pause in the research lesson. Even though we had done the planning together during the break we all saw something slightly different and we all had a slightly different opinion […] it was really nice to discuss those opinions to be able to come up with one that was for the whole team. I really enjoyed that part. Tom had some really good ideas about his interpretation of what he saw but it was not the same as my interpretation […] by discussing it I could see what he saw that I had overlooked or misinterpreted… I really enjoyed that part.
>
> [Ruth, secondary teacher. Cycle 1 post-PD programme interview]

In Cycle 2, Tom reflected on the value of the opportunity to revise the research lesson mid-way through and highlighted the intensity and focus of the discussion. His views were shared by all members of the TDT.

> I really enjoyed that the discussion we had was at a very high level. And it was really helpful to have you there facilitating. The discussion we had about which way the lesson should go was quite intensive, but the method that we came to was a really simple and effective way of scaffolding the algebra which we did not think of with all that planning time before. Being able to consider their responses was very valuable.
>
> [Tom, primary teacher. Cycle 2 post-PD programme interview]

Importantly, Tom identified an 'in-the-moment' learning event where the TDT produced a refinement to the lesson that Tom described as "scaffolding the algebra". He explained that this was able to happen because the TDT had the opportunity to discuss the work of the pupils mid-way through the lesson. He also recognised the value of an external facilitator.

7.3.2 The opportunity to analyse the pupils' work

The detailed information obtained from observing the pupils through the visualisers contributed both to the planning of the second half of the research lesson and to the shared learning in the post-lesson discussion. Adam explained that he valued the opportunity to see all of the work at the same time without the pupils being present. He also considered this type of observation to be a useful tool for professional learning:

> You don't get the chance in the lesson and when you can see everyone's books and no one is writing you can compare everyone's books at the same time and the part of the lesson that you stopped it at [...] you know what anticipated response you are expecting… if they are what you are anticipating that's great… if they are more than you are anticipating that's better because you have got things that you can talk about… yeah it's definitely a really, really useful tool for CPD.
>
> [Adam, secondary teacher. Cycle 1 post-PD programme interview]

Similarly, Ruth indicated that being able to observe the pupils' books in this way helped her consider decisions about the sequencing of the second part of the lesson. She noted that she would not have come up with this sequence on her own.

> It was useful to be able to observe the work of some of the pupils in detail during the pause, this then helped me think about the sequencing for the second part of the lesson when we were discussing it […] during the break in the lesson. I do not think I would have come up with this on my own if I had been teaching the lesson but I guess I would get better at it the more I taught the lesson.
>
> [Ruth, secondary teacher. Cycle 1 post-PD programme interview]

Ruth also indicated that in order to get better at sequencing she would use the task again. This suggests a change in her thinking about the selection of tasks. In the pre-study interview, Ruth said that previously when she was selecting tasks to teach problem solving she would look for something new rather than use a task she had used previously except when she remembered a particular task that had "gone down well".

In Cycle 2, Clare valued the affordance of seeing the pupils' books mid-lesson because it allowed her to consider how the second half of the lesson might be augmented.

> I had never appreciated how useful it would be to see all of the work midway through the lesson. Normally when I look at the books after the lesson I think about the things that went wrong. This opportunity […] enables me to think about what I can do in this lesson rather than wait until the next lesson.
>
> [Clare, primary teacher. Cycle 2 post-PD programme interview]

7.3.3 The opportunity to rehearse and practise

Marie saw the pause in the research lesson as an opportunity to practise building up a picture from observing the pupils' work and that this was a useful professional learning experience.

> Yes, I think one of the difficulties with, obviously doing it in lessons, it is that you can't be seeing 30 bits of work at the same time and seeing how they've all started, and how they've got to a methodology that you want to pick up and talk to the class about […] Rather than just seeing a snapshot every time you look over their shoulder to see their book. Yeah, to try and build that picture of what's going on in the class is difficult. So as an opportunity to practise, I think it was, yeah, it was really useful to be able to see, see that, that live, work going on.
>
> [Marie, secondary teacher. Cycle 1 post-PD programme interview]

Charlotte found it useful to revisit the original plan and to have time not only to re-think the second half of the lesson but also to rehearse and practise the orchestration sequence.

> I really liked the chance to revisit the second half of the lesson from looking at their books. It was really helpful for me to have help with planning and then […] having the time to go through the new sequence together.
>
> [Charlotte, primary teacher. Cycle 2 post-PD programme interview]

7.3.4 Time to plan and learn

Ruth explained that having the time to reflect on the work done in the lesson so far was very useful in helping to plan the second half of the lesson.

> Having the time to think about the work done so far was really useful. We could then think about how we could change the sequence to improve the lesson.
>
> [Ruth, secondary teacher. Cycle 1 post-PD programme interview]

Tom talked about not being pressurised, suggesting that in other situations where he was observing pupils' work he did not feel he had sufficient time to do so.

> I like the fact that we were able to come out of the lesson, and not have to do it in a kind of pressurised on the spot. On your own feeling like you know, trying to manage a hundred things at once. Being able to come out and spend a little bit more time as a team to look over it. I think it was a really useful way of understanding how people think about the mathematics that we were looking at and how we might piece it together and gives you experience for the future if you're going to do a similar thing.
>
> [Tom, primary teacher. Cycle 2 post-PD programme interview]

Tom explained the value of looking at the work "as a team" and that he found the ideas and thinking of colleagues in piecing the work together very useful. He also envisaged that this learning would be beneficial in the future, suggesting that this activity helped him develop knowledge of possible pupil responses and the experience of anticipating responses.

Charlotte also appreciated the opportunity and time during the pause to discuss the children's responses and to work with these rather than just continuing with the instructions in the lesson plan.

> I think it gives more time for exploration and discussion. So as a result of the planning in the break for me as the teacher it's allowed me to work more with the children's responses, and think I can go in that direction and use this response rather than trying to get them back on to whatever it says on the lesson plan.
>
> [Charlotte, primary teacher. Cycle 2 post-PD programme interview]

Tom valued the pause in the lesson as a useful opportunity to "think deeply" about the lesson and how to use pupils' responses. He also related it to other PD developments in school.

> I can see the value in this research lesson with a pause in it as a CPD programme to help teachers do this better because we thought so deeply about the lesson itself, because we thought so carefully about the different ways it could have gone and we came to make a decision on that. And because generally it has made me, all of us, think more deeply about how we do use those responses. Although it's not exactly the same thing, in school, we've been working on whole class feedback. So it's not breaking up a lesson in the middle of it. But

> it's definitely spending a lot more time talking to the class about what we saw in their books as a whole.
>
> [Tom, primary teacher. Cycle 2 post-PD programme interview]

The beneficial effects of the pause were also evident in the discussion that took place. The transcript of the pause discussion in Cycle 1 (Appendix 9) illuminates the work of the TDT in deciding how the second half of the lesson should be planned. The TDT's forensic analysis of the work of different pupils informed a rich discussion on how to establish an orchestration sequence of the pupils' work that the teacher would use to develop the second part of the lesson.

Four out of the five participants were fully engaged in the specific activity of deciding the order in which the pupils' solutions would be discussed. However, the motivations for doing so were different. For some, the objective appeared to be to ensure that the orchestration led to the production of the equation $3T + 2B = 23$ whereas for others the main ambition was to produce the four solutions and compare the methods by which they were derived. Ruth provided significant input to the discussion which revealed the extent of her thinking which was formed during her observation of the pupils' work during the lesson and in the pause PD session. She explained the strategy for selecting the order in which the pupils' responses were to be considered, which was to "build" on the previous response. However, as mentioned earlier, the sequence which was ultimately devised was used to work towards solving the problem and completing a table of solutions rather than the development of the mathematics to be learned from exploring the problem.

Notwithstanding the observations above that suggest that the participants developed different understandings of the purpose of orchestrating the learning, I assert that they recognised the value of the pause as an effective professional learning opportunity in which the technique of orchestrating the learning could be developed. As such, I consider this component to be one that contributes significantly to the coherence of the PD programme and which validates the design intentions.

7.4 The TDT's analysis of the orchestration teaching sequence

This professional learning activity was part of the initial PD programme design and was originally part of the post research lesson discussion. As discussed earlier, in Cycle 2 this activity was developed into a separate PD session. Its purpose was twofold. Firstly, the session was designed to bring together the key concepts of the PD programme and to extend the professional learning of the participants. In PD Session 2, the participants observed a video of a teacher who demonstrated an orchestration sequence for the three cards problem. Whilst the benefits of this type of video-based CPD activity are well known, (Marsh & Mitchell, 2014) a disadvantage is that the observers sometimes do not gain fully from the experience because of the unfamiliar context in which the video was produced. By using the material that the participants had contributed to and developed themselves, I considered it more likely they would engage with the contents of 'their' video more objectively.

The second rationale for this PD activity was to use the observed outcomes from the lesson and the participants' reflections to inform my understanding of the contributions made by the PD components and in particular the pause in the research lesson.

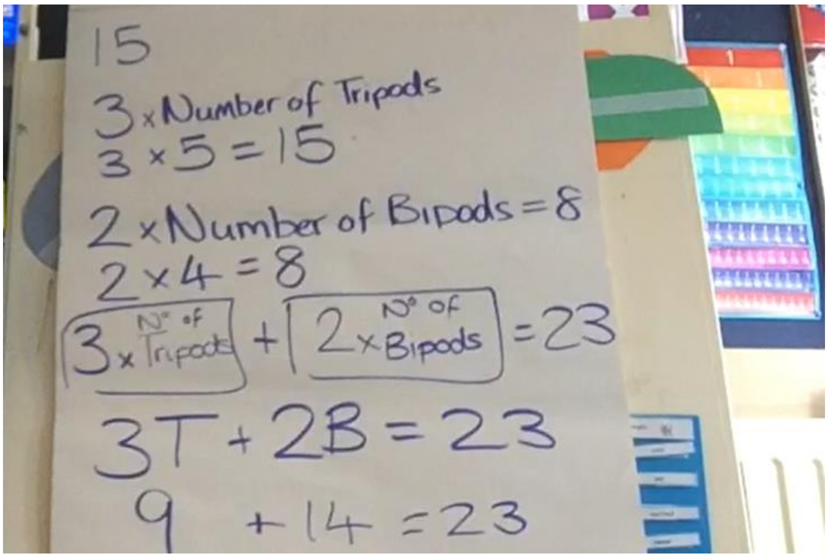
The period of nearly eight weeks between the research lesson and the reflection on the orchestration sequence was determined by the timing of the summer break. Had I had the opportunity to decide on the interval I would have chosen a shorter period of time. However, I judged that the participants would be still familiar with the key moments within the orchestration sequence although it is worth noting that for Charlotte this was the first time that she had seen this sequence from the perspective of watching herself teach. The transcript of the orchestration sequence together with images of the worked solutions can be found in Appendix 10. In summary, the transcript revealed the following key observations.

- The teacher consciously collated the methods in a particular sequence in accordance with the plan that had been devised in the pause PD session.
- The first method that was shared concluded with an empty number sentence which was then repeated in the second method using a different solution, but did not include addition or equal signs and the total 23.
- The introduction of the equation was almost incidental and was generated by using a mixture of the language used by the pupils and the new terms 'T' and 'B' introduced by the teacher (although the teacher did not define T or B).
- The construct of the terms '3T' and '2B' appeared correctly[39] in terms of the multiplier and multiplicand.

[39] Only in the sense of the expression being read as 3 multiplied by T.

- The teacher used the pupils' methods to orchestrate the learning and frequently used rhetorical questions to confirm their thoughts and ideas to the rest of the class. This resulted in the final orchestration sequence containing the equation 3T + 2B = 23 as planned and as shown below in Figure 7.2.

Figure 7.2 *Charlotte's final orchestration page*



Due to various circumstances only two members of the TDT including the teacher watched this episode. Following the observation session, the teachers discussed their views and observations. The discussion was facilitated by the following questions:

1. How well did the sequencing of the selected responses support the development of the lesson?
2. In terms of a professional learning experience, what are your reflections now?

Tom indicated that the mathematical content appeared to come out naturally and that the lesson had "flow". He also attributed the success of the lesson to a combination of the original lesson plan and the revised lesson plan produced in the pause.

> The algebra seemed to come out quite naturally and had a flow to it. It felt like that she (teacher) was walking the pupils through their work. I think some of it probably came from the way that we had planned beforehand.
>
> [Tom, primary teacher. Cycle 2 orchestration discussion]

He went on to explain that the teacher's summary on the flip chart was almost exactly as they had planned and that this indicated to him that the pupils had understood. Importantly, he noted that this summary was produced by the teacher picking the key points from their explanations and then recording these whilst at the same time reiterating what the pupils had said.

> As they explained their answers, Charlotte was writing them down and then by picking out the key parts of their solutions, […] what we got on the flip chart was almost as we had planned for exactly and in that sequence. As a result, I thought many of the pupils seemed to understand where each part of the solution had come from […] As Charlotte was explaining their thought process back to them you could see that they knew the logic behind it.
>
> [Tom, primary teacher. Cycle 2 orchestration discussion]

Charlotte suggested that the sequencing provided a scaffold and described the process as guiding the learning.

> It was almost like, we wanted them to have what I was writing down as their thought process. So it was like scaffolding it for them. So it's like that step between them not doing it themselves yet, but they understand the thought processes. So they're kind of in between whether they need guiding to. We know we want them to be able to do that for themselves in the end.
>
> [Charlotte, primary teacher. Cycle 2 orchestration discussion]

Charlotte's and Tom's comments indicate that they both judged the sequencing of responses as effective and that the pupils could see the connection between their own work and the way in which the teacher subsequently developed their thinking to produce 'new learning'. Both Tom and Charlotte went on to talk about how the equation $3T + 2B = 23$ was formed. Tom noted that Charlotte was consistent in her positioning of the multiplier and multiplicand in line with the convention that would be used in the topic of algebra.

> We introduced a challenging concept from such a simple problem and we made the relationship […] the terms 3T and 2B fall out naturally. It came out naturally because they started off with so many groups of tripods and bipods, and

> Charlotte was consistent in the way she wrote that it was always three legs multiplied by this many tripods.
>
> [Tom, primary teacher. Cycle 2 orchestration discussion]

Charlotte said that she would have written 3T even if the pupils had formed the equation the other way round.

> Previously even if they had formed the terms of the equation the other way round […] you know like 'T3' I would have said it the other way around, I'd have written it that way because that's how my brain wants it to be.
>
> [Charlotte, primary teacher. Cycle 2 orchestration discussion]

It was clear from these comments that both Charlotte and Tom have observed and understood how the design of the task and the sequencing of the pupils' responses have led to the "natural" formation of the equation $3T + 2B = 23$. Charlotte said that her own knowledge of the construct of algebraic terms would have led her to simply reverse the terms had they occurred in the reverse configuration. She also felt that the children were comfortable with the way that their work was discussed.

Tom pointed out that the orchestration sequence would not have looked the same if they had continued with the original lesson plan and he therefore credited the perceived success of the lesson to the revisions made during the pause in the research lesson.

> I think there was a bit of a moment of discovery for all of us in the pause part of the research lesson. As a result of this planning we knew the direction we wanted it to go. And we knew the type of conversations we wanted them to have but we didn't really have any idea how it was going to go or necessarily how we're going to get there. But that did not matter because of the pause we had a plan for the work that we would discuss with the pupils and in a particular order which worked really well. I do not think we would have had the same result if we had stuck with the original plan.
>
> [Tom, primary teacher. Cycle 2 orchestration discussion]

When asked about the value of this review session as a professional learning activity, Charlotte indicated that observing the orchestration sequence had influenced her views on the value of using the pupils' work and she now felt more confident interacting with the pupils' responses to develop the learning. Her use of the phase “the same mathematics in the problem” suggested that she now recognised the merit of the process of task design and task analysis.

> I think observing this video has made me think more deliberately. I perhaps value thinking more about what I will do with what the children come up with, whereas before I thought, well, this is what I want the end to look like, this is what I want them to come up with, rather than kind of using what they’ve got, and then getting there, if that makes sense? So I feel more confident in being able to interact with their responses to let the lesson go where it needs to go, rather than just getting to the end, irrespective of what they do. Before I would kind of rein them back into what I thought they need to be doing. Whereas now it's like, well, let them go this way because I know we will get to discuss the same mathematics in the problem.
>
> [Charlotte, primary teacher. Cycle 2 orchestration discussion]

In a similar way, Tom acknowledged the importance of taking the time to work with the responses provided by the pupils in order to let the lesson develop. He also indicated that the task should lead to the mathematics that is to be taught.

> I would echo that. One thing I’ve learned is the same sort of thing that Charlotte's saying there that there’s real value in taking the time over the responses which you actually get whether it’s the ones you're expecting or not and not feeling bound by where you want the lesson to go. But recognising that you need to work with what the children are telling you in that lesson. And that the task we had planned should lead you to the mathematics you want.
>
> [Tom, primary teacher. Cycle 2 orchestration discussion]

Irrespective of how well the team judged the success of the orchestration, both Charlotte and Tom considered that there is real value in working with the pupil responses, whether they were anticipated or not, and therefore they appreciated the opportunity to explore and practise this teaching technique within a PD programme. Interestingly, Tom also connected this practice to a development in his school where they are currently exploring whole-class discussions of the work in the children’s books.

In consideration of the commentary above, I have concluded that the review of the orchestration teaching sequence was an effective CPD session that both complemented and brought together the key concepts of the PD programme. Importantly, it also supported the design intention of the pause which was to enable the participants to develop their expertise in sequencing and orchestrating anticipated responses.

7.5 Teacher Design Teams

As discussed in Chapter 2, Teacher Design Teams were established to ensure that the participants recognised that the purpose of their engagement in the programme was principally for the development of their own professional learning rather than the exploration of a research question in a Lesson Study programme. In each cycle, the TDT members collaborated to design and research educational resources and practices and indicated the extent to which they had grown professionally (Binkhorst et al., 2017). Furthermore, the participants' engagement in the design process created a sense of ownership of the resources produced. It was intended that this sense of ownership would increase the likelihood that the resources would become part of the participants' practice.

At this juncture it is important to recap my role as part of the TDT. As explained in Chapter 3, through the course of the study I adopted three distinct roles. Initially I took the position of complete observer, maintaining the position of researcher. I then moved to the position of participant-as-observer; this is where I became involved in the participants' goals mainly by supporting the development of the research lesson. Finally, during the pause and the post-lesson discussion, I took on the role of observer-as-participant whereby I maintained a strong research focus. Each of these roles defined my relationship with the TDT and allowed me to move in and out of the TDT. However, even when I became part of the TDT (participant-as-observer) I did not consider myself to be a substantive member because often I could not fully commit to the other members' values and goals (Adler & Adler, 1994).

All of the participants made observations about the advantages of being part of a team with the important TDT characteristics of relationships and trust. Two participants also remarked on the benefits of being part of a team with other TDT characteristics. For example, Clare indicated that having a common school approach towards teaching mathematics was helpful to them working as a team, whereas working on CPD with colleagues who had different approaches or used different schemes might not be as effective.

> I think it helps that we kind of knew each other, and we've worked together before and so it broke the ice a little bit. And also that, although we're quite similar because of the way in which we teach maths at the school. So I think that helps that we're all on the same page. Whereas sometimes you can go into a group, and schools might do things very differently, might have very different understandings of what's important in teaching maths.
>
> [Clare, primary teacher. Cycle 2 post-PD programme interview]

This reflection accords with the work of Handelzalts et al. (2019) who assert the importance of TDTs being formed from the same school and with members who have the same motivations.

Tom explained the benefit of the team having ownership of the development process:

> We knew that we were going into it designing a lesson that one of us was going to teach and at the beginning we did not know which one of us it was going to be and I also think there was a genuine interest in the maths and the problem. I was interested in the problem and wanted to build a lesson using that problem. I would say that was a big part of it. And I think it was helpful as well when there were times when we were discussing you made it clear that you would remain silent and if we were silent it was up to us to start ourselves going again.
>
> [Tom, primary teacher. Cycle 2 post-PD programme interview]

Tom also recognised the importance of the relationship between the TDT and the facilitator. This aligns with the findings of Voogt et al. (2011) who showed that the interactions of TDTs with external facilitators can contribute positively to the quality of the design and to teachers' learning.

The views from the participants indicate that the use of TDT in this study was an effective approach and was able to accommodate participants from different organisations. In Cycle 1, the TDT was already established as all participants were members of the same mathematics department, whereas in Cycle 2 the teachers came together from two different schools (in the same federation and so used the same scheme of work) and year groups. In both cycles, the participants contributed freely to the team discussions and often without hesitation. In their discussions on how to sequence the responses, they demonstrated confidence in their own ideas and equally they showed willingness to consider the proposals of others. Whether this was the case because of the existing relationships between the participants in the TDT or because of the relationship between the participants and the programme is unknown.

However, the most noticeable aspect of the use of TDT in Cycle 1 was the development of different roles within the team. The transcript of the pause discussion reveals that when the team was planning the second half of the lesson, the teacher, Joe, only asked questions to seek clarification from other members of the group. He did not make any suggestions about how the second half of the lesson should proceed and offered no view on the suggestions of others. He listened carefully to the discussion, and at the points where he perceived that the group had come to a consensus about the next step, he reiterated the approach. This finding may indicate that in this situation Joe had become uncertain of his teaching in this situation and so was comfortable accepting different suggestions and approaches. Whilst the interactions in TDTs are particular to the relationships within them, the observation of Joe's behaviour aligns with the finding of Handelzalts et al. (2019) who suggest that teachers who collaborate on the renewal of their curriculum may initially feel a loss of individual freedom to act on their personal preferences, without overview by colleagues. Moreover, group settings may more readily reveal any uncertainties in an individual's teaching.

7.6 The use of a modified Lesson Study process

I decided to incorporate the professional learning programme of Lesson Study because several of its components supported the design intentions of the PD programme and also the pause could be readily located within a research lesson. As already reported, the known benefits of Lesson Study were observed in this research. For example, in Cycle 1, Dave reflected that the research lesson enabled him to understand more about teaching mathematics through problem solving.

> It's like with all of these things, to really understand, you have to experiment and practise. Being able to do this in a research lesson where the learning is more important than the outcomes is so useful.

[Dave, secondary teacher. Cycle 1 post-PD programme interview]

In Cycle 2, Clare talked about the value of the collaborative work on planning the research lesson and then being able to see how the lesson unfolded as a result of the planning.

> Having the time to work collaboratively was really useful. Being able to listen to each other's thoughts within a structured process helped me think carefully about the purpose of the lesson. And then to be able to see the lesson in action and to see if our planning worked was really powerful.
>
> [Clare, primary teacher. Cycle 2 post-PD programme interview]

Also, Lesson Study can result productive learning even when unintended outcomes occur. In Chapter 6, I explained that Ruth may have misunderstood the purpose of orchestration: she viewed the process as one where she would orchestrate according to how she saw the mathematical connections rather than by sequencing the approaches presented by the pupils. Nevertheless, her interpretation still could be a valid approach to teaching mathematics. When a research lesson does not go to plan or the outcomes are not as intended, this can be due to many factors that are not associated with the design or research topic being explored. This can result in the approach being dismissed for the wrong reasons or there being insufficient CPD time to explore the issues raised (Seleznyov, 2018).

Several variants of Lesson Study exist internationally and in the UK, many of which have been borne out of responses to logistical challenges or particular interpretations. Within the UK variants, the components commonly retained from the authentic Japanese Lesson Study are those that relate to the design and planning of the research lesson often referred to as the research proposal, the observation of the research lesson, and the post-lesson discussion. It is therefore important that any judgement of the effectiveness of Lesson Study recognises the impacts of mutations, faithful or otherwise.

In this study, a number of modifications to Lesson Study were made as part of the PD design and these were discussed in Chapter 2. In summary, the modifications were:

- The planning time allocated to the research lesson was significantly reduced as there was no curriculum analysis (*kyozaikenkyu*) and the problem to be used in the research lesson had already been identified.
- The research lesson contained a pause during which a PD session took place.

- In addition to myself, only the members of the TDT observed the research lesson.
- The post-lesson discussion focused predominantly on the second half of the research lesson.
- The final commentary from a 'knowledgeable other' was replaced by an analysis of the orchestration sequence by the TDT in each cycle.

The rationale for these modifications was explained in Chapter 2 together with the known positive effects of Lesson Study as a professional learning experience. The most conspicuous modification in this study related to the number and nature of the observers in the research lesson. This decision to restrict observers was based on the reasoning that the discussion during the pause and the post-lesson discussion would be more focused and would not be diluted by inputs from observers who had not been part of the PD sessions. The teacher of the lesson would also be free from external pressure[40].

In Cycle 1, Marie commented that in her previous experience of post-lesson discussions within Lesson Study the focus had often been on irrelevant pedagogies, whereas in the current situation "we only thought about the areas of the lesson that we needed to". In Cycle 2, Paul compared this Lesson Study programme with previous programmes. He commented on a past experience when he taught a research lesson and noted a number of differences. He recalled the nature of the critical comments from external observers (that is, external to the research team and the school). Importantly, he noted that the comments were more about general pedagogy and minor organisational issues, such as the colour of the paper being used, and did not focus on the areas that were important.

> I think the experience we've had, working together, compared to the ones that we had previously was different because in the past we had new people involved. I can remember when I taught the lesson in the last Lesson Study. And there was quite a lot of people that attended, it kind of felt more critical. This one was constructive. And we only thought about the areas of the lesson that we needed to […] there was no critique like, oh, we could have done this better […] I remember comments that were nothing to do with the research lesson like the colour of paper that I used.

[40] Because of our accountability system (Ofsted), teachers in the UK often find it hard to openly critique lessons even when it has been explicitly explained that the evaluation only relates to the plan of the lesson and not the teacher delivering the lesson.

[Paul, primary teacher. Cycle 2 post-PD interview programme]

The importance of the type of relationship formed within the TDTs was highlighted by the participants and particularly by those who had a previous experience of Lesson Study. For these participants, the effectiveness of the post-lesson discussion was enhanced by the modification that the research lesson would be observed only by the members of the TDT and the researcher. Also, the post-lesson discussions in both cycles focused in detail on the mathematics produced by the pupils and the effectiveness of the sequencing of anticipated responses. In the Cycle 1 post-lesson discussion, rich dialogue took place about the effectiveness of the research lesson, as can be read in the excerpt below.

Marie: I really liked the end of the lesson where Joe used the table to draw out the equation. It looked like the pupils all understood where the equation had come from which was really good.

Adam: I think the combination of actually deciding to use the table with the selected pupils' work, in a particular order, and having the opportunity to do this half-way through the lesson, which we would not normally be able to do, I thought was quite powerful.

Joe (T): I still don't feel having taught the lesson that they understand the equation. I think their understanding was superficial.

Adam: Was it that they were beginning to start to understand because the next step might be some kind of proof or substituting values into the equation?

Joe (T): Maybe and when move onto the next problem that does not have a solution that would convince me that they would have understood today.

Ruth: I quite liked the way it flowed from one response to the other however because we have got two things going on. Because they are Y7s we have got to teach through problem solving and teach for problem solving so in my head someone like Thomas who has started the problem he was demonstrating his problem-solving skills, but going to the algebra I am a little bit like Joe (T). Nellie was so close but I think the use of letters was probably a jump too far.

Dave: I think also that before we got to the equation we should have looked at the '3M' part or the '2N' part and take each term into consideration separately.

(T) denotes the teacher of the research lesson.

[Extract from Cycle 1 post-lesson discussion]

In this discussion, Marie thought that the pupils understood where the equation had come from and that the objective of the lesson had been achieved. Joe (T) felt that as the pupils did not understand the construct of the equation, the objectives had not been met. Ruth liked the "flow" of the lesson but also thought the step of introducing the algebra was "a jump too far". She also introduced the idea that the method of teaching mathematics through problem solving for Year 7 pupils might need to be underpinned first by some teaching of problem-solving skills[41]. Finally, Dave progressed the discussion by returning to the planning of the lesson and suggested (with hindsight) that the equation should have first been developed in stages.

This excerpt illuminates the depth of focus and sharing of critical reflections in the post-lesson discussion. These rich qualities of the discussion I suggest were due in part to the design intentions of the programme where only the members of the TDT and the PD facilitator were present.

I do not suggest that this modification should apply to all Lesson Study programmes, as I recognise that the inclusion of observers outside the research team can often provide appropriate challenge and additional expertise. However, in contexts where the main purpose of the Lesson Study programme is to promote the professional growth of the participants, rather than to research and improve the curriculum or whole-school aspects, I argue that the Lesson Study is likely to be more effective if observers are limited to the participants and an external facilitator.

7.7 Summary

In this chapter I have reflected on the design intentions of the four key PD programme components and the two CPD approaches, and compared them to outcomes drawn from the

[41] Ruth used the term 'teaching for problem solving' which was part of the training in PD Session 2

participants' views and their actions within and responses to the different PD components in the course of their participation. In summary, the observations in this chapter suggest that the enacted design of the PD programme sessions generally met the design intentions, and where they did not it was possible to remediate some aspects of the design in the subsequent implementation cycle. The effect of the pause in the research lesson was positive and clearly contributed to the practical development of the teaching technique of orchestration. The use of visualisers, initially devised in response to Covid-19 constraints, was highly successful and as such became part of the PD design. The use of Teacher Design Teams within the modified Lesson Study process was an effective combined CPD approach which facilitated high-quality discussion and learning during the pause in the research lesson and the post-lesson discussion in terms of content, focus and purpose.

Chapter 8 – Characteristics of the PD programme

Introduction

In this chapter I describe the positive characteristics of professional learning that were identified by the participants and that emerged in my observations of the PD components. I begin by presenting the views of the participants about the programme, and in particular their views about the suitability of this programme for teachers with differing experiences and backgrounds. I then revisit the framework for analysis elaborated in Chapter 4 which was developed to indicate the characteristics of professional learning evident in the PD programme as designed and also to identify the effective characteristics from the observed outcomes. Next, I address the concept of coherence and why it is important. I consider the practical approaches used to monitor the coherence between the design intentions and the PD programme as enacted. Finally, I return to the barriers described in Chapter 6 to present further discussion on issues of coherence between the PD programme and the participants' professional contexts.

8.1 Participants' views on the characteristics of the PD programme

The participants compared this PD programme favourably with other professional learning programmes. Their comments frequently highlighted the value of working together, sharing and supporting each other and, as already discussed, their enjoyment of focusing on the mathematics rather than administrative activities such as target-setting and moderation. For example, in Cycle 1, Adam explained that he valued talking about mathematics and remarked that often the CPD and department meetings he is involved in take him away from his "core" purpose. He also enjoyed being part of a TDT and listening to the ideas of others.

> Thinking back on a lot of the maths CPD, it's more been delivered at you. You've not really had much opportunity to contribute or listen to more people from the audience or participants. It's just been one person normally at the front talking about whatever the latest thing is, so I really enjoyed it. I'll be honest, […] I like talking about maths and I think it's something that we don't do enough of as practitioners. It was good to have someone such as Dave coming up with

> his ideas because he sees things differently to the way I do. And that's great, because it sparks me off as well. And just that discussion, where we get the chance to sit down as a team and talk about maths, well it doesn't happen as a rule. When we have department meetings, it's always mainly about some administrative thing or some task, which takes us away from the core of what we're about.
>
> [Adam, secondary teacher. Cycle 1 post-PD programme interview]

Paul described how the PD programme differed from his previous CPD, giving the specific example of a Mathematics Mastery training programme, and he highlighted the practical, hands-on experience and the degree of challenge offered by the PD on teaching mathematics through problem-solving.

> A lot of the CPD that I've been on is led by someone talking, [...] halfway through there might be an activity, then you talk a little bit more maybe a group discussion, whereas this was practical hands on, really involved and we were thinking all the time and challenged from the first hour till the last second. And I've never been a part of anything that's been remotely like that. So I can't think of one CPD that I've been part of that that was similar in any way. The Mathematics Mastery training is very, very simple. They take you through a toolkit, they look at examples of planning, they take you through of one of their lessons. And that's pretty much it. Whereas this is this is about learning to adapt to the responses we get from the children [….] It's very much focused around what they're going to do within lesson and not what we're going to do throughout the lesson.
>
> [Paul, primary teacher. Cycle 2 post-PD programme interview]

Similarly, Charlotte indicated that even when her previous CPD experiences were concerned with the teaching of mathematics, the courses had more of a training focus rather than a shared approach to developing teaching and learning. She contrasted those experiences with the current PD programme which she described as having more of an action research focus involving talking about exploring and what the teaching and learning would look like in the classroom.

> I found the programme really helpful in terms of having the opportunity for conversation around mathematics and how we can approach it. A lot of the time, the CPD is […] here's some information, here's the way that you can approach it, go and have a go on your own. So for example in the Mastery training we just get told how to deliver each part of the mastery lesson. Whereas this was more, let's explore something, let's discuss it, let's have that conversation about what that would look practically like in a classroom.
>
> [Charlotte, primary teacher. Cycle 2 post-PD programme interview]

In both cycles the participants indicated that learning about task design and the orchestration technique were essential components in the planning and design of the research lesson. The following comment by Paul reflects a view shared by all of the participants:

> I don't think there's any part of it that would be less important or more important, we opened with theories which led into the development of the problem and then the lesson. We taught the lesson, we started with the theory, we've looked at the problem, we'd looked at every response we could think of… it was almost like we were attached to the lesson, we knew where every part of it.
>
> [Paul, primary teacher. Cycle 2 post-PD programme transcript]

In Cycle 2, Tom explained that the planning time for the research lesson was important and that the inputs from the other PD sessions in the programme supported the planning process.

> I think in order for the lesson to go as well as it felt like it did, we needed to have that planning time in the sessions […] and its funny really because it's not really that big of a task when you think about it. But then when we started talking about […] it was a very big task with lots of interesting things at play […] and so I think all the sessions helped us understand this more.
>
> [Tom, primary teacher. Cycle 2 post-PD programme interview]

Tom commented on the nature of the activities within the programme and remarked that the problem-solving task was good for discussion. He described his experience of working collaboratively to explore the task and the learning that led to a consensus about the mathematics that the problem could be used to teach.

I found the number cell one very interesting because when we were discussing it we seemed to all have some different ways of approaching the problem […] we then all came to the idea there was kind of one way we wanted the children to go, and I think we knew the order as well. And that got us thinking about, well, how do we then make it into more of a problem to solve and what mathematics would they learn? We talked about which numbers we'd put in the cells and whether it was better to have a three, four or five, and then how they worked out the patterns in those. So I think it was a really good task for discussion.

[Tom, primary teacher. Cycle 2 post-PD programme interview]

Tom also suggested that this programme could be used throughout the school and that the components of task design and orchestrating the learning would be central.

I have also thought about how this programme could be used throughout the school although it would be difficult for a school to fit in the amount of time as it's basically going to be all of the CPD time for, say, a month or two, something like that. I think it'd be important that everyone sees that. And also they see the transferable element to it. Whilst it might be focused around one research lesson, it's deeper than that. It's about task design and about orchestrating the learning and I think that's very important.

[Tom, primary teacher. Cycle 2 post-PD programme interview]

The various comments set out above reveal the participants' recent experiences of CPD and the differences between types of professional learning. Despite the shift in recent years from knowledge-based teacher development in favour of a more practice-oriented approach to teacher learning (Schrittesser et al., 2014), for these participants it appears that professional development focused more on training than learning is still very much in use.

8.1.1 Perceived usefulness of the programme for teachers with different levels of experience

Whilst the overall view of the participants was that this PD programme was relevant and appropriate for their own professional development, two of the Cycle 1 participants gave caveats about the usefulness of the programme for teachers who were at different career stages. Ruth, an experienced teacher, felt that the programme was more suited to teachers with experience. In particular, she referred to her accumulated experience of different responses by pupils doing mathematics problems and said that without this experience she would not have learned as much as she did from this PD programme.

> For me I think it is about knowledge. I do not think I would have learnt as much as I have if I was new to teaching but because I have a few years working with individuals. […] I often think where on earth did you get that answer from? […] we have taught children for such a long time and still wonder where has that mistake come from, having that experience knowing that children do think out of the box. Children are not adults, they do not think the same as us, they sometimes do not have our experiences so having that knowledge of how children think about problems has helped me really concentrate on the processes that I want them to go through.
>
> [Ruth, secondary teacher. Cycle 1 post-PD programme interview]

To clarify, I asked Ruth if she thought early career teachers would find this programme more challenging.

> I feel it's, yeah for a new teacher they would find it hard. I look back at me when I was younger and the focus was learning the curriculum, writing lesson plans, there was so much I needed to learn. But knowing and having a bit of experience and knowing what the classroom is like has helped me to adapt to what we have been working on a lot quicker and to think I get that, that helps ... but that is my personal view of where I was as a new teacher. There is so much to learn as a new teacher I think that this would be quite hard…
>
> [Ruth, secondary teacher. Cycle 1 post-PD programme interview]

From Ruth's perspective, the acquisition of several years' experience of observing and learning about pupils' responses to problems, including their errors and misconceptions, is an important prerequisite for developing the teaching technique of orchestrating the learning. By contrast, Marie was a recently qualified teacher and said she felt well placed to benefit from the programme. She imagined that it might be hard for a teacher who had been teaching a long time to adapt to a new approach.

> I think it's very difficult to be, you know, 20 years down the line and be feeling like you're back at the stage where you've got to learn how to how to teach again, which is just a strange thing. And I think for me, and for Dave, I think it was great that we experienced it right at the start, because we had no real first point of call of how we were going to teach forever.
>
> [Marie, secondary teacher. Cycle 1 post-PD programme interview]

Different levels of teacher experience in a PD group is an important (perhaps glaringly obvious) consideration in the design of professional learning programmes. The significance of this issue is most striking in the context of whole-school programmes that are delivered uniformly across a wide range of staff with different levels of experience and expertise, and different beliefs. It is not surprising then that many of these programmes, especially those designed to improve pupil outcomes, do not lead to the anticipated level of improvement or change. Therefore, with regard to the design principles of effective CPD (Desimone, 2009), it would be useful to consider how the features of content focus, active learning, coherence, duration, and collective participation could be aligned to the previous experience of the participants.

8.2 Analysis of the PD programme characteristics

In Chapter 4, I showed how the design of the PD programme was analysed using the triple-lensed framework and Desimone's (2009) specified features. The PD programme components were grouped into three categories to which the lenses were applied to identify the effective CPD characteristics evident in the design of the PD programme. In the subsections below, I characterise the PD programme as enacted. I do this by viewing the programme outcomes through each of the lenses in the analysis framework and Desimone's features.

8.2.1 The PD programme – transmissive or transformative?

Kennedy (2005) poses five questions that are used as tools for the analysis of models of CPD. The five questions are:

1. What types of knowledge acquisition does the CPD support, i.e. procedural or propositional?

2. Is the principal focus on individual or collective development?
3. To what extent is the CPD used as a form of accountability?
4. What capacity does the CPD allow for supporting professional autonomy?
5. Is the fundamental purpose of the CPD to provide a means of transmission or to facilitate transformative practice?

In summary, the answers to these questions in relation to this PD programme suggest that the programme was designed to provide propositional knowledge for the purpose of individual development that would lead to transformative practice. The outcomes described in Chapters 6 and 7, suggest that this PD programme is most closely aligned with the community of practice and action research models. The establishment of the TDT supported their development as a team who were bound by mutual engagement (Wenger, 1999). The participants appreciated having control over the agenda, for example in deciding who would teach the research lesson and when this decision would be made. The ownership of the design of the research lesson, and importantly the modifications made to the second half of the research lesson, also added to the community of practice. In the post-research lesson discussion, the participants were comfortable in discussing their critical reflections about the lesson and as such exhibited some of the characteristics of action research. I suggest that this 'comfortableness' was due to the development of a community of practice that had the features of TDTs and benefited from the existing relationships in the team (recognised in the selection of the participants for the study).

The TDT's analysis of the orchestration sequence highlights the character of the CPD as a means of facilitating transformative practice and was an activity developed to add to the learning ecology of teaching mathematics through problem solving. The analysis took place in both cycles but was more effective in Cycle 2 when the participants were able to review the orchestration sequence on video. My observations of the participants' discussion during this session affirmed the value of this activity.[42] The transcript of the orchestration sequence (Appendix 10) showed that they took ownership of their learning and therefore the activity could be characterised as having a transformative effect. Importantly, and as recognised by Kennedy (2005), a transformative effect may not constitute a definable model in itself but is one that "recognises the range of different conditions required for transformative practice" (p.

[42] It would be interesting to consider the nature of the contribution of this PD component had the orchestration sequence not been deemed a success by the participants.

246). This interpretation aligns with the outcomes of this study: I consider that the development of communities of practice and opportunities for action research are desirable (and maybe even essential) conditions to facilitate transformative practice.

8.2.2 The personal, social and occupational aspects of the PD programme

Bell and Gilbert's model describes three interrelated features of professional learning –the personal, the social and the occupational (Bell & Gilbert, 1996; Clarke & Hollingsworth, 2002). The ***personal*** aspect was evident in the participants' active engagement in the PD programme and the impetus for change expressed in their desire to explore how the learning from the PD programme could be incorporated into their practice. However, this active engagement also revealed barriers that were momentarily visible. For example, Dave in Cycle 1 was uncertain about the value of teaching mathematics through problem solving. Knowing this beforehand would have been useful in the PD design. A problem of course is that Dave may not have known this himself until he was participating in the programme. In Ruth's case, whilst the programme crystallised her understanding of the importance of the problem-solving task and how it can be designed to avoid or trigger misconceptions, her particular view did not fully encompass how a task could be designed to connect and develop mathematical concepts and ideas. Meanwhile, Marie's perspective was more related to teacher efficacy and whilst it is unclear if the view she took was affected by the challenges presented by task design or by the introduction of the specific teaching technique, it did highlight a perceived need for improvements in her subject knowledge.

The ***social*** aspect of the model was evident through the establishment of a TDT and communities of practice. Whilst this learning environment did lead to the mediation of new knowledge within the community (Falk & Dierking, 2000) without additional evidence there was no way of knowing that this would then extend to actual changes in practice. The ***occupational*** aspect of the model was evident in the opportunities for teachers' personal development of mathematical subject knowledge, supported by the PD's links between theory and practice. For example, the participants said they valued engaging in task design, analysing pupils' work, and considering the research papers on multiplication and Diophantine equations. They also provided examples of acquiring new pedagogical knowledge. In addition to their own reflections during the post-study interviews, their new knowledge was strongly evident in Cycle 2 during the discussion of the orchestration sequence.

8.2.3 The sphere of professional learning

Reid's framework focuses on the spheres of action in which professional learning takes place and specifies four quadrants in which the various forms of CPD can be categorised (Frazer et al., 2007). This PD programme was designed to be located in the **'formal and planned'** quadrant and this location was largely borne out by the observations from the study. The exception was the PD session on the participants' observation of the orchestration sequence. Although the event itself was planned, the learning from it was not planned or prescribed in detail because it was dependent on the output of the research lesson and the interactions between the TDT.

8.2.4 The effective features of the PD programme

Desimone's (2009) features of CPD – appropriate duration, active learning and collective participation – were all evident in the PD programme as designed. The programme extended over a period of five weeks[43]. The participants were ***actively learning*** in the process of task design and they ***collectively participated*** in the design and modification of the research lesson. This experience together with the analysis of the orchestration sequence also enabled the participants to engage in ***action research***. The designed tasks together with the associated teaching approaches provided relevant ***content*** and ***focus*** and, as indicated by the participants, resulted in a deepening of their subject and pedagogical knowledge.

With regard to the feature of ***coherence*** **(**Desimone et al., 2002), the PD programme was designed so that the learning from PD sessions would support the approach of teaching mathematics through problem solving and that this would be evidenced in the planning and teaching of the research lesson. I have already explained an issue that arose in Cycle 1 due to the participants regarding the principles of task design and the orchestration technique as separate and independent components within the programme. However, modifications to the PD sessions in Cycle 2 to address this issue appeared to create a better understanding of the relationship between the principles of task design and the teaching technique of orchestrating

[43] In Cycle 2 this was longer due to the addition of the separate PD session on the analysis of the orchestration session.

the learning. This would indicate that the programme in Cycle 2 had more coherence than in the first cycle. I recognise the speculative nature of this claim due to the number of degrees of freedom that are present within the analysis, not least the fact that there was a different set of participants, with different knowledge and experience, in each cycle. Nevertheless, the development does point to the potential benefits of the cyclic nature of design research.

8.3 The importance of coherence

During the course of reviewing and consulting the literature for this study I have encountered the vast array of terms used to define and describe effective CPD, an array which Coffield (2000) argues has led to 'conceptual vagueness' in the discourse about professional development. The term 'coherence' is another which could be regarded as conceptually vague. Fullan and Quinn (2015) state that coherence is *not* about structure, strategy or alignment (although these aspects can help) and is more about the "integration of diverse elements, relationships, or values", as defined by Merriam-Webster (2002). Similarly, Desimone et al. (2002)[44] suggest that coherence is a characteristic of effective PD which incorporates experiences that are consistent with the teachers' goals for professional development. This is a broad definition that refers to the relationship between the programme's objectives and the participants' goals. Even then, the definition potentially has a number of hidden inferences and tautologies. For example, consider the question 'If a teacher attends a PD programme whose components align with their goals such as better outcomes for pupils, yet the specific content within the programme does not lead to this realisation for the teacher, could the programme be described as coherent?'. It is therefore important to ensure that the things to be cohered are directly relatable and that the ambition of each can be clearly defined.

Therefore, whilst I support the view espoused in the literature that coherence is a prerequisite of all effective PD programmes, I also argue that its meaning can only be realised by the clear identification of the components that are to be cohered with the design intentions of the programme. In the subsections below I discuss two aspects of coherence: the coherence between the PD programme and the design intentions; and the coherence between the PD

[44] In proper acknowledgement to Desimone, this definition of coherence is one of five key features of professional development and is complemented by other characteristics that require different aspects of coherence such as *collective participation* and *content focus*.

programme and the participants' classroom environments. The focus on these two aspects resonates with the views of Penuel et al. (2007) who found that teachers' perceptions of the coherence of the PD programmes were significantly related to the degree to which the PD programme components promoted the implementation of the targeted knowledge and skills.

8.3.1 Coherence between the PD programme and the design intentions

In Chapter 7, I indicated that the PD programme components overall supported the design intentions and as such gave a perspective on the coherence between the PD programme and the PD design. However, as discussed earlier, I identified a specific lack of coherence between the enacted PD programme and the design intentions concerning the relationship between the principles of task design and the technique of orchestrating the learning. This relates to phenomena known as design by intention and design by implementation (Swan & Swain, 2010). In the design of the PD sessions, I made assumption that the participants would see the progression between the sessions whilst maintaining a clear understanding of the overall purpose of the programme. In several cases the data did not support this assumption. Examples have already been given of participants not interpreting the activities within the PD programme as intended. It is therefore important to stay as 'close' as possible to the participants both in the design intentions and in the implementation of the PD programme. As Swan and Swain remark, the researcher's role needs to develop in order to accommodate 'mutations' of the design in the hands of the teachers. This could suggest that the coherence between the design intentions and the PD programme as implemented is partly dependent on the coherence between the PD components themselves.[45]

8.3.2 The role of formative tools to assess coherence

Ensuring coherence between the components of a PD programme requires continued attention to both the design of the programme and the focus of the intended learning in relation to the aims of the programme. In this study, pre-PD questionnaires were used to identify barriers that could be resolved by adjusting the programme activities without changing its principal ambitions. Formative questionnaires were also used in the PD sessions to develop the programme between research cycles. Further monitoring methods could have

[45] Even if all of the PD components can be shown to be coherent, this does not necessarily mean that the PD programme will be coherent with the design intentions.

been incorporated into both the maintenance and delivery of the programme. For example, in some PD programmes, technology such as 'Slido' (an audience interaction tool) is used to capture and collate formative data and instant feedback. However, as evidenced in this study, the use of such tools, for example the Johari Window[46] in Cycle 1, is not guaranteed to enhance the effectiveness of the programme.

Just as in learning situations for children, ongoing auditing processes are useful in longitudinal programmes of adult professional learning to monitor the environment for misconceptions and errors that might lead to misalignment between the intended and observable outcomes. Formative evaluations of PD programmes therefore should include mechanisms to identify these moments producing misconceptions and errors. These will inform decisions on how to modify and/or augment upcoming components of the programme to ensure that the content focus and active learning remains in alignment with the expectations of the participants and the ambition of the programme. Whilst monitoring can be relatively straightforward, particularly with the use of technology, the modification of the programme components is not always easy, especially in programmes that contain new approaches which may challenge teacher beliefs.

A problem with formative tools is that they are visible within the programme and are potentially disruptive to the delivery of the programme. Also, they may be unreliable if the relationships between the participants and the facilitators are insecure. Importantly though, regardless of the approach, obtaining formative data is useful only if it can be used to respond positively and constructively to the issues raised and without affecting the principal objectives of the programme. For example, in my experience of the National Numeracy Strategy, consultants delivered five-day training programmes typically to several groups of headteachers over consecutive weeks. As a result of reading the formative and summative evaluation data in these programmes, consultants were often left concerned and anxious at the prospect of having to deliver further programmes. Furthermore, because the programme was highly prescriptive, the consultants remained unsettled as they felt unable to modify the content of the programme in the light of evaluations.

An alternative solution may be to develop PD programmes that enable opportunities for the participants' beliefs and views to move in and out of alignment with the central objectives of

[46] The Johari Window was also used as a data collection method.

the programme without destabilising the relationship between the participant's belief system and the ambition of the programme. This would enable more interactivity between the learning ecology and the participants. As an example, this programme could have been designed so that first the participants are presented with a number of conceptual approaches to teaching mathematics through problem solving. They could then use these to develop their own theoretical framework for teaching mathematics through problem solving. PD programmes with such a characteristic would exhibit the property that I define as ***dynamic* coherence**. I consider this learning to be an important outcome from this study: the principle of dynamic coherence could be developed further to contribute to the known characteristics of effective professional learning programmes.

8.3.3 Coherence between the PD programme and the participants' context

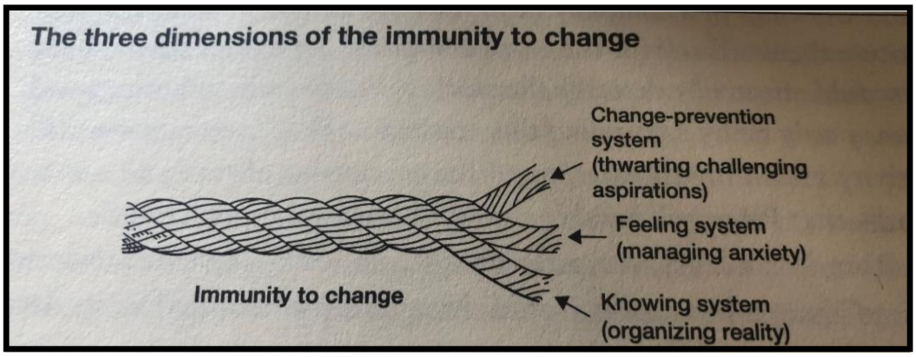
In Chapter 6, I identified a number of barriers to change in teachers' practice that I had not fully anticipated and which became momentarily visible during the programme. Whilst teachers' time constraints in school were an expected barrier and one that is often a factor in teacher change, the pupil-related factors perhaps also should have been anticipated and addressed in the programme design. Therefore, more attention could be given to the content focus (Desimone, 2009) of the PD sessions, specifically the skills required of pupils to explore the problem at hand, and the potential impact of pupils' attitudes towards problem solving. This may have helped to reduce or eliminate some of the pupil-related barriers raised.

It is likely that the barriers which became visible during the PD programme had already been formed in the participants' daily professional context. As Pedder et al. (2008) suggest, the barriers presented by teachers can indicate a lack of coherence between the programme intentions and the participants' current classroom practice (Desimone et al., 2002; Pedder et al., 2008). Therefore, in order to design a PD programme that could address these barriers it would have to cohere with the participants' daily classroom practice. This would be a challenging task given the contextual pressures operating on teachers' practice.

Pedder et al. (2008) observed that schools rarely evaluate CPD programmes in relation to the school context and, therefore, commonly the PD ambitions do not cohere with practice and the programmes do not result in substantive or permanent change. In other words, the PD programmes are unable to support teachers to transfer and develop the transient or immediate changes that they evidence in the PD programme into longer-term ‘occupancy’ in the teachers’ practice. In this context I define occupancy as an observed effect where it is clear that the teacher’s change or development is embedded and sustained. Whilst the programme in this study was designed to influence views about teaching mathematics through problem solving, I was aware at the design stage that it would be uncertain whether the programme would result in significant and permanent changes in participants’ practice. The programme introduced the participants to a practice that required the development of new skills which would need developing over time. To do this, the participants would either have to elect to continue to develop the practice in their own teaching and/or have access to further professional development of this type. Both of these routes present challenges particularly within the current national context outlined in the first chapter.

The first challenge is that we know that teachers often are uneasy about adopting new methods without significant reassurance or evidence that their endeavours are worthwhile (Buehl & Beck, 2015) and therefore have a natural immunity to change. A useful tool for discussion of this longstanding issue in education is the immunity to change model devised by Kegan and Lahey (2009). The entwined nature of the three dimensions of this immunity to change is neatly explained in the diagrammatic representation below (Figure 8.1).

Figure 8.1 *The three dimensions of immunity to change (Kegan & Lahey, 2009)*



In the model, the first strand relates to the inhibitors that currently exist within the system as part of the teacher’s everyday practice. In this study, these were barriers such as finding suitable tasks, pupils’ attitudes and skills, teachers’ subject knowledge, and teachers’ opportunities to practise and develop. The second strand represents the feelings experienced by the teacher whilst considering change, with the knowledge of the barriers that exist. The most common feelings expressed by participants in this study related to uncertainty about the ultimate value of the approach together with their degree of confidence in being able to teach mathematics through problem solving effectively. This 'feeling system' interconnects with the third strand which is the pressure to maintain that which we know. As Adam indicated in this study, the new approach would not be considered as an effective way of 'catching up' pupils whose progress had been affected by the pandemic. The model makes clear how these three strands can intertwine to produce a resistance to change that is powerful and difficult to undo.

A second key challenge is that although research-led programmes such as Lesson Study and Collaborative Lesson Research[47] clearly have the potential to be powerful tools in the development of professional learning for teachers of mathematics, we know that many schools are unable to sustain programmes of this nature. This scenario resonates with research on the methods of identifying and developing longitudinal professional development programmes for teachers of mathematics. Hawley and Valli (1999) place strong emphasis on teachers' *self-identified* needs as the basis for professional learning. They argue that teachers should not only be involved in the identification of their learning needs but should also contribute where possible to designing the ways in which their needs might be addressed. From this discourse, it would be reasonable to suggest that the effectiveness of this PD programme could be enhanced by the inclusion of events that support precise identification of the participants' learning needs. These could then be addressed either during the existing programme or as part of a subsequent programme.

A third challenge to coherence relates to policies and guidance from external sources. In the U.S. context, Garet et al. (2001) point to the need for CPD to cohere with state and district standards and assessments that are promoted through mechanisms such as textbooks, published schemes, state policies, and professional literature. This would seem to be a reasonable ambition. However, for this ambition to be successful there must first be some coherence between the policy development for teaching mathematics and the framework for educating and training teachers on the teaching principles defined in such policy. For UK schools this presents an immediate challenge for the development of a framework for longitudinal professional learning with a coherent focus.

[47] Collaborative Lesson Research-UK is a registered charity run by educators for educators. Its purpose is to support activity that many might call Lesson Study. https://www.collaborative-lesson-research.uk/

8.4 Summary

In this chapter, I have discussed the participants' views of the PD programme including favourable comparisons with characteristics of previous PD they had experienced. They raised questions of the programme's suitability for teachers with different levels of experience which revealed an interesting polarity in views that might be related to the age and experience of the participants. Using the framework for analysis developed for the study, I have identified a range of characteristics of effective professional learning that were incorporated into the design of the programme and others that were observed during the delivery of the programme. These outcomes, together with those identified in the previous two chapters, point to additional considerations that are important in the design of professional development programmes for teachers:

- issues relating to design by intention and design by implementation
- being able to respond to hidden barriers that become momentarily visible.

Finally, in this chapter I have considered the issue of coherence from a number of perspectives. I have argued that the coherence between the PD programme and the participants' classroom context is a key factor in the potential of the PD to convert transient change into occupancy within teachers' classroom practice.

Chapter 9 – Conclusion

Introduction

I began this thesis by stating that this study was not seeking to demonstrate that teaching mathematics through problem solving was the most effective way to teach mathematics nor that the teaching approaches presented in the programme were the only ones that could be used to teach mathematics through problem solving. In Chapter 1, I described the national context in which this study took place and I was aware that the PD programme might not lead to the participants' wholesale adoption of teaching mathematics through problem solving as defined.

However, within this context, I was confident that my theory of learning developed from the principles of task design in conjunction with the teaching technique of orchestrating the learning had the potential to provide an effective learning opportunity for the teaching of mathematics through problem solving. I was also excited by the potential impact of the innovations of the use of visualisers and the pause in the research lesson. Therefore, the research questions were designed to reflect the programme's ambitions and have been developed and refined throughout the course of the study.

In this concluding chapter, I revisit the research questions to summarise the main outcomes from the study documented in the preceding chapters. For each of the research questions, I consider the extent to which the data has informed the answer to the question and I summarise my learning and the knowledge produced. I then discuss the main implications of this research and the limitations of the study. Finally, I consider the future opportunities for research in this area.

9.1 Research outcomes

Zagzebski (2017) proposes that knowledge is "a cognitive contact with reality arising out of acts of intellectual virtue" (p. 109). In this study, I have contributed to the literature and confirmed knowledge that is already known and which has now been validated in the participants' context (reality). It is important to state that this confirmatory knowledge is often partial so care must be taken to ensure that the context and conditions in which confirmatory knowledge is obtained can be recognised and validated in the existing knowledge domain. I have done this by establishing this knowledge in the context of the participants' professional learning and workplace environment using the principles of task design (Kieran et al., 2015) and through the proposed approach to teaching mathematics through problem solving. In the following subsections I summarise the learning relating to each of the research questions.

9.1.1 Research Question 1

How does the use of task design in conjunction with the teaching technique of 'orchestrating the learning' support teacher learning in the teaching of mathematics through problem solving?

In Chapter 6, I showed that the effect of using task design in this PD programme confirms what is already known: that it can improve teachers' subject and pedagogical knowledge. The addition of the teaching technique of orchestration, which involves planning the sequencing of anticipated pupil responses, appeared to extend this knowledge. The participants indicated that they had developed their understanding of the mathematics that could be learned from the task and the types of questions and teacher connectives that would be effective in developing the pupils' understanding.

There was no evidence that associating task design with the orchestration technique lessened the known positive effects of task design. Indeed, the data from the participants indicated that using the orchestration technique in conjunction with the design and analysis of mathematical tasks helped the participants develop their ability to think more effectively about the range of anticipated pupil responses and how they would interact with those responses 'in the moment' in the classroom. This is a feature of teacher practice that was defined as contingency. I have therefore concluded that:

- As evidenced in the literature, engaging teachers in the process of task design can develop their subject and pedagogical knowledge.
- The experience of the task design process alongside the teaching approach of orchestrating anticipated responses enabled the teachers to understand more about what pupils actually do when confronted with a mathematical task. Importantly, the

use of the visualisers enabled the teachers to analyse the work of the pupils in detail and so made an important contribution to developing their understanding.

- The participants demonstrated an increased confidence in planning for and anticipating responses and the teachers' comments on the outcomes of the research lessons indicated that the use of task design together with the development of the orchestration were effective components in the PD programme designed to support teachers in their development of teaching mathematics through problem solving.

9.1.2 Research Question 2

How does the introduction of a 'pause' in the research lesson support the development of 'orchestrating the learning' as a method of teaching mathematics through problem solving?

Kazemi et al. (2016) explain how rehearsal before enactment can lead to beginning teachers developing a better understanding of complex teaching methods. From the evidence presented in Chapter 7, it was clear that the pause in the research lesson provided the participants with an 'in context' training opportunity to rehearse in specific circumstances (Guzman, 2009), an opportunity which the participants found highly valuable. Therefore, the use of the pause in the research lesson was an effective tool in supporting the development of the orchestration sequence.

The participants recognised the challenge of orchestrating pupil responses to produce the intended learning. This challenge revealed a complexity that may not have been fully considered in the PD programme as there are frequently several effective ways in which the orchestration can be planned to deliver the intended outcomes of the lesson. Nevertheless, the event contributed to the design intentions of the programme and created a professional learning environment to develop the skills of sequencing and orchestrating anticipated responses. The use of visualisers supported this pause activity very effectively by enabling the participants to examine the pupils' workings in real time and in a level of detail that the teachers had not previously experienced. The quality of the information obtained during the pause also effectively supported the post-research lesson discussion.

The observed characteristics of this PD session also aligned with several of the known features of effective professional development, in that the session provided the opportunity for the participants to work collaboratively and develop their learning in a professional and relevant context.

In summary, I conclude in response to Research Question 2:

- The innovation of the pause in the research lesson was an effective design component that supported the learning of the orchestration teaching technique and contributed to the coherence between the PD programme components.
- The use of visualisers was a valuable addition to the PD programme and specifically contributed to the quality of the participants' discussion and learning during the pause and the post-lesson discussion. However, it is important to acknowledge that whilst the use of the visualisers enabled a detailed level of focus on the pupils' work, this level of observation is often achieved in Japanese research lessons without the use of this technology.
- The characteristics of the pause in the research lesson aligned with the features of effective professional learning.

9.1.3 Research Question 3

How do the designed features of the PD programme 'teaching mathematics through problem solving' contribute to knowledge about professional learning programmes and environments for teachers of mathematics?

The contribution of the designed components of the PD programme have already been discussed above. With regard to the CPD approaches of Lesson Study and Teacher Design Teams, I have shown that it is important to analyse their purpose and to establish the circumstances in which they can operate effectively. The adaptations made to Lesson Study and the application of the characteristics of Teacher Design Teams both added to the corpus of practical knowledge on developing professional learning environments for teachers.

I have also indicated how the use of formative evaluations during longitudinal PD programmes can strengthen the coherence between the different components of the programme and the coherence between the programme and the design intentions, but that there are important implications to consider when using such evaluation tools. I have suggested that it may be possible to promote greater coherence by developing PD programmes that can accommodate the different belief systems and attributes of teachers. I have defined this property as dynamic coherence.

In Chapter 8, the participants offered conflicting views about the suitability of this programme for teachers with different levels of experience. On the one hand it was thought that beginning teachers would need to have previous experience of how children approached mathematical problems and knowledge of the different types of responses they would give. An alternative view suggested that this PD programme would be challenging for experienced teachers who were not used to teaching in this way and had developed practices using more 'traditional' approaches. Clearly both views have merit in that they are linked to the belief systems of the participants and therefore highlight the importance of coherence between the PD programme and the participants' beliefs and contexts.

In summary, I conclude that:

- Whilst the PD programme was well received and the PD components were recognised as having a positive effect, it is important to acknowledge that the features of this programme were designed around many of the key principles of Lesson Study. As such it should be recognised that positive effects could be due to the professional learning environment of Lesson Study in addition to the contribution of the PD components.
- The use of PD mechanisms and processes should be analysed and designed in the context of the ambition of the PD programme.
- The issue of coherence between PD components could be explored further by considering the design of programmes that are less dependent on its components and enable the participant's beliefs and views to move in and out of alignment with the central objectives of the programme, a characteristic I have defined as dynamic coherence.

9.2 Implications of the study

The combination of 'knowing how to' and 'knowing why' is sometimes termed practical knowledge. The notion is more precisely characterised by Bereiter (2014) as "explanatorily coherent practical knowledge" (p. 5). Practical knowledge provides explanation, but unlike formal theoretical knowledge, its main purpose is not to hypothesise or predict but to provide reason-based practical guidance. In the following subsections I explicate the practical knowledge obtained from this study by discussing the implications for the continued development of teaching mathematics through problem solving, the future use of Lesson Study and its variants, and the challenge of introducing and embedding the teaching of mathematics through problem solving in the current UK educational and professional learning context.

9.2.1 Teaching mathematics through problem solving

The first implication from this research concerns the future development of teaching mathematics through problem solving and is relevant to professionals involved in curriculum design. In this study, the observations of the orchestration teaching sequence and the views of the participants in the post-lesson discussion support the position in the literature that teaching mathematics through problem solving is an effective way for children to learn mathematics. However, whilst the study's outcomes confirm the merit of this particular teaching approach, it was also clear that the participants were unlikely to incorporate the approach substantively into their practice.

It is my view that whilst there have been great innovations and developments in mathematics curriculum resources and associated teaching approaches, these have yet to produce a coherent framework within which the body of professional knowledge for mathematics teaching can develop, rationalise and converge. The participants in this study were all using a particular published mathematics scheme which has many references to problem solving and contains several problems that could be utilised to teach mathematics through problem solving. There are also other published schemes and curriculum materials that incorporate the concept of problem solving but do not extend to the use of specific problems to teach mathematical content in the way described in this study.

Trainee teachers enter the world of teacher education often with many ideas about how to teach mathematics (Feiman-Nemser et al., 1989). As children, and as students of the subject themselves, they have developed their own webs of interconnected ideas about mathematics, about teaching and learning mathematics, and about schools (Ball, 1988). My own experience is that beginning teachers typically have firm or strong views but often become muddled as they go on to learn approaches to teaching that they did not themselves experience as children. Therefore, in the absence of high-quality early career development, these teachers form an over-simplified view of teaching and often default to teaching approaches they experienced when they were learners[48]. Equally, experienced teachers who have passed through a corridor of continual change and uncertainty may find it difficult to switch to a new approach from their most recent and established practice. As Adam, a participant in this study, said, “Again, it's all about that reluctance to do anything that's out of the comfort zone and it’s as much for the teachers as for the students themselves”.

It is therefore not surprising that influencers within the education marketplace can successfully promote a variety of resources and associated teaching approaches. In my view, what is required is this: the designers of schemes and resources who espouse different methodologies need to operate within frameworks that are research based and where different approaches can converge. For example, some mathematics teachers are currently incorporating the Rosenshine principles of instruction into their lessons (Rosenshine, 2012). The 10 principles were developed from three research sources and, whilst they are mostly credible[49], in some cases they have been reduced to a teaching sequence known as ‘I do, You do, We do’. An understanding of Rosenshine’s principles as intended could converge with the principles of teaching mathematics through problem solving by changing the sequence to ‘You do’, ‘We do’, and then ‘I do’. However, just as in task design, it is important to understand the designer’s intention when adopting generic teaching approaches into one’s practice. Also, this type of convergence would require extensive collaboration between curriculum designers and the results would have to align with current national and local policy ambitions.

There is also the need for alignment between teaching *for* problem solving and teaching *through* problem solving (discussed in Chapter 2). In order to learn mathematics through problem solving, the problems must be accessible to the learner and therefore the learners must possess the relevant problem-solving skills. At the moment, it is unclear if and how these skills should be taught. For example, in Chapter 2, I set out a framework of problem-solving strategies, and within each strategy resides a number of problem-solving skills that children should learn. Accepting this framework would require teachers to work with tasks and approaches to teaching that would enable the pupils to develop these skills. In this study,

[48] A notable HMI, Arthur Owen, once told me that he believed that around 90% of newly qualified teachers of mathematics revert to methods they were taught as children within the first two years of their teaching career.
[49] The second of the 10 instructional principles, ‘Present new material in small steps with student practice after each step’, has recently been deleted from the NCTEM guidance on teaching for mastery.

this approach was defined as 'teaching for problem solving' and is one where the teaching approach is focused on the development of the problem solving rather than the solution to the problem or the mathematics that can be learned from exploring it.

As also discussed in Chapter 2, teachers could use problems to introduce pupils to domain-general strategies such as 'be systematic' and 'try specific examples' (not included in the framework in this study). However, Foster (2023) suggests that these general strategies have not been effective and are difficult to teach, so he advocates "the explicit teaching of domain-specific problem-solving tactics that are applicable over narrow ranges of mathematical content". (p. 3). This approach seeks to enable all pupils to become powerful problem solvers; if successful, it would equip such problem solvers with the strategies and skills specified in the framework in this study. Thus, there are multiple ways in which pupils can learn the problem-solving skills required to access and solve problems. Both approaches seek to promote the development of problem-solving skills but clearly each has its own resource requirements and each places different cognitive demands on the teacher.

In summary then, this research suggests that for teaching mathematics through problem solving to become more widespread in the practice of teachers, there would need to be developments in published schemes of work and resources and in further professional learning for teachers. These advancements would need to involve the development and design of tasks associated with mathematical content, and training for teachers on how such tasks would be used to teach mathematics through problem solving.

9.2.2 Professional learning for teachers of mathematics in the UK

It is now widely accepted that professional learning which engages teachers in research or enables them to act as researchers is a positive characteristic of CPD (Joubert et al., 2010; Sims et al., 2021). This approach is often described as evidence-based practice where teachers explore reliable research from outside their setting or use research methods to explore their own practice, or both (Williams & Coles, 2007). At the same time, we know that the PD landscape for teachers of mathematics still contains the range of models in the framework described by Kennedy (2005) including those which are not necessarily conducive to promoting exploratory evidence-based practice. On one side of the range is the 'transmissive' model, such as used when introducing a new approach through a commercial scheme or programme on an aspect of teaching mathematics such as mastery or proportional (and now additive) reasoning. On the other side is the ‘transformative’ model such as the approaches in Lesson Study and communities of practice which are established to explore aspects of teaching or solutions to identified areas for improvement. The development of Kennedy’s framework has also led to the emergence of a ‘transmission/transformative dualism’ (Kennedy, 2014), with schools providing models of practice such as Rosenshine’s (2012) principles of instruction or the Frayer Model (Frayer et al., 1969)[50] for exploration or adoption through research.

Given the current PD landscape in the UK where individual teacher autonomy has been eroded by a neoliberal political agenda (Adams & Povey, 2018), how can developments in mathematics teaching such as teaching through problem solving be introduced and sustained in the professional learning of teachers? In their review of the characteristics of teacher development that increase pupil performance, the Education Endowment Foundation (Sims et al., 2021) hypothesise that for PD to be effective it has to incorporate a set of mechanisms that are able to achieve four purposes, and they describe how the PD can fail if one or more of these purposes is not achieved. The four purposes are:

- to instil new evidence-based insights
- to motivate goal-directed behaviour
- to develop different techniques that teachers use to put these insights to work
- to help embed this new practice.

The PD programme in this study provided the participants with evidence-based insights on the effectiveness of teaching mathematics problem solving. Resources and training on putting these insights to work were designed to motivate the participants to consider this new approach to teaching mathematics. Opportunities for participants to develop their practice were built into the programme. However, as stated earlier, it was apparent that the programme alone could not achieve the purpose of embedding this approach into the teachers’ practice. In my view, this obstacle was largely due to the lack of coherence between the purpose of the programme and the participants’ classroom contexts.

A recurring observation in this study was that teachers’ time constraints in their school contexts were a significant barrier for the participants and that this lack of time prevented

[50] The Frayer model was originally designed as a strategy to support concept mastery and is now used by some schools to support the development of vocabulary in subjects.

them from exploring and developing tasks to teach the mathematics through problem solving. It was also evident that the participants found themselves challenged by the proposed teaching approach and so it is likely that time for more professional learning and training opportunities would be required. I therefore contend that for this PD programme to cohere with the participants' professional contexts they would need to have access to and familiarity with a set of appropriate resources (tasks) – or mathematics schemes that already contained high quality resources – that are linked to learning outcomes. In addition, their settings would need to reflect a commitment to professional learning and have a leadership vision that supported the development of teaching mathematics through problem solving. Only then would PD programmes of the type used in this study and others (such as Lesson Study, Collaborative Lesson Research (CLR) and the Content Focused Coaching programme (Gibbons & Cobb, 2016)) have the potential to successfully embed the teaching of mathematics through problem solving into the practice of teachers. This issue returns us full circle to the first implication of the study and suggests that unless the participants' settings can be developed as described above, such PD programmes will fail to cohere with the participants' classroom contexts. Therefore, I argue that a prerequisite for the implementation of programmes of this nature should be a pre-programme audit of the teachers' context, the results of which would inform the design of the PD programme.

9.2.3 The use of Lesson Study as a professional learning tool in the UK

The final implication from this research concerns the CPD approaches used in professional learning, which the EEF refers to as 'forms' and cites Lesson Study as an example. An outcome from this research is that Lesson Study and its variants can make a positive contribution to the professional learning environments designed to improve classroom teaching. However, despite the plethora of books and research papers on Lesson Study and its various components (Doig & Groves, 2011; Hart et al., 2011; Lewis et al., 2009; Ono & Ferreira, 2010; Hiebert et al., 1999; White & Lim, 2008), it has been argued that some aspects of Lesson Study still seem not to be well understood outside of Japan (Fujii, 2014). If this assertion is valid then it would be sensible to consider a different perspective on how Lesson Study might be used. Commentators on Lesson Study have analysed its key features in order to present it as a model for professional learning which can be implemented in other settings. For example, Lewis and Hurd's (2011) description of each step in Lesson Study is summarised as follows:

- Goal setting: identifying the gap between the long-term goals and current reality.
- Lesson planning: teachers collaboratively plan a research lesson.
- Research lesson: one team member teaches the research lesson.
- Post-lesson discussion: observers share data from the lesson in a formal post-lesson discussion.
- Reflection: a report is written on the original research lesson proposal, the student data from the research lesson, and reflections on what was learned.

Those who ‘know’ Lesson Study will be conscious of the brevity of the above statements and therefore the potential for misinterpretation without careful consideration of the detail that accompanies these statements. There have been extensive efforts and research into developing an understanding of Lesson Study and the purpose of each component in order to replicate the process – on the premise that the model must be adopted authentically if it is to work effectively. This study's findings support the view that there are other ways in which Lesson Study, or research-led variants of it, such as Collaborative Lesson Research (CLR), can be successfully modified to create conditions for effective professional learning. In this study it was the exploration of the research questions that led to the use of Lesson Study components rather than the development of task design and orchestrating the learning using Lesson Study. This then led to the modification of certain Lesson Study components:

- the use of a discussion paper to replace the analysis of the scheme of work (*kyozaikenkyu*)
- the pause in the research lesson and the use of visualisers to support the analysis of the pupils’ work
- the focus of the post-lesson discussion on the second half of the research lesson.

Rather than trying to fully replicate and implement the Lesson Study model (i.e., faithful adoption), I suggest that professional learning programmes could be built from components of Lesson Study that incorporate the principles of design research, in the first instance. By this I mean that the identification of potential solutions to long-term goals (which is the first step in Japanese Lesson Study) is supported by first of all devising a research proposal for the PD design which could then be used to define the ‘type’ of Lesson Study to be used. This may stem the erosion of professional autonomy that in my experience has led to challenges in sustaining Lesson Study in the context of UK schools.

9.3 Limitations of the study

Like all postgraduate research, this study is limited in scope. In addition, I note specific limitations concerning: the participants in the study and the fact that the data produced is entirely based on the participants' self-report of their experiences; the use of design research within the context of the participants; and the potential issue of misalignment between the data and the research questions, deriving from the use of conjecture mappings with coding.

The deliberately selected participants in this research should not be considered a representative sample of teachers as they had more experience than the typical practitioner with regard to this type of professional learning and the focus on problem-solving approaches in mathematics. This influenced the design of the PD programme in that I was able to incorporate the components of Lesson Study more easily and begin with a more advanced introduction to the teaching of mathematics through problem solving. This factor could have influenced the data that was collected and so introduce bias to the findings: in other words, the positive outcomes observed could have been influenced by the starting points of the participants as well as their experience of the programme. Having said this, the issue highlights the importance of ensuring coherence between the PD programme and the participants themselves. If the participants had been new to Lesson Study and teaching mathematics through problem solving, I would have devised a slightly different programme.

Also, as indicated earlier, the participants had a unique relationship with the researcher. As the researcher is a figure of authority within their school setting and a presumed 'expert' in this area of mathematics professional learning, there was a potential risk of an imbalance in the power dynamic within the relationship that could result in spurious outcomes. As a result, the participants could have felt unable to give their true views on the value of teaching mathematics through problem solving. For example, in order to avoid committing to the teaching approach, they could have identified other secondary barriers which although important were not the primary reasons for their reluctance to adopt this approach. I understand that this would have been less likely if the participants were not known to me and had not been part of previous Lesson Study programmes led by me.

I also recognise the potential impact of the different positions I took during the study. At different times I adopted three out of the four roles defined by Buford Junker's typology of researcher roles; moving from one role to another could blur the objectivity of each role. For example, moving from the role of observer as participant in the PD programme to the role of complete observer in the data collection and analysis could have influenced the coding processes resulting in bias in the data.

The methodology of design research is often situated in domain-specific learning processes that produce a theoretical framework which is accountable to the activity of design (Cobb et al., 2003). In this study, the domain was the professional learning of teachers of mathematics and therefore the context of the learners could impact on the designed innovation. For example, the use of the pause in the research lesson to support the development of the teaching technique of orchestration could be considered a useful addition to research lessons with this focus. However, in order for teachers to develop competence in orchestration, many more training opportunities might be required which may not be readily available in the same way due to the magnitude and intensity of the resource required to develop this competence. Therefore, unless the participant is committed to developing this practice in the 'normal' environment, or a strategy can be devised to incorporate the pause into the normal environment, then the positive outputs from the PD programme may be lost.

The study comprised two research cycles with participants from different phases of education. As discussed in Chapter 5, a design modification was made to the PD programme sessions in Cycle 2. The modification was made because of the apparent lack of explicit connection drawn between the principles of task design and the teaching technique of orchestration. As the modified content was presented to teachers with different mathematical and teaching experiences, I recognised that it would not be possible to evaluate whether the different outcome was a result of the modification. However, I do consider the modification to be an appropriate development within the PD programme irrespective of the background experiences of the teachers involved.

I used the processes of conjecture mapping and coding to connect the data to the research questions. This was not a controlled mechanical process and it is possible that my approach could have incorrectly aligned some of the data with the relevant research question. However, I am aware that the literature on qualitative research suggests that the believability of the findings should be prioritised over their detachability from their empirical context (Bochner, 2018). Further, whilst I recognise that the identification of inductive codes could be considered as arbitrary, the purpose of generating the inductive codes was to bring together the relevant deductive codes into themes for analysis.

9.4 Future research possibilities

In this study, conjecture mappings were used to develop a framework for analysis by defining high-level conjectures that when embodied in a PD design would lead to specific mediating processes which would in turn support desirable outcomes. In the last chapter, I introduced the idea of ***dynamic coherence*** which builds on the Merriam-Webster definition of coherence: “the integration of diverse elements, relationships, or values”. Although I have not developed the concept any further in this study, I believe that dynamic coherence in professional learning programmes could be explored in future studies. This could be done via conjecture mappings that include mediating processes to allow programme participants to successfully move in and out of alignment with the central objectives of the programme without destabilising the relationships between the participants’ expectations of the PD programme and its design intentions. In other words, the coherence between the programme and participants would be dynamic not static.

As discussed in Chapter 2, discourse is at the centre of teaching and learning (Xu & Mesiti, 2022). There are potential benefits of pedagogical models for organising classroom discussions but these models are still in need of further development. Research shows that the orchestration of whole-class discussions is one of the key challenges facing teachers – this is especially the case where the mathematics task in focus is cognitively challenging and open (Ball, 1993; Ni et al., 2014; Stein et al., 2008). As Xu and Mesiti (2022) suggest, teachers who view mathematics as a coherent body of knowledge are more likely to orchestrate the classroom discussion so that the mathematics is presented as a connected domain. Therefore, in the PD programme an alternative introduction to the technique of orchestration could have been to first develop a network of anticipated responses connected to mathematical concepts, thereby giving flexibility to the planning of the sequencing process. This could also provide other opportunities for the participants to develop ‘contingency’ which, as discussed in Chapter 2, refers to the teacher’s ability to respond in the moment to the ideas and approaches presented by the pupils (Rowland et al., 2005). I also consider that the continued exploration of the orchestration technique as presented in this study, with further development, could contribute to the evolution of practice which recognises the importance of seeing mathematics as a coherent and interconnected set of concepts.

Problem solving and wizardry

Finally, I would suggest that the majority of professional development programmes of this nature contribute positively in some way to the professional learning of teachers. As in this programme, some of that learning translates to developments in practice. Whilst it was never my expectation that all of the participants would adopt the approach of teaching mathematics through problem solving as a regular feature of their practice, I had hoped that the programme would be successful in demonstrating the value of teaching mathematics through problem solving. I had hoped that this success would have been evidenced by some level of commitment from the participants to incorporate this approach or aspects of it into their practice. Whilst this programme was clearly of value and the designed features have been shown to contribute to the professional learning of teachers, other questions have galloped into view – questions which pertain more to the context in which the professional learning takes place, in addition to its nature and quality. For example, one question that has been forming during my journey through this study is:

Given the relatively well developed research infrastructure in the UK, ***how can*** *PD programmes of this nature successfully reside in this space and influence the teaching and learning of mathematics?*

Alchemists of the past were preoccupied with the possibility of turning one substance into another, with particular interest in the transformation of lead into gold. However, it is well known that you cannot make a silk purse from a sow's ear – or so I thought! In 1921, Arthur D. Little, Inc., of Cambridge, Massachusetts did just that. The idea was to prove that something said to be impossible was, with sufficient effort, money and ingenuity, attainable. Whilst it may be relatively easy to begin to explore the question above, I suspect the answer to the 'how can' part will involve not just alchemy but wizardry too!

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Appendices

Appendix 1 - Remote Learning

This appendix describes the modifications made to the PD programme following the decision to proceed with the study during the Covid-19 pandemic. The modifications made were in consideration of the rapid review of remote learning carried out by Perry et al. (2021) and in particular their findings on the affordances and limitations of remote and blended approaches to teacher education.

The OECD report 'Professional growth in times of change' (OECD, 2020) observed that scheduling conflicts and lack of time are among the most widely reported barriers to teachers' participation in professional learning. Clearly, the use of remote learning to deliver CPD significantly ameliorates the impact of these particular barriers and the OECD report refers to a number of countries that have increasingly relied on the development of online teacher communities to facilitate teacher professional learning.

Whilst digitalisation expands the opportunities for teacher professional learning, sessions increasingly are offered outside of school hours and therefore there is a risk that they add to teachers' already large workloads and may negatively affect their well-being. The OECD's 'Survey of Adult Skills' found that more than 60% of teachers surveyed reported that they had engaged in open/distance education only outside working hours and another 20% said that it was mostly outside working hours.

Of course, remote learning is not new. For example, it has been the core delivery model for the Open University in the UK over the last 40 years. Across parts of western Australia and New Zealand, the College of Radiologists has delivered CPD programmes by video conference to rural radiologists and associated professionals since 2003. The Open University used the term 'distance learning' which may or may not have a different connotation to 'remote learning'. There is ongoing debate about the definitions of 'remote', 'distance' and 'blended' modes of learning (Hobbs & Bolan, 2021). However, a review of the literature by Lorraine Sherry (1995) identified the core definition of distance and remote learning as the one articulated by Perraton in 1988: the "separation of teacher and learner in space and/or time".

APPENDIX 1

In this study, the headteacher of each participating school agreed that the PD programme sessions and the research lesson could take place in school time. During the first cycle, the participants accessed the PD sessions from home as this was their normal place of work at the time. Thus, each participant was physically distant from other members of the TDT. In the second cycle, the participants accessed the PD remotely in school and decided to do so as a group due to the easing of pandemic restrictions. Only the pre- and post-study interviews were carried out during the participants' own time and the option of doing these remotely gave them considerable flexibility in the choice of location and time.

In their rapid review of remote learning, Perry et al. (2021) identified six modes of online or blended teacher education:

- lectures, workshops, seminars, discussion groups or conferences, including one-off sessions and series
- coaching and mentoring
- classroom observations with feedback and/or discussion
- resource bases or repositories, with varying degrees of user interaction and content creation
- platforms and self-study programmes, ranging from less to more structured programmes that give learners access to curated/designed resources, learning content, assessments and/or directed activities
- virtual reality spaces or simulations.

The reviewers contend that irrespective of the modes of delivery, there are key elements which are likely to be critical to their success. These elements include the currently known characteristics of effective face-to-face CPD (Cordingley et al., 2015). The review collated the principles of effective CPD into five themes and analysed them against the different modalities. The five themes are:

- pupil orientation
- collaboration and support
- presence, participation and facilitation
- community formation
- diagnostics, differentiation and teacher starting points.

APPENDIX 1

MODIFICATIONS AND INNOVATIONS IN THIS STUDY

The Covid-19 pandemic arrived during the design phase of the study. The decision to proceed was made in the knowledge that it was possible to deliver the programme remotely but that all the programme components including the research lesson and data collection processes would need to be modified. Some of the required modifications, such as those to the pre-study interviews and the delivery of the three taught PD sessions, were relatively straightforward. Other components such as the research lesson and the pause within the research lesson presented more of a challenge.

Participants could easily observe the research lesson remotely but observing the pupils' work whilst the participants were not in the classroom was a substantial challenge. The solution was the use of remote visualisers in the research lesson, as will be discussed below. However, during Cycle 2, some of the restrictions were lifted which meant that the research lesson, the pause and the post-lesson discussion could be carried out in face-to-face mode.

At the time of the initial design of the PD programme, the rapid review of remote learning by Perry and colleagues had yet to be published. However, several of the PD components contain modifications which utilised remote technology. These are described below and are considered in relation to the five themes identified in the rapid review.

Use of break-out rooms

In teacher education, a common approach to achieving interactivity is to foster collaboration between participants. However, the concept of interactivity is not confined to person-to-person interaction – it can occur between oneself and an idea, concept, hypothesis or artefact and therefore may not necessarily be affected by any physical separation. The notion of separation then applies to the current definition of remote learning only in terms of the teacher and the learner having different locations, whereas in the past the idea of separation also meant the absence of or delay in interactivity. Some of the evaluations of distance learning in the past indicated that a major drawback of using broadcast media, such as radio and television, for instruction was the lack of an immediate two-way communications channel. This drawback was compounded by shortcomings in the delivery by facilitators who were expert in the subject matter but were not necessarily the best at transmitting their knowledge via these media. However, whilst the issue of interactivity has now been addressed by the introduction of computer-based communication systems supported by the World Wide Web, many challenges remain concerning the design and transmission of information to learners and consideration of how it is subsequently received and decoded by those learners.

Ensuring conditions for interactivity in professional learning by itself does not automatically lead to high-quality collaboration – which ultimately needs to be sustained by the participants themselves. As stated by Darling-Hammond et al. (2017):

> High-quality PD creates space for teachers to share ideas and collaborate in their learning, often in job-embedded contexts. By working collaboratively, teachers can create communities that positively change the culture and instruction of their entire grade level, department, school and/or district.

In this study, space for sharing and collaboration was achieved by implementing two strategies using the ‘breakout room’ facility within the online conferencing platform, Zoom. The first strategy was for the TDT to work independently on developing the research lesson plan without the presence of the facilitator. This approach can be useful for some collaborative tasks where the participants may be nervous of making a contribution in the presence of an ‘expert’ and their trepidation could potentially affect the quality of the discussion (a bit like Paul Hollywood watching you make filo pastry!).

The second strategy during the session in the breakout room was to give the TDT ownership of the time spent on the task. Normally, the facilitator specifies in advance the period of time allocated for the task (which, incidentally, most course participants initially ignore). It was anticipated that not setting a time limit and allowing the participants to decide when to return from the breakout room would increase the focus and quality of discussion. However, both remote and face-to-face CPD of this type is ultimately time-limited, so giving the TDT control over time spent on the task required careful planning and flexibility.

Using remote visualisers to analyse pupils’ work

As discussed earlier, due to Covid-19 regulations at the time of the study, it was not possible for the participants to observe the pupils’ work in the research lesson by moving around the classroom at agreed planned moments. In fact, in the first cycle, the regulations meant that only the teacher delivering the lesson was allowed to be in the classroom with the pupils.

In response to this problem, remote visualisers were incorporated into the research lesson so that members of the TDT could observe the work of the pupils. The technology required pieces of hardware installed on pupils' desks. Whilst the use of visualisers presented a number of logistical challenges – which were overcome and refined throughout both research cycles – it also led to a number of unexpected advantages. The most significant of these was the opportunity for participants to develop a detailed understanding of the pupils' reasoning as the work was being produced (with little interference from the observer) which would inform the strategies that could be used in teaching.

Stupel and Ben-Chaim (2017) observed that in the literature on the use of problem-solving approaches to connect mathematical concepts, there is often surprise and even astonishment at the lack of evidence of teachers using different methods to develop pupils' reasoning and mathematical thinking. In the authors' analysis of trainee teachers' views following engagement in the 'special square' problem, they noted the trainees' enjoyment from finding different solutions and their surprise at the existence of other solutions. Importantly, the trainees commented that whilst they were familiar with the mathematics required to solve the problem, they had no previous experience of thinking in this way.

In Japan, teachers have a different arc of experience. Unlike the trainees engaged in the special square problem, beginning teachers in Japan have past experience of multiple solutions to problems – firstly when they were pupils themselves. This approach is then revisited through their study of approved textbooks which contain tasks that have been carefully designed, in some cases over several decades, and then refined through the nationally recognised professional learning programme known as Lesson Study. An objective of the PD programme in the present study was to create a learning opportunity for participants to appreciate the importance of having detailed understanding of the ways in which problems could be approached by pupils and subsequently how they could be used by the teacher.

As a result of using the visualisers, the participants were not only able to see what the pupils had done in real time, they were also able to review their work as a TDT. Also, as discussed in Chapter 8, the different approaches used in the two cycles provided future options for analysing the pupils' work. Thus, the visualisers made it possible for teachers to obtain detailed information to which they would generally not have had access before this PD opportunity. In addition to deepening their own professional learning, the information and insights added to the richness of the discussion during the pause in the research lesson and in the post-lesson discussion.

Refinements in PD programme design

One of the five characteristics of design research methodology is its cyclic nature (Cobb, 2003). Whilst Bakker (2018) suggests that all five characteristics (see Chapter 3) do not all need to be present, the cyclic component would appear to be essential in design research involving invention and revision as iterative processes. Being able to review the facilitation of the PD sessions in the first cycle provided the opportunity to make assessments and discuss potential modifications to the facilitation in Cycle 2. Specifically, by reviewing the presentations, discussions and facilitation of the sessions, it was possible to evaluate aspects of the PD sessions such as:

- consistency in language
- precision in mathematical language
- use of questions
- facilitation of discussion
- interpretation of participant contributions.

In addition, there was the option of analysing each session retrospectively to look for visual cues and associated comments from the participants that might indicate their feelings or beliefs about the content or nature of the particular PD session. Analysing this data alongside the presentation of the PD session provided further information on the quality of facilitation. It also provided the opportunity to review the quality of 'chairing skills' in the pause in the research lesson and the post-lesson discussion.

Additional data

Having recordings of the pre-PD and post-PD interviews allowed the researcher not only to produce transcripts for analysis but also to gather additional non-verbal data from the participants and to reflect on the interview design and technique. The non-verbal information potentially offered further insights into some of the verbal responses made. For example, in an audio-recording, a long pause can be interpreted in many different ways, but being able to watch the respondent during such a pause can provide clues about their opinions or attitudes which add nuance to the meanings of the statements they make. With regard to the design of the interviews, in Cycle 1 the narrative interview technique was employed for the pre-PD interviews. This was subsequently changed to a semi-structured format, partly as a result of observing the Cycle 1 interviews. Similarly, by observing my own performance as the researcher-interviewer during the interview process it was possible to review the ways in which the questions were asked and to reflect on my disposition and body language as I listened to the interviewees' responses to the questions.

CHALLENGES OF REMOTE LEARNING

In this final section, I discuss two of the challenges that were observed as a result of the use of remote technology in this PD programme.

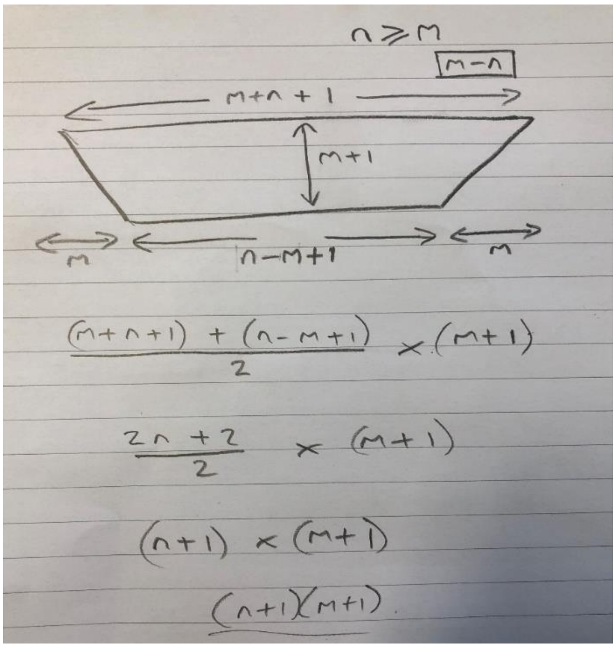
Participant engagement

The first challenge concerns the capacity of PD remote learning to engage the participants. The intrinsic interest and motivation that participants bring to PD programmes is either sustained, enhanced or diminished by the quality of engagement generated during the PD programme. Degree of engagement is sometimes hard to detect (Nicolini et al., 2003). Whilst there is some debate on its effectiveness, reflection is still regarded as an important tool in professional learning that can support participation and engagement within 'communities of practice' (McArdle & Coutts, 2010).

Teachers seldom work in settings conducive to sustained reflective practice. In their own classrooms, they are usually left to their own devices except when they are engaged in processes such as appraisal or school reviews; even then it can be argued that such events are more about analysis and evaluation than reflection. Effective CPD often includes moments where teachers are supported and encouraged to reflect, but it is important that facilitators of PD understand the conditions that sustain reflection and especially those that may lead to changes in behaviours, actions and beliefs.

This point brings the discussion back to the design of the PD programme in an online environment and in particular one of the themes in the principles of effective CPD – presence, participation and facilitation. Whilst Park et al. (2013) suggest that it is possible to achieve all of these within an online space, it is clear that the lack of face-to-face interactions, either between the participants and the facilitator or between the participants themselves, can in some situations diminish the effectiveness of collaboration.

In Cycle 1, in the first PD session, each participant individually explored the half-time scores problem (explained in Chapter 3) before discussing the problem together as a TDT. One participant was interested in the generalised form of the sort 'sum of the scores'. As a result, he produced the following diagram for discussion.



Proof using the sort displayed in Chapter 2, Figure 2.5

After several attempts to explain this method, including the manipulation of some hastily made cards, it became clear that the other TDT participants and the PD facilitator (me) had concluded that we would have enjoyed and understood the explanation much more if we were all in the same room at the time. As one member of the TDT said: "I wish I could see what you are doing with those cards!".

With hindsight, or indeed with more consideration to the participants, the exploration of the task and the subsequent discussion of this method could have been more effective if the participants also had access to the physical cards (as recommended in the introduction to the problem). This would have potentially improved the participation and facilitation but it also points to the challenge of providing bespoke resources to participants who are physically separate in different locations. On the other hand, this separateness meant that there was 'space' for each participant to reflect on the method without (in the moment) having to declare their understanding.

Teachers' starting points in the PD

The second challenge related to the importance of establishing an understanding of teachers' starting points (an aspect of effective CPD that is referred to as 'diagnostics' by Cordingley et al., 2015) with regard to:

- professional identities, practices and motivations
- beliefs
- approaches to learning
- existing knowledge and skills.

In this study, the task of assessing these starting points was undertaken with the use of pre-study questionnaires and pre-study interviews together with my knowledge from previous experience of working professionally with the participants. The combined data was used to inform the design of the taught components of the PD programme and in particular to avoid (as far as possible) unnecessary conflicts pertaining to the participants' current beliefs and practices that did not need to be challenged as part of this programme.

Irrespective of the validity of this audit and the subsequent design, there remains the question of how these teacher diagnostics and potential differentiation of teachers can be accommodated within the design of remote teacher education programmes. A possible answer to this question is found by considering the reasons for including diagnostics in any PD programme: one of the reasons, as in this study, is to attain teacher agency.

Agency can be defined as 'the capacity to act'. For an individual to exercise agency, not only do their belief systems need to be in alignment with the action, but they must also have the skills required to act. Given that the capacity to act is therefore contingent, agency can be thought of as a state to achieve rather than a quality that can be possessed. Biesta and Tedder (2007) refer to this as an ecological understanding of agency. Here, even 'high capacity' individuals may fail to achieve agency if the conditions are too difficult.

Teachers make sense of their practice both by taking action and reflecting on action. Often the reflection happens in isolation from others and so the reasons for any changes made as a result of reflection on action are largely unknown. As such, teachers can leave PD programmes uncertain of the knowledge they are leaving with or the contribution they have made, because of internal conflicts between what they have been presented with and what they thought they knew. This outcome is often observed in 'delivery' or 'empty vessel' CPD models (Dadds, 1997).

Importantly then, in order to strive for teacher agency, PD programmes should incorporate the known 'antidote components' of active participation, collaboration and action research. Whilst careful facilitation can support these components, there is a need to tread very carefully when the PD sessions are provided remotely. High-quality professional learning involves the participant 'going' to places of uncertainty, a journey which can only be undertaken in a climate of trust and respect. In a virtual forum, it is often challenging for a facilitator (or participant) to audit the emotional state of the room due to the absence of visual clues or difficulty in generating a dynamic or robust online discussion. For example, simply coming off mute or putting a virtual hand up in a remote learning situation can add a level of stress that could affect the articulation of a view or opinion.

Summary

In this review of remote learning, I have described the advantages and limitations of the use of digital technology to carry out the PD programme during the Covid-19 pandemic. The resulting challenges have been attended to and, where possible, the affordances have been incorporated into the study design. In particular, the use of visualisers to observe pupils' work led to unanticipated benefits for the quality of the participants' analysis and learning and brought a valuable new dimension to the study.

Appendix 2 – Participant pre-study questionnaire

A professional development programme for teaching mathematics through problem solving

Sheffield Hallam University

Participant questionnaire

For each of the statements below indicate the extent to which you agree or disagree with the statement.

Statement	Strongly agree	Agree	Neither agree nor disagree	Disagree	Strongly disagree
Thinking and reasoning processes are more important than specific curriculum content.					
Mathematics should always be taught through real life problems					
Mathematical subject knowledge is the most important attribute for teaching					
The most important aspect of mathematics is to know the rules and to be able to follow them					
My role as a teacher is to facilitate pupils' own inquiry.					
Pupils learn best by finding solutions to problems on their own.					
Pupils should be allowed to think of solutions to practical problems themselves before the teacher guides them to develop solutions					
Pupils should learn from seeing different ways to solve a problem, either by pupils presenting their solutions or					

by the teacher presenting alternative solutions					
Students should learn basic skills before being asked to solve non-routine mathematical problems.					
Teaching secure use of procedures is more important than facilitating classroom discussion					
Contexts should regularly be used at the start of topics to generate a discussion of strategies.					
There is usually a best method for solving a mathematics problem and my job is to make sure students learn that method.					
The most important part of teaching mathematics is explaining ideas and procedures clearly					

For each set of statements below consider how *important t*hese are in your teaching. Please rank the activities in order of importance. Rank 1 is the most important and rank 4 is the least important

S1	
Statement	Rank
Pupils should be to explain their answers to others	
Pupils should be able to construct a written record that shows the learning process	
Pupils should be able to follow the reasoning of another pupil	
Pupils should be able to evaluate other procedures than their own	

APPENDIX 2

S2	
Statement	Rank
Problem solving is more important than mathematical content (algorithms, rules and formulae).	
Pupils should always begin to solve a problem independently	
Problem solving skills should always be taught in the context of actual problems.	
Problems should be given to pupils that use several aspects of mathematics	

S3	
Statement	Rank
Pupils must be able to decide on their own procedures or methods	
Pupils must explore alternative methods for solutions	
I should teach the most efficient way to solve a particular kind of problem	
I should direct pupils away from non-standard or inefficient methods	

S4	
Statement	Rank
I provide opportunities for students to make conjectures about mathematical ideas.	
I encourage discussions where students question each other and explain their thinking	
I correct any mistakes or misunderstandings when students speak in class straightaway	
I keep students' talk in whole class discussion on topic to make sure key teaching points are made	

APPENDIX 2

Name ____

Date completed ____

Appendix 3 – A discussion about Tripods and Bipods

Tripods and bipods and Diophantine equations

> The spaceship problem
>
> Some tripods and Bipods flew from Planet Zero. There were at least two of each of them.
>
> Tripods have 3 legs
>
> Bipods have 2 legs
>
> There were 23 legs altogether
>
> How many Tripods were there?
>
> How many Bipods?
>
> Find two different answers

This problem is taken from the NNS booklet ‘Mathematical Challenges for able pupils in Key Stages 1 and 2.’ and is one of a family of problems that involves combining multiples of given numbers, for this problem 3 and 2, to make a given total, in this case 23. Essentially there are 2 variables to manipulate, the number of Tripod’s legs and the number of Bipod’s legs in order to make the total. There are other problems that belong to this family of problems, see below.

When faced with a Spaceship-type problem for the first time, many pupils will use a ‘trial and error’ approach, testing various combinations chosen at random. This unsystematic approach may well hit upon the answers, but offers no insight into how related problems might be solved which are less prone to lucky hits.

A more systematic approach would be to list multiples of 3 and 2 and use this data to search for combinations that total 23. For the Spaceship problem the list might include:

Tripod Legs	3	6	9	12	15	18	21	24	27	30
Bipod Legs	2	4	6	8	10	12	14	16	18	20

APPENDIX 3

Here we are manipulating the 2 variables (Tripod's legs and Bipod's legs) independently. The lists are too long and the search for the total 23 may still be tackled in an unsystematic way with results missed. An alternative listing might involve:

Tripod legs	3	6	9	12	15	18	21
Extra legs	20	17	14	11	8	5	2

This time only one variable, the Tripod's legs, is being manipulated independently. The other variable, the extra legs, is dependent on the Tripod legs, i.e. these figures are determined by the total number of legs. Here the lists are of optimum length as there are only 23 legs in total. As the 'extra legs' are Bipods and must be multiples of 2, the solutions can be found more systematically. We can sort out these cases that are even from those that are odd, in this case – 20 legs (10 Bipods); 14 legs (7 Bipods); 8 legs (4 Bipods); and 2 legs (1 Bipod). Such an approach would lend itself to solving other problems that fall into this family of problems.

The essence of these problems is a single underlying equation. For the Spaceship problem we were seeking multiples of 3 and multiples of 2 that sum to 23. If we use N and M for these multiples we are trying to solve the equation:

$$3N + 2M = 23$$

In this equation N and M are positive whole numbers and represent the numbers of Tripods and Bipods respectively. Such an equation is called a Diophantine equation after the Greek mathematician and arithmetician Diophartus (c.300 AD).

For this Diophantine equation, $3N + 2M = 23$ there are a number of (but finite) solutions $N = 1$, $M = 10$; $N = 3$, $M = 7$; $N = 5$, $M = 4$; and $N = 7$, $M = 1$. If desired the number of solutions could be reduced to 2 by adding a condition in the question.

Understanding the structure of these equations can lead to the development of similar problems types of problem and can also be used to decide whether there are any solutions to a given problem. For example:

Suppose the Spaceship had Tripods and Hexpods (6 legs), would it be possible to have 230 legs? This time the Diophantine equation is:

$$3N+6M = 230$$

It would be useful to consider how the lesson would be constructed to arrive at the point where;

$N+2M = 230/3$

Remember that N and M are positive whole numbers, so N+2M, the left hand side of this equation, is also a positive whole number. However, 230/3 is not an integer. Consequently, the equation has no solution, and the problem has no solution, we need not bother listing and sorting

Appendix 4 – Formative questionnaire

Formative questionnaire for CPD session 2 – Task Design and an introduction to the problem for the research lesson

For each of the questions choose the most appropriate statement that reflects your answer

Question 1 (S1)

In PD Session 1, the classification of problem-solving skills (generating data, sorting and classifying, patterns and relationships, reasoning and proof) was shared after you had considered the problem-solving strategies pupils should be taught and learn.

A. I would have preferred to see the classification first to provide a stimulus to my thinking.
B. It was useful for us to think about problem-solving skills before the classification was shared.

Question 2 (S2)

The Johari Window was introduced to examine how we, in our role as a research team, were responding to the PD programme and to consider how our own beliefs were impacting on the intended learning from the session. How helpful was this in identifying issues in the PD programme?

A. Did not help with reflecting on the PD programme.
B. Did not help but nor did it impact negatively on the PD programme.
C. Was useful as a tool to think about how the PD programme was impacting on my own beliefs in relation to the objectives of the PD programme.

Question 3 (S3)

When we were exploring the task for the research lesson in the breakout room, were given no time limit on how long we could spend working on the plan.

A. I would have preferred a time limit being given to the activity.
B. I was happy that we could choose when to return to the PD session.

Question 4 (S4)

The Tripods and Bipods problem was presented and participants were asked to do the problem as group (Teacher Design Team). After this a paper on the problem setting out possible solutions and linking it to an aspect of linear equations (Diophantus) was presented the to the team. How helpful was the paper?

A. The paper was helpful but highlighted gaps in my own subject knowledge and made me feel insecure
B. The paper was helpful in highlighting aspects of the problem that I had not considered or was unaware of.
C. The paper was helpful in highlighting aspects of the problem that I had not considered or was unaware of but I would have preferred to have been given the paper at the same time as being asked to do the problem.

Appendix 5 – Framework for developing problem solving skills

A Framework for developing the teaching of Problem Solving

Points of Departure

PD1: Can we classify the range of problem solving strategies that should be taught?

1. Generating Data and Listing	**2. Sorting and Classifying**
Pupils should be taught to: • Generate data from a given rule or a set of conditions • Derive a set of numbers or shapes that meet a list of criteria • Find the largest and/or the smallest cases or values for given circumstances and conditions • Systematically list and record all the possibilities in a set given a number of conditions	**Pupils should be taught to:** • Sort objects, numbers or shapes by deciding whether they meet given criteria • Classify a set of objects numbers, or shapes using a number of criteria or properties • Identify criteria to describe sets of numbers, objects or shapes that have been sorted or classified • Use sorts and/or classifications to complete sets with missing items
3. Identifying Patterns and relationships	**4. Explaining and reasoning**
Pupils should be taught to: • Organise data to generate and complete patterns • Use symmetric properties in shapes, sets of numbers and calculations to establish relationships and enumerate lists • Organise information into tables, charts and diagrams in order to recognise and discover patterns and relationships • Describe relationships and patterns. Manipulate these to generate new ones	**Pupils should be taught to:** • Use calculations to support explanation and argument • Look for a counter example to define the conditions and limits of a rule • Use a relationship or pattern to justify or confirm others. • Use properties and relationships to reason and deduce. • Use a diagram to support an explanation • Generalise in order to prove

In addition to the classified set of specific strategies pupils should be taught how to develop resilience and perseverance when engaging in mathematical problems that require the application of several strategies and approaches.

APPENDIX 5

PD2: Would it be helpful to classify these strategies against different type of problems as set out below?

Type of Problem/Problem Solving Strand	**Generating Data**	**Sorting and Classifying**	**Patterns and Relationships**	**Explaining and Reasoning**
Word problems 1 step				
Word problems multi step				
Puzzle-type problems that require some reasoning, they are content free but involve some train of logical thought				
Puzzle-type problems that require some reasoning, they are content specific and employ deduction from known facts				
Problems that require the interpretation of pictures tables of graphs, interpolating and extrapolating, within and beyond what is known				
Problems that require the identification of patterns from which relationships or totals can be found				
Problems involving combinations and permutations that lead to generalisations				
Problems that require proof by contradiction or counter example				
Problems that involve statistical techniques				

APPENDIX 5

PD3: Would it be useful to describe specific teaching approaches using particular activities in relation to the four strands?

Problem Solving Strategies	**Generating Data and Listing**	
Pupils should be taught to:	Teaching Approaches	Teaching Activities
Generate data from a given rule or a set of conditions.		
Derive a set of numbers or shapes that meet a list of criteria		
Find the largest and/or the smallest cases or values for given circumstances and conditions		
List all the possibilities in a set given a number of conditions		

Problem Solving Strategies	**Sorting and Classifying**	
Pupils should be taught to:	Teaching Approaches	Problem
Sort objects, numbers or shapes by deciding whether they meet a given criteria		
Classify a set of objects numbers, or shapes using a number of criteria or properties		
Identify criteria to describe sets of numbers, objects or shapes that have been sorted or classified		
Use sorts and/or classifications to complete sets with missing items		

APPENDIX 5

Problem Solving Strategies	**Patterns and Relationships**	
Pupils should be taught to:	Teaching Approaches	Problem
Organise data to generate and complete patterns		
Use symmetric properties in shapes, sets of numbers and calculations to establish relationships and enumerate lists		
Organise information into tables, charts and diagrams in order to recognise and discover patterns and relationships		
Describe relationships and patterns. Manipulate these to generate new ones		

Problem Solving Strategies	**Explaining and Reasoning**	
Children should be taught to:	Teaching Approaches	Teaching activities
Use calculations to support explanation and argument		
Look for a counter example to define the conditions and limits of a rule		
Use a relationship or pattern to justify or confirm others.		
Use properties and relationships to reason and deduce.		
Use a diagram to support an explanation		

Appendix 6 – Cycle 1 Lesson Plan

Cycle 1 Lesson Plan

Lesson Study Research Lesson	
Team/Department: Maths	
Research Question: How does sequencing anticipated responses support teaching mathematics through problem solving?	
Class Context A general overview of the group, including any contextual information	
A specific mixed ability Y7 group of 14 pupils. 4 HAPs, 6 MAPS, 4 LAPS. (4 PP. 1 SEN)	
Assumed Prior Knowledge What prior learning has taken place that will be pertinent to the research lesson	
The planning team considered arithmetic skills.	
Departmental Discussion Points on the development of the research lesson	
NCTM suggests 'Problem solving means engaging in a task for which the solution method is not known in advance.' We have done a lot of work as a department around the introduction to solving problems. We have found that some children are more willing to 'try' the maths on a whiteboard where they can erase mistakes. Working with others serves to aid some children and will allow them to get	

started if they struggle to see a starting point with an investigation. This is normally because they lack in confidence. However, the emphasis on grouped work and tables has meant some students sometimes try to hide from facing the work and allow others to do it for them. So this needs to be carefully managed and not used for all of the tasks.

Allowing students, a chance to try and complete a task before been shown the most efficient method should build confidence in their mathematical ability and develop their problem solving skills.
'When students discover mathematics concepts for themselves and refine problem solving skills in small groups they learn mathematics and self-reliance' (Simon, M 1986)

Ensuring students develop strategies to solve problem we looked at investigations with classes throughout the year. We initially found that some students were reluctant at first to solve problems without the help of the teacher and thus were more dependent than others. Some of the most-able students were often the students who were struggling when it came to these lessons and we believe this was because this was the first time these students had truly struggled and been tested rather than been spoon fed.

The task:
The task itself will be to develop a Diophantine equation : Spaceship Problem Some tripods and Bipods flew from Planet Zero. There were 23 legs altogether. How many Bipods and Tripods were there?
This will allow a range of student responses to allow us to build up the response to the Diophantine equations and explore the relationship with the variables.

The structure of the lesson:

Students will be working independently to answer the task. They will have visualisers to allow the teacher to see their work. This will allow whole class discussion. Students may not fully complete the task but by sharing their work so far students can decide which strategy is best as a class and thus developing a faster way to problem solve. This will also allow us as staff to develop anticipated responses (for any we have not thought about). The lesson itself will follow the Japanese Structure of Hatsumon- The teacher will explain the main task and allow students to ask questions about the structure of the lesson with prompts about the task itself. Students then work in groups to solve the problem. Whole class presenting (Nariage). Matome-summary. We have adapted the Japanese structure so that the teacher can direct the flow of the lesson rather than leaving it to chance.

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The Research Lesson

APPENDIX 6

(The lesson should be planned in detail)				
Lesson Phase/ Time	**Teacher Actions – The Lesson Plan** Be precise with activities and language. What questions will you ask? What examples will you use? How will it be modelled?	**Why are you doing this?** How does the teacher actions link to the research?	**Pupil Responses** What are the possible responses that we might expect	**Possible Pupil Misconceptions** What are the potential misconceptions pupils may have and how will you react?
5 min	The teacher is going to welcome the class and allow students to read the task. 'Please come in and read the question. Write any questions about the task that could help you or another student? Do not attempt the task itself.'	This allows students time to think about the task and cause some students to ask questions about the task.	Pupil should sit quietly and think about the problem.	
	Teacher asks 'Do you have any questions for me about the task?'	To allow students to clarify the questions to eliminate a misunderstanding of the wording of the problem.	'Is there only one of each?' (No) 'Can you have zero bipods' (No) 'How many legs do bipods/tripods have?' (What is the number of wheels on a bicycle/tricycle? A triangle means three angles so how many legs do you think is a tripod?) 'Do bipods mean they have two legs?' (Yes) 'Do tripods have 3 legs?' (Yes) 'Can you have more than one bipod? Tripod?' (Yes) 'Is there more than one answer?' (Yes)	Students may have singular bipod and tripods.

	Allow students time to answer independently on the task.	To allow students time to write down their process of solving this problem.	Students to respond by writing the possible strategies: I found 2,4,6,8,10,12,14,16,18,20,22 3,6,9,12,15,18,21. (Listing the multiplication tables without much reference to the question.) This is what I found out… 2 x 10=20 3 x 1=3 20+3=23 So therefore there were 20 bipods and 1 tripod There are 4 tripods if you count their legs it would make 12 but if you count all the legs it would make 18 but if you use your 3 and 2 timetables it would make 23 so the answer is 20 bipods and 1 tripod. (Misunderstanding the calculations they have done so need clearer knowledge of 2 legs x 10 = 20 legs) This is what I found out Tripods 5 legs x 3=15 legs Bipod 2 legs×4 = 8 legs (Shows the units) 10 bipods, 10×2=20 and 1 tripod, 20+3=23 7 tripods, 7×3=21 and 1 bipod, 21+1=23 (Shows understanding and calculation but no reference to answering the question. This just shows the number of legs is equal to 23 in both calculations.) My answer is 5 tripods and 4 bipods. I have found 3 ways to do it.	Pupils may do calculation errors. Misunderstand multiples. Not fully explain their written methods correctly. Misunderstand the equations they have written. May stop after getting only one answer.

			10×2+1×3=20+3=23 7×2+3×2=23 4×2+5×3=23 (No real answer to the question just 3 calculations and using a single tripod as an answer) I found 2,4,6,8,10,12,14,16,18,20,22 3,6,9,12,15,18,21. So 20+3=23 21+2=23 14+9=23 15+8=23 (Again doesn't answer the question but lists the multiplication table to show 4 possible calculations that are relevant to the question) I found Tripod legs 3 6 9 12 15 18 21 legs 20 18 16 14 12 10 8 6 4 2 So 14+9=23 15+8=23 So 7 bipods and 3 tripods And 5 tripods and 4 bipods (Lists the tables to compare so a total of 23 can be calculated quicker starting at 20 and 21 in the 2,3 multiplication tables, interpret the calculation to give an answer) 7 tripods and 3 bipods . I found this out because.. 3×3=9 2×7=14 9+14=23 My second answer is…. 3×5=15	

			2×4=8 8+15=23 There are 5 tripods and 4 bipods (Demonstrates secure multiplication knowledge and interprets the calculation to give the two possibilities) I realised that a group of bipods have an even amount of legs so we must have an odd amount of tripods. So 7 bipods and 3 tripods And 5 tripods and 4 bipods (Understands the number work with bipods having an even number of feet so understands must have odd number of tripods because odd+even=odd) 2N + 3M =23 N=1 then 3M=21 M=7 (Must have more than one of each type) N=2 then 3M=19 (No whole answer) N=3 then 3M=17 (No whole answer) N=4 then 3M=15 M=5 (4bipods and 5tripods) N=5 then 3M=13(No whole answer) N=6 then 3M=11(No whole answer) N=7 then 3M=9 (7bipods and 3tripods) N=8 then 3M=7(No whole answer) N=9 then 3M=5(No whole answer) N=10 then 3M=3(Must have more than one of each type) N=11 then 3M=1(No whole answer) N=12 then 3M=-1 (Cannot have a negative solution for the number of aliens)	

			(Student creates a Diophantine equation and explores some relationship between the variables. Shows some generating data technique.)	
	Teacher to share feedback in the order labelled from the previous line in the student response.	This allows the build-up of answers from students and allows the calculations to be discussed.	Pupils see their mistakes and build their knowledge of how to solve this problem as well as understanding of where the Diophantine equation comes from.	Pupils may misunderstand where the equations come from and only understand the number work. So a summary is needed.
	Then shares the Diophantine equation 2N+3M = 23 Because 2N will always be an even number 3M must be odd. So 3M=21,19,17,15,13,11,9,7,5,3 but because 3M represents the multiples of 3 we can only have 3M=21, 3M=15 3M=9 and 3M=3. However, you can not have M=1 because you can not have 1 tripod and also you can not have 7 tripods because this would give you 1 bipod. That means you have a final solutions of 2N+15=23 So N=4 so 4 bipods and 5 tripods additionally you have 2N+9=23 So N=7 where a second answer is 14 bipods and 7 tripods.	This demonstrates the model answer and reaffirms the use of the Diophantine equation. Students copy down the model answer so they can apply to the extension task to see if students use this technique to interpret the Diophantine equation.	Pupils understand what the N and M represent in the Diophantine equations. They start to understand the relationship between the variables.	Some students may still struggle with the solving of the process of N= and M= and the interpretation. Clear teacher explanation is needed.

	Then ask Suppose the Spaceship had Tripods and Hexpods (6 legs), would it be possible to have 230 legs? ‘For this I want you to write an equation to start with’	To assess whether students understand what the most efficient way to solve the problem is (Can they create a Diophantine equation)	Pupils to write 3N+6M = 230	Pupils may not write algebraically well and use words.
	Write this on the board: 3N+6M = 230 Ask students to find the values of N and M so you have the number of tripods and hexpods.	This allows an explanation of no answer possible if students think about the relationship between the variables.	Some students may then use N=1, N=2… Some students may realise 3N can be odd and even and 6M is always even and you cannot have Odd+Even=Even. So would look at just wen 3N is even. N=2,4…	Students may revert back to original strategy of problem solving.
	Write this on the board in silence 3N+6M = 230 3(N+2M)=230 N+2M = 230/3	This allows students to watch the factorising and solving process to see a simpler equation to look at the relationship between N and M.	Students watch the teacher and think about what the teacher is doing. Students then ask questions which leads onto the final explanation.	Students may not see or understand the HCF or the simplification of the equation.
	Explain N+2M = 230/3 N is a whole number and 2M is a whole number and 230/3 gives a remainder so there is no integer solution.	This allows students to understand why the Diophantine equation is useful so they don’t have to spend a long time with generating data to think there is no solution. This proves there is no solution.	Student should understand there is no solution.	

Appendix 7 – Cycle 2 Lesson Plan

Cycle 2 Lesson Plan

THE PROBLEM:	**Tripods and Bipods Some Tripods and Bipods flew from Planet Zero. There were 23 legs altogether. There were at least two Tripods and two Bipods. How many Bipods and Tripods were there?**	**Design Considerations** Why 23 legs? Why not a number of legs that gives no solution – see discussion paper
The mathematics		**The problem solving strategies**
Develop problem solving skills **Introducing this idea of a linear equation expressed as** **3T + 2B = 23 beginning with**		**Generating data** **Sorting and listing** • **Reasoning - Use calculations to support explanation and argument**

3 legs x 3 tripods + 2 legs multiplied by 7 bipods = 23 legs **Leading to** **3 legs x 5 tripods + 2 legs x 4 bipods = 23 legs** **Leading to** **3 legs x a number of tripods + 2 legs x number of bipods = a number of legs** **3T + 2B = 23**	**See attached discussion paper 1**

MATHEMATICS PROBLEM SOLVING LESSON PLAN

INDEPENDENT WORK ON THE PROBLEM	ORCHESTRATING THE LEARNING *(TEACHER COLLECTS INFORMATION ON CLIPBOARD USING SEATING PLAN)*	
We anticipate that the children will:	SEQUENCING	KEY CONNECTIONS
Document all the possible ways you think the pupils will approach the problem. 1. Misconception (is it a misconception?) – assume there can be only one of each 2. Draw tripods and bipods and count 3. Write out the numbers 3 and 2 multiple times and count 4. List 3 and 2 times tables and look for two numbers within that sum to 23	2 and 3 5 then 4 Leads to 6	Orchestrate 2 and 3 but make connection between pictures and number Orchestrate 4 and 5 by using pupil methods to generate equation for one

5. Make number bonds to 23 and look for multiples of 3 and 2 6. Systematic trail and improvement start with a multiple of bipods or tripods to calculate a number of legs and then subtract that number of legs from 23. Then establish if answer is a multiple of 2 (for bipods) or 3 for tripods		solution and then ask children to use same format for the other or for their own solutions.

PRELEARNING OR SCAFFOLDS THAT MAY BE NECESSARY:	**Opportunities to develop understanding of problem solving strategies**
What prior knowledge will the pupils need to access the problem? The pupils will need arithmetic skills and to be comfortable with the problem having more than one solution	Key questions or modification to the task (not additional tasks) that deepen the learning. How do we know that there has to be an odd number of Tripods? How do we know that we have found all the possible solutions? How will we explain why when we replace number or words with a single letter the multiplication sign disappears?

	3 x 5 + 2 x 4 =23 **3 legs x 5 tripods + 2 legs x 4 bipods = 23 legs** **3T + 2B = 23**

Appendix 8 – Previous Lesson Plan

South Yorkshire Maths Hub
Lesson Study

School: St Thomas of Canterbury Primary Catholic Voluntary Academy			
Research Team			
Whole School Research question	How can we develop the teaching of problem solving skills within a 6 part lesson?		
Date of Research Lesson	15.3.17	Year Group	4

About the topic
In 2014, 1 in 6 15 year olds failed to meet the minimum standard in international mathematical problem solving tests[1]. Children are often able to perform mathematical operations, but do not know which operations to apply when posed with problems. Our research question concerns developing problem solving skills in the context of fractions, as children have struggled with fractions for a long time, and little progress has been made in over 30 years [2.] Many can master the part-whole concept, especially if already understood with cardinal numbers. The operations themselves however, pose problems, and children more often than not apply procedures they do not conceptually understand[3]. A short summary of the common misconceptions is provided below. **Half means just one whole cut into two pieces** For example – many children will wrongly say that this circle has been cut into thirds. ??? **Fractions of the whole are whole numbers in themselves.** For example, to think that when a cake is cut into half you get two cakes (which implies you get more, when in fact it's just 2 halves of the whole, which is less). **Fraction symbols incorrectly identified.** For example to read $^1/_3$ as three quarters or to write three quarters as 3' $^1/_4$'s or simply not being able to read fraction symbols. **The bigger the number on the bottom, the bigger the fraction.** This results to wrongly ordering unit fractions. For example to think that $^1/_6$ is bigger than ½ . **The size of a fraction depends solely on the number at the bottom (denominator) and you can ignore the number on the top (numerator).**

For example: to think that $^1/_4$ is bigger than $^7/_8$. $^3/_4$ is always more than ½ ,, not making reference to the whole.

Fractions are added together by adding the top numbers together then adding the bottom numbers together.

For example to think that $^3/_5 + ^2/_4 = ^5/_9$.

If you ask a mathematician what maths is about, they will mention, among other things, solving problems and pattern recognition. Lynne McClure in 2013 found that when British primary school children were asked, they said that maths was about 'learning rules' and 'remembering facts' to help them 'pass tests'. [4] The new curriculum states that "The national curriculum for mathematics aims to ensure that all pupils… become fluent… reason mathematically… and can solve problems." A high-quality mathematics education necessarily develops the ability to reason mathematically and solve problems, because problem solving is the *whole point* of learning maths. Through the use of problems and puzzles key ways of working in maths can be introduced with great effect.

Considering that most mathematicians stated that they were introduced to the inspiring puzzles outside of their school life – it would make sense that we endeavour to inspire more children through incorporating problem solving and inspirational puzzles into teaching mathematics [5]. Fiori made a strong case for problem-solving to be a daily feature of mathematics teaching [6].

Our lesson will consider teaching *for* problem solving rather than *through* or *about* problem solving. This means we will be explicitly modelling and teaching a problem solving skill in the context of mathematical concepts they have previously been taught. The scope of what can be considered a 'problem solving skill' is wide, and may include visualisation, generalisation, inductive reasoning, proving concepts, sorting and classifying, working systematically, generating data, estimating and using a 'trial and error' approach. Our lesson will teach the skills of 'sorting and classifying' and 'working systematically,' although the skills applied in the lesson will not be restricted to just those two. Becoming confident in these skills is a difficult process that requires a variety of exposure to problems, and explicit teaching of skills. Jennie Pennant identifies 3 main ways in which we as teachers can support this process: "through our choice of task, through structuring of the problem solving process, and through explicitly and repeatedly providing children with opportunities to develop key problem-solving skills"[7]. The first two areas of teacher influence will be explored in the lesson. How to repeatedly implement the teaching of these skills will be informed by discussions after the lesson.

Mathematical problems tend to fall into five categories: finding all possibilities, logical, visual, rule and pattern recognition and word problems. Exposure to similar types of problems over time can aid the development of problem-solving skills. As 'find all the possibilities'' is a commonly used deepening question in our 'next-step' morning learning (e.g. The whole is 15, there are two parts. Find all the possibilities), a we chose a problem that would require the generation of data, and working systematically to ensure that all solutions had been found as this concept would be familiar to the children.

In the context of our school we teach prescriptively using the six part lesson structure. Part of our discussion centred on whether our children are given enough opportunity to develop problem solving skills through productive struggle, and if we could adapt our teaching of maths to include more opportunities, within which part would the problem solving activity be best suited?

References

[1] The Guardian (2017) *Students in England performance in international problem-solving test* | Education | The Guardian. [ONLINE] Available at: https://www.theguardian.com/education/2014/apr/01/students-england-international-problem-solving-test-pisa-oecd. [Accessed 13 February 2017].

[2] Lortie-Forgues, H., Tian, J., & Siegler (2015) *Why Is Learning Fraction and Decimal Arithmetic So Difficult?*: Carnegie Mellon University

[3] Siegler, R.S., Carpenter, T., Fennell, F., Geary, D.C., Lewis, J., Okamoto, Y., Thompson, L. and Wray, J., 2010. *Developing effective fractions instruction for kindergarten through 8th grade*

[4] Lynne McClure. 2013. *Problem Solving and the New Curriculum*. [ONLINE] Available at: https://nrich.maths.org/10367. [Accessed 5 March 2017].

[5] Boaler, J. Kovalelvskaya, S et al. (Sep 2015). *The Elephant in the Classroom: Helping Children Learn and Love Maths.*

[6] Fiori, N. (2007). *In search of Meaningful Mathematics: The Role of Aesthetic Choice.* Doctoral Dissertation, Stanford /university.

[7] Jennie Pennant. 2014. Developing Excellence in Problem Solving with Young Learners. [ONLINE] Available at: http://nrich.maths.org/10865. [Accessed 5 March 2017].

Assumed prior knowledge
This will be a list of the content that the children have covered and the skills that you expect them to have. We should also indicate which knowledge and skills will be built on and identify and explain any new learning in terms of knowledge and skills When the children solve the problem, they will have recently completed a 20 lesson unit of Mathematics Mastery lessons. During this unit pupils begin by revisiting previous learning, considering what a fraction is and how it can be represented. They then progress to find equivalent fractions, are introduced to mixed numbers and improper fractions, add and subtract fractions, calculate fractions of quantities and finally solve problems involving fractions. Throughout the unit pupils will be using a variety of representations, to increase their flexibility and depth of understanding with fractions. During the lesson, the children will need to apply many of these skills. They will recognise fractions in abstract pictorial figures, identify the value of these fractions, find equivalent fractions and add fractions to make the whole of 1. It is anticipated that children will have a grasp of these concepts, but may still make procedural errors at certain parts of the lesson. These responses have been anticipated and discussed in the lesson plan. The children are comfortable working within the six part lesson structure as this is used consistently throughout school. This is important as the children will recognise different parts of the lesson and know how to respond or what is expected of them, for example the expectation of the use of the mathematical language shared at the beginning of the lesson.

APPENDIX 8

About the research question/Considerations

This will be a reflection (narrative of the current thinking with the research question) with regard to personal experience and associated research about teaching pupils to develop their own strategies to solve problems

At our school, the use of a '6-part lesson' is embedded into our practice. The automaticity of this allows our teachers to keep a very fast pace, regularly give feedback and assess learning. This has allowed our lessons to flow quickly and has allowed more opportunities to question creatively and precisely. It also gives the children absolute certainty as to what is coming next in each part of a lesson every day.

Each part of the lesson has a specific purpose which is outlined below. The adaptation of this structure to best develop problem solving skills will be considered when planning the lesson in detail, and whether indeed this structure will be an aid or a hindrance in the teaching of problem solving skills.

Do Now - A quick task that all children can access with little to no teacher input. May be used to increase fluency in a previously taught concept, or to remind children of a relevant concept previously taught (allowing for cumulative learning over time). It should also allow all children to experience success at the start of every lesson, reinforcing the idea that all children can achieve in mathematics.

New Learning - This typically introduces the key mathematical vocabulary, and clear modelling of tasks to be completed throughout the lesson. This typically lasts 10-15 minutes and involves partner discussions and questioning to identify misconceptions before the Talk Task.

Talk Task - The correct and precise use of mathematical language is integral to a deep, lasting understanding of maths. Talk Tasks foster this with partner or group tasks involving children talking in full sentences about maths. Children are listened to and assessed on their ability to justify their reasoning to both their partner and an adult.

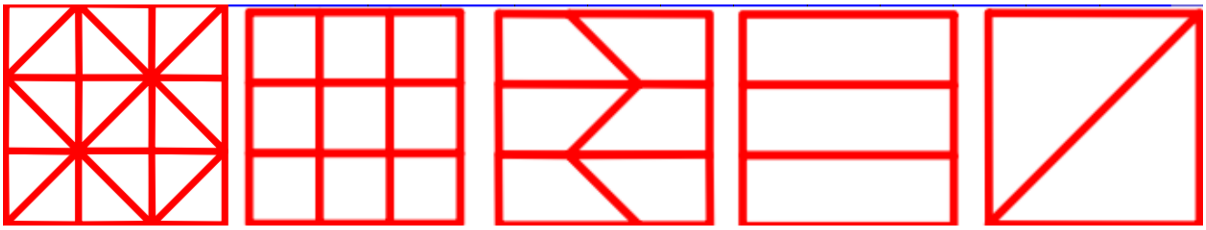
Develop Learning - The develop learning part of a lesson is intended to deepen children's understanding of the concepts taught in the New Learning and practised in the Talk Task with precise questioning, variations of the concept given (for example a different visual representation of the same concept), linking new concepts to those previously taught or exposing possible misconceptions children may have.

Independent Task - This part of the lesson allows pupils to practise and explore the concepts taught in the Develop Learning independently, demonstrating what they have understood and learnt. Careful questioning and observation of children during this task should allow those that grasp quickly to be quickly moved on to a more challenging task that deepens their thinking (rather than moving on to other material) and to inform how the plenary can be used to aid those that struggle.

Although children are expected to work 'independently', the use of mathematical language with their classmates is still expected and encouraged.

Plenary - The closing part of the lesson is used to assess which children may need same-day-intervention to address some misconceptions or gaps in knowledge. It should summarise the learning that has taken place during the lesson. They may be planned before a lesson, but will often be adapted to fit the needs of the learners during the lesson.

We adapted an ‘nrich’ problem found here https://nrich.maths.org/2124 for the purposes of the lesson study. Children will use the figure to find every fraction pairs (two fractions, the sum of which is 1) can using the figures (image below) provided. Children will be asked to explain how they know they have found every possibility.



How this research lesson will address the research question?

This should explain the new teaching strategies that will be used in the lesson to respond to the research question. The strategies should link to the focus of the lesson

Being experienced in teaching the Maths Mastery curriculum through the six part lesson structure we were able to discuss and analyse the strengths and weaknesses of working prescriptively to this curriculum. One of the main points raised was due to the quick pace of the lessons there seemed to be a lack of teaching *for* problem-solving and this raised questions such as; do our children have the opportunity to develop problem solving skills? Does the pace of a 6-Part Lesson impede the development of these skills? Can it be adapted?

A problem is something that one cannot immediately solve. This is important as we know that independent problem solving skills are essential for thinking mathematically and for students studying in the 21st century [8]. In order to develop these skills children need time to practise them; testing out ideas through trial and error and adjusting their thinking which allows for the development of metacognition and confidence in thinking mathematically. As teachers we are able to support our students in developing these skills, to do this we need to teach for problem solving. Explicitly teaching the skills needed to enjoy taking risks and tackling maths problems successfully.

Mason et al. (1982) suggested that it is experience the sense of slight struggle which we experience when working with challenging problems that encourages us to think mathematically [9].

As such our lesson will 'draw out' this productive struggle element of problem-solving across the lesson. Children will be guided to 'have a go' at the problem (the first barrier to problem-solving) with a Do Now task accessible to all - dividing each figure into two parts. The skill of labelling the fractions will be modelled through mime in the New Learning, allowing children questions and address any misconceptions surrounding fractions quickly so that the skill of sorting can be focussed on. Sorting the figures will then be modelled by the teacher, and the children will be given time to sort their own working as a group during the Talk Task. Then the questions of 'how do you know if we have every possibility?' and 'how do we know if we have the same answer twice?' will be addressed by using the children's responses and modelling of working systematically with the data we have during the Develop Learning. Finally the solving of the problem and the solution will be modelled for the class during the Plenary, after the children have had a go independently during the Independent Task.

Rather than a specific mathematical concept/procedure being modelled, the structure of the lesson will be used to guide the class chronologically through a process one might use to tackle the problem. The children will be assessed during group and independent work, and children's responses will be used as teaching points during the direct teaching parts of the lesson.

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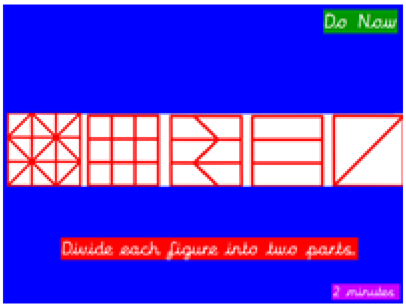
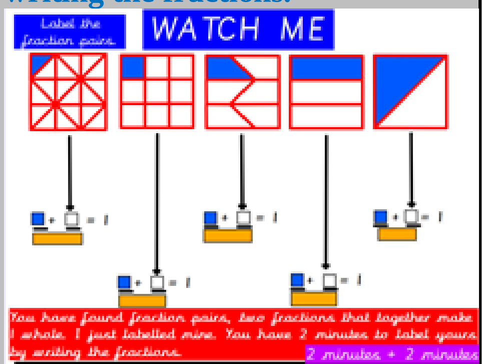
[8]Higgins, J. (June 2011). Mathematics Needs - Summary. *Advisory Committee on Mathematics Education*

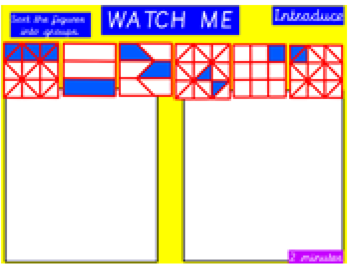
[8] Mason, J., L. Burton, et al. (1982). *Thinking Mathematically*, Prentice Hall.

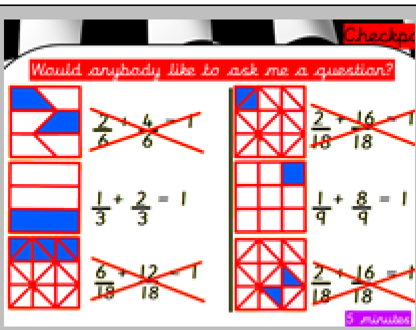
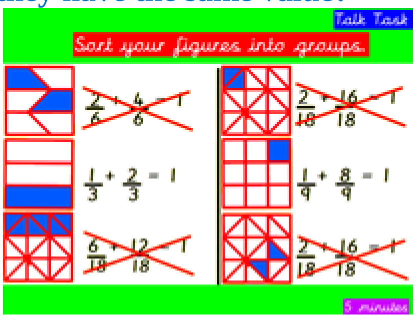
The Research Lesson	
Goals of the lesson	To develop problem solving skills; focusing on working systematically and sorting.
The Problem	
Lesson Structure	6 part lesson. Do now, New learning, Talk task, Develop learning, Independent Task, Plenary. Grouping: Children will work in pairs and as a 6 in tables. Resources: Many laminated figures, colouring pens. Adult Support: 2 Additional Adults (other members of planning team)

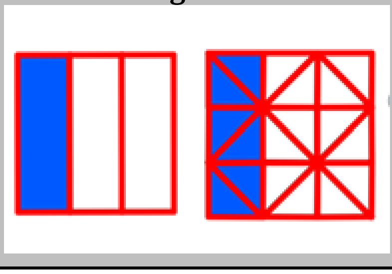
Mathematics Mastery Lesson 15.3.17

Key Learnin g	**To systematically sort fraction pairs to aid problem solving.**	**Anticipated Responses/Considerations**
Do now	**Introduce the children to the star words. Explain to them that if they hear any of the star words, they can raise their hand and say that I have used the star words and explain what the context is.** **Divide each figure into two parts.**	**Why share the language used at the start of the lesson?** **We discussed the importance of sharing the star words first ('fraction pairs' 'equivalent fractions' 'sorting into groups' 'working systematically' 'all the possibilities'), as it would highlight to the children that this lesson will be looking at problem-solving skills rather than mathematical concepts.**

	 **Children divide each figure into two parts by colouring parts in.** **Children have 2 minutes to complete this. All adults to ask…** **How have your divided their squares and why have you done this? What have you created?**	**Children will be encouraged to use these words by adults in the classroom and their peers.** **What if children do not divide them into equally sized parts?** **Children will be told to 'colour in the lines.'** **What if children are just attempting to create visual patterns?** **This is fine - At this stage it is only important that children can generate data independently. They do not yet have to see them as representing fraction pairs.**
Check point	**Watch me: Label the fraction pairs.** **Silently count the number of equal parts in each figure, reveal the denominator. Then count the number of blue parts and reveal the blue numerator. Then count the number of white parts and reveal the white numerator. Repeat for each figure, beginning with 1/18 + 17/18.** **You have found fraction pairs, two fractions that together make the whole of 1. You have 2 minutes to label your by writing the fractions.**  **All adults make a note of this and ensure we check in with**	**Why reveal the fractions in that order?** **We chose the order of 'revealing' the fractions to reflect the thinking process we believed the children would find most helpful – 'how many equal parts has the whole been divided into?' à 'how many of these parts have we got?'. We chose to model this process for each figure rather than finding all the denominators, and then all the numerators, as this would allow them to see the modelling of an individual fraction pairs more times.** **Which figures and fractions should be used?** **We decided to use 1/18+17/18, 1/9+8/9, 1/6+5/6, 1/3+2/3 and 1/2+1/2 as it would most obviously demonstrate that as a denominator increases the value decreases. It would also imply to children that 'little triangle means eighteenth/square means 9^{th}' etc. which may give some children an important**

	those children at the next checkpoint.	**visualization tool to aid in labelling. Should children's responses be taken or pre-used figures? We decided at this stage to use our own figures to ensure that there was absolute clarity as to the value of each fraction. What if children recognize equivalent fractions early? Praise the child and mention that this will be addressed later. What if children are unable to label the fraction pairs? All adults to assist with this – it is more important that they are labelled correctly and quickly so that the development of sorting skills can be focussed on. Vinculums and addition signs will already be drawn on, with clear boxes for the denominator and numerator to scaffold this for all children.**
New learnin g (max 15 mins)	**Sort the figures into groups. Children watch mimed sorting of different figures on the board. These will be sorted into two different groups of equal value. Children can visually see that some of the fractions are equal in value to the others.**  **Adults to walk round checking that children have sorted correctly. If they have made a mistake we can address that on the whiteboard and use this is a teaching point.**	**What if children cannot see the fractions as having equal value? This can quickly demonstrated by manipulating their figures, rotating and placing them on top of each other. They can be told 'it doesn't matter if they don't have the same shape, it just matters that the same amount is coloured.'**

Check point	 Would anybody like to ask me a question?	What if children think that the fractions have been crossed out because they are wrong? The middle will be circled instead – 'these three are the same, I like this one the best. I like it best because it has the smallest denominator.' What if children think that the smaller the denominator, the smaller the value of the fraction? Children can easily be shown the difference in size between a 3rd and an 18th using the figures provided.'
Talk task	Sort your figures into groups of equal value. Children will be sorting their figures out into groups of equal value in the same way that has been modelled in the checkpoint. Adults to walk round and assist the children that may need help and guiding those who don't, to ask those that have finished can you explain how you know they have the same value? 	What if children sort their fractions by another set of criteria? Children may sort by shape, size, number of sides, the colour they used etc. These may be used as teaching points, but children will be told that is one way to sort, but we want to sort them into groups of equal value.
Check point	Find an example where two children have coloured in two different examples of equal value using different figures and ask the children: What is the same and what is	What if children struggle to see that 2 figures are of equal value? This can quickly be demonstrated by placing figures on top of each other.

	different? e.g. 	
Deepen ing unders tandin g **(max 10 mins)**	**Start with a bar that is divided into fifths. Silently model systematically dividing it into fraction pairs starting with 1/5+ 4/5.** **What fraction pair is next?** **When we have shown them that 2/5+3/5= as a fraction pair ask the children** **Have we found all the possibilities?** **Carry on with 3/5+2/5= and ask the children** **Do you notice anything?** **Do we need to carry on?** **If yes, carry on and show them that 4/5+1/5 is the same as 1/5+4/5**	**What if children do not know that 1/18 and 17/18 is the same as 17/18 and 1/18?** **If this does not come up in questioning, this will need to be explicitly mentioned.** **What if children think that all visually different variations of a fraction pair count as different examples?** **If this does not come up in questioning this will need to be told the figures are to help us find the fraction pairs. We are sorting the fraction pairs, not the figures.**

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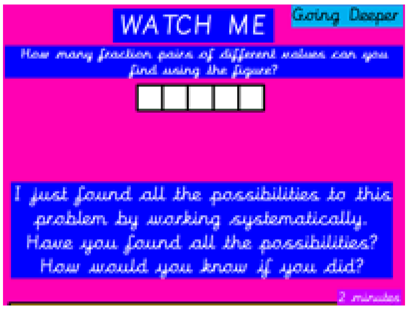
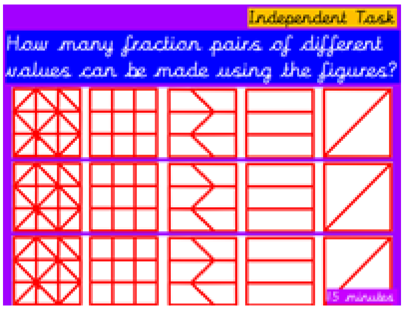
<td></td>
<td>

Bring the children's work up from the talk task and ask the children: What is the same and what is different?

We have to start this with 1/18 + 17/18 does anybody have a different figure with the same value. The answer is no, move through quickly getting to 2/18. Children should be able to see fairly quickly that they have a square that is the same. Then move straight into the independent task.</td>
<td></td>
</tr>
<tr>
<td>Independent Activity. (max 20 mins)</td>
<td>
</td>
<td>What if children think they just need to keep adding one to the denominator?
Children may need reminding that there are only halves, 3rds, 6ths, 9ths and 18ths in the figure.
What if children are working unsystematically?</td>
</tr>
</table>

	The children problem using the systematic skills taught throughout the lesson. I will prompt them to being with either 17/18=1/18 or the other way round.	**Children will be asked why did you start with that one? Children will be using all 5 different figures during the independent task so that new data need not be generated.**
Plenary	**Modelling of the solution, using children's responses.**	

Appendix 9 – Transcript of the pause discussion - Cycle 1

(T) denotes the teacher of the lesson

Adam: Would you begin by returning to the two questions (statements)? For me picking up from the break would we want to find out if we had got all the possible solutions from the data that was put up on the board?

Dave: Erm I think I would have tried to finish the table first and then come back to the questions (sentences). I would use the table to show the patterns that would come up or as Adam says go to the statements to see what they have written and then go to the table and fill it in with their answers including the incorrect one from Aidan […] but make sure it is filled in so that the patterns are easily identified and then come back to how many answers they thought there were

John: Do we need to know if the pupils have got any more suggestions that have not yet been identified?

Adam: Would that come from how they have completed one of the two statements (questions) […] there might be some that say there are more possible answers.

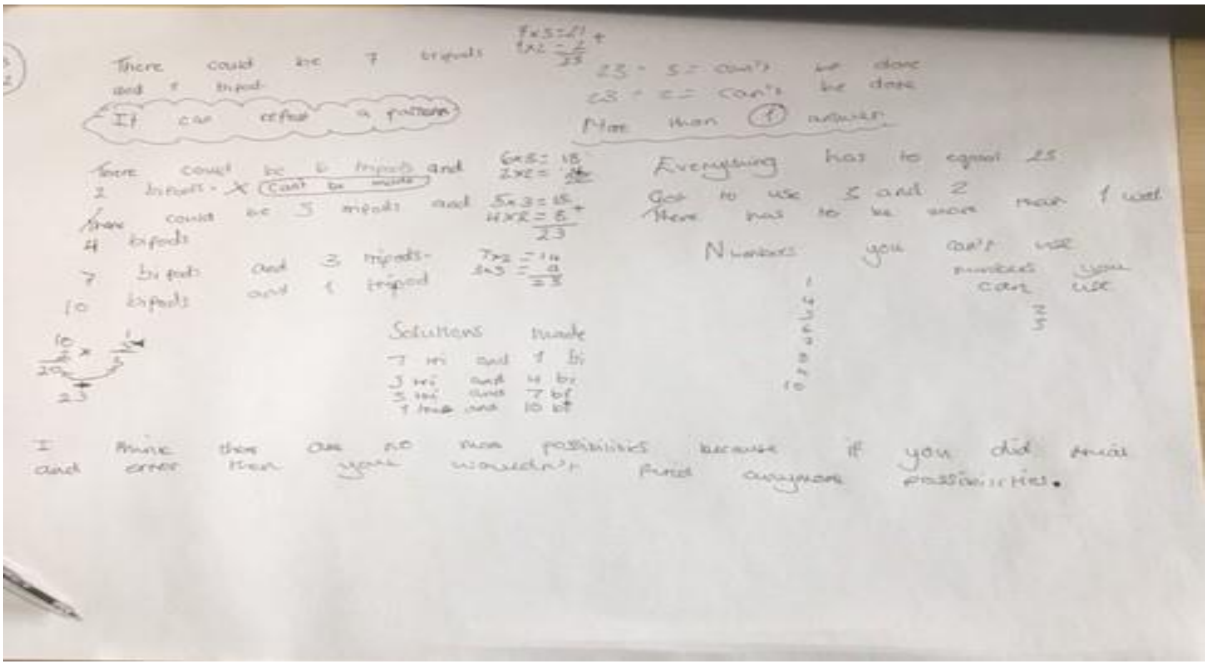
Dave: I see what you are saying, the table is the thing that makes everything really clear and you want to discuss other methods before we get to the easily identified pattern in the table, so yeah maybe look at other methods before the table. To be fair some have not shown any working but the ones who have listed the multiples and circled the answers […] it's probably best to go to them before you go the patterns in the table maybe?

Marie: Yeah I think starting with the multiples would be a good place to start and then yeah the table shows the patterns in a clearer way rather than…

Adam: It will do when the table is reordered.

Ruth: If I was doing this I would use the pupils work in some kind of an order so that each time I used the answer from a pupil it would improve on the answer from the previous one. So the first student I would start with is Thomas.

APPENDIX 9



Thomas's work

Ruth: Thomas showed that he was looking for patterns so he started with an answer which was 7, 1 then he did 6 and 2 then he did 5 but instead of doing 3 he did 4 because he realised that 6 and 2 did not work, so he realised that his patterns did not work. Then you have got Samuel who used multiples of 5 which is what we talked about when we were planning the lesson. He combined a tripod and a bipod to get 5 but that is really limiting because it's really hard to get all four answers from that one. Then you have got Aidan and Sienna who did 'lovely' multiples of 2 and 3 and number bonds to 23 and I think Sienna's work is fantastic the way in which she has presented it and the last one I have got is Nellie who was leading on to write the equation but she just doesn't have the skills to write the equation but she writes a wordy sentence, she writes an essay but what she was actually doing leads on to the equation […] she had worked out different multiples of 3 and then subtracted this from 23 ..isn't that the best one for leading on to the equation? I would finish with Nellie and use her to demonstrate how we can put her solution into a formula

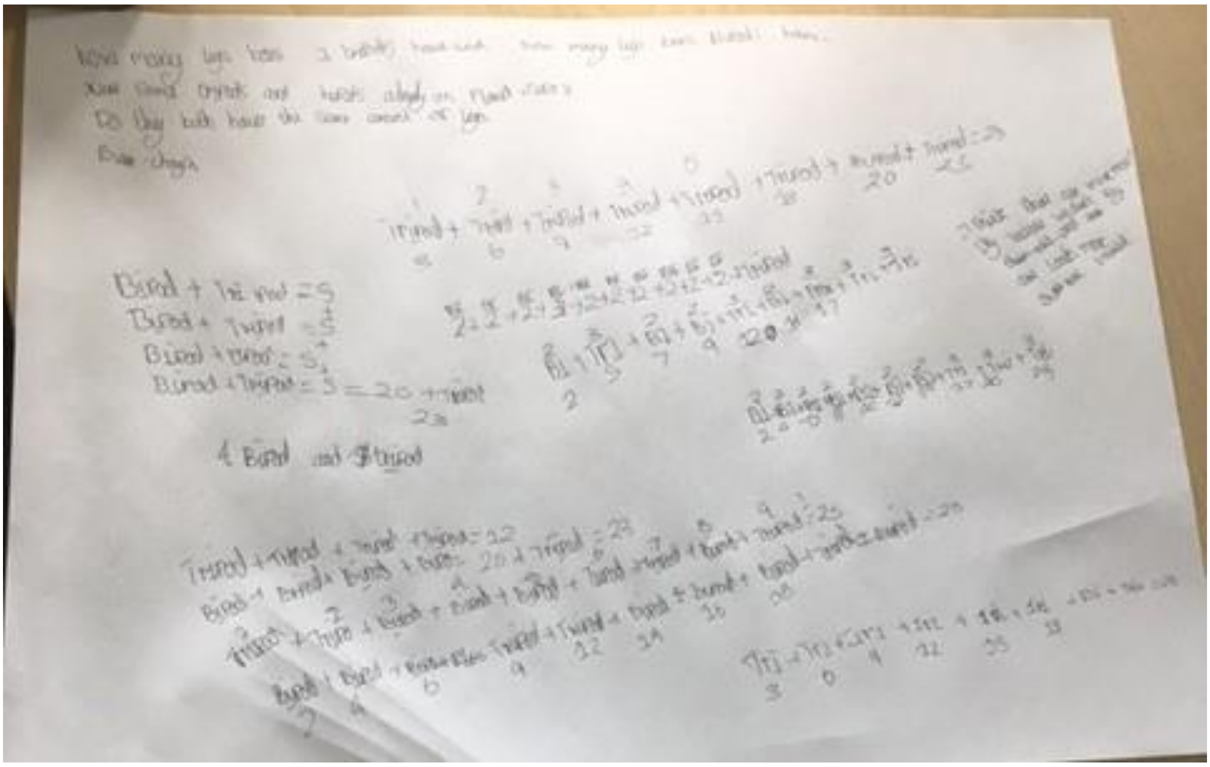
Dave: Where does the table fit in here?

Ruth: Err with number 3 Aidan and Sienna's number bonds leads onto the table.

(T): So are we going to start with Dave? I will show his work and what do you want me to ask, why he started there or why he made a change?

Ruth: Yeah that would be good, so you could say, so you have found a solution and you have thought of a pattern. You did 7 and 1 then 6 and 2 why did you go to 5 and 4 instead of 5 and 3…hopefully you can pick out his comment that it doesn't work.

Ruth: I would then look at Samuel's work who wrote multiples of 5.



Samuel's Work

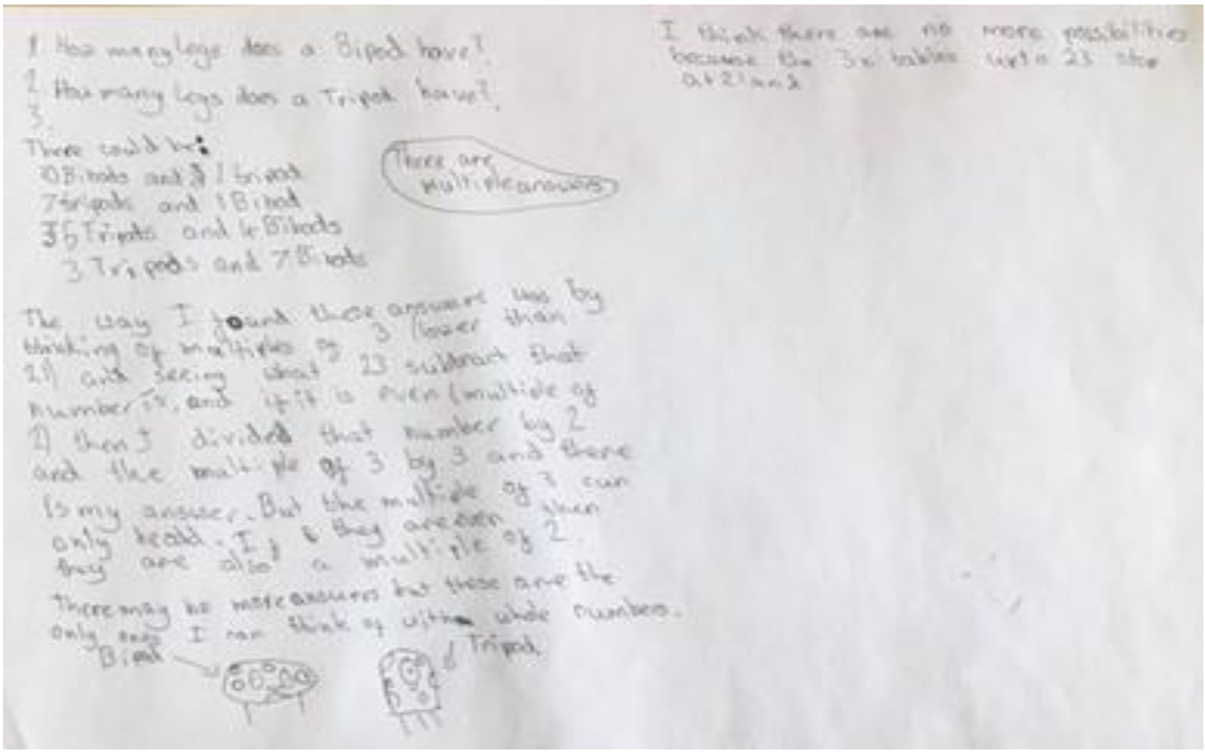
Ruth: He combined 1 bipod and 1 tripod to get 5. So he was going up in 5's he then changed his strategy again […] I mean both Thomas and Samuel had different starting points I think they both realised that their starting point was limiting and they both changed to using multiples of 3 and multiples of 2.

John: What are we trying to get them to do? Are we trying to get them to solve the problem in a particular way?

John: There was one pupil who chose a number of tripods and then took that away from 23 and then tried to divide by 2… I just wonder if this table should try and include every method? We could for example build the table and record 1,2,3,4,5,6,7,8 tripods to realise that you do not need to go up to 8….so if Marie (T) started with one of Thomas's solutions and put it into the correct column in the table.

(T): So I would take Thomas's first then what about Aidan's incorrect answer?

Dave: I think we should get all of the correct answers first and complete the table in the right columns and then go to Nellie's solution with that worded approach. Then she could talk you through why she did not get the right answer.



Nellie's work

Ruth: Yes, she has got the answer but then […] it is because she has done her multiples of three and then subtracted them from 23 and I thought that was so close to an equation and this would lead into those last couple of slides which is where we want the pupils to be able to see the equation

Adam: Yeah that is good

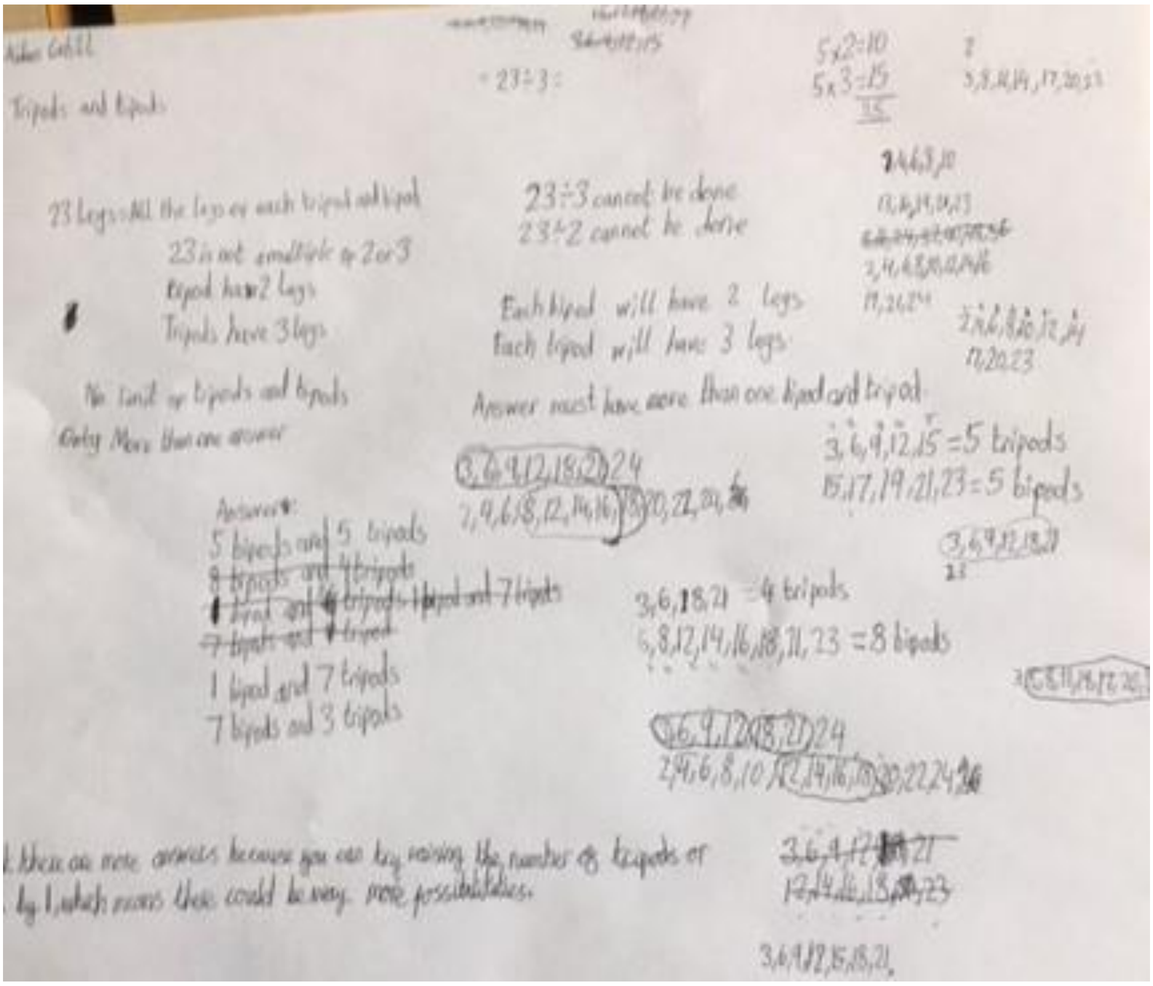
Dave: Yeah good

(T): So I am going to start with Dave and show his solution for 7 and 1 and then ask why did he change his method?

Dave: Well he then went to 6 and 2 next and that did not work so you need to highlight that […] then what you can say is that you have changed, then you can go to Samuel because he has got the 5 and 4 which can go in the table.

Ruth: And again he changed his strategy because he realised that if he just used multiples of 5 he didn't think he would get all of the answers

Dave: And then you can go to Aiden's?



Aiden’s work

Dave: As well as his incorrect answer he has the 7 and 3 answer which is correct which we could also use as we are using an incorrect answer

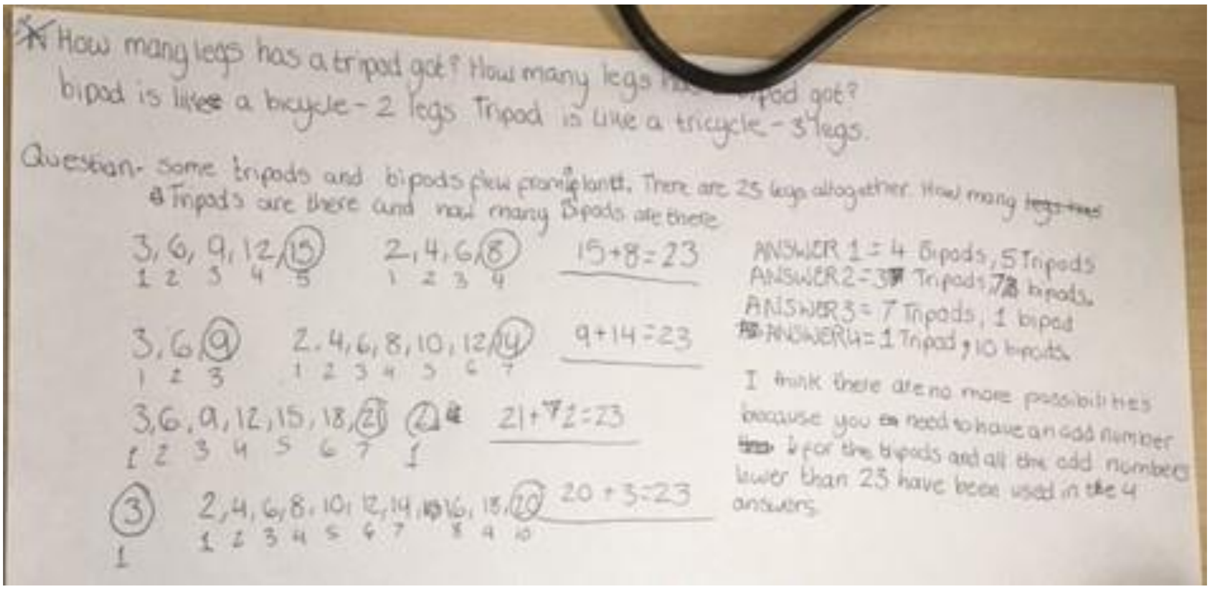
(T): And then I am going to go […]

Dave: To Sienna for the final one?

(T): I thought I was going to […]

Ruth: Well Sienna for the final answer

Dave: And then to Nellie for the ones in between?



Sienna's work

Ruth: She has 4 answers and she has used the multiples of 2 and 3 and number bonds to 23.

Dave: So she will have the one tripod and 10 Bipods to fill in […] so that is all of them.

Adam: Can I just ask then at any point are we going to refer to the two statements that we asked them to complete?

Dave: Oh yeah.

Ruth: We have already asked them to complete the sentence now we could ask do you still agree with the statement you completed in session one because at least one kid has written [...] I think there are many more possibilities

Adam: We could ask if they have changed their mind how has the table helped them in doing that?

Ruth: Oh Good

(T): So I am going to Thomas first, then Samuel, then Aiden then Sienna then I am going to Nellie. I am going to jump in between… this is the only bit where John said share the 1 to 8 that I am unsure of…

Dave: So after Sienna's there are missing gaps in the table so there we would say what would go in these.

Adam: Is that when the girl that developed the word equation comes in?

Dave: Yeah Nellie

(T): And then I show the table or do I fill in the table

Adam: I guess this will depend on time but showing the table filled in will lead to the equation.

(T): Do I show the table to develop the algebra or develop the algebra for the workings on the whiteboard?

John: Oh I see

Ruth: Showing the completed table on the PPT will make it easier for them to see the connection between M and N.

John: Will we show the numerical calculations alongside the M and N formula?

(T): So we're there then?

Ruth: Yes

Appendix 10 – Transcript of Orchestration Sequence

Chloe:	Why did you start with an odd number of tripods?
Lawrence and Luca:	We wanted to find the number of tripods because the tripods has to be an odd number?
Chloe:	Why does the number of tripods have to be odd?
Lawrence: Luca	If it isn't an odd number then the number of bipods would be odd and bipods have two legs.
Chloe:	So we know that any number of bipods will give us an even number of legs? Looking at your solution what did you do to start with in terms of thinking about the number 23?
Lawrence: and Luca	So we started thinking about what numbers would work and we began with 15 because 5 multiplied by 3 equals 15.
Chloe:	So you started thinking about multiples of 3 and then what did you do?
Lawrence: and Luca	So we then thought that 15 plus something must make 23.
Chloe:	So you started with 15, was that just a random number you thought of?
Lawrence: and Luca	No we thought what numbers we could rule out. So it could not be 1 because you have to have more than one and it could not be 9 because that is too many legs and 7 only leaves one leg.
Chloe:	So you ruled our 1 because it did not follow the rule, 9 because it was too many and 7 because that does not follow the rule?
Lawrence: and Luca	Yes.
Chloe:	So what did you do then?
Lawrence:	We worked out the missing number would be 8. We divided 8 by 2 and Luca which is 4 and so we thought we had 4 bipods.
Chloe:	So you've got 4 bipods and you had 5 tripods. And I think looking at everyone's solutions, everyone got 4 bipods and 5 tripods.

Chloe:	So you were thinking about the number 23 and what we needed to add together once you started with 15. So you were thinking, something add something equals 23? And so 5 lots of 3 plus something equals 23?

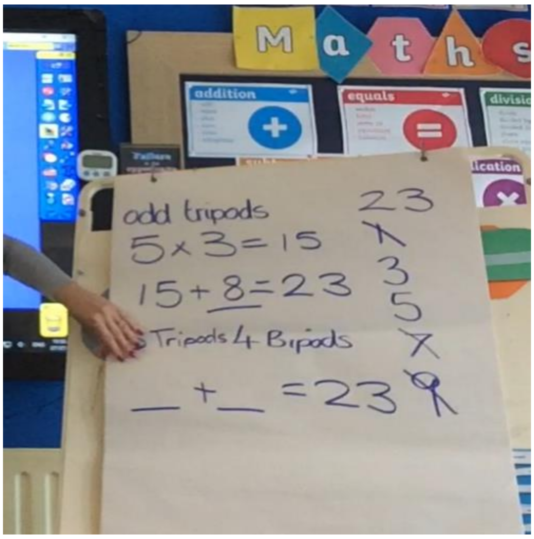


Figure 10.1 Orchestration Sequence 1

Figure 10.1 shows the collation of the first sequence following the discussion with Lawrence and Luca. Chloe next asks George and Will to explain their work.

Chloe:	George and Will. What did you do? (pause). What was your first bit of thinking?
George and Will:	Our first bit of thinking was that we could have two Tripods and 2 Bipods so 6 and 4 that makes 10.
Chloe:	So you did 2 Tripods which would give us 6 legs and 2 Bipods which would give us 4 legs. (pause)
George and Will:	And so 6 and 4 that makes 10 legs
Chloe:	And then what?
George and Will:	So we then got another 5 Bipods which is 10 and so we have now got 20 legs
Chloe:	So these are Bipods and these are Bipods and these are Tripods so we have got 20 legs all together?
George and Will:	Yes so then we added another Tripod so that makes 23 legs

Chloe: So you started with the minimum number of each we could have and then you kept adding multiples of 2 and 3?

George and Will: Yes, so we then got 3 Tripod and 7 Bipods

Chloe: So we have got 3 Tripods and 7 Bipods and I think that is the other solution that many of you found in this room Yes?

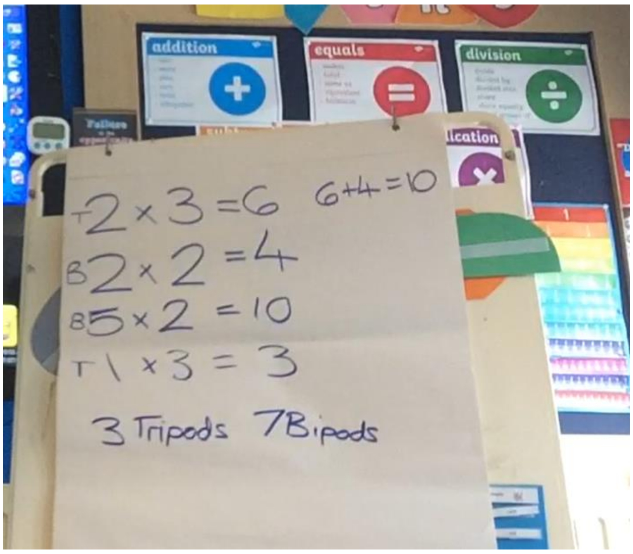


Figure 10.2 Orchestration Sequence 2

Chloe then explains that lots of people started with the number 15.

Chloe: Lots of you started with the number 15. Where did that come from? Amy would you like to explain?

Amy: Well I just knew it had to be 3 times something so I chose just 5

Chloe: So we always know that we are starting with 3 lots of…

Amy: … something

Chloe: So it's going to be 3 lots of our number of tripods. That is always going to be something that we need to work out yes? And lots of you we're thinking about multiples of 3 and is that how you arrived at 15? Is that what you did Ethan?

Ethan: Yes

Chloe: Why did you not start with 3 tripods?

Ethan:	It's just that 15 it's a good whole number. It's a big number and you can work on from it.
Chloe:	So it was a multiple of 3 that you quite liked and it was a good chunk of the 23 yes? OK
Chloe:	So we had 5 lots of 3 to make 15 so where did you go from there Ava?
Ava:	So we found out that we had 15 so then we thought how many two's do we need to add on to make 23.
Chloe:	So you found how many groups of 2 were needed so you did two lots of the number of Bipods so you knew that had to be 8 once you had got this 15.
Chloe:	So you did the inverse, you divided the 8 by 2 and that left you with how many Bipods?
Ava:	Four.
Chloe:	If we were going to put this in to a calculation we could say that we that we were doing three lots of the number of Tripods plus two lots of the number of Bipods
Chloe:	You are all looking at me blank. What we are doing is three times the number of Tripods plus two times the number of Bipods must be equal to twenty-three?
Chloe:	So if we did three times the number of Bipods plus two lots of the number of Bipods that must equal 23. Is that what we have all worked out?
Chloe:	So if I called the number of Tripods 'T' and wrote '3T' plus '2B' equals 23 would that make sense to you?
Chloe:	So what is this equation telling me Adam?
Adam:	So this shows you that you can work it out […] if you had 3 tripods you would write 9 and that would mean that you need 14 legs for the Bipods

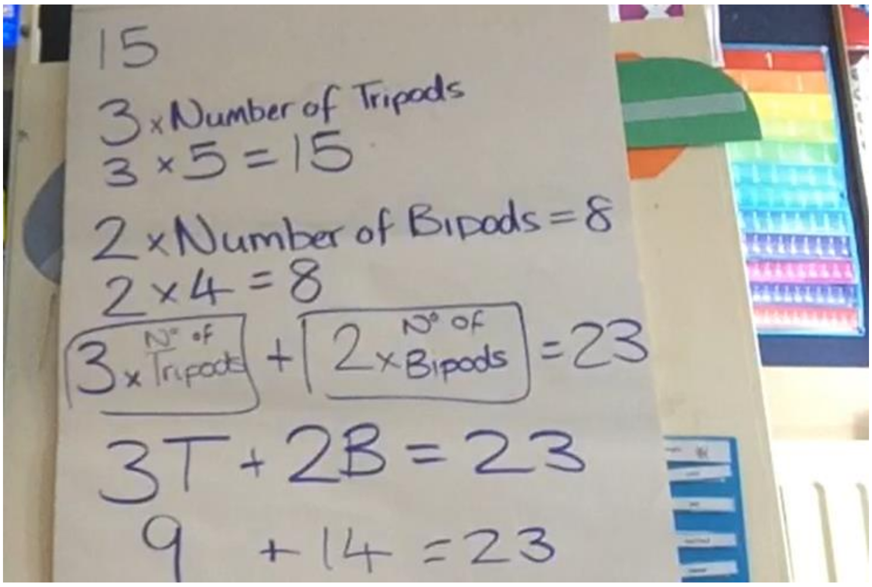


Figure 10.3 Orchestration Sequence Final

Appendix 11 – Anonymised example of coded responses from pre-study interview using TBI framework (Luft 2007)

Participant 1 [redacted]				
Q1. How do you maximise learning in your classroom?				
Traditional	Instructive	Transitional	Responsive	Transformative
		Task design is very important in order to secure feedback to address misconceptions	… and then to see where the lesson goes next (I suppose)	It is not just about coming to an answer it is important for children to explain their reasoning
Q2. How do you describe your role as a teacher of mathematics?				
	I need to know the subject inside out and the children know that I can answer their questions well	Direct teaching is not always the best way it is better if the children come to their own conclusions and can explain their reasons themselves	Sometimes I am a direct teacher sometimes I am leading them in their own discovery	

Q3. How do you know when your pupils understand?				
	Look in books	If a task is well designed if they have answered it well it tells me they understand	Deeper understanding comes from being able to talk around concept. I use discussion and independent tasks to assess	Reflects on experience of self as a learner. Children need to be able to talk about a concept and use it to solve problems

APPENDIX 11

Example of participant Pre-study interview analysis using Luft's framework (2007)

Participant 1 [redacted]				
Traditional	Instructive	Transitional	Responsive	Transformative
Q4. How do you decide when to move on to a new topic with a class?				
	Some of it comes from using the Maths Mathematics Mastery (scheme of work)	There is the opportunity for assessments within units and at the end of units […] I would say that too often it is led by wanting to get on to the next unit		
Q5. How do you know when learning is taking place in your class?				
	Circulation in the room looking and listening.	I know when children understand work they did not understand before	I know when there is experimentation taking place – example given of	

	See it in the quality of written work in books		an enrich task using square numbers You could hear from the questions that the children were asking - you can see the light bulb moments	
Q6. How do plan to teach the topic of problem solving?				
		The mathematics mastery programme – the problem solving come at the end. Some lessons involve building the skill and	The most successful lessons are where we are trying to find out	

Appendix 12 – Developing algebraic relationships using number cells

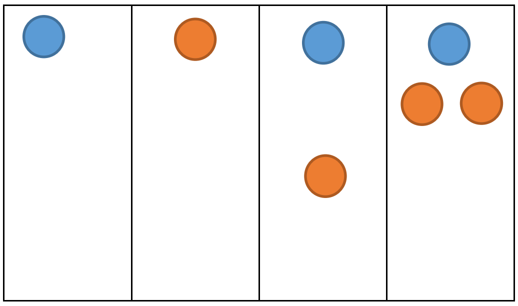
The number cell below is completed by first selecting two numbers, 5 and 6. Each preceding number if found by adding the last two numbers together.

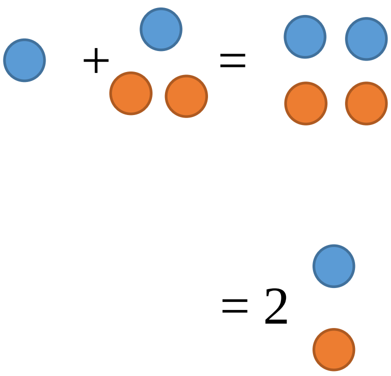
5	**6**	**11**	**17**	**28**	**45**

Point of departure 1

Below is a number cell with two numbers missing. Find the two numbers?

3	**?**	**?**	**93**





The above diagrammatic representation can show that the contents of the first cell plus the contents of the last cell sum to twice the contents of the third cell.

3 + 93 = 96

= 2(48)

APPENDIX 12

3	**?**	**48**	**93**

Diagrammatically it can also be shown the contents of the last cell minus the contents of the first cell is equal to twice the second cell.

93 – 3 = 90

90 = 2(45)

Here is a five cell

?	**?**	**26**	**?**	**69**

Find the missing numbers

Point of departure 2

a	**b**	**a + b**	**a +2b**	**2a + 3b**

69 – 26 = a + 2b

a + 2b = 43

a + b = 26

(a + 2b) – (a + b) = b

43 – 26 = 17

b = 17

Point of departure 3

a + b = 26

2a + 2b = 52

(2a + 3b) – (2a + 2b) = b

b = 69 – 52

b = 17

a + b = 26