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CLASSROOM NOTES

## A model of invasion by bodysnatchers from the far reaches of space

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**ABSTRACT**
A discrete model of invasion by subversive alien bodysnatchers is constructed and analysed numerically as a proposed undergraduate project. Several extensions of the model are demonstrated, including a spatial system of ten planets arranged on a one-dimensional lattice, and we discuss some of the practical considerations that arise when supervising such a project. Exercises for further student investigations of the model are suggested.

**KEYWORDS**
Mathematical modelling, Lyapunov exponents, student project, numerical investigation.

## 1. Introduction

Are you concerned about aliens from the far reaches of space abducting your friends and replacing them with insidious replicas? Have you noticed colleagues displaying a dearth of justifiable emotions such as adulation of your work and personality? They may have been replaced by alien duplicates as part of a conspiracy to seize control of Earth. Plots by militant extraterrestrial lifeforms to destroy humanity frequently appear in science-fiction [1]. One particular threat, known as bodysnatchers following their depiction in the 1950's science fiction novel [2] and film [3], is the possibility of aliens secretly abducting individual humans and replacing them with their own agents who have adopted the likeness and mannerisms of the victim but lacking in emotion (as portrayed in [4]). This term may also refer to parasites [5], or psychic machine gods beyond our comprehension [6], that can exert control over the minds and actions of others.

Fortunately for humanity, investigations of such threats can be conducted by undergraduate applied mathematicans, by constructing a population model and analysing its dynamics. Developing programming abilities using MATLAB or similar packages is an increasingly useful skill for such students. Whilst a introduction to the syntax and possibilities of these packages is often taught in a traditional lecture course, understanding how to bring the power of computational and numerical methods to bear on a problem may be better suited to problem-based learning: for example, by investigating a dynamical system as the subject of a final year

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undergraduate research project. In addition to the general benefits of project-based learning for understanding the subject material [7], this can introduce a student to the entire research process - from proposing and justifying the model, to developing the code, and visualising and interpreting results. Computational investigation can allow opportunities for students to contribute to current research [8]. It also provides context for the practical challenges that researchers face and which often present difficulties for a student when they start out on a research degree: how to deal with limitations of computation time, what level of accuracy is appropriate, what kinds of numerical errors are likely to occur and how to spot and account for them, how to wisely allocate time and effort, and how to deal with a problem that (unlike most lecture-course exercises) has not been deliberately hand-crafted with an intended result in mind. Such final year projects can be both effective learning tools *and* preparation to bridge the gap between student and practitioner.

Following investigations of zombies and cannibals [9,10], the purpose of this article is to demonstrate the construction of a model of potential interest to undergraduate students. As the final year project is often the first and only sustained independent piece of investigative work for undergraduates, a humorous or unusual topic can be helpful to encourage the instrinsic motivation that will be required [11]. For a modelling project, supernatural threats can be an entertaining application of epidemiology (consider Adams' excellent interweaving of a zombie attack narrative with a series of dynamical systems problems [12]). Furthermore, students can engage with encoding assumptions from entertainment outside of their degree course in the model, bridging the mental gap between "degree knowledge" and real-life intuition.

We shall illustrate some computational techniques from the theory of discrete dynamical systems that can be applied to analyse the model's dynamics. Suggestions are included for further extensions to the model that could form the basis of a short research project. When it comes to the subject of invasion by alien bodysnatchers from outer space, mathematics is no substitute for paranoia and interplanetary nuclear weapons, but we hope to demonstrate some of the potential applications of a mathematical analysis of such hypothetical situations.

## 2. Project outline

A modelling and population dynamics project could begin by introducing the student to the general predator-prey model (or its continuous-time equivalent):

$$N_{t+1} = f(N_t)N_t - g(N_t, P_t)P_t \quad (1)$$

$$P_{t+1} = \lambda g(N_t, P_t)P_t + \mu P_t \quad (2)$$

where $N_t$ and $P_t$ are the prey and predator populations at the $t^{th}$ time-step and $0 < \lambda < 1$ and $0 < \mu < 1$ are the predator's ecological efficiency and survival rate respectively. The student can investigate the effect of different choices for the prey growth function $f(N)$ and the predator's functional response $g(N, P)$ (some options are described in Section 3.1 as we construct our own model). Often, they will uncover some aspect that particularly interests them and that can set the direction for the rest of their project. However, they may require prompting with a more specific model.

In this paper, we will walk through the construction of an example discrete-time model in a science-fiction setting. Depending on the student's level of experience with population modelling, the envisioned pedagogical use is to provide an example of a model that the project student could be introduced to, ask them to reproduce the main results, and then invite them to attempt exercises from Section 6 or some model variants of their own devising. By explaining our modelling rationale and suggesting some alternative functions throughout, sufficient context is given that a different "base" model could be provided to the student if desired, and that the student should have enough understanding to formulate and justify some variations. Experiences at our institution have resulted in student projects focussing on strange attractors, comparative studies of different choices for $f(N)$ and $g(N, P)$ in the standard predator-prey model (1)-(2), and investigating the role of a time-delay in the prey reproduction mechanism.

As illustrated in Figure 1, we will organise our example in three broad stages of a student project in population dynamics and chaos:

- **Phase I:** Introduce the student to the basic concept of a population or predator-prey model, and suggest some reading from standard mathematical biology texts (e.g. [13]). This can be accompanied by assigning a writing task (e.g. describe the logistic map) to motivate their reading and get them acclimatised to mathematical writing from the start in their project, and a short programming task to ensure they are acquainted with a suitable language for numerical analysis.
- **Phase II:** Select a model and undertake analysis using standard techniques:
  - Analytical results such as equilibria and boundaries of the phase space (Section 3.2).
  - Numerical simulation to calculate the largest Lyapunov exponent and identify attractors, chaos and periodicity (Section 3.3-3.5).
- **Phase III:** Extension of the model, for example space or stage-structuring of the populations, or investigating further questions such as those suggested in the exercises section (Sections 3.5, 4 and 6).

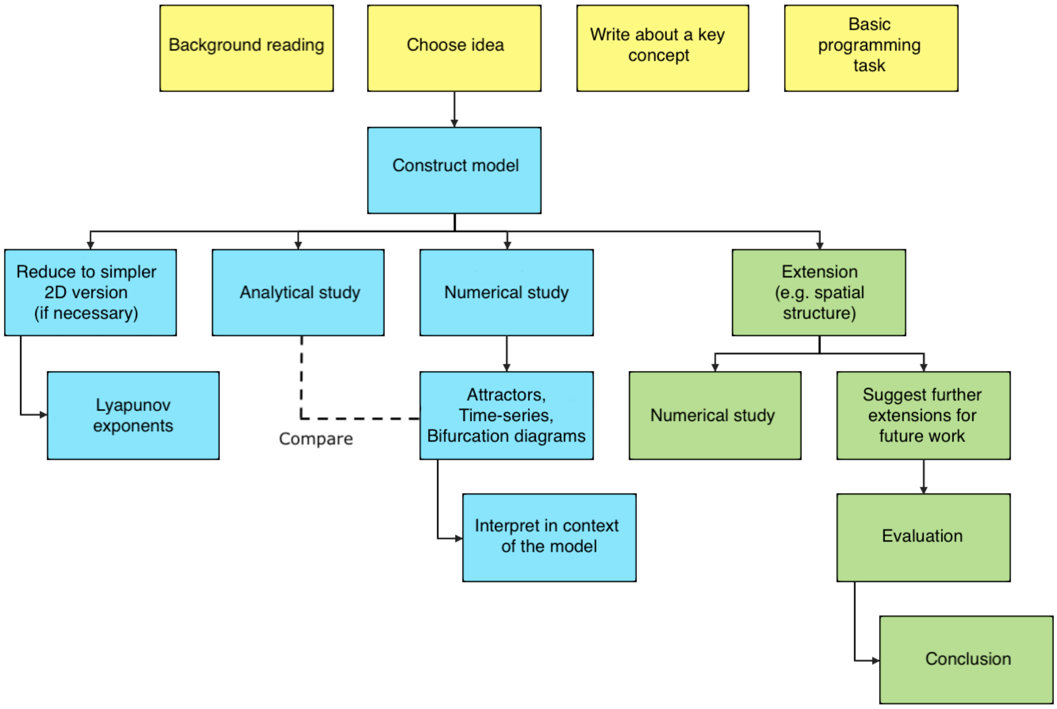



**Figure 1.** Flowchart of typical stages in a modelling and dynamical systems student project:
- Phase I (Yellow)
- Phase II (Blue)
- Phase III (Green)

## 3. A discrete-time model of bodysnatchers on one planet

### *3.1. Designing the model*

We will model the activity of a particular kind of bodysnatcher threat: an alien species that abducts individual humans, replacing them with replicas of their own species. However, we shall highlight how minor modifications could produce a model for alternative forms of bodysnatcher such as a psychic menace that indoctrinates members of the human population through mind-control, or a parasitic hivemind that infects humans through fungal spores to the same end.

We begin developing our model of a bodysnatcher invasion by first considering the simplest case: a single planet, with two populations at time-step $n$ being the humans $x_n$ and the bodysnatchers $y_n$. We choose to construct a discrete-time model, as these are both easier to work with and can feature interesting dynamics including chaos even in only a one-dimensional system. To derive the formulae for how both populations will update every time-step, let us consider the process in discrete substages.

(1) We begin the $n+1^{th}$ time-step with a human population $x_n$ and bodysnatcher population $y_n$.

(2) First, the human population is updated, accounting for reproduction and natural mortality in the absence of bodysnatchers. Standard, simple growth functions include Malthusian growth (unbounded and suitable only for small populations with an abundance of resources), logistic growth, or Ricker growth. To limit the human population by competition for resources, a reasonable choice with the potential for interesting dynamics [14] is the logistic map with growth parameter $r$ and carrying capacity $c$:

$$x_n' = \frac{r}{c} x_n \big(c - x_n\big) \tag{3}$$

$$y_n' = y_n \tag{4}$$

(3) Second, bodysnatcher activity takes place.

The major choice to be made here is the form of the functional response. In general, this term $g$ governs the number of prey $N$ killed per predator $P$, which in this model is analogous to the number of humans converted or replaced by the bodysnatchers. The simplest choice for a functional response is Lotka-Volterra [15], where the number of "prey" killed scales linearly with their population. To prevent the "predators" from acting with an unlimited appetite, we can instead use Holling Type II [16] (derived by considering the maximum number of prey a predator can eat regulated by the time it takes to handle a kill) or Beddington-deAngelis [17,18] which is further limited by the amount of predators present in order to account for interference competition between predators as they hunt. Many more variants have been proposed, and students should be encouraged to consider what are the most important limitations on the hunting mechanism for their model, so that an appropriate functional response can be selected.

For our purposes, let's assume that bodysnatchers generally continue to act out the lives of the humans they replace. So unlike a traditional predator-prey hunting model, on a planetary scale an individual bodysnatcher will stay in one place rather than overtly hunting over the entire planet. As a result, we can expect the effectiveness of an individual bodysnatcher to, on average, be limited by the number of unconverted humans in it's vicinity on the planet. That is, their abduction activity will be most effective when their *density* in the overall planetary population is low. If the bodysnatchers already constitute a high fraction of the population on a planetary scale, most will be ineffective as their neighbours (family, friends and colleagues) may already have been replaced. This suggests that we model the bodysnatcher activity using a ratio-dependent functional response [19]:

$$g(N,P) = g\left(\frac{N}{P}\right) = \frac{\alpha(N/P)}{\beta(N/P)+\gamma} = \frac{\alpha N}{\beta N + \gamma P} \tag{5}$$

To maintain simplicity, we will choose $\alpha = \gamma = 1$. To scale the effectiveness of the bodysnatchers, $\beta$ is replaced by a parameter that measures the level of alertness of the humans regarding the bodysnatcher threat in their midst, called resistance ($0 < R < 1$). The greater the numbers and alertness of the human population, the fewer successful attacks are managed by the bodysnatchers.

$$g(N,P) = g\left(\frac{N}{P}\right) = \frac{N}{RN + P} \tag{6}$$

Of the number of subsequent abductions, a fraction $0 < B < 1$ result in successful replacement by a bodysnatcher individual. This parameter controls the reliability of the process by which the bodysnatchers are able to construct a synthetic replica to the deceased human, and is equivalent to ecological efficiency in a standard predator-prey model.

In addition, during this stage some bodysnatchers are discovered and executed by the human population. To regulate this without introducing additional free parameters, we re-use resistance, normalised by the carrying capacity of the planet, so that if the human population (after reproduction) is maximal, then $R$ is the fraction of bodysnatchers who are identified and eliminated. We will assume that the number of bodysnatchers that perish naturally or in other ways is negligible by comparison[1] and that they are unable to reproduce biologically with the humans or each other. Consequently, the only terms in the bodysnatcher function are loss to human discovery and gain due to this functional response, and there is no (for example) logistic growth term due to breeding.

Combining these processes of bodysnatcher abduction activity and their

[1] To model a type of bodysnatcher who seizes control of the host biological human (rather than replacing them with a replica), this would need to be amended unless the parasite's influence granted longevity.

persecution by the humans results in the following equations:

$$x''_n = x'_n\left(1 - \frac{y'_n}{Rx'_n + y'_n}\right) \tag{7}$$

$$y''_n = \left(1 - R\frac{x'_n}{c}\right)y'_n + B\left(\frac{x'_n}{Rx'_n + y'_n}\right)y'_n \tag{8}$$

(4) This completes one full iteration.

$$x_{n+1} = x''_n \tag{9}$$

$$y_{n+1} = y''_n \tag{10}$$

Combining substages 1-4, and ensuring only non-negative values (an essential constraint for a model of ecological populations), we obtain the following formulae for one iteration:

$$x_{n+1} = \max\left\{0, \frac{r}{c}x_n(c - x_n)\left(1 - \frac{y_n}{R\frac{r}{c}x_n(c - x_n) + y_n}\right)\right\} \tag{11}$$

$$y_{n+1} = \max\left\{0, y_n\left(1 - \frac{Rr}{c^2}x_n(c - x_n) + \frac{B\frac{r}{c}x_n(c - x_n)}{R\frac{r}{c}x_n(c - x_n) + y_n}\right)\right\} \tag{12}$$

In this section, the carrying capacity of the planet will be $c = 10000$.

### *3.2. Analytical study*

There are three sets of fixed points for this two-species system. If the humans are eliminated, the bodysnatchers will remain in a steady state. Thus there exists an infinite set of fixed points of the form[2]:

$$(0, y) \qquad \forall y \geq 0 \tag{13}$$

If instead the bodysnatchers die out, the formula for the human population reduces to the standard one-dimensional logistic map which has the additional fixed point:

$$\left(c\left(1 - \frac{1}{r}\right), 0\right) \tag{14}$$

[2]If the type of bodysnatcher being modelled is one which takes over a biological human's body, then this set of fixed points will eventually decay to $(0, 0)$.

Finally, there may exist interior fixed points for $x \notin \{0, c\}$, and $y \neq 0$. The $x$-values are provided by the solutions to the cubic equation:

$$x^3 - 2cx^2 + c^2x - q = 0, \quad \text{where} \quad q = \frac{c^3 B}{R^2 r^2} \tag{15}$$

The corresponding values of $y$ are given by:

$$y = \frac{Bc}{R} - R\frac{r}{c}x(c - x) \tag{16}$$

We will comment on the stability of this fixed point in Section 3.3.2.

### *3.3. Lyapunov exponents*

To classify the dynamical behaviour of the map, we calculate the characteristic Lyapunov exponent, $\lambda_1$. This quantifies the "stretching" of nearby orbits under the action of the map, and so provides a quantifiable measure for the "sensitivity to initial conditions" which is a fundamental criterion of chaotic dynamics. For a discrete orbit $(\underline{\mathbf{x}}_n)_n$ under a general vector-valued map $\underline{f}$, this largest exponent is defined by:

$$\lambda_1 = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln(|\underline{f}'(\underline{\mathbf{x}}_k)|) \tag{17}$$

This measures (on a logarithmic scale) the average absolute gradient of the map along the path of the orbit, akin to determining how far the orbit would be separated from an arbitrarily close point at the next iteration of the map. Calculating this quantity is non-trivial for most systems with more than two dimensions, however a procedure for practical calculation in the two-dimensional case is provided by Sprott's textbook [20], which we reproduce here:
Let

$$J_n = J(\underline{\mathbf{x}})|_{(x_n, y_n)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{18}$$

so that $A, B, C, D$ are the entries of the Jacobian matrix evaluated at the $n$th iteration. Then we define a rescaled variable (this is necessary as otherwise the relative sizes of certain quantities could tend to infinity as the calculation progresses.) :

$$y'_{n+1} = \frac{C + Dy'_n}{A + By'_n}. \tag{19}$$

The maximal Lyapunov exponent can then obtained by:

$$\lambda_1 = \lim_{n \to \infty} \frac{1}{2n} \sum_{k=1}^{n} \log\left(\frac{(A + By'_k)^2 + (C + Dy'_k)^2}{1 + {y'_k}^2}\right) \tag{20}$$

Calculating Lyapunov exponents provides a powerful tool for classification while introducing students to practical issues involved with numerical analysis. They must decide how many transients are necessary to be reasonably confident of having arrived at the attractor, and choose how many iterations to use to get a "good enough" approximation of the true value of the Lyapunov exponent, all the while ensuring that calculations do not take too long to perform. The is also opportunity for students to take the initiative in pursuing leads as an independent researcher should - starting with a broad scan of the parameter space, identifying areas of interest and zooming in for a closer look.

We begin with an overview of the parameter space. As $B$ controls the fraction of successfully replaced abductees, it must be within the range $0 < B \leq 1$. The human growth parameter $r$ must lie within $1 < r < 4$ to give non-zero final behaviour in the absence of bodysnatchers, and the resistance level is also in the range $0 < R < 1$. We sample at 100 increments for each of these three ranges, to produce Figure 2 from a three-dimensional scatter plot of $10^6$ data points. Note that a common pitfall for students new to computational modelling is to fail to include a test for $|\underline{\mathbf{x}}_n| < \epsilon$, with $\epsilon$ being some small value, when classifying the extinction case, as it may be that the system is very slowly converging to the origin when growth rates are insufficiently large. Students should be encouraged to make additional elementary deductions on their own such as how to determine if the orbit has arrived at a fixed point by testing if $|\underline{\mathbf{x}}_{n+1} - \underline{\mathbf{x}}_n| < \epsilon$.

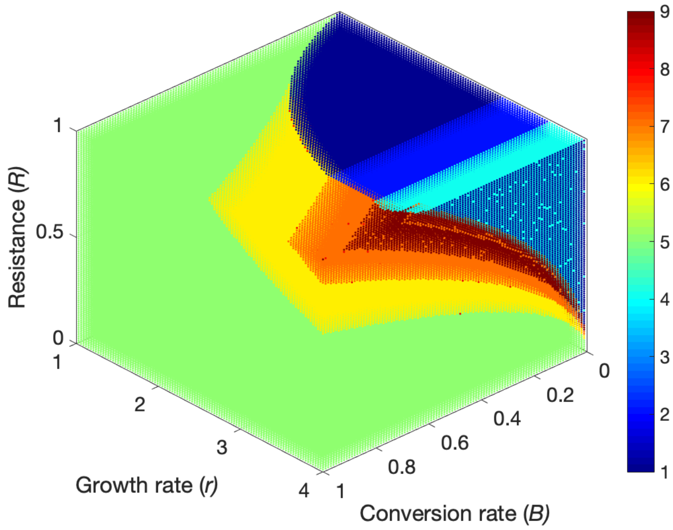



**Figure 2.** Parameter space classification overview

The legend for Figure 2 (and others illustrating the parameter space, unless stated otherwise) is as follows:

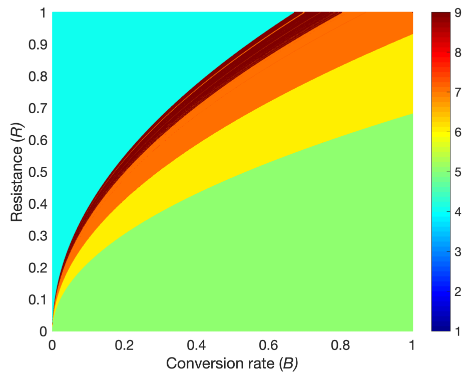
(1) Death of bodysnatchers only; The human population has maximal Lyapunov exponent $\lambda_1 < 0$; Period 1
(2) Death of bodysnatchers only; $\lambda_1 < 0$
(3) Death of bodysnatchers only; $\lambda_1 \approx 0$
(4) Death of bodysnatchers only; $\lambda_1 > 0$
(5) Death of humans only; $\lambda_1 \approx 0$; Period 1 (This is the only possible outcome for

bodysnatchers-only.)

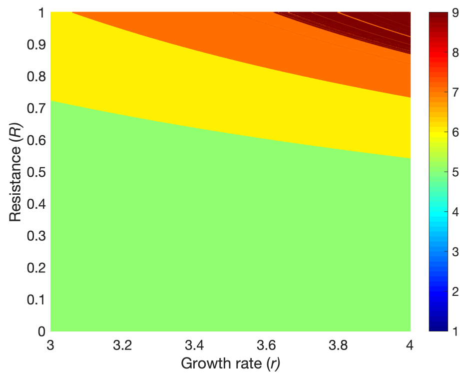
(6) Both populations survive; $\lambda_1 < 0$; Period 1
(7) Both populations survive; $\lambda_1 < 0$
(8) Both populations survive; $\lambda_1 \approx 0$
(9) Both populations survive; $\lambda_1 > 0$

All states are clearly visible in Figure 2, except for (3) and (8) which occur only at bifurcation points in this model, and do not lay claim to sustained regions of the parameter space.

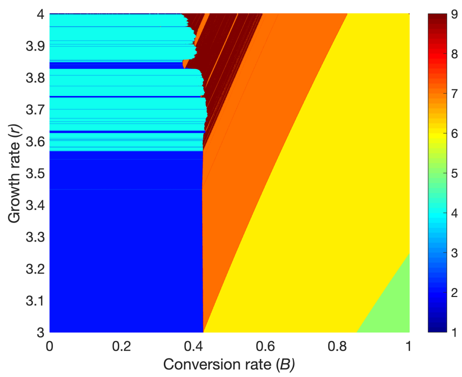
Motivated by these results, we take three more detailed cross-sections of the parameter space (Figure 3): a $2000 \times 2000$ examination of the $(B, R)$-parameter space across the full range, with the human growth parameter fixed at $r = 3.8$ (Figure 3(a)), a $2000 \times 2000$ slice of the $(r, R)$-parameter space at $B = 0.7$ (Figure 3(b)) with the growth parameter being constrained to the range associated with period and chaotic behaviour $3 < r < 4$, and finally a $2000 \times 2000$ slice of the $(B, r)$-parameter space at $R = 0.8$ (Figure 3(c)).



(a) Human growth rate: $r = 3.8$



(b) Conversion rate: $B = 0.7$



(c) Resistance: $R = 0.8$

**Figure 3.** Parameter space classification for three slices

All three images illustrate the classic period-doubling route to chaos within a space of coexistence between the two populations, most clearly as either the human growth rate or the resistance level increases.

#### *3.3.1. Fixed r*

If the conversion rate $B$ from abducted humans to new bodysnatchers is too low, or if the strength of human resistance $R$ is too great, then the bodysnatchers will fail to sustain themselves and will die out (Figure 3(a)). In this event, the system collapses to the well-studied one-dimensional logistic map, with the resulting human behaviour determined by the capacity and growth parameters. For $r = 3.8$, this results in two-dimensional chaos smoothly transitioning to one-dimensional chaos as the boundary of bodysnatcher survival is crossed. For any non-zero conversion rate $B$, a minimum level of resistance is required to prevent the human population from being completely killed off, and the greater the conversion rate the greater this threshold for coexistence (Figure 3(a)).

#### *3.3.2. Fixed B*

The smooth boundary between bodysnatcher-only (green) and period-1 coexistence (yellow) is determined by the nature of the solutions to the cubic equation for the interior fixed points: below the boundary two of the three roots are complex, and when it is crossed by increasing the resistance and the two roots become real, one of them is attracting. Hence the final behaviour of the system immediately transitions to this real fixed point solution. The second root that becomes real simultaneously is never stable in this region of the parameter space, and the third solution to the cubic equation is always real but also always unstable in this range of $(r, R)$ with $B = 0.7$. As resistance increases further in Figure 3(b), the system bifurcates from this coexistance fixed point to periodic and eventually chaotic behaviour. This occurs due to the fixed point losing its stability, although it remains as a real solution. The status of this fixed point across the parameter space is shown in Figure 4, and is determined numerically by evaluating the Jury conditions for stability [21] across this space:

For $z \in \mathbb{C}$ we define the function $F$ by:

$$F(z) = z^2 - tr(J^*)z + det(J^*), \tag{21}$$

where $tr$ and $det$ denote the trace and determinant, and $J^* = J|_{(x^*,y^*)}$ is the Jacobian matrix of the map evaluated at the fixed point $(x^*, y^*)$. Then the Jury conditions state that the fixed point is linearly stable precisely if all three of the following are satisfied:

$$F(-1) > 0, \qquad F(1) > 0, \qquad 1 - det(J^*) > 0. \tag{22}$$

While the conversion rate is fixed at $B = 0.7$, increasing the growth rate of the humans lowers the minimum resistance necessary for allowing the humans to survive and coexist, and then also lowers the resistance levels required to trigger periodic and chaotic behaviour (Figure 3(b)). It is not surprising that $r$ and $R$ have similar effects in this respect, as increasing either results in larger human populations.

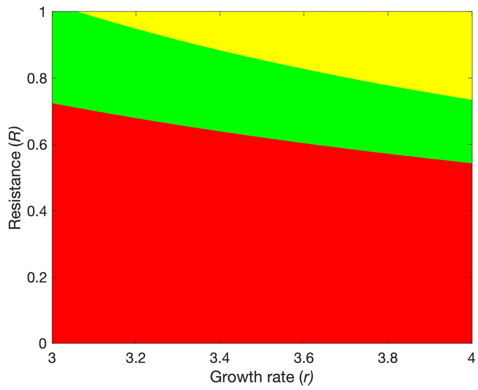



**Figure 4.** Existence and stability of one interior fixed point solution when $B = 0.7$:
Red = Complex
Yellow = Real but unstable
Green = Real and stable

### *3.3.3. Fixed r*

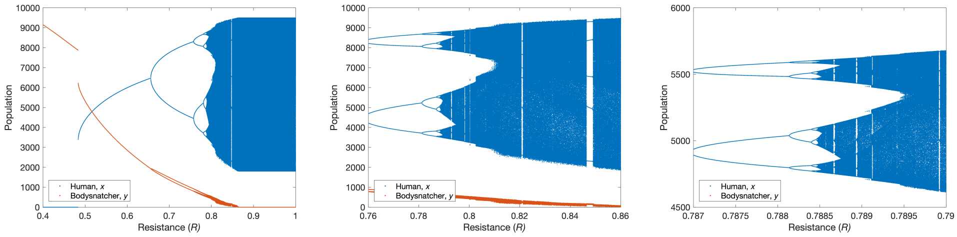
When the resistance level is fixed at $R = 0.8$, we see again the minimum threshold of conversion rate $B \approx 0.427$ required for the bodysnatcher population to become self-sustaining (Figure 3(c)). If this is reached, the system immediately changes from human-only behaviour (typically periodic if $r < 3.56$, and possibly chaotic if the $r$ value is sufficiently large) to periodic or chaotic coexistence. As the conversion rate increases, the human population suffers greater depletion, and so a larger growth rate is required to achieve the same dynamic behaviour.

### *3.3.4. Summary*

Overall, provided that the growth parameter $r$ for the human population is within the window necessitated by the logistic map, for coexistence (whether ordered or chaotic) we require that the rate of conversion $B$ is sufficiently large to sustain the bodysnatcher population, and then that the human resistance $R$ window lies within an interval dependent on this value - too low resulting in the humans being replaced to extinction, and too high causing the bodysnatchers to perish and the humans to survive independently.

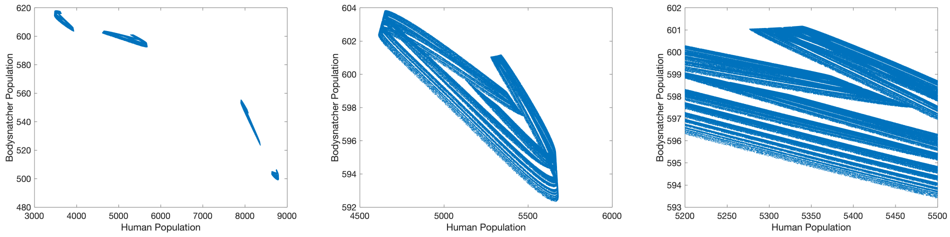
## ***3.4. Further numerical analysis***

To illustrate these dynamics further, we take a vertical slice of the parameter space from Figure 3(a) at $B = 0.5$, and produce the corresponding Feigenbaum diagrams, showing the final behaviour of an orbit over 1000 iterations after $10^6$ transients (Figure 5). An example of a strange attractor for human-bodysnatcher co-existence from the chaotic region of the parameter space is given in Figure 6.




(a) Full range (b) Zoom 1 (c) Zoom 2, human population

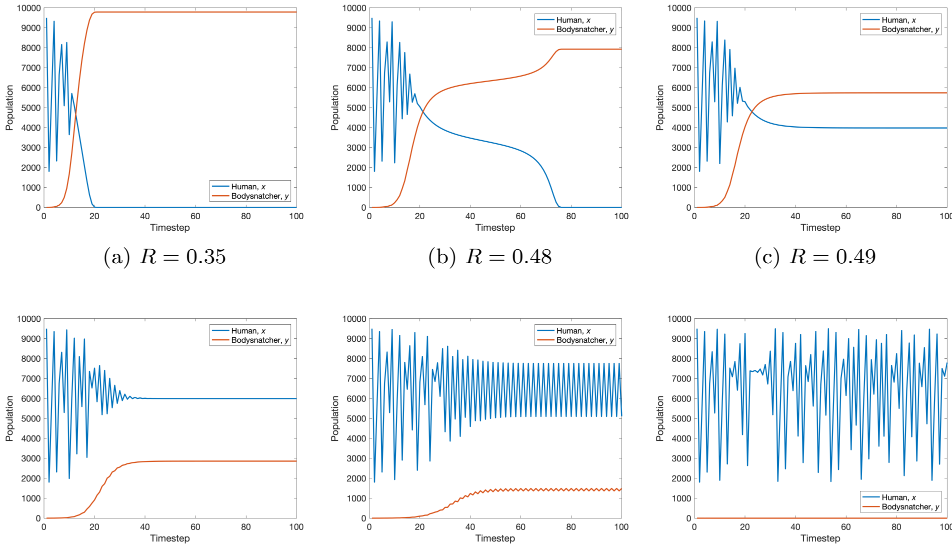
**Figure 5.** Feigenbaum diagram of attractors across a range of $Rr = 3.8, B = 0.5$, and $10^6$ transients




(a) Full attractor (b) Zoom 1 (c) Zoom 2

**Figure 6.** Example of coexistence strange attractor in $(x, y)$ phase space ($r = 3.8, B = 0.5, R = 0.79$ and $10^7$ transients)

Finally we show the time-series of the system over the first 100 iterations from a selection of values of resistance $R$ in Figure 7. We can see the transition from bodysnatchers-only when resistance is low (Figure 7(a),(b)), to fixed-point coexistence (Figure 7(c),(d)), periodic coexistence (Figure 7(e)) and finally human-only chaos when the resistance is too great to allow the bodysnatchers to survive (Figure 7(f)). We note that all of these occur for human growth parameter $r = 3.8$, and so the introduction of bodysnatchers at a variety of intermediate resistance levels is able to stabilise the human population from its naturally chaotic dynamics to ordered behaviour - either fixed (Figure 7(c),(d)) or periodic (Figure 6(e)). This has some potential benefits to human society, making demand for food and other supplies more predictable, and ensuring that the potential workforce or customer base does not dip too low in certain years. Should we suspect a government or megacorporation conspiracy as being the source of the bodysnatcher threat?




(a) $R = 0.35$ (b) $R = 0.48$ (c) $R = 0.49$

(d) $R = 0.60$ (e) $R = 0.70$ (f) $R = 0.95$

**Figure 7.** Time-series of first 100 iterations ($r = 3.8$, $B = 0.5$)

### *3.5. Dynamic resistance*

Compared to the human growth parameter and the conversion rate, it seems plausible that the resistance level of the population could shift rapidly in response to circumstances during the simulation, as the general populace's activity and mindset adjusts to awareness of the bodysnatcher invasion and potential calamity. To model this, whilst also eliminating a parameter of the model to be varied, we propose a rule for how resistance could be updated at each iteration. First, the ideal resistance level is calculated as the fraction of the population that are bodysnatchers.

$$R_d = \frac{y_n}{x_n + y_n} \tag{23}$$

However, information and awareness is slow to spread, and humans may not be willing or able to immediately adjust their outlook. Thus, at each time-step, we merely move halfway towards what would be the "appropriate" resistance level for the current severity of the threat.

$$R_{n+1} = \frac{1}{2}\Big(R_n + R_d\Big) \tag{24}$$

We will compare this resistance rule with an alternative delayed resistance that recognises the unique method of a subversive bodysnatcher invasion. Unlike zombies or a more overt threat, when most of society has already been overcome and the bodysnatchers are everywhere, they exert their malign influence through the media and friends in an effort to convince the remaining humans that everything is alright.

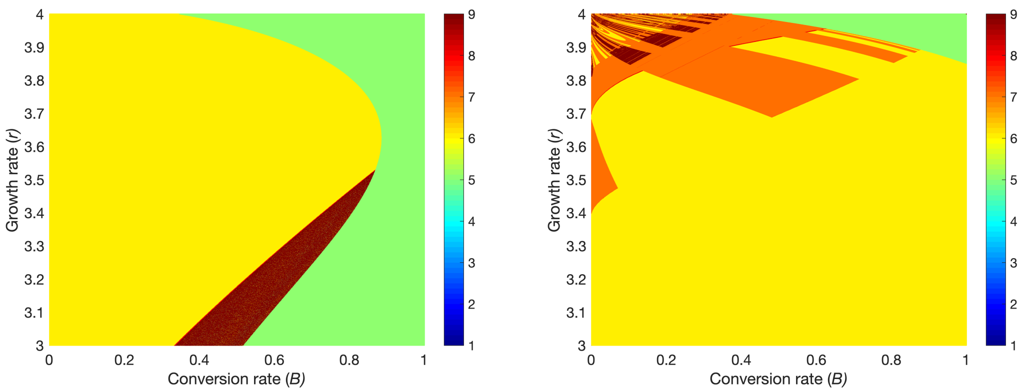
To portray this feature of our subjects, resistance should decrease when the proportion of bodysnatchers is very large. This can be achieved using any uni-modal map as a function of bodysnatcher density, so we can again employ the logistic map and choose the most interesting scaling parameter of 4. While the resistance will decrease to zero as the bodysnatchers become dominant, note that this significantly increases the resistance at intermediate fractions.

$$R_d = 4 \times \frac{y_n}{x_n + y_n} \times \left(1 - \frac{y_n}{x_n + y_n}\right) \tag{25}$$

Again we will include a moderate time delay.

$$R_{n+1} = \frac{1}{2}\Big(R_n + R_d\Big) \tag{26}$$

Modulating the parameters of a simple map in this way is a very straightforward means of generating interesting dynamical results - for example varying the growth parameter of a logistic map [22], or (similar to this formulation of dynamic resistance) using a logistic map to generate the control parameter of a second logistic map [23].




(a) Proportional

(b) Logistic

**Figure 8.** Parameter space classification for dynamic resistance

The behaviour of the system with these two schemes is shown in Figure 8, over the same region of parameter space as Figure 3(c) that had $R$ fixed at 0.8. In both cases, dynamic resistance has removed the possibility of the bodysnatchers being eliminated and thus allowing their survival beneath $B \approx 0.4$, all the way to a conversion rate of almost zero! This is because when their population is small, resistance is reduced and the humans relent from pursuing them. It would be interesting to consider if this possibility should return if the time-delay was made sufficiently large so that after a major invasion the humans would hunt aggressively for a long time.

In both Figure 8(a) and 8(b), the role of conversion rate $B$ remains intuitive, as smaller is better for the humans. However, to ensure the survival of humanity, an intermediate growth rate $r$ is preferable for the proportional case (Figure 8(a)), while in the logistic case we merely require that it is sufficiently small (Figure 8(b)). In the proportional case, for $r \leq 3.5$ there is a “lip” where the transition as the conversion rate decreases is from bodysnatchers-only to coexistence-chaos. However, this region is very speckled with green - indicating that the boundary here is likely about the minimum human population size, as tiny changes within this part of the parameter space may result in the 2-d chaotic orbits bringing the humans beneath the cut-off and transitioning immediately to a bodysnatcher-only case. In the logistic case, a fractal pattern of two-dimensional period-doubling (orange) to chaos (dark red) is stretched over the parameter space from the top-left (0,4) corner. In this case we can obtain coexistence-chaos from arbitrarily-small conversion rates as long as $r$ is sufficiently close to 4.

Interestingly, when conversion is high and growth is low, it is beneficial for human survival over much of the parameter space to utilise logistically-updated resistance rather than the proportional resistance scheme. The resistance function will be higher at low-intermediate fractions of bodysnatchers in the logistic case, and thus it is more preventative, whilst it becomes less effective if the bodysnatchers are able to obtain a high fraction. This indicates that to save humanity in this scenario it is more important to adopt preventative action, stopping the bodysnatchers from ever becoming a serious threat. This is confirmed by examining the fraction of bodysnatchers over the final 10,000 time-steps of the simulations - in the logistic case it is 0.3-0.4 at the highest conversion rates in the coexistence region, compared to 0.6-0.7 near the border in the proportional case.

## 4. A space-structured extension to multiple planets

### *4.1. Extending the model to a lattice of ten planets*

A natural extension for this sort of model is to study the impact of an alien offensive on a human civilisation consisting of multiple planets. We will briefly consider a system of concentric layers of planets, represented by a one-dimensional axis $i = 1, \ldots, 10$. At time-step $n$, $x_{n,1}$ is the population of humans on Earth, and $x_{n,i}$ for $i > 1$ are the populations on more remote planets. The carrying capacity decreases in size as we move further to the outer reaches of space. In this model, in the absence of bodysnatchers, human reproduction occurs on each planet and is followed by movement by diffusion between neighbouring positions on the lattice. Unfortunately this galactic paradise is interrupted by the arrival of an extremely small population of alien bodysnatchers on the most distant planet. They seek to gain control of Earth and the highly-populated core worlds, and move towards them whenever possible.

A single iteration of the model at the $n + 1^{th}$ time-step is as follows, starting with human and bodysnatcher populations $x_{n,i}$ and $y_{n,i}$ respectively on the $i^{th}$ planet:

(1) First, the human population is updated:

$$x'_{n,i} = \frac{r_i}{c_i} x_{n,i} \big(c_i - x_{n,i}\big) \tag{27}$$

$$y'_{n,i} = y_{n,i} \tag{28}$$

(2) Second, bodysnatcher activity takes place:

$$x''_{n,i} = x'_{n,i} \times \left(1 - \frac{y'_{n,i}}{R_i x'_{n,i} + y'_{n,i}}\right) \tag{29}$$

$$y''_{n,i} = \left(1 - R_i \frac{x'_{n,i}}{c_i}\right) y'_{n,i} + B\left(\frac{x'_{n,i}}{R_i x'_{n,i} + y'_{n,i}}\right) y'_{n,i} \tag{30}$$

Note that these two equations occur simultaneously.

(3) Third, movement may occur. A fraction (0.1%) of the human population of each planet departs for each available destination (neighbouring planets). Meanwhile, being specifically motivated to reach the core worlds, a larger fraction (1%) of the bodysnatcher population of each planet will attempt to move one step closer to Earth. Thus they experience uni-directional diffusion.

$$x_{n+1,i} = \begin{cases} 0.999 x''_{n,10} + 0.001 x''_{n,9}, & \text{if } i = 10 \\ 0.998 x''_{n,i} + 0.001 x''_{n,i-1} + 0.001 x''_{n,i+1}, & \text{if } 2 \leq i \leq 9 \\ 0.999 x''_{n,1} + 0.001 x''_{n,2}, & \text{if } i = 1 \end{cases}$$

$$y_{n+1,i} = \begin{cases} 0.99y''_{n,10}, & \text{if } i = 10 \\ 0.99y''_{n,i} + 0.01y''_{n,i+1}, & \text{if } 2 \leq i \leq 9 \\ y''_{n,1} + 0.01y''_{n,2}, & \text{if } i = 1 \end{cases}$$

(4) Fourth, we update the local resistance level of each planet using the delayed logistic model from Section 3.5:

$$R_{d,i} = 4 \times \left( \frac{y_{n+1,i}}{x_{n+1,i} + y_{n+1,i}} \right) \left( 1 - \frac{y_{n+1,i}}{x_{n+1,i} + y_{n+1,i}} \right) \tag{31}$$

$$R_i \mapsto \frac{1}{2}\left(R_i + R_{d,i}\right) \tag{32}$$

Earth has a capacity of $c_1 = 100,000$, and the capacity $c_i$ of the remaining nine planets decrease linearly in accordance with their increasing distance from Earth. The rate of this decrease is determined by the capacity-scaling parameter $s$ according to:

$$c_i = 100000 \times \left( 1 - \frac{s-1}{s} \times \frac{i-1}{9} \right) \tag{33}$$

Thus, when $s = 1$, all 10 planets will have the same capacity, but when $s = 100$, the most remote planet will have capacity $c_{10} = \frac{1}{100} \times 100,000$. At the beginning of all simulations, each planet has a human population set to half of the planet's capacity, all resistance levels are set to zero, and for each planet the human growth constant is equal to the global growth parameter for the simulation ($r_i = r \ \forall i$). A single bodysnatcher ominously appears on the outermost planet (i.e. $y_1^{10} = 1$).

### *4.2. Results*

We briefly examine the outcome of the system across a three-dimensional parameter space: 100 values of the global conversion parameter $B$ in $[0, 1]$, 100 values of the capacity-scaling parameter $s$ in $[1, 100]$, and 100 values of the human growth parameter $r$ in $[1, 4]$. This data is visualised in Figure 9. Of these $10^6$ data points, the humans and bodysnatchers coexist at the end of $10^6$ iterations for a fraction 0.660 and the humans are wiped out in the remaining 0.340 fraction. With these parameter choices, the bodysnatchers are never totally eliminated. As with the case of a single planet, it is generally favourable for human survival to have a large growth parameter $r$ and a low conversion rate $B$ to minimise the effectiveness of the bodysnatchers. The capacity-scaling parameter $s$ appears less influential.

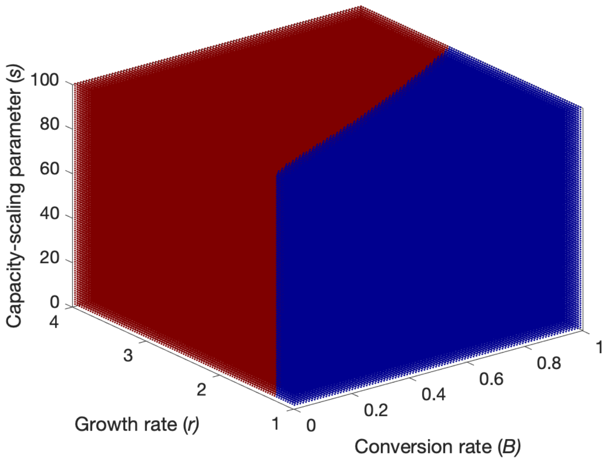



**Figure 9.** Simulation outcome: Blue = Bodysnatcher victory. Red = Coexistence.

As with the two-dimensional model, chaotic and ordered regions of the parameter space can then be identified by calculating the maximal Lyapunov exponent, although the method to implement this in higher dimensions will require additional thought [20]. It is also possible to obtain the full spectrum of Lyapunov exponents using the Householder QR-decomposition method [24].

## 5. Conclusion

As we have seen, an alien threat can be used to motivate a two-dimensional population dynamics model that is simple and yet distinct from a standard predator-prey system. There are many possible extensions of such a model, and here we have demonstrated two examples - rules for the human's response to the invasion to behave dynamically, and a spatial expansion to a one-dimensional lattice of planets with non-uniform parameters and a specific alien invasion strategy. In the following exercises (Section 6), we suggest some of the additional ways that complexity could be added in to the model, but there are many others that would make a suitable undergraduate research project. We have demonstrated some of the analytical and numerical techniques that can be applied when studying such a system: the use of Lyapunov exponents to classify ordered and chaotic behaviour in a two-dimensional case.

When modelling fictional species or situations, it is both a challenge and a creative opportunity to justify the model rules and parameter choices. How can we know what realistic behaviour should look like? This has allowed us to construct our own rules for a dynamic version of the resistance level of mankind. It is a simple abstraction of the degree to which the human civilian population is aware of the alien threat and how seriously they take it, in turn determined as a function of the density of the bodysnatcher populations. Although we have assembled a narrative to justify each stage of these rules, establishing exactly what parameter values should be used and evaluating the result for plausibility is challenging. Several of the suggested exercises are designed to encourage students to think of their own rule-set that could be justified as a model of alternative population behaviour.

## 6. Suggested exercises

We here provide some suggestions of extensions to the model or additional investigations that could be conducted as part of an undergraduate student project.

### *6.1. Alternatives to the basic model*

(1) How could additional "max" functions be included in the final formulae for the human and bodysnatcher populations to more rigorously ensure that the substeps do not permit any pathological behaviour? In particular, in the current set of equations there is the possibility of negative human populations during the human reproduction substage.

(2) In equations (7)-(8), the bodysnatcher activity takes place as a single stage, with their abduction activity and their being pursued by humans taking place simultaneously. How would the equations be formulated if these were separated into two subsequent substages - with the capture of bodysnatchers taking place after they have concluded their infiltrations?

(3) Considering the footnotes in Section 3, how could you redesign the model to account for an alternate version of bodysnatchers such as a psychic hivemind that takes control over biological humans? This could include replacing the death rate for the bodysnatchers due to human hunting with a natural mortality rate, or redesigning the functional response so that their conversion activity becomes more efficient when bodysnatchers occur in large numbers due to their combined psychic influence. In addition, when the humans reproduce it is assumed that they are limited by resources such as food, water and housing amongst only other humans. If the bodysnatchers are also biological humans then they will also necessarily be using these resources. How would you have to alter equation (3) to account for this?

(4) How do the results of the two-dimensional model change if the Ricker map

$$x'_n = x_n \, \mathrm{e}^{r\left(1-\frac{x_n}{c}\right)} \tag{34}$$

is used to model human reproduction in equation (3), rather than the logistic map?

(5) Consider again the case of "delayed dynamic resistance". Let's introduce a new variable $0 < \delta < 1$ that regulates the reaction speed at which the human population's resistance level is updated:

$$R_d = \kappa \times \frac{y_n}{x_n + y_n} \times \left(1 - \frac{y_n}{x_n + y_n}\right) \tag{35}$$

$$R_{n+1} = (1-\delta)R_n + \delta R_d \tag{36}$$

How is the fraction of $(B, r)$-parameter space for which coexistence occurs impacted as $\kappa$ and $\delta$ are varied?

(6) When modelling disease dynamics, it is common to use age-structuring of populations if particular age categories respond differently (for example, if the young or elderly are more susceptible) to the disease. Consider separating the human population into three age categories (young, adult, elderly) so that only the adult population reproduces, adding to the young category, and that a fraction of each subpopulation moves into the next group at each time-step. By numerical investigation, determine how the bodysnatchers should distribute or focus their efforts between different age classes to maximise their numbers or their likelihood of a successful takeover.

### *6.2. Expanding the spatial model*

(1) Investigate the effect of varying the human and bodysnatcher migration rates and directions. Considering potential limitations such as fuel storage on their transport ships, what other rules could reasonably govern human migration? Perhaps the survivors wish to flee to societies that have high resistance? What would be the effect if human survivors flee closer to the Earth to take refuge from the bodysnatcher incursion?

(2) Can you design a third population - the Inquisitors, a militarised group that humanity can turn to in times of crisis? When designing this group's interactions, we should incorporate a trade-off that acts in a fundamentally different form to the advantages of employing them. For example, whilst the presence of inquisitors on a planet rapidly decreases the bodysnatcher population, perhaps it also suppresses reproduction or productivity of the standard human population. In the spatial model, we could then compare the effectiveness of a variety of inquisitor deployment strategies:

- No movement.
- Evenly distributed.
- All go to the location with the highest fraction of bodysnatchers.
- All go to the location with the largest population of bodysnatchers.
- Distribute in proportion to fraction of bodysnatchers in the population.
- Distribute in proportion to the absolute populations of bodysnatchers.

How effective are each of these strategies at reducing the fraction of parameter space where the bodysnatchers succeed in overcoming the human civilisation? Are the strategies for saving humanity and eradicating the bodysnatchers co-incident or not?

(3) An extension to the multiple planet model could be the option for humans to implement a quarantine of particularly affected planets for a limited period of time, greatly reducing the movement allowed to and from that planet. Assuming first that only one planet can be quarantined at any one time, investigate the role of the severity and duration of the quarantine measures. How should the humans decide where to impose it? You could further test the impact of a time-delay between the conditions for local quarantine being met, and the measures being enacted, to advise the human government on the relative importance of a fast, severe, or long planetary lockdown.

(4) Extend the spatial model to a two or three-dimensional array of planets. You could then explore the possibility of games on this grid, as the bodysnatchers attempt to manoeuvre through the space to reach Earth, while the inquisitors position themselves to prevent this as much as possible. What sort of strategies could you investigate here? Consider the analogues to real-world disease quarantine, and strategies for preventing forest fires.

(5) In truly dire times, the inquisition has one last weapon at its disposal - the ability to destroy all life on a given planet. However, this can be utilised only once and Earth may not be targeted. Furthermore, it is generally poor for the public image of the inquisition, and so using it will result in no additional inquisition recruitment on any planet for a subsequent number of iterations. By numerical investigation, make a case for where and when the weapon should be deployed for maximum effect (aside from the very first iteration of course!). You could consider additional constraints on the weapon's use such as a minimum resistance level required across all planets in order for the general public to accept this strategy.

(6) The discrete population dynamic model demonstrated in this paper is just one paradigm for modelling such a space-structured scenario, and becomes less suitable as the spatial grid increases in size and complexity. An alternative approach with simplified dynamics but suitable for a spatial structure of vastly increased size could be to construct a discrete cellular automata (for example on a $100 \times 100$ grid of planets or regions), with different states representing normal civilians, inquisitor occupation, abandonment, and bodysnatcher infestation of varying intensities. Simple rules govern the updating of each cell based on the states of itself and its neighbours. Choosing reasonable rules for how the states should influence each other and constructing such a model in Microsoft Excel could form an alternative undergraduate project, with the possibility of live demonstrations of the evolution of the state space from a variety of starting conditions with a variety of inquisitor strategies.

## Declaration of interest statement

No potential conflict of interest was reported by the author.

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