# An alternative route to the Mandelbrot set: connecting idiosyncratic digital representations for undergraduates

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Richard Miles

Mathematics undergraduates often encounter a variety of digital representations which are more idiosyncratic than the ones they have experienced in school, and which often require the use of more sophisticated digital tools. This article analyses a collection of digital representations common to undergraduate dynamical systems courses, considers the significant ways in which the representations are interconnected and examines how they are similar or differ from those students are likely to have experienced at school. A key approach in the analysis is the identification of mathematical objects corresponding to manipulative elements of the representations that are most essential for typical exploratory tasks. As a result of the analysis, augmentations of familiar representations are proposed that address the gap between local and global perspectives, and a case is made for greater use of isoperiodic diagrams. In particular, these diagrams are proposed as a new stimulus for students to generate their own explorations of fundamental properties of the Mandelbrot set. The ideas presented are expected to inform the practice of teachers seeking to develop visually rich exploratory tasks which pre-empt some of the issues of instrumentation that mathematics undergraduates experience when introduced to new digital tools. The overarching aim is to address significant questions concerning visualisation and inscriptions in mathematics education.

*Keywords*: Digital representation, visualisation, instrumentation, undergraduate tasks, Mandelbrot set.

## 1. Introduction

Drew Berry's *Visualizing the code of life* (Berry, 2017) epitomises the power of digital technology to reify complex scientific concepts and processes. For many mathematics undergraduates, computer generated images and animations depicting fractal structures seem to hold a particular allure. Digital images such as these have been a driving force for mathematical enquiry in their own right and, from a panorama of visualisations, O'Halloran (2015) singles out the familiar representation of the Mandelbrot set to illustrate how visualisations can lead to new mathematical knowledge.

Looking at the image that Mandelbrot (1982) produced of his eponymous set alongside the embryonic version produced by Brooks & Matelski (1981)[1] (see Figure 1), it is easy to imagine the different meanings that students and teachers alike might construct for a single mathematical object depending on the technology available. This close association between "internal" and "external" visualisation in tandem with technology is described in detail by Borba & Villarreal (2006, p. 99). The complexity of what can be meant by *visualisation* also presents itself through this example. To avoid ambiguity, the term *digital representation* is used here to refer to computer generated images representing mathematical objects such as those shown in Figure 1.

First courses in dynamical systems are amongst the richest in terms of digital representations and suitable text books (such as, Alligood *et al.*, 1996; Holmgren, 1996; Scheinerman, 2012; Layek, 2015; Devaney, 2018; Lynch, 2018) enthusiastically encourage students to engage with a range of computational techniques and visual explorations. On the specific topic of fractals, explorations intended for both secondary school and undergraduate students have been considered in some detail (for example, see Doyle, 1992; Naylor, 1999; Choate *et al.*, 1999; Frame & Mandelbrot, 2002; Fraboni & Moller, 2008;

[1]The image produced by Brooks & Matelski (1981) originates from a 1978 conference. Mandelbrot (1980) includes related fractal images of this type but they do not correspond as directly to that of Brooks & Matelski (1981).

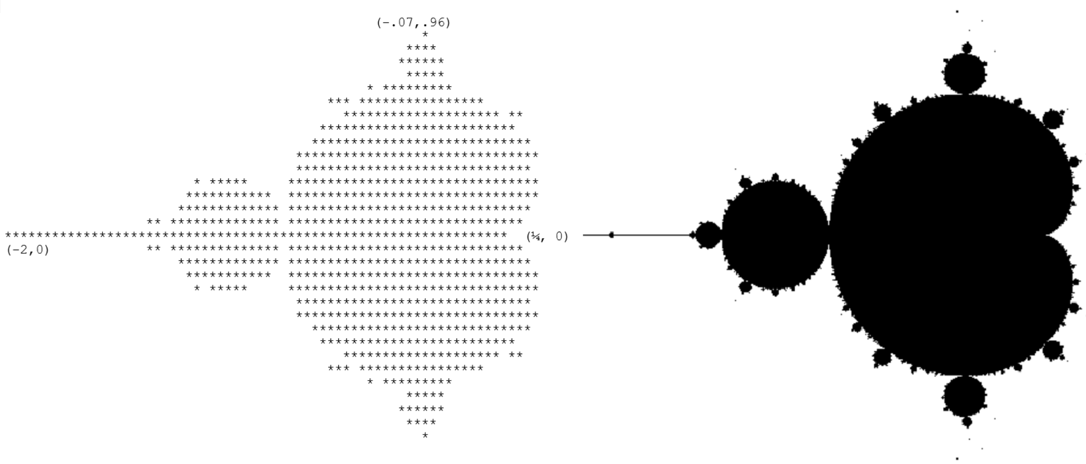


FIG. 1. Images of the Mandelbrot set: recreations of the original digital representations from Brooks & Matelski (1981), on the left, and Mandelbrot (1982), on the right.

Peitgen & Richter, 2013; Peitgen *et al.*, 2013). Novel investigations involving bifurcation diagrams (for example, see Devaney, 1991; Klebanoff & Rickert, 1998; Ross & Sorensen, 2000; Burns, 2002) may also be suitable for some undergraduates, but can be very challenging for others. More generally, a major difference between many of the digital representations that students encounter at university and ones that they have experienced at school (for example, those produced by a graphics calculator, by dynamic geometry software or by a smartphone app) is that the former are more likely to be derived from specific technologies capable of dealing with research level applications (Oates, 2011; Thomas *et al.*, 2017). Thomas *et al.* (2017, p. 109) argue that this has the potential to cause a "discontinuity between school and university, making the transition less than smooth for students". This situation may be complicated by issues of instrumentation that can arise from introducing new and complex technologies in one-semester courses (Stewart *et al.*, 2005). In order to begin to smooth this discontinuity in relation to an area of mathematics decorated with idiosyncratic digital representations, this article analyses a collection of digital representations that often feature in exploratory tasks relating to periodic behaviour in dynamical systems. It considers significant ways in which the representations are interconnected, how they are similar or differ from those students are likely to have encountered at school, and identifies the essential mathematical objects required to manipulate the representations. By identifying such corresponding objects, the aim is to provide the simplest basis upon which subsequent instrumental orchestrations (see Trouche, 2004) can be tailored to individual teaching situations, independent of a particular appropriate technology. As a result of the analysis, existing digital representations relating to periodic point classification are augmented so as to bridge the gap between local and global perspectives (Vandebrouck, 2011; Thomas *et al.*, 2017). Finally, I make the case for greater use of isoperiodic diagrams which, despite being simply defined and easily manipulated (see Section 3), do not yet seem to feature in the undergraduate literature. Crucially, I reveal here how these can be used to provide students with a new visual stimulus to motivate their own explorations of important properties of the Mandelbrot set.

This work, therefore, contributes to the broader aim of addressing one of Presmeg's significant questions (Presmeg, 2006, 2014) concerning visualisation in mathematics education, namely, "How can teachers help learners to make connections between idiosyncratic visual imagery and inscriptions, and conventional mathematical processes and notations?" (Presmeg, 2006, p. 25). The results are important for university mathematics teachers seeking to develop exploratory tasks using digital representations whilst at the same time pre-empting some of the difficulties their students encounter when introduced to a variety of new digital tools.

## 2. Background to the analysis: visualisation, digital representations and instrumentation

Mathematicians, mathematics educators and psychologists use the word visualisation in a number of different ways depending on context and focus (Clements, 2014). As pointed out by Arcavi (2003, p. 215), visualisation describes "both the product and the process of creation, interpretation and reflection upon pictures and images". O'Halloran (2015) considers visualisation from a multimodal perspective and uses the term "mathematical image" (O'Halloran, 2015, p. 63) to distinguish representations of mathematical objects such as those shown in Figure 1 which some authors refer to as visualisations. Presmeg (2006) considers further nuances of terminology and, in line with Roth (2003) and Cobb (2002), describes digital representations within the more general context of inscriptions. Here, digital representations are regarded as a type of inscription that is fundamental to student visualisation and exploration of dynamical systems.

The ideas in this article are framed within a broader instrumental approach that focuses on how users' own mental schemes and techniques may be combined with a digital tool to form an *instrument* (Rabardel, 2002; Artigue, 2002). Trouche (2004) addresses the teacher's role in guiding students' learning processes in relation to such instruments using the metaphor of *instrumental orchestration*. Elements of the resulting framework have been expanded and applied in a number of practical situations (for example, see Drijvers *et al.*, 2010, 2013; Thomas *et al.*, 2017).

There are a variety of ways to produce the representations discussed here using a software package such as SageMath, MATLAB, or Mathematica and/or an appropriate programming language or, alternatively, using a bespoke graphical user interface. Specialist digital dynamical systems tools have also been made freely available to students and teachers (for example, see Back *et al.*, 1992; Boeing, 2016). The DSWeb project (SIAM, 2019) gives an idea of the level of sophistication possible when bespoke tools are constructed by teachers and by their students. A particularly interesting case of students' own tool development, featuring exploration and visualisation of the Mandelbrot set, is described by Marshall *et al.* (2014). A discussion of all the advantages, disadvantages and choices surrounding the digital tools just listed is beyond the scope of this article and reliance on a particular technology is deliberately avoided. Instead, the digital representations themselves are considered as key components for exploratory tasks and it is assumed that the chosen digital tools can be configured to enable the most straightforward and essential manipulation of the representations. These assumptions can be seen as the basis for a typical *exploitation mode* (Drijvers *et al.*, 2010, p. 215) within an instrumental orchestration.

In their comparison of multimodal social semiotic and constructionist perspectives on the use of digital representations, Morgan & Kynigos (2014) highlight the central role of manipulation. The authors refer to this colloquially as "tinkering with digital representations" (Morgan & Kynigos, 2014, p. 360). According to their perspective, manipulation and the subsequent changes can be treated as part of the representation itself. A precise taxonomy of such interactions has been produced by Sedig & Sumner (2006). For example, this taxonomy differentiates between *manipulative* interactions such as using a cursor to grasp a digital representation and perform some action on it, and more *conversational* interactions such as entering commands into an input box. Trouche (2004) uses the general term *enablements*

TABLE 1. *Six digital representations relating to periodicity.*

| Representation | Dimension | Axes | Enabling objects |
|---|---|---|---|
| cobweb plot | 1 | state-state | map, initial point |
| bifurcation diagram | 1 | parameter-state | family of maps |
| classification plot | 2 | state coordinate | map |
| neighbourhood diagram | 2 | state coordinate | map, fixed point, neighbourhood radius |
| isoperiodic diagram | 2 | parameter coordinate | family of maps |
| Mandelbrot set | 2 | parameter coordinate | family of maps |

to refer to elements of a digital tool that allow users to make such adjustments. Irrespective of a chosen technology, it is possible to identify the mathematical objects corresponding to enablements that facilitate the most essential manipulation of a given representation. I refer to these objects here as *enabling objects*. Corresponding examples will be contrasted with more generic enablements in the next section.

## 3. Analysis and results: connecting digital representations related to periodicity

In what follows, a *dynamical system* may be thought of simply as a system of states determined by a map $f : \mathbb{R}^n \to \mathbb{R}^n$, where $n = 1$ (a *one-dimensional system*) or $n = 2$ (a *two-dimensional system*). The focus here is on examples defined by polynomial coordinate functions that are governed by one or two parameters, as these are most often used as the basis of exploratory tasks. That is, examples that correspond to a family of maps, such as the logistic family $f(x) = ax(1-x)$, the Hénon family $f(x,y) = (a - x^2 + by, x)$, or the complex quadratic family, defined as a family of two-dimensional systems by $f(x,y) = (x^2 - y^2 + a, 2xy + b)$, where $a$ and $b$ are real parameters. For all such examples, the defining family of maps and real parameters feature as enabling objects of the digital representations considered. These representations together with their specific enabling objects are listed in Table 1. Details of the axes are also included, as these are related to the combinations of enabling objects. For example, the space of parameters may be represented on one axis (corresponding to a one-dimensional parameter space and labelled as 'parameter-state') or on both axes (corresponding to a two-dimensional parameter space and labelled as 'parameter coordinate'). In both these cases, a family of maps is subsequently required as an enabling object. In contrast, if both the axes correspond in some way to state space (labelled either as 'state-state' or 'state coordinate' depending on the dimension of the system), then only a single map is required.

The six example text books listed in the introduction were selected based on accessibility of the content for undergraduates studying a first course in dynamical systems. Among the representations listed in Table 1, the cobweb plot, bifurcation diagram and Mandelbrot set feature in consistent forms in all of these text books. Neighbourhood diagrams are found to have more variable forms and purposes and the representation considered here is based on Alligood *et al.* (1996, Ch. 2). Isoperiodic diagrams do not feature in any of the text books listed, and the classification plot is introduced later in this section.

Before analysing the facets of each representation in Table 1 in more detail, it may be helpful to recall how the most common representations are likely to arise in a natural teaching sequence. Students are generally introduced to one-dimensional systems prior to two-dimensional ones; therefore, they usually encounter cobweb plots and bifurcation diagrams (representing aspects of periodicity for one-

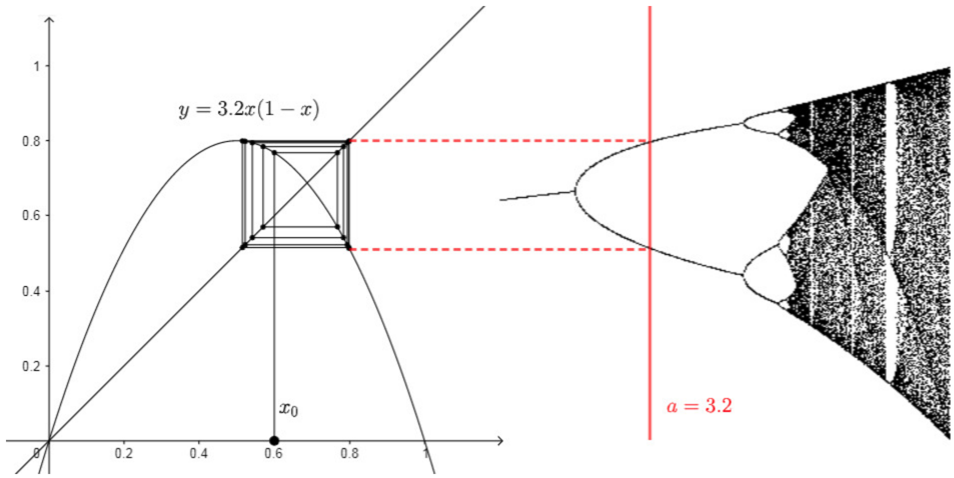


FIG. 2. A cobweb plot showing an attracting period 2-orbit for the map $f(x) = 3.2x(1-x)$ linked via the red dashed lines to the corresponding orbit in relative position in the bifurcation diagram of $f(x) = ax(1-x)$.

dimensional systems) prior to neighbourhood diagrams (representing local expansion and/or contraction around a fixed point in two-dimensional systems). The familiar representation of the Mandelbrot set is likely to be first encountered independently of a link with periodicity, as such a link does not present itself directly from the definition. The link that is usually made relates to "periods of bulbs" (Frame & Mandelbrot, 2002, p. 64), and this leads to well-established exploratory tasks for students. Another well-known exploration (Devaney, 1991) connects the Mandelbrot set to bifurcation diagrams. However, neither of these lines of investigation are likely to arise spontaneously for students based on the familiar representation of the Mandelbrot set alone, without additional prompting from the teacher.

The cobweb plot augments a Cartesian representation of the defining map for a one-dimensional system using a two-dimensional representation of a one-dimensional trajectory (shown for a logistic map in the left half of Figure 2). The enabling objects are a single defining map $f$ (for example, entered via an input box) and an initial point $x_0 \in \mathbb{R}$ (for example, adjusted via a control that can be dragged along the $x$-axis). The bifurcation diagram is a more bespoke representation, where the parameter space usually appears on the $x$-axis and the state space appears on the $y$-axis (shown for the logistic family in Figure 2, right). This means that the bifurcation diagram requires the defining family of maps as an enabling object. However, no further enabling objects are required as the $y$-values (representing asymptotic orbits) of the displayed co-ordinates depend only on the independent $x$-value (representing the defining parameter). With only a single enabling object, the usefulness of this representation is greatly increased by generic enablements such as range and zoom controls that allow a user to explore particular parameter intervals in more detail.

With its grounding on the global Cartesian representation of a function, the cobweb plot satisfactorily addresses the issue of linking the concepts of repelling and attracting periodic orbits to one of the most common digital representations that students will have encountered before in school. Thomas *et al.* (2017) provide insight into what is known about such encounters, and consider the typical technologies used. For example, most students are likely to have graphed functions using technology such as Geogebra which has been used to generate the graph in Figure 2. Therefore, the additional cobweb is built on familiar terrain. Although the bifurcation diagram is more idiosyncratic than the cobweb plot, it also uses the idea of displaying points based on some form of coordinate dependency. So, there is also a Cartesian perspective familiar to students underpinning this representation. There are also natural ways in which the cobweb plot and the bifurcation diagram can be visually connected, as suggested by the red lines in Figure 2. These representations do not usually appear juxtaposed in this way, yet a dynamic link between the two (facilitated by the additional enabling object of a parameter value) generates a new hybrid digital representation that emphasizes the global Cartesian perspective.

Beyond the problems associated with the shift in technologies from school to undergraduate level highlighted in the Introduction, one of the most significant issues identified by Thomas *et al.* (2017) (see also Vandebrouck, 2011) is the leap from global and pointwise representations of functions in school to local perspectives at undergraduate level. Neighbourhood diagrams embody this leap. These representations are often used to reveal expansion and/or contraction close to a fixed point in a two-dimensional state space and facilitate classification of these points as saddles, sources and sinks. Beyond the defining map $f(x,y) = (f_1(x,y), f_2(x,y))$ and a given fixed point, a neighbourhood diagram also requires a neighbourhood radius as an enabling object (for example, implemented via an input box or slider). Although intersections of the curves $f_1(x,y) = x$ and $f_2(x,y) = y$ correspond to fixed points, this global perspective is often suppressed (for example, see Alligood *et al.*, 1996, Sec. 2.2). Generic zoom enablements of digital tools make inclusion of these curves much more meaningful, as the local and global perspectives can then be easily compared. The left pane of Figure 3 shows this augmentation for the Hénon map $f(x,y) = (\frac{1}{2} - x^2 - \frac{3}{2}y, x)$ at a source.

How else might we construct a digital representation that can be tied to the familiar global picture revealing the location of fixed and periodic points but which also facilitates classification of the associated periodic orbits? Beginning with the correspondence between intersections of the curves $f_1^k(x,y) = x$ and $f_2^k(x,y) = y$ and points of period $k$, pixel colour coding around each point $(x,y)$ on or close to the curves according to the moduli of the eigenvalues $\lambda_1$, $\lambda_2$ of the Jacobian matrix $Df^k(x,y)$ gives a visual means to classify the periodic orbit as a source, sink or saddle. For example, for $j = 1,2$, using green to show both $|\lambda_j| > 1$, red to show both $|\lambda_j| < 1$ and blue to show only one of $|\lambda_j| > 1$, respectively. This produces a *classification plot* which requires only a given map as an enabling object. The right pane of Figure 3 is a classification plot showing a period 3 saddle orbit for the Hénon map $f(x,y) = (\frac{5}{4} - x^2 - \frac{3}{4}y, x)$. Colours are chosen according to the regime just described and the colour intensity (that is, an RGB vector scaling) is based on the proximity of $f^k(x,y)$ and $x$, or of $f_2^k(x,y)$ and $y$.

The use of pixel colouring according to corresponding point properties is also a key idea behind representations of two-dimensional parameter spaces. The bifurcation diagram for one-dimensional systems provides a means to make immediate qualitative comparisons within a family of dynamical systems. With an analogous purpose in mind, Gallas (1993) introduced isoperiodic diagrams for the two-dimensional parameter space of the Hénon map. The essence of his idea is to colour code points in this space according to the length of a dominant attracting periodic orbit (if one can be computationally detected) for the corresponding Hénon map. The resulting digital representation (see the left pane in Figure 4) provides a highly structured picture of parameter space and this only requires a two-parameter family of maps as an enabling object. In a similar way to the a bifurcation diagram, the usefulness of this digital representation is enhanced by generic enablements such as range and zoom controls. There is nothing special about the Hénon map here, apart from the number of dimensions of the parameter space, so isoperiodic diagrams allow students to visualise parameter spaces for two-parameter families much more generally. For example, an isoperiodic diagram for the complex quadratic family can be produced using the same tool as for the Hénon map. The result is shown in the right pane of Figure 4 where there is an instantly recognisable resemblance to the standard representation of the Mandelbrot

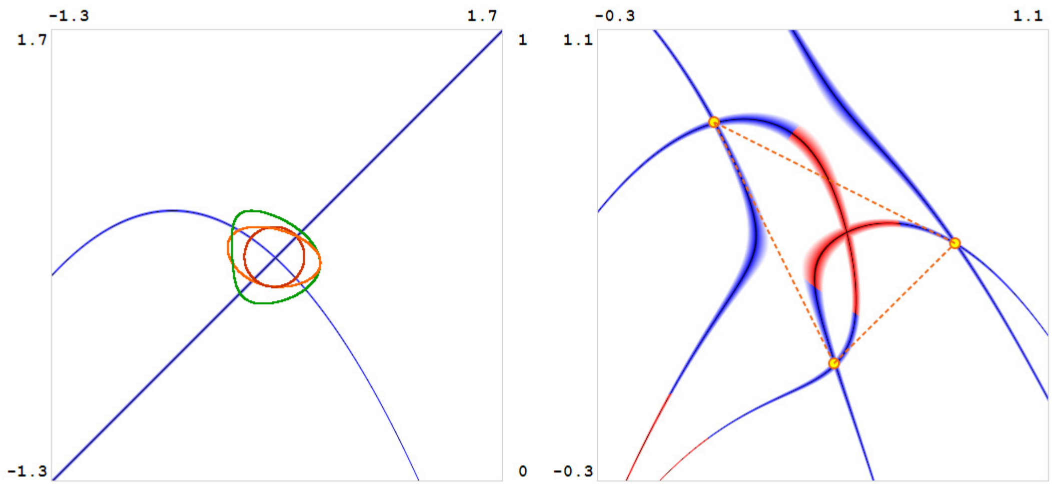


FIG. 3. The left pane shows two iterates of distortion for a neighbourhood centred around a source for the Hénon map $f(x,y) = (\frac{1}{2} - x^2 - \frac{3}{2}y, x)$. The right pane is a classification plot, using colour coding based on eigenvalues of $Df^k(x,y)$ to identify a period 3-saddle orbit for $f(x,y) = (\frac{5}{4} - x^2 - \frac{3}{4}y, x)$

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set. Thus, the isoperiodic diagram provides students with a visual stimulus to formulate their own questions concerning periods of bulbs in the Mandelbrot set. Beyond investigations of the complex quadratic family, isoperiodic diagrams also provide students with a means to investigate and interlink the Mandelbrot-like sets that arise from multiple critical points for more general complex polynomial maps. For example, I have used variations on the following task prompt to initiate such explorations.

*The Mandelbrot set is defined in terms of a single critical point for the complex quadratic map* $z \mapsto z^2 + c$*, where c is a complex parameter. How might 'generalized' Mandelbrot sets be defined for a complex cubic map where the constant term is given by a parameter? For example, for a map such as* $z \mapsto z^3 - z + c$*? How do these generalized Mandelbrot sets relate to the associated isoperiodic diagram, and what can you deduce about the structure of such sets in relation to parameter space?*

This example underscores how important it is that digital representations are equipped with well-chosen enablements. In particular, the implied notion of a 'generalized' Mandelbrot set requires not only a defining map but the addition of a critical point as an enabling object. Such deep explorations are far more difficult for students to undertake without appropriately configured digital representations at hand. Therefore, in this example, isoperiodic diagrams and 'generalized' Mandelbrot sets offer technological and pedagogical affordances (Trouche, 2019, p. 395) that enable students to experiment efficiently, simulate, explore similarity and change, and form conjectures to support their conceptual understanding of dynamical systems.

## 4. Conclusions and challenges

Digital representations are motivating and powerful components of mathematical exploration. This article provides a basis for university mathematics teachers to consider how they arrange digital representations within their own exploratory tasks for students. However, with the wealth of capabilities offered by

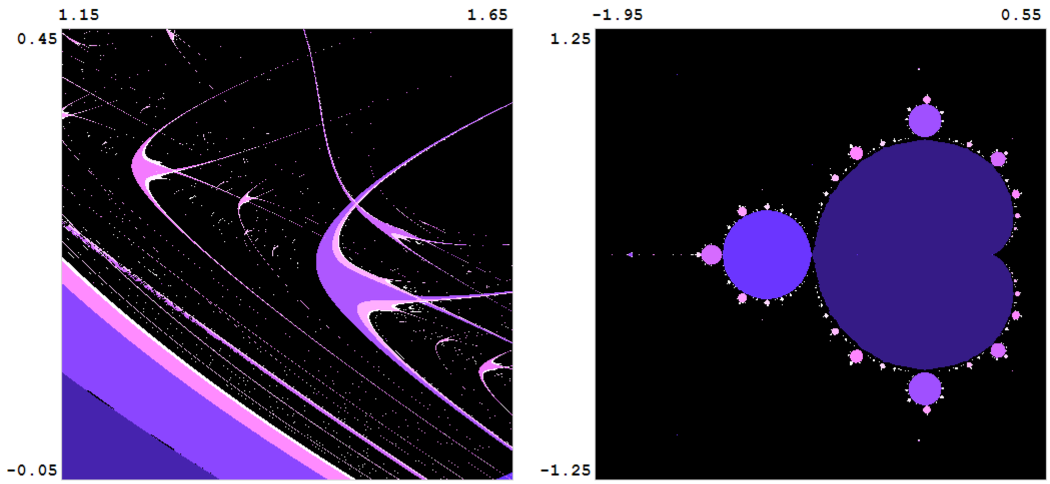


FIG. 4. Isoperiodic diagrams for the Hénon family $f(x,y) = (a - x^2 + by, x)$ (left pane) and the complex quadratic family $f(z) = z^2 + a + bi$ (right pane). In both cases, the parameter space of $(a,b)$-coordinates is represented.

digital tools deployed at undergraduate level comes an inevitable level of complexity. Here, the essence of a significant collection of digital representations has been extricated from these potentially complex technologies. This enables the representations to be systematically compared and connected with each other and with the familiar representations that undergraduates will have encountered in school. The innovations that are proposed capitalise on existing visual links and address the global-local leap that students are presented with at undergraduate level. In addition, by exploring the affordances of isoperiodic diagrams, a new visual route of exploration for students is revealed which empowers them to formulate their own conjectures concerning important properties of the Mandelbrot set.

An alternative lens that could be used in this article is provided by the notion of *webbing* (Noss & Hoyles, 1996), as this addresses the guiding aim of helping learners to make connections, as described in Presmeg's question in the Introduction. Although this perspective is not pursued here, it could potentially be harmonised with instrumental approaches (Trouche & Drijvers, 2014). Placing the analysis within an instrumental approach also raises further questions and challenges. For example, how can intended instrumental orchestrations be best achieved with the aid of one (or more) of the many digital tools available? In particular, configuring such tools so that students are able to produce intended representations using pre-selected enablements can be challenging. This configuration requires not only adaptable capabilities from the chosen tools but also requires teachers to develop expertise in manipulating these capabilities towards their own didactical intentions. The complexities of this issue recall the connection between instrumental orchestration and technological pedagogical content knowledge raised by Drijvers *et al.* (2013). I have often arranged exploratory tasks around the representations considered here with the aid of bespoke browser-based tools[2] which I have gone on to progressively refine based on successive iterations of the tasks. In line with a concluding conjecture of Drijvers *et al.* (2013), integration of these tools has stabilized to a point where my own orchestrations have shifted focus, placing mathematical exploration above technology mastery. For example, I now devote less time to procedural

[2]Figure 3 and Figure 4 were produced using these bespoke browser-based tools. Figure 2 was produced using a combination these tools and Geogebra.

tasks at the start of computer-based classes and aim to move students more quickly on to tasks of the type described at the end of Section 3, where it is necessary for them to experiment using the appropriate digital representations themselves.

Using the term *instrumental genesis*, Trouche (2004) describes the process of how instruments may be constructed as augmentations of the body, comprising both an artifact component (for example, a digital representation) and a psychological component linked to students' activities, knowledge and existing methods of working. Given the diversity and complexity of tasks involving digital representations that my students now seem confident in approaching, a potential outcome of the shifted focus described above is a more affirmative perception of instrumental genesis taking place. Thus, to record this more concretely, an important future research direction presents itself concerning students' fluency, synthesis and perspectives on the role of the digital representations considered here, probing the choices, inferences, and conjectures that they make when using these representations in their own explorations.

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