Mathematical Analysis in Investment Theory: Applications to the Nigerian Stock Market

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Mathematical Analysis in Investment Theory: Applications to the Nigerian Stock Market

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DECLARATION

I certify that the substance of this thesis has not been already submitted for any degree and is not currently considering for any other degree. I also certify that to the best of my knowledge any assistance received in preparing this thesis, and all sources used, have been acknowledged and referenced in this thesis.

Abstract

This thesis intends to optimise a portfolio of assets from the Nigerian Stock Exchange (NSE) using mathematical analysis in the investment theory to model the Nigerian financial market data better. In this work, we analysed the 82 stocks which were consistently traded in the NSE throughout 4years from August 2009 to August 2013. We attempt to maximise the expected return and minimise the variance of the portfolio by using Markowitz's portfolio selection model and a three-objective linear programming model allocating different percentages of weight to different assets to obtain an optimal/feasible portfolio of the financial sector of the NSM. The mean and the standard deviation served as constraints in the three-objective model used, and we constructed portfolios with the aims of maximising the returns and the Sharpe ratio and minimising the Standard Deviation (Variance) respectively. In another development, we use Random Matrix Theory (RMT) to analyse the Eigen-structure of the empirical correlations, apply the Marchenko-Pastur distribution of eigenvalues of a purely random matrix to investigate the presence of investment-pertinent information contained in the empirical correlation matrix of the selected stocks. We use a hypothesised standard normal distribution of eigenvector components from RMT to assess deviations of the empirical eigenvectors to this distribution for different eigenvalues. We also use the Inverse Participation Ratio to measure the deviation of eigenvectors of the empirical correlation matrix from RMT results. These preliminary results on the dynamics of asset price correlations in the NSE are essential for improving risk-return trade-offs associated with Markowitz's portfolio optimisation in the stock exchange, which we achieve by cleaning up the correlation matrix. Since the variance-covariance method underestimates risk, we employ Monte-Carlo simulations to estimate Value-at-Risk (VaR) and copula for a portfolio of 9 stocks of NSE. The result compared with historical simulation and variance-covariance data. Finally, with the outcome of our simulation and analysis, we were able to select the assets that form the optimal portfolio and the weights allocation to each stock. We were able to provide advice to the investors and market practitioners on how best to invest in the sector of NSE. We propose to measure the extent of closeness or otherwise in selected sectors of the NSE and the Johannesburg Stock Exchange (JSE) in our future work.

Key terms: Portfolio optimisation; Weight allocation; Diversification of assets; Sharpe ratio; Random matrix theory and Correlation matrix; Inverse participation ratio; Eigenvalue and eigenvector; VaR; Copula

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Dedication

I wish to dedicate this dissertation to God almighty who has shown me His steadfast love throughout this period and to my lovely family; my wife Mrs Amara Nnanwa, my children; Daphne, Wendy and Valerie for all their moral support especially in my difficult times.

Contents

Chapter 1 Introduction

1.1 Introduction to emerging market.

An Emerging market is one which is not fully developed but progressing towards becoming a well-developed market. Its market does not have a reasonable level of efficiency, standard rules of operation like what is obtainable in America, Europe, and Japan, where you have a well-developed economy/ market. Although the emerging market is not developed as that of America, Europe, and Japan, as we mentioned earlier, it must process those characters of the developed ones. These include a regulatory body that oversees the activities of the market, stipulated rules for the market operations, the existence of some form of market exchange, liquidity in local debts and equity. The emerging market also has some physical, financial infrastructures such as stock exchange, a unified currency, and banks.

There are still controversies around the countries that belong to the list of emerging markets. However, one looks at it, countries that exhibit those characteristics we mentioned above belong to the group of an emerging market. Some Asian countries like China, Malesia, Indonesia, Thailand, South Korea, Philippine, Taiwan , UAE etc., South American countries like Peru, Mexico, Colombia, Argentina, Venezuela, and some eastern European countries like Ukraine, Poland, Romania, Bulgaria, Greece and Czech Republic are among the group of emerging markets. Others are some African countries like South Africa, Nigeria, Morocco, Ghana, Kenya, etc. Though these markets mentioned above are not well developed, the investors like to invest in the market because of its high prospective return rate. The high rate of return guaranteed, due to what is obtainable in the economy of those countries where the market is situated. Secondly, the emerging markets experience faster economic growth like what in the case of China and Nigeria. Investment in these markets is exciting due to its higher returns, but it comes with huge problems. The level of risk of the investments in the emerging market is so high due to the political instability is such countries. Also, problems associated with national infrastructures and high volatility rate in its currency exchange rate. In this work, our interest is to work on some of the shortcomings in the emerging market and improve the investment, with applications to the Nigerian Stock Market.

This PhD work is titled 'Mathematical Analysis in Investment: Applications to Nigerian Stock Market'. It aims to develop models that will fit the peculiarities of the Nigerian financial market data much better than current models. The research contributes to an understanding of the necessary foundations and dynamics of the Nigerian financial system.

It is relevant to current efforts by economic policymakers such as the Securities and Exchange Commission (SEC) to improve the performance of the Nigerian financial markets and key market sectors. The theoretical model building will drive the research while the applications in view will be used to test the validity and stability of the models in different market contexts.

1.2 Rationale for the Research

The study is on the current results in the topic themes of mathematical analysis and mathematical finance to develop new results in investment systems and trading, and portfolio theory and optimisation, with illustrative examples from the Nigerian Stock Market (NSM).

1.3 Background notes on the Nigerian Stock Exchange

The Nigerian Stock Exchange (NSE) was founded in 1960 as the Lagos Stock Exchange and started operation in Lagos in 1961 with 19 securities listed for trading. It later became the Nigerian Stock Exchange in 1977 with branches in some commercial cities in Nigeria. It has about two hundred (200) listed companies, whose data from their performances are published daily, weekly, monthly, quarterly and annually. The Exchange maintained an All-share index, formulated in 1984; its index is value-weighted and is computed daily with the highest value #66,371.20, which was on March 3rd, 2008. The Securities and Exchange Commission regulates the Nigerian Stock Exchange; they make sure that there are no unfair manipulations of trading practices and the bridge of market rules.

1.4 Research aims and objective

Like we said earlier, this PhD work aims to develop models that will fit the idiosyncrasies of the Nigerian financial market data much better than current models, and to apply the models to investment prospects in the Nigerian stock market (NSM).

1.4.1 Objectives

Objectives 1: To review the mathematical foundations of portfolio theory and optimisation based on the Markowitz (1956) model, to determine the strengths and weaknesses of the model

Objectives 2: To examine the mathematical properties of different risk measures which underpin investment and portfolio theory, and hence explore how the Markowitz and related frameworks can be extended to accommodate such risk measures

Objectives 3: To more generally examine the mathematical analysis of investment systems to determine how the interactions between securities in the portfolio of the investment will help to optimise the portfolio.

Objectives 4: To apply the results where possible to investment portfolios and investment schemes which are relevant to the stylised facts of the NSM

1.5 Research Questions

Research Question 1: What are the mathematical underpinnings of Markowitz's model, including such constructs as utility functions, non-linear analysis? e.g. convex and coherent risk measures, and constrained optimisation theory? What are the weaknesses and strengths of the model under different investment scenarios, like multi-objective portfolio optimisation with constraints?

Research Question 2: What are the mathematical properties of different risk measures used in investment and portfolio theory? How do these properties inform the extension of Markowitz's and related theories under such risk measures, for example, to dynamically optimal portfolios relevant to medium-to-long term investment decisions with time-varying risk measures? What are the necessary and sufficient conditions for the existence of unique and tractable optimal portfolios in such situations?

Research Question 3: Which key concepts and frameworks underpin the mathematical analysis of investment systems? What conditions make such systems valid or invalid? In other words, what is the character of the investment systems under different assumptions about investment strategies and scenarios? What are the impacts of extenuating factors such as the Correlation matrix; the Inverse participation ratio; Eigenvalue and eigenvector; Portfolio optimisation through Random matrix theory?

Research Question 4: How can some of the results from 1-3 above demonstrated empirically using suitable examples, including those from the NSM?

For RQ1, we did an extensive review of the existing literature on the key concepts underpinning the objective, for example, The Mathematical analysis underpins utility functions, convex, concave and coherent risk measures, and its link to investment theory.

Markowitz portfolio optimisation, its weaknesses, the attempts made by other researchers to correct the deficiency of Markowitz portfolio theory (MPT) and finally, we looked at a set of solutions that will give the best trade-off among the objective of the problem, using the multi-objective approach with different constraints.

For RQ2, we considered medium-to-long term investment decisions with time-varying risk measures. Dynamically, optimal portfolios linked to investment theory (for example, we looked at risk management, using different measures of financial risk).

For RQ3, an extensive review of the existing literature and was focused on the key concepts underpinning the RQ3, for example, Random matrix theory (RMT), Correlation matrix, cleaning up technic, etc. We used a hypothesised standard normal distribution of eigenvector components from RMT to assess deviations of the empirical eigenvectors to this distribution for different eigenvalues.

For RQ4, we applied some of the results got from RQ 1 to 3 to the NSM. We achieved this by obtaining and use of relevant data from NSM, including interviews of key market participants and policymakers such as stockbrokers, investment banks, CBN, NSE and SEC, and analysis of the financial product.

1.6 Technical note

Markowitz's modern portfolio theory is all about a strategy in investment that helps the market practitioners to construct an optimal portfolio, by looking at the relationship between the return and the risk. The theory implies that the optimal portfolio does not only depend on the return it gives, also on the risk taken while investing. Therefore, portfolio selection imposes a more significant challenge to optimisation problem since the risk of every portfolio influences the interactions or correlations of different stocks in that portfolio. The correlation matrix obtained from the historical empirical data from the market will help in the risk management and portfolio optimisation. The RMT will give the spectrum of the eigenvalue, which contains essential information about the market. This information can be used to remove the noise through the filtering process.

1.7 Definitions of some key concepts and terms

In this section, we give the definitions of some terminologies we will be using in the course of this research, as in Chidume (2014). Some of these include the spaces we are expected to work. Examples are normed linear spaces, inner product, and Hilbert spaces, Banach spaces, etc., and the operators like convex and concave functions. This became so important since we cannot adequately handle risk measures and its management in our chosen area and direction of the research, without bringing in the consequences of Hahm - Banach theorem.

Definition 1.7.1 Vector space

Let X be a nonempty set, K a scalar field. Suppose the functions $\bullet\,, +$ and defined on X; that is, $X \times \mathrm{X} \rightarrow \mathrm{X}$, such that:

X is an abelian group.

$k \bullet (x + y) = k \bullet x + k \bullet y$ for all values $x, y \in X, k \in K$.

$(\alpha + \beta) \bullet x = \alpha \bullet x + \beta \bullet x$ for each $x \epsilon X$ and $\alpha, \beta \in K$.

$(\alpha\,\beta) \bullet x = \alpha(\beta\, x)$ for all $\alpha, \beta \in K$ and $x \in X$.

Then X is called a linear space or a Vector space over $K.$ If K is a set of real numbers or complex numbers, then X is called real linear space or complex linear space respectively.

Definition 1.7.2 Normed linear space

Let X be a linear space over K. A norm is a real - valued function $\|\cdot\|$, that is $\|\cdot\| : X \to [\,0\,,\infty)$ such that for arbitrary $x, y\ \in\ X,\ k\ \in\ K$ and the following conditions are satisfied,

$\|x\|\ \geq 0\ and\ \ \|x\| = 0$ if and only if $\ x = 0$.

$\|k\,x\| = |k|\ \|x\|$ for all $k \epsilon K$ and $x\ \in\ X$

$\|x + y\|\ \leq\ \|x\| + \|y\|$ for arbitrary $x, y\ \in\ X$.

Then a Linear space with a norm define on it is called a normed Linear space.

Hint: if X is normed Linear space, the norm $\|\cdot\|$, always induce a metric ρ on X given that $\rho\,(\,x, y) =\ \|x - y\|$ for all $x, y\ \in X$. Remember that a sequence $\{\,x_n\}$ in a metric space is said to be complete if every Cauchy sequence in X converges to a point (element) in X.

Definition 1.7.3 Banach space

A Banach space is a complete normed space. In other words, a normed space X is a Banach space if every Cauchy sequence in X converges.

Definition 1.7.4 Inner product space

Let E be a linear space. An inner product on E is a function $\langle,\rangle : E\ \times E\ \to \mathbb{C}$ defined on $E\ \times E$ with values in $\mathbb{C}$ (where $\mathbb{C}$ is the set of complex numbers) such that the following three conditions are satisfied

$\langle x, x\rangle\ \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$

$\langle x, y\rangle =\ \overline{\langle y, x\rangle}$, (where the bar indicates complex conjugation)

$\langle \lambda x + \mu y,\ z\rangle =\ \lambda\langle x, z\rangle +\ \mu\langle y,\ z\rangle$.

Then $(E, \langle x, x\rangle)$ is called an inner product space.

Hint: Concerning the norm defined in 1.7.2, we say that a sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in inner product E if and only if $\langle x_n - x_m, x_n - x_m\rangle^{\frac{1}{2}} =\ \|x_n - x_m\|\ \to 0$ as $n, m\ \to\ \infty$. An inner product space E is complete if every Cauchy sequence in E converges to a point of E.

Definition 1.7.5 Hilbert space

A complete inner product space is called Hilbert space, and it is a complex Hilbert space or a real Hilbert space if its linear space is complex linear space or real linear space respectively.

Definition 1.7.6 Convex sets

A subset $\mathbb{C}$ of a linear space X is said to be convex if for each $x, y \in \mathbb{C}$, the line segment $(\lambda + (1-\lambda)y\,)$ for each $\lambda \in (0,1)$ belongs to $\mathbb{C}$. Therefore, a convex set contains all the lines joining any two points of the set. The intersection of a finite or infinite number of convex sets is convex. And finally, the convex hull of a set $\mathbb{C}$ is the intersection of all convex sets which contains the set $\mathbb{C}$.

Definition 1.7.7 convex function

Let $\mathbb{C}$ be a nonempty convex set in $\mathbb{R}^n$. The function $f: \mathbb{C} \to \mathbb{R}^1$ is said to be convex on $\mathbb{C}$ if $f(\lambda + (1-\lambda)y\,) \leq \lambda f(x) + (1-\lambda)f(y)$

For each $x, y \in \mathbb{C}$ and $\lambda \in (0,1)$. The function f is said to be strictly convex if the above inequality holds as a strict inequality for each $x, y \in \mathbb{C}$ and $\lambda \in (0,1)$. There are two basic properties of convex function that made it very useful in applied Mathematics, that is; (1) its maximum is attained on the boundary of its domain of definition. (2) if it is a strictly convex function, it admits at most one minimum.

Definition 1.7.8 Concave function

Let $\mathbb{C}$ be a nonempty convex set in $\mathbb{R}^n$. The function $f: \mathbb{C} \to \mathbb{R}^1$ is said to be Concave on $\mathbb{C}$ if $f(\lambda + (1-\lambda)y\,) \geq \lambda f(x) + (1-\lambda)f(y)$

For each $x, y \in \mathbb{C}$ and $\lambda \in (0,1)$. The function f is said to be strictly Concave if the above inequality holds as a strict inequality for each $x, y \in \mathbb{C}$ and $\lambda \in (0,1)$.

Definition 1.7.9 Monotone functions

Let F be a function such that $f: \mathbb{R} \to \mathbb{R}$, is said to be

An increasing function if for all values of $x, y \in \mathbb{R}$ with $x < y$ implies that $f(x) < f(y)$.

A decreasing function if for all values of $x, y \in \mathbb{R}$ with $x > y$ implies that $f(x) < f(y)$.

A non-increasing function if for all values of x $x, y \in \mathbb{R}$ with $x \geq y$ implies that $f(x) \leq f(y)$.

A non-decreasing function if for all values of $x, y \in \mathbb{R}$ with $x \leq y$ implies that $f(x) \leq f(y)$.

1.8 Organisation of the thesis

This thesis is organised into seven chapters to address the objectives and the research questions. It is as follows: Chapter 1 deals with the introduction and the fundamental concepts of the research, the objectives and the research questions, background of the study and the definitions of the terms used in work.

Chapter 2 reviews the related literature, showing how portfolio optimisation originated, its challenges, and effort made by researchers to improve on the existing results. We also looked at what the literature holds in terms of risk of a portfolio, different types of risk, mathematical properties of risk and its evolution, different ways researchers developed to manage the risk of a portfolio.

In Chapter 3, we gave the outline of the thesis, our data collection, and our research methodology.

Chapter 4 deals with the Mathematical Foundations of Markowitz's Portfolio Optimization Theory. We explore the indicated construction of the optimal portfolio, portfolio selection and weight allocation (allocations of the weights on different assets in the portfolio to give a better yield) convex analysis used in the Markowitz's portfolio theory. The focus was on the strengths and weaknesses of the model under different investment scenarios, like multi-objective constrained optimisation, and the methods used to overcome the shortcomings. The approach is primarily a critical literature review that informs work in this chapter of the thesis.

In Chapter 5, we worked on Random Matrix Theory and Applications. We reviewed different theories, strategies, scenarios, assumptions, and concepts, underpinning Random matrix theory on investment systems, with a focus on the use of hypothesised standard normal distribution of eigenvector components from RMT to assess deviations of the empirical eigenvectors to this distribution for different eigenvalues. We also use the Inverse Participation Ratio to measure the deviation of eigenvectors of the empirical correlation matrix from RMT results. These preliminary results on the dynamics of asset price correlations in the NSE are essential for improving risk-return trade-offs associated with Markowitz's portfolio optimisation in the stock exchange; we achieved by cleaning up the correlation matrix.

Chapter 6 deals with risk management, portfolio optimisation under different risk measures. We studied some risk measures and the kinds of investment and portfolio optimisation schemes in which they are used. The emphasis was on how to build on the existing results under different risk measures and the conditions for the existence of unique solutions. We investigated both the theoretical results and their empirical applications. For example, using copula functions to examine the skewness, thus the joint occurrence within an n-dimensional random vector $U = (u_1, u_2, \dots, u_n)$ with copula C is completely controlled by the copula function. C assigns probabilities to the joint occurrence of the individual stocks which we saw as random vector X where $X = (x_1, x_2, \dots, x_n)$, and $U = (u_1, u_2, \dots, u_n) = C(F_1\ (X_1), F_2\ (X_2), \dots, F_n\ (X_n))$ where F_i denotes the respective cumulative distribution function of the stocks, that is, the random vector X and $i = (1,2, \dots, n)$ for all $n \epsilon N$

Finally, Chapter 7 deals with empirical applications of the Research Results in the Nigerian Stock Market, we considered the types of assets that exist in the NSM, the nature of investment returns and risks associated with them, and features of the NSM that may require unique investment approaches. We then used insights from these ideas to produce empirical applications of the research results with the assets from NSM as inputs. We concluded, made our recommendations and made hints on our future research.

1.9 Contributions to knowledge.

In this section, we share the outcome of our work and some of the deliverables made. In our work, we focused on the optimisation portfolio which includes; maximisation of the returns, minimisation of the associated risk, proper selection of the assets that make up the portfolio and effective weight distribution among the assets to form an optimal portfolio.

From the literature, it shows that we are the first to apply Random matrix theory to Nigerian stock market and we were able to deliver the following; 1) T.C Urama, C.P Nnanwa, and P.O. Ezepue, Application of Random Matrix Theory In Estimating Realistic Implied Correlation Matrix (2018) submitted to Physica A. 2) C. P. Nnanwa,

T. C. Urama, P. O. Ezepue, Random matrix approach to cross-correlation in an emerging African market. (2018) Submitted to Journal of Financial Markets. 3) C. P. Nnanwa, T. C. Urama, P. O. Ezepue, Portfolio optimisation of financial services stocks in the Nigerian Stock Exchange. American Review of Mathematics and Statistics. December 2016, Vol. 4, No. 2, pp. 1-9. 4) Urama, T.C., Ezepue, P.O. and Nnanwa, C.P. (2017) Analysis of Cross-Correlations in Emerging Markets Using Random Matrix Theory. Journal of Mathematical Finance, 7, 291-307. https://doi.org/10.4236/jmf.2017.72015. 5) C. P. Nnanwa, T. C. Urama, P. O. Ezepue, Random matrix theory analysis of cross-correlation in the Nigerian Stock Exchange, Proceedings ISF (2017) accepted to appear and finally, with two conference presentations done in Australia and the UK.

1.10 Summary and Conclusion

In this chapter, we tried to bring out the critical aspects of the research, these include; the introduction and the background of the study, others are, the aims and objectives, definitions of some key concepts we will be using most often than not in the course of the work and also the plan for the study. Finally, we gave the indicative structure of our intended thesis and even some of the contributions to the knowledge we hope to achieve.

Chapter 2 The General Literature review

2.1 Introduction

In this section, we aim at discussing in detail what the literature holds in investment and portfolio optimisation. Also, we will look at the views of different researchers about Markowitz's portfolio theory, its benefits, weaknesses, limitations and assumptions, and the attempts made by the researchers to improve on the weaknesses and removal of the limitations and assumptions.

Furthermore, we had a literature review in all the technical chapters that are peculiar to the topic treated in the chapter.

2.2 Literature review

In the early 1950s, Harry Markowitz, a PhD research student, designed a financial model otherwise called mean-variance (MV) portfolio optimisation. This method was developed to help the investors know which asset that will be selected in a portfolio, how the selection will be done and the weight of each asset in the collection.

2.2.1 Mathematical foundations of portfolio theory

In the paper titled Portfolio selection (1952), Markowitz's outlines the importance of diversification of portfolios. He pointed out that there is a rule to invest in a portfolio which implies that the investors should diversify and at the same time should maximise expected return. However, he said that the portfolio with the maximum expected yield is not necessarily the one with minimum variance, he went further to explain that there is a rate at which the investors can gain expected return by taking on variation or reduce variance by giving up the expected return. The mean-variance portfolio optimisation helped the investors to quantify the risk of the portfolio compared with the specific risk of the assets in the collection. It tried to solve the problem of portfolio selection for a risk-averse investor; in this respect, it chooses the portfolio with the minimum risk with the same return or the portfolio with the maximum profit with the same risk.

Some researchers had shown that Markowitz's diversification of portfolio has significant benefits for investment; a reduction in loses and also in the portfolio volatility. Though the report did not make it explicit that diversification of portfolio will guarantee a profit or prevent against loses, but it will at least protect some of the profits the investment has accumulated.

Michaud (1989b) outlined some of the benefits of Markowitz's portfolio model (mean-variance optimisation MV). He listed the benefits as the satisfaction of client objectives and constraints; control of portfolio risk exposure; implementation of style objectives and market outlook; efficient use of investment information; and appropriate portfolio changes.

Jorion (1992) in his work reported that studies over a while suggested that the international diversification into foreign bonds has some benefits which are measured by comparing the performance of a passive world index with that of a US index. Jorion (1992) describes the mean-variance optimisation as the cornerstone of the modern finance theory; he believes that it is a powerful tool for efficiently allocating wealth to different investment alternatives. He further said that the method takes care of the investor's preferences and expectations of return and also it considers the risk for all assets, and the overall portfolio risk reduced through diversification.

Generally, research has shown that Modern portfolio theory when holding various uncorrelated assets combined in a portfolio, the return is improved, and the risk reduced to a bearable minimum level. Also, the risk level of the individual security in a portfolio does not matter as long as its return varies from the other securities in that portfolio.

Rotblut (2010), shows that MV produces the ideal amount of return for a given level of risk. This line is known as an efficient market frontier; It determines if a portfolio is taking on too much risk in respect to a given level of return and also can reveal if a portfolio is achieving a level of performance for the amount of risk it is taking.

Finally, if an investor is risk-averse, MV allows him to allocate the majority of the portfolio to bonds and bond maturity funds; and will allocate the smaller portion of the portfolio to other classes of assets while investors with a higher risk tolerance may choose to allocate the majority of their portfolios to stocks.

b) Weaknesses, limitations and assumptions of Markowitz mean-variance

However, research has shown that the Markowitz mean-variance has some weaknesses and several constraints. These limitations have taken centre stage of research. Researchers like: Fuerst (2008), Norton (2009), Ceria and Stubbs (2006), Goldfarb and Iyengar (2003), Jorion (1992), Konno and Suzuki (1995), Michaud (1989a) (1989b), Bowen (1984), Ravipti (2012) etc. discussed the weaknesses, limitations and assumptions in their works. Markowitz himself in Markowitz (1952) says that they tried to avoid mathematical proofs and could only get a geometrical presentation for 3 or 4 security cases. Therefore, these two are the main limitations of MV; the model did not allow n-security in a portfolio. Michaud (1989b), shows that the fundamental flaws of the mean-variance optimiser are its estimation error. It tends to overweigh those securities with a high estimate of return, negative correlations, and small variances or underweight those securities with a low rating of return, positive correlations, and large deviations. He pointed out that the statement; 'Optimizer, in general, produce a unique optimal portfolio for a given level of risk' is highly misleading. The ill-conditioning of the covariance matrix is yet another problem of MV; it makes the optimisation to be highly unstable by making a small change in the input assumption to lead to a significant difference in the solution.

Konno and Suzuki (1995) in their research show that based on the assumption that investor is risk-averse; MV believes that the distribution of the return is multivariate normal, or the utility of the investor is a quadratic function of the rate of return. But unfortunately, they noticed that neither of the two holds in practice. Haung et al. (2008) in their words say that when probability distributions of security returns are asymmetric, variance becomes a deficient measure of investment risk because the selected portfolio based on variation may have a potential danger to sacrifice too much-expected return in eliminating both low and high return extremes.

c) Some of the improvement of the Markowitz mean-variance

Since the discovering of the Markowitz's MV limitations and weaknesses, a lot of researchers have been working on the model to improve and develop it in different directions. Authors like Jobson, Korkie and Ratti (1979), Jobson and Korkie (1980), Frost and Savarino (1988), Jorion (1992), Michaud (1998), (1989b), Polson and Tew (2000), etc. worked on the estimation error. Others like Britten-Jones (2002), Kandel and Stambaugh (1996), Zellner and Chetty (1965), Klein and Bawa (1976) and Brown (1978) worked on the Markowitz's model by using Bayesian approach and predictive probability to improve and develop the model in various ways. Huang et al. (2008) and Markowitz (1993) tried to develop the model to Mean-semi variance while other like; Konno and Suzuki (1995), Liu, Wang and Qiu (2003), and Pornchai et al. (1997) further developed the model by adding skewness in the mean-semi variance model. Authors like: Vercher et al. (2007), Leon et al. (2002) and Carlsson et al. (2002) in their works tried to replace the uncertain returns of the securities with fuzzy numbers.

2.2.2 Mathematical properties of risk measures

There are two basic rules about winning in trading as well as in life; 1). If you don't bet, you can't win and 2). If you lose all your chips, you can't bet'.

Larry Hite. Schwager (1998)

Therefore, life is all about taking the risk. In this section, we aim at discussing in detail what is in the literature about the properties of different risk measures as regards investment and portfolio theory.

The measurement of risk in the investment management cuts across the wide disciples like; economic theory, statistics of actuarial sciences and the probability theory Lleo (2009).

a) Definition of Risk measure in portfolio management.

In financial mathematics, a risk measure is used to determine the amount of an asset or set of assets (traditionally currency) to be kept in reserve. The purpose of this reserve is to make the risks taken by financial institutions, such as banks and insurance companies, acceptable to the regulator (Wikipedia). A risk measure is defined as a mapping from a set of random variables to the real numbers, where this set X; of random variables represents portfolio returns. Then, the risk measure $\rho(X)$ is a function ρ such that $\rho : X \rightarrow \mathbb{R}$ (set of real number), if it is

Monotonous: $for\ every\ x, y\ \in X\ such\ that\ x \geq y\ implies\ \rho(y)\ \geq \rho(x),$

Sub-additive: $for\ every\ x, y, x + y\ \in X\ \ implies\ \rho(x + y) \leq\ \rho(x) +\ \rho(y),$

Positively homogeneous: $for\ x\ \in X, h > 0, hx \in X\ implies\ \rho(hx) = h\rho(x),$

The translation is invariant: $for\ x \in X\ , a \in \mathbb{R}\ implies\ \rho(x + a) = \rho(x) - a,$

Before now, most investors and market practitioners were making use of the risk measures like; Variance and standard deviation risk measure, Value at risk, Measure of downside risk, conditional value at risk, and Extreme theory, until the evolution of the current and more sophisticated risk measures.

Note that the axioms of risk measure in the portfolio analysis is as thus; the set X is the random collections of returns of portfolios. That is $x, y \in X$, while $\rho(X)$ is risks associated with X and $\rho(y), \rho(x) \in \rho(X)$. Then, the monotonicity means that if the return of portfolio x is greater than the returns of y then, the risk of portfolio y; $\rho(y)$ is greater than $\rho(x)$ which the risk of x. The second one which is subadditivity, means that the sum of the individual risks of portfolios x and y; $(\rho(y) + \rho(x))$ is always less than or equal to the risk of the both $\rho(x + y)$. The axiom of positively homogenous implies that the risk of a portfolio increases with the same rate when there is a positive increase in the portfolio. And finally, the transitivity shows that addition of cash into a portfolio decreases the risk by the same amount added.

b) Markowitz's Risk measure.

The Standard deviation is seen as the oldest risk measure, having been introduced by Markowitz (1952). Standard deviation σ (SD) as a symmetric risk measure gives the accurate total risk of the return, because it includes both the upside deviation and downside deviation in its computation of the risk of a return at the same time, this characteristic of SD becomes one of its shortcoming. Due to the symmetric nature of SD, when the return distribution becomes skewed, it affects the accuracy of the risk measure.

To correct this shortcoming of the SD, Capital asset pricing model (CAPM) was introduced. It describes the relationship between risk and expected return, which is used in the pricing risky securities.

$$\overline{r_a} = r_f + \beta_a \left(\bar{r}_m - r_f\right),$$

where $\bar{r}_m = Expected\ market\ return,\ r_f = Risk\ free\ rate$

and $\beta_a = Beta\ of\ the\ security$

CAPM splits the total risk into systematic risk, thereby provides a more excellent decomposition of risk.

c) Value at Risk (VaR) as a risk measure.

Value at Risk (VaR) is a type of risk measure which is widely used by market practitioners; it represents the maximum loss within the confidence level 1 - α that a portfolio could incur over a specified period, Lleo (2009). VaR of a portfolio can be computed using three main methods called; delta-normal, historical simulation and Monte Carlo simulation. Beder (1995) said that VaR is seductive but dangerous because it is often non-convex and non-smooth as a function of investment position; therefore, difficult to optimise using scenarios.

Delta-normal is an analytic method that provides a mathematical formula for the VaR, and it assumes that the risk factors are log-normally distributed, and the securities return is linear in the risk factors. The method involves going back in time to compute the variances and correlations for all the risk factors and portfolio risk generated. It is generated by the combination of linear to many factors that are assumed to be normally distributed and by the forecast of the covariance matrix which required: 1) for each risk factor, estimates of volatility and correlations and 2) positions on risk factors. The normality and the linearity assumption for the risk factors is the major shortcoming of the method, Jorion (1996).

When we compute VaR from the historical assets returns of a portfolio by applying the current portfolio allocation to derive the portfolio's return distribution, it is called Historical simulation method. Although, this method is suitable for fat-tailed and skewed distribution, and it does not assume any particular form for the return distribution which is a good advantage, but the fact that it assumes that the past return distribution is an accurate predictor for the future return patterns is a significant shortcoming. If the asset returns are normally distributed, the result obtained is the same with the delta-normal, and it required: 1) for each risk factor, a time-series of actual movements, and 2) position on risk factors, Jorion (1996).

Monte Carlos simulation is a probabilistic method that obtains the VaR of a portfolio numerically by generating the returns using many random simulations. It has advantages over the two mentioned earlier because it allows the assets to be non-linear and the risk factors do not follow a specific type of distribution. It has two steps: 1) the choice of the distributions and parameters like risk and correlations are derived from historical data, and a stochastic process is specified. And 2) fictitious price paths are simulated for all variables of interest.

VaR is generally known for its blind spot as a shortcoming; therefore, this gave rise to the development of Expected shortfall (ES) which Acerbi and Tasche (2002) defines it as

$$ES_{(\alpha)}(X) = \frac{-1}{\alpha} \int_{0}^{\alpha} F\overleftarrow{X}(p)\, dp$$

And Conditional Value at Risk (CVaR) is defined by Rockafellar and Uryasev (2002a) as

$$CVaR_{(\alpha)}(X) = -E\left[X \mid X \leq F\overleftarrow{X}\,(\alpha)\right]$$

Where $\alpha \epsilon (0, 1]$ is known as the confidence level. These refinements of VaR are closely related risk measures but they are not the same. They try to solve the problem of the blind spot in the tail of VaR

Later in the development of research, new risk measures were developed. These risk measures include monetary risk measure, Coherent risk measure; convex risk measure and Spectral risk measure.

d) Monetary risk measure

Let X be a linear space of bounded functions containing the constants. A mapping $\rho : X \rightarrow \mathbb{R}$ is called a monetary measure of risk if it satisfies the following condition for all $x, y \in X$;

Monotonous: $if\ x \geq y, implies\ \rho(y) \geq \rho(x)$,

The translation is invariant: $if\ a \in R\ implies\ \rho(x + a) = \rho(x) - a,$

Monetary risk measure is a class of risk measure equates the risk of an investment with the minimum amount of cash, or capital, that one needs to add to a specific risky investment to make its risk acceptable to the investor or regulator. Artzner introduced it, et al. (1999), they saw it as a function of the absolute loss that an investor could potentially incur on a position. In other words, Monetary risk measure is a distance between an investment's potential loss and acceptable level of loss;

and also risk is expressed as a monetary amount in Nigerian Naira, U.S. dollars, British pounds, Euro, etc.

Any monetary risk measure ρ is Lipschitz continuous with respect to the supremum norm. $|\rho(X) - \rho(Y)| \leq \|X - Y\|$.

Let ρ be a monetary risk measure with acceptance set $\mathcal{A}_\rho = \{X \in \mathcal{X} | \rho(X) \leq 0\}$. Then, $\mathcal{A}_\rho$ is non-empty, and satisfies the following two conditions: $inf\{m \in \mathbb{R} \mid m \in \mathcal{A}_\rho\} > -\infty$, $X \in \mathcal{A}_\rho, Y \in \mathcal{X}, Y \geq X \Longrightarrow Y \in \mathcal{A}_\rho$. Moreover, $\mathcal{A}_\rho$ has the following closure property: for $X \in \mathcal{A}_\rho$ and $Y \in \mathcal{X}$, $\{\lambda \in [0,1] | \lambda X + (1 - \lambda)Y \in \mathcal{A}_\rho\}$ is closed in $[0,1]$. ρ can be recovered from $\mathcal{A}_\rho$: $\rho(X) = inf\{m \in \mathbb{R} \mid m \in \mathcal{A}_\rho\}$.

ρ is convex risk measure if and only if $\mathcal{A}_\rho$ is convex ρ is positive homogeneous if and only if $\mathcal{A}_\rho$ is a cone. In particular, ρ is coherent if and only if $\mathcal{A}_\rho$ is a convex cone. If we assume that $\mathcal{A}$ is a non-empty subset of $\mathcal{X}$ which satisfies

$inf\{m \in \mathbb{R} \mid m \in \mathcal{A}\} > -\infty$, $X \in \mathcal{A}, Y \in \mathcal{X}, Y \geq X \Longrightarrow Y \in \mathcal{A}$. Then the functional $\rho\mathcal{A}$ has the following properties: $\rho\mathcal{A}$ is a monetary risk measure. If $\mathcal{A}$ is a convex set, then $\rho\mathcal{A}$ is a convex risk measure. If $\mathcal{A}$ is a cone, then $\rho\mathcal{A}$ is positively homogeneous. $\mathcal{A}$ is a subset of $\mathcal{A}_{\rho\mathcal{A}}$. If $\rho\mathcal{A}$ satisfies $\{\lambda \in [0,1] | \lambda X + (1 - \lambda)Y \in \mathcal{A}_\rho\}$ then $\mathcal{A} \equiv \mathcal{A}_{\rho\mathcal{A}}$. Note: the proof of the Hints will be given in the subsequent and relevant chapters of the thesis.

e) Coherent risk measure

Eventually, Coherent risk measure was developed, and Artner et al. (1999) defined it as a Subclass of Monetary risk measure, which satisfies the coherent properties. It includes; monotonicity, subadditivity, homogeneity and translation invariance. Lleo (2009) has shown that standard deviation calculated using a distribution of asset returns is not a monetary risk measure, and consequently will not be a coherent risk measure since coherent is a subclass of monetary risk measure. Also, Artzner et al. (1999) have shown that VaR lacks the property of subadditivity and therefore, cannot be coherent. Acerbi and Tasche (2002) show that Expected shortfall (ES) is a coherent risk measure and if CVaR and ES coincide when P&L distribution is continuous, and they are coherent, but when it is not continuous, CVaR is not coherent. Rockafellar and Uryasev (2000a) tried to solve the problem by introducing a standardised α-tail cumulative density function, which ensures that CVaR is coherent for discontinuous distribution functions. Later, some researchers extended the coherent risk measure by redefining the measure in different ways, Bielecki and Pliska (2003) introduced a set of log-coherence properties which can be applied to measure risks based on the distribution of instantaneous (log) returns, and this was called log-coherence. Cherny and Madan (2006) introduced and defined a utility function P(.) out of coherent and called it a coherent utility risk measure. Csoka et al. (2007) and Artzner et al. (2007) introduce a general equilibrium perspective and a multi-period measurement process for coherent risk measure respectively, and they are called coherence and general equilibrium and multi-period coherent risk measure.

f) Convex Risk measure

Follmer and Schied (2002a) argued that the risk of some instruments might not increase linearly with the size of the position, and this contradicts the Artzner et al. (1999) homogeneity properties, they thereby proposed the relaxation of the homogeneity and sub-additive properties by replacing them with convexity. This gave rise to loosening the original property of coherence. Therefore, all coherent risk measure is necessarily convex, but the converse is not necessarily true. In 2002, Szego showed that VaR is neither coherent nor convex since it lacks the property of subadditivity. Eventually, more people worked on the class of risk measure; some of them were: Ben-Tal and Teboulle (2007), Kloppel and Schweizer (2007), and Jobert and Rogers (2008). Owing to the translation invariance axiom required incoherent and convex risk measures, the practitioners did not widely accept them, because the practical work done these days on capital allocation assumes incoherent risk measure. Consequently, Rockafeller et al. (2006) came up with deviation risk measures, an alternative class of risk functional which do not require the translation invariance axiom. Farinelli et al. (2008) introduced a promising practical, effective one-sided risk measure used in new-type performance ratio, which does not require translation invariance also.

Furthermore, Power CVaR (PCVaR) was developed. Chan and Yang (2009) developed a new class of risk measures which is convexity and monotonicity, through a nonlinear weight function, can flexibly reflect the investor's degree of risk aversion and can control the fat-tail phenomenon of the loss distribution.

In another development, Kountazakis (2011) studied coherent and convex risk measure on non-reflexive Banach spaces. They extend the results of Konstantinides and Kountzakis (2011) in a case where the space of the positions E is a non-reflexive Banach space. In that paper, the author showed that the asset is some interior point e ϵ $int(p_o)$ namely an interior point of the wedge under which E is partially ordered, e is either a reference “cash stream” as in Stoica (2006) or a “relatively secure cash stream” as in Jaschke and Kuchler (2001). They prove the dual representation of the (Po , e)-coherent and the (Po , e)-convex risk measures defined on E, which make it different from the theorems of representation of Artzner et al (1999), Delbean (2002) and Follmer et al (2000a).

g) Spectral Risk measure

Two researchers; Kusuoka (2001) and Acerbi (2002), while working independently, tried to parameterise coherent risk measure using a risk-aversion function, this introduced the class of risk measure called spectral risk measure. It is a risk measure given as a weighted average of outcomes where adverse outcomes are, typically, included with larger weights. Spectral risk measure is a function of portfolio returns and outputs the amount of the numeraire (usually a currency) to be kept in reserve. It is always a coherent risk measure, but the converse does not always hold. An advantage of spectral measures is how they can be related to risk aversion, and particularly to a utility function, through the weights given to the possible portfolio returns.

Definition, consider a portfolio X, and then a spectral risk measure $M_{\phi} : \mathcal{L} \to \mathbb{R}$, where ϕ is non-negative, non-increasing, right-continuous, an integrable function defined

$$\int_0^1 \phi(p)dp = 1$$

on $[0,1]$ such that is defined by

$$M_{\phi}(X) = -\int_0^1 \phi(p)F_X^{-1}(p)dp,$$

where F_X is the cumulative distribution function for X. If there are S equi-probable outcomes with the corresponding payoffs given by the order statistics $X_{1:S}, \dots, X_{S:S}$.

Let $\phi \in \mathbb{R}^S$, the measure defined by is a spectral measure of risk if satisfies the conditions: Non-negativity: $\phi_S \geq 0$ for all $S = 1, \dots, S$. Normalization: $\sum_{S=0}^{S} \phi_S = 1$

Monotonicity: ϕ_S is non-increasing, that is $\phi_{S1} \geq \phi_{S2}$ if $S_1 < S_2$ and $S_1, S_2 \in \{1, \dots, S\}$. These are properties of spectral risk measure, every spectral risk measure $\rho : \mathcal{L} \to \mathbb{R}$ satisfies:

Positive Homogeneity: for every portfolio X and positive value $\lambda > 0$, $\rho(\lambda X) = \lambda \rho(X)$.
Translation-Invariance: for every portfolio X and $\alpha \in \mathbb{R}$, $\rho(X + \alpha) = \rho(X) - \alpha$
Monotonicity: for all portfolios X and Y such that $X \geq Y$, $\rho(X) \leq \rho(Y)$
Sub-additivity: for all portfolios X and Y, $\rho(X + Y) \leq \rho(X) + \rho(Y)$
Law-Invariance: for all portfolios X and Y with cumulative distribution functions F_X *and* F_Y respectively, if $F_X = F_Y$ then $\rho(X) = \rho(Y)$
Comonotonic Additivity: for every comonotonic random variables X and Y, $\rho(X + Y) \leq \rho(X) + \rho(Y)$.
Note that X and Y are comonotonic if for every $\omega_1, \omega_2 \in \Omega$: $\left(X(\omega_2) - X(\omega_2)\right)\left(Y(\omega_2) - Y(\omega_2)\right) \geq 0$. It was noticed that this class of risk measure contains expected shortfall and CVaR (when the returns of distribution of CVaR is continuous), in which risk aversion function is constant within the interval $(0, \alpha)$. Tasche (2002) show that spectral risk measure not only satisfying the four properties of coherent risk measure (making it a subclass of coherent risk measure) but also added two more properties which is the law of invariance and commonotonic additivity. Pflug (2000) and Rockafeller and Uryasev (2000a) introduced an optimization method for CVaR, while Acerbi and Simonetti (2002) extended the optimization method for CVaR by applying spectral risk measure to portfolio selection.
In another development, a two-parameter beta-VaR measure which shows the role played by the beta probability distribution in its definition and, also a one parameter alpha-VaR measure, is the restriction of the two-parameter beta-VaR measure were the two new spectral risk measures introduced by Cherny and Madan (2006).

2.2.3 Mathematical analysis of investment systems.

In this section, we aim at reviewing the mathematics of the investment systems, some key concepts of mathematical analysis and stochastic (calculus) processes as relate to investment systems and draw their relevance as regards to emerging markets.

The primary purpose for taking up any investment or indulging in the mathematics of investment is to increase the value of the money or any valuable thing invested (otherwise called capital) in a business. Therefore if the amount someone invests in business is, (P) over a period of time, (t) to increase to the value S, one can now talk about the interest on (P) (or the discounted valve on (S)) is $(S-P)$. So, we can see the rate of the interest or discount value over a period of time is determined by, $r = \frac{S-P}{P}$, and $u = \frac{S-P}{S}$ respectively.

a) Investment system

Zhu (2007), saw an investment system as a set of rules of buying and selling of investment properties such as stocks, bonds, real estate, commodities and their derivatives for capital appreciation. Zhu also presents the investment system as a practice of evaluating and comparing the effectiveness of investment systems using their actual or simulated history of its performances. On the other hands, they argued that these historical performances do not always reflect the true potential of the investment system because investment sizes often skew the results.

In this work, Zhu considered the process of testing an investment system over a set of historical data, they denoted $\{g_n : n = 1, \ldots, N\}$ with $g_1 < g_2 < \ldots g_N$ (where $g_n < 0$ represents a loss), as the outcomes of the trades generated by the system in terms of percentage gain. It was their intention to use the test to identify the frequencies p_n associated with each gain outcome g_n . Again, they denoted s as the denote the size of each trade as the percentage of the available capital and *M* as the total number of trades in the test, having in mind that the frequencies $\{p_n\}$, the return depends on the size of each trade. Therefore, the number of trades with gains g_n is Mp_n. If $\mathrm{G}(s)$ is the average exponential rate of growth of the investment capital per trade with a trading size s percent of the available capital, they had that;

$$G(s)^M = \prod_{n=1}^{N}(1 + sg_n)^{Mp_n}$$

and

$$G(s) = \prod_{n=1}^{N}(1 + sg_n)^{Mp_n}$$

where a maximum of $G(s)$ gives a good indication of the potential profitability of the investment system. They used its natural log for analysis since the natural log is an increasing function, the maximum of $f\,(s)$ will give an equivalent indication of the effectiveness of the investment system. Therefore, the log of $G(s)$ is;

$$lnG(s) = f(s) = \sum_{n=1}^{N} p_n ln(1 + sg_n)$$

the range of s is $[0, 1]$ however, to explore the full potential of the investment system, they allowed s to take all the values in $(-1/g_N, -1/g_1)$, the domain of f , with the interpretation that $s > 1$ represents trade on margins and $s < 0$ represents shorts

b) Utility distribution

If we consider a sequence of assets; x_i , and a unit of P is invested in the ith asset which gives a random return expectation of x_i as $E(x_i) = \mu_i$ where there is a riskless asset with a return ρ, the implication of the capital asset pricing model of Sharpe (1964) and Linetner (1965) is that

$$\mu_i = \rho + \tau\beta_i$$

where β_i is the covariance between x_i and the return on the market portfolio.

Chamberlain (1983) pointed out that there must be sharp restrictions on either the quadratic utility or the distribution of return (eg., multivariate normality). Consequently, the theory which underlines the result above is highly criticised. They went further to say that, even if the restriction conditions are satisfied that observing the return on the market portfolio will be difficult. Since $\mu_i = \rho + \tau\beta_i$ is derived from mutual fund separation where investors divide their wealth between the riskless asset and single fund, Chamberlain (1983) said that the consequence of this is that all investors should have perfectly correlated returns on their portfolios which seems likely not to be true.

Ross (1976) and (1977) on Sharpe (1964) and Linetner (1965), proposed an alternative basis for $\mu_i = \rho + \tau\beta_i$, which rests on the distributional restrictions implied by a factor structure;

$$x_i = \mu_i + \beta_{i1}f_1 + \ldots + \beta_{ik}f_k + \vartheta_i$$

for $(i = 1, 2, 3, \ldots)$, where ϑ is the idiosyncratic disturbances which are uncorrelated with each other and with the factors f.

Chamberlain (1983) said the factor structure implies that the variance of the portfolio's return can be decomposed into two components which are: a market component generated by the $f's$ and an idiosyncratic component generated by the $\vartheta's$. They saw $f's$ as the representative of the economy wide shocks that affect all asset returns while ϑ_i; is the uncertainty that is specific to the ith asset.

Ross showed that $\mu_i = \rho + \tau\beta_i$ or its K-factor generalization is approximately true if there exist numbers $\tau_1, \ldots, \tau_k$ such that

$$\gamma \equiv \sum_{\mathrm{i}=1}^{\infty} (\mu_i - \rho - \tau_1\beta_{i1} - \ldots - \tau_k\beta_{ik})^2 < \infty$$

is the implication of the absence of arbitrage opportunity equilibrium.

Chamberlain and Rothschild (1983) tried to define and put the asset on the market in a sequence, where one unit of money invested in the ith asset gives a random of x_i, and the portfolio formed by investing α_i in the ith asset has a random return as

$$\sum_{i=1}^{N} \alpha_i x_i$$

the vector represents the portfolio $(\alpha_1, \ldots, \alpha_N)$.

$L_2(P)$ is the underlying probability space which denotes all random variables with finite variances defined on that space and x_i are assumed to have finite variances, therefore the sequence $\{x_i\}_{i=1}^{\infty}$is in $L_2(P)$. With this, we can easily see that the mean of the x_i is $\mu_i = E(x_i)$, variance is $\sigma_{ii} = V(x_i)$ and the covariance is $\sigma_{ij} = Cov(x_i, x_j)$.

If the span of $x_1, \ldots, x_N$ which is the linear subspace that consists of all the linear combination of $x_1, \ldots, x_N$ be denoted by $\boldsymbol{\mathcal{F}}_N = [x_1, \ldots, x_N]$, but $\boldsymbol{\mathcal{F}}$ is the infinite union of $\boldsymbol{\mathcal{F}}_N$, that is,

$$\bigcup_{N=1}^{\infty} \boldsymbol{\mathcal{F}}_N = \boldsymbol{\mathcal{F}},$$

such that $\boldsymbol{p}$ which is a random return on a portfolio formed from some finite subset of the assets is a member of $\boldsymbol{\mathcal{F}}$**,** that is $\boldsymbol{p \in F}$**.**
Remember, if we look at $L_2(P)$ under the mean-square inner product;

$$(p,q) = E(p,q) = Cov\,(p,q) + E(p,q)E(p,q),$$

It is easy to see that it has the Hilbert space framework, with the norm as;

$$\|p\| = \left(E(p^2)\right)^{\frac{1}{2}} = \left(V(p) + \left(E(p)\right)^2\right)^{\frac{1}{2}},$$

for all $p, q \in L_2(p)$. Remember that $\overline{\boldsymbol{F}}$, which is the closure of $\boldsymbol{F}$ is a Hilbert space since it is a linear subspace of $L_2(p)$. For $\boldsymbol{p \in \overline{F}}$, there is a sequence $\{p_N\} \in \boldsymbol{F}$ with the norm $E((p_N - p)^2)$ that tends to zero as N tends to infinity;

$$E((p_N - p)^2) \to 0 \;\; as\, N \to \infty.$$

This implies that there is a finite number of portfolios whose random returns are good approximations of p. If we have $(x_1, \ldots, x_N)$ then, $\sum_N$ becomes the covariance matrix of $(x_1, \ldots, x_N)$ and $\sum_{i=1}^{N} \alpha_i$ is the cost of the portfolio, then $p = \sum_{i=1}^{N} \alpha_i x_i$ with norm as

$$\|p\|_2 = \left(\sum_{i=1}^{N} \alpha_i\right)^{\frac{1}{2}}$$

and also $q = \sum_{i=1}^{N} \beta_i x_i$, with norm as

$$\|q\|_2 = \left(\sum_{i=1}^{N} \beta_i\right)^{\frac{1}{2}}.$$

Looking at the equality in the probability space; $L_2(P)$ which is Hilbert space, is $E((p - q)^2) = 0$, therefore it implies that $p - q = 0$, showing that $p = q$. If $p = q$, therefore $V(p - q) = 0$ and this gives that $\alpha_i = \beta_i$.

Remember that Riesz representation theorem stated thus; If L is a continuous linear functional on a Hilbert space $\mathcal{H}$, then there is a unique $q \in \mathcal{H}$ such that $L(p) = (q,p)$ for every $p \in \mathcal{H}$. And the Projection theorem stated thus; If $\mathcal{G}$ is a closed linear subspace of a Hilbert space $\mathcal{H}$, the every $p \in \mathcal{H}$ has a unique decomposition as $p = p_1 + p_2$, where $p_1 \in \mathcal{G}$ and $p_2 \in \mathcal{G}^{\perp}$ (i.e., $\mathcal{G}^{\perp} = \{p_2 \in \mathcal{H}\,; (p_2, q) = 0 \; for\; every\; q \in \mathcal{G}\}$).

Chamberlain and Rothschild (1983) came up with the result; If $p_N = \sum_{i=1}^{N} \alpha_{iN} x_i$ converges to p (in mean-square), then p is referred to as a limit portfolio. They further stated that if it is not possible to obtain a riskless, positive, return at zero cost, then the linear functional that gives the cost of a portfolio is continuous. From Riesz's theorem, it follows that this linear functional can be represented by a limit portfolio and this limit portfolio generates the mean-variance frontier.

Zhu (2012) worked on a discrete model for the financial market, they used a finite set Ω to represent all possible economic states and assume that the natural probability of each state is described by a probability measure P on the power set of Ω, and the they assume that $P(w) > 0$ for all $w \in \Omega$. If $RV\,(\Omega)$ is the finite dimensional Hilbert Space of all random variables defined on Ω, with inner product

$$\langle \xi, \eta \rangle = E^P(\xi\eta) = \int_{\Omega} \xi \eta dP = \sum_{w \in \Omega} \xi(w)\, \eta(w) P(w).$$

For $\xi \in RV\,(\Omega)$ they used $\xi > 0$ to signal $\xi(w) \geq 0$ for all $w \in \Omega$ and at least one of the inequalities is strict. We consider a discrete model in which trading action can only take place at t = 0, 1, 2, . . .

Chamberlain (1987) developed bonds on asymptotic efficiency for a particular class of non-parametric models; they assume that $\{(x_{i1}, x_{i2}, y_i)\}_{i=1}^{\infty}$ is independent and identically distributed according to some unknown distribution F_0, and that $E(x_{i1}u_i) = E(x_{i2}u_i) = 0$, where the residual u_i is defined by $u_i = y_i - \beta_{01}x_{i1} - \beta_{02}x_{i2}$. They want to impose the restriction that $\beta_{02} = 0$. Suppose that they first obtain $(\hat{\beta}_{1n}, \hat{\beta}_{2n})$ from the unrestricted least squares regression of y on x_1, x_2, and then obtain a restricted estimator of $\theta_0 = \beta_{01}$ by choosing $\hat{\theta}_n$ to

$$\min_{\theta}\left(\begin{pmatrix}\hat{\beta}_{1n}\\ \hat{\beta}_{2n}\end{pmatrix}-\begin{pmatrix}\theta\\ 0\end{pmatrix}\right)' A_n\left(\begin{pmatrix}\hat{\beta}_{1n}\\ \hat{\beta}_{2n}\end{pmatrix}-\begin{pmatrix}\theta\\ 0\end{pmatrix}\right),$$

where A_n converges with probability one to A, a positive-definite matrix and the optimal choice for A is the inverse of the asymptotic covariance matrix of $(\hat{\beta}_{1n}, \hat{\beta}_{2n})$. In the homoskedastic case with $E(u_i^2|x_{i1}, x_{i2}) = \sigma^2$ we set

$$A_n = n^{-1}\sum_{i=1}^{n} x_i x_i'$$

where $x_i' = x_{i1}, x_{i2}$ and $\hat{\theta}_n = \beta_{yx_1}$ which is the least square coefficient in the regression of y on x_1. But more generally, in the heteroskedasticity,

$$A_n = \left[\left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \hat{u}_{in}^2 x_i x_i'\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right)^{-1}\right]^{-1},$$

where $\hat{u}_{in}^2 = y_i - \hat{\beta}_{1n}x_{i1} - \hat{\beta}_{2n}x_{i2}$

from the theory of minimum-χ^2 or minimum-distance estimation that the limiting distribution of $\sqrt{n}(\hat{\theta}_n - \theta_0)$ depends upon A and not upon the sequence of estimators $\{A_n\}$. If $E(u_i^2 x_i, x_i) = E(u_i^2)E(x_i, x_i)$, then $\hat{\theta}_n$ has the same asymptotic variance as β_{yx_1}; but in general, it is strictly more efficient.

Chamberlain (2000) worked on choosing a two assets portfolio at a date T, with the returns at t per unit invested at $t-1$ is regarded as y_{1t} *and* y_{2t}. His portfolio choice is

$$Z \equiv \{(y_{1t}, \ldots, y_{Kt})\}_{t=0}^{T}$$

having observed the value of the variables $y_{3t}, \ldots, y_{Kt}$ as the future forecasting values of the returns. If the portfolio with two assets is held on until H, with one unit of the amount invested on it; a and $1-\mathrm{a}$ respectively.

If $w = \{(y_{1t}, y_{2t})\}_{t=T+1}^{H}$ and $h(w,a)$ denote the value of the portfolio at $t = H$:

$$h(w,a) = a\prod_{t=T+1}^{H} y_{1t} + (1-a)\prod_{t=T+1}^{H} y_{2t},$$

The problem posed here is how an investor is going to choose a. If the investor regards (z,w)as the outcome of the random variable (Z,W) with distribution Q and u as his utility function, then choose a decision rule d that will maps observations z into actions a:

$$\max_{d} E_Q \left[u\left(h\big(W, d(Z)\big)\right)\right].$$

c) Ito Calculus

Ito (1944) and (1951) invented stochastic calculus and stochastic differential equation respectively since then many researchers have been working on the area extensively. It is assumed that the stochastic finance theory presumes that the stock price, currency exchange rate and interest rate follow Ito's stochastic differential equation. Liu (2013) questioned and argued the rationality of the above presumption. Their line of argument follows thus:

If they assume that the stock price X_t follows the stochastic differential equation

$$dX_t = eX_t dt + \sigma X_t dW_t$$

where e is the log-drift, σ is the log-diffusion, and W_t is a Wiener process, they tried to describe what will happen with such an assumption. It follows from the stochastic differential equation; $dX_t = eX_t dt + \sigma X_t dW_t$ that X_t is is a geometric Wiener process, i.e.,

$$X_t = X_0 \exp((e - \sigma^2/2)t + \sigma W_t$$

from which they derive

$$W_t = \frac{\ln X_t - \ln X_0 - (e - \sigma^2/2)t}{\sigma}$$

whose increment is

$$\Delta\, W_t = \frac{\ln X_{t+\Delta t} - \ln X_0 - (e - \sigma^2/2)\Delta t}{\sigma}$$

and they got

$$A = \frac{-(e - \sigma^2/2)\Delta t}{\sigma}$$

Their line of argument continued, that the real stock price X_t is a step function of time with a finite number of jumps, despite the fact that it looks like a curve. Without loss of generality, they assume that X_t is observed to have 100 jumps during a fixed period. Then, they divided the period into 10,000 equal intervals and, they observed 10,000 samples of X_t, which follows from $\Delta\, W_t = \frac{\ln X_{t+\Delta t} - \ln X_0 - (e - \sigma^2/2)\Delta t}{\sigma}$; that $\Delta\, W_t$ has 10,000 samples that consist of 9,900 A's and 100 other numbers:

$9{,}900 \begin{cases} A, \\ A, \dots, \\ A \end{cases} and\ 100 \begin{cases} B, \\ C, \dots, \\ Z \end{cases}$. They stated that nobody can believe that those 10,000 samples follow a normal probability distribution with expected value 0 and variance Δt, (Figure illustrating that to be shown later). They claimed that this fact contradicts the property of Wiener process, which the increment ΔW_t is a normal random variable with expected value 0 and variance Δt. Thus, they conclude that the stock price X_t does not follow the stochastic differential equation.

They went further to presume that some think that the stock price does behave like a geometric Wiener process (or Ornstein-Uhlenbeck process) in macroscopy although people recognise the paradox in microscopy. However, as the very core of stochastic finance theory, Ito's calculus is just built on the microscopic structure, of Wiener process (that is the differential dW_t), rather than macroscopic structure. They claimed that Ito's calculus is precisely, dependent on the presumption that dW_t is a normal random variable with expected value 0 and variance dt.

This, they saw as an unreasonable presumption which causes the second order term in Ito's formula,

$$dX_t = \frac{\partial h}{\partial t}(t, W_t)dt + \frac{\partial h}{\partial w}(t, W_t)dW_t + \frac{1}{2}\frac{\partial^2 h}{\partial w}(t, W_t)dt$$

They concluded by saying that the increment of the stock price is impossible to follow any continuous probability distribution, and due to the basis of the above paradox, they do not think that Ito's calculus can play the essential tool of finance theory because Ito's stochastic differential equation is impossible to model real stock price.

d) Uncertainty Distribution

Black and Scholes (1973) constructed a theory for determining the stock option price, otherwise known as the Black-Scholes formula. Since then, stochastic financial mathematics was founded based on probability theory. Sometimes in practice, one may notice that there are small numbers of samples or no samples at all that are available to estimate a probability distribution. To solve this problem, Liu (2007) developed Uncertainty theory, which is a branch of axiomatic mathematics for modelling human uncertainty based on normality, duality, subadditivity, and product axioms. The theory has been applied to many fields such as uncertain programming, uncertain inference control, and uncertain optimal control.

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element Λ in $\mathcal{L}$ is called an event. A set function $\mathcal{M}$ from $\mathcal{L}$ to [0; 1] is called an uncertain measure if it satisfies the following axioms; Liu (2007): ΩM

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ;

Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ;

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M} = \left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Where $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. To obtain an uncertain measure of the compound event; Liu (2009a), defined a product uncertain measure there by producing the fourth axiom of uncertainty theory:

Axiom 4. (Product Axiom) Let$(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots, n$, then the product uncertain measure $\mathcal{M}$ is an uncertain measure on the product σ-algebra $\mathcal{L}_1 x\, \mathcal{L}_2\, x, \ldots, x\mathcal{L}_n$ satisfying

$$\mathcal{M} = \left\{\prod_{i=1}^{n} \Lambda_k\right\} = \min_{1 \leq k \leq n} \mathcal{M}_k\{\Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}.$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$for $k = 1, 2, 3, \ldots$

Therefore Liu (2007) gave the following definitions:

Definition: An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, for any Borel set B of real numbers, the set.

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

Is an event. To describe an uncertain variable in practice, Liu (2007) defined the concept of uncertainty distribution.

Definition: An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x \to -\infty} \Phi(x) = 0, \qquad \lim_{x \to +\infty} \Phi(x) = 1$$

Thus, this shows that the uncertainty distribution $\Phi: \mathbb{R} \to [0, 1]$ of an uncertain variable, ξ is defined as $\Phi(x) = \mathcal{M}\{\xi \leq x\}$ and the expected value of its variable is the definition given above.

2.2.4 Practical applications of these ideas in different contexts,

In this section, we aim at reviewing the empirical applications of some of the ideas we have talked about to some emerging markets, with emphasis on the stylised facts of asset returns and their implications for volatility estimation as applied to investments, trading, and risk management.

Cervelló-Royo et al. (2015) present empirical evidence which confronts the classical Efficient Market Hypothesis, which states that it is not possible to beat the market by developing a strategy based on a historical price series. They propose a risk-adjusted profitable trading rule based on technical analysis and the use of a new definition of the flag pattern, which defines when to buy or sell, the profit pursued in each operation, and the maximum bearable loss. Empirically, they used a database comprised of 91,307 intra-day observations from the US Dow Jones index and parameterised the trading rule by generating 96 different configurations and reported the results of the whole sample over three sub-periods. They also replicated the analysis on two leading European indexes: the German DAX and the British FTSE, and the returns provided by the proposed trading rule are higher for the European than for the US index. This highlights the more significant inefficiency of the European markets.

a) Random matrix theory (RMT)

El Alaoui (2015) studied cross-correlation of the structure of a portfolio of equities among stocks of Casablanca Stock Exchange by using random matrix theory (RMT) to analyse eigenvalues and see if there is a presence of pertinent information using Marčenko–Pastur distribution. They inferred that Marčenko–Pastur distribution presented the theoretical interval of RMT predictions to observe which eigenvalues are deviating by plotting their empirical distribution. And also, the deviating eigenvalues might contain important information about the market which represents about 11% of the studied eigenvalues of Casablanca Stock Exchange stocks. They advised the Portfolio managers to consider that there is a sharp asymmetry in the left, which means that the market reacts more to the adverse events than good events when they construct their portfolios. To help the practitioners reduce their errors of predictions, they prescribed the cleaning procedure of the correlation matrix, which reduces the gap between predicted and slightly realised risks. Furthermore, they claimed that the analysis of eigenvectors components distributions of eigenvalues showed that normal distribution fitting is not very suitable for elements that are outside of the range of RMT predictions, which shows that they are not noisy elements. Finally, they confirm that the inverse participation ratio gives more precision about the deviation degree of eigenvalues elements to understand better the correlation structure of the portfolio.

b) Book-to-market (BM) ratios

Ko et al. (2014) studied Taiwan stock market; they argue that a sophisticated investor can do better (i.e. obtain higher returns) than a simple buy-and-hold strategy by timing the market with the help of some technical analysis. They show that an application of a moving average timing strategy to portfolios sorted by book-to-market (BM) ratios could generate higher returns than the buy-and-hold strategy and confirm that the moving average timing strategy does substantially out-perform the buy-and-hold strategy. Based on trading signals issued by the moving average rule, they prescribe to the practitioners to go for a zero-cost portfolio constructed by buying the highest BM portfolio, and short-selling the lowest BM portfolio and the application of their result demonstrates that such a new investment strategy can produce significantly positive returns. They further verified their results by extending the empirical study with a different currency, alternative lag lengths, transaction cost, sub-period analysis, business cycles and market timing.

c) GARCH (p,q) model

Bollerslev (1986) proposed a GARCH (p,q) model for measuring and forecasting financial market volatility. Ezepue and Omar (2013b) discussed the possibility of developing a dynamic portfolio optimisation schema which incorporates know, for a GARCH (1,1)– based, volatility measures of risk. Ezepue and Omar (2013c), look at multidisciplinary stochastic – time series and control engineering research in stock market characterisation and development (SMCD). In that paper, they illustrate the fundamental ideas of all shares index in the Nigerian stock market (NSM) for the period 2000–2010, using ARCH–GARCH volatility modelling. Here, the researcher will look at GARCH (1,1) model which in the literature on volatility models is considered to be a versatile risk model that applies to many situations in which financial quantities vary over time Omar (2012).

Ezepue and Omar (2013a) review the recent publications on the global financial crisis policy papers on stock and capital market development as relating to volatility modelling of the Nigerian stock market. In their work, they apply what they called five candidate GARCH volatility models to the All shares index and returns of the NSM for 2000 – 2010 so that they can identify volatility driver and select a suitable best – fit – model for the pre and post-crisis periods. Though some authors like Aliyu (2009a),(2009b), (2012); Adamu (2008); Adebiyi et al (2009) ; AFDB (2007); Chinzara (2008); Ezepue and Omar (2012); Musa et al (2013); Okpara (2010); Omotosho and Daguwa (2012); Umar and Abdulhakeem (2010) and Yahaya (2012), have worked in this area of Nigerian financial system but Ezepue and Omar (2013a) did not stop at the exploration of the ideas with singular focus on the topics like the authors mentioned above. Instead, they went further to develop the concept to a comprehensive – Economy stock market characterisation.

2.3 Summary and conclusion

In this chapter, we try to bring out what is in the literature as regards the work we want to do. We also discussed risk and its type, risk management and evaluations, and return of an investment. Furthermore, we looked at the portfolio, the value of a portfolio at any given time, the interactions of different assets in a portfolio. Finally, we analysed the effect of portfolio diversification.

Chapter 3 Overview of the study and the methodology

3.1 Introduction

In this chapter, we are going to discuss our research methodology; a brief outline of the methods which we used while carrying out the research work. We gave a brief insight on how we obtained our data and, will enumerate the models we proposed using in the course of the research. Furthermore, a brief background study of the work shown in the following sections in this chapter.

3.2 Data to be used

We collected the data which we are working with from the Nigerian Stock Exchange (NSE) and some relevant institutions like Central bank of Nigeria and some stockbroker firms. Other datasets were obtained from the Johannesburg stock exchange (JSE).

3.3 Research Methodology

To justify the research objectives and questions, we adopted some methods and strategies. These strategies and techniques help to bring out the insight we have in work, and some this was discussed in subsequent sections.

3.4 Risks obtainable in the investment system.

Every investment one makes, one is expected to have some returns. But these returns are not guaranteed. Therefore, the possibility of variation of the return that comes in from the expected return is known as **Risk**. Consequently, we can see risk as to the potential for variability in Returns**.**

Some investments are considered as a **Low - Risk investment**, while others are as a **High - Risk investment**. The Investment with Fairly stable returns is said to a low - risk investment, examples are government securities, while the one that has a fluctuating return is considered as a high - risk investment and examples are Equity shares.

This variation in returns (in holding securities, such as shares and Debentures, etc.), otherwise called risk is caused by several factors which are grouped into two major groups namely; **Systematic -Risk** and **Unsystematic - Risk**.

3.4.1 Systematic - Risk

Systematic - Risk comprises of factors that are external to the company, and it affects many securities simultaneously, and they are mostly not controllable. Take, for instance, factors like economic, political and social changes will affect the variability in the securities returns. If the economy moves into recession, the profits made by the companies will experience a downward shift, thereby bringing a decline in the stock price of most companies. Few factors made up this systematic risk, and they include; Interest rate, Market risk and Purchasing power risk.

3.4.2 Unsystematic - Risks

Unsystematic - Risks are those risks which the company can control the factors that induce the risks. Since the risk is controllable by the company, it is, therefore, unique and peculiar to the company due to the company's practises. Examples of such factors are a scarcity of raw materials, management inefficiency and labour strikes. This unsystematic risk factors can be further grouped into two ways; the operating an environment of the company which is called **Business Risk** and the financial pattern or policies adopted by the company which is called **Financial Risk**.

Having gone through this, we can now see that the total risk components of security are the systematic risk and the unsystematic risk.

3.5 Risk Measurement and Evaluation

We have seen that the risk in investment is associated with the returns; the expected return of an investment is the probability-weighted average of all possible returns. Therefore, the risk of an investment cannot be evaluated or measured adequately without referring to the return.

Let us denote the possible returns as x_i , the probabilities of the possible returns as $p(x_i)$, while $E(X)$ which is the sum of the products of the possible returns with their respective probabilities. Therefore, the expected returns is calculated thus,

$E(X) = \sum_{i=1}^{n} x_i \, p(x_i)$ where $i = 1, 2, 3, \ldots, n$.

For instance, if an investor has a share which he is expecting $30\%, 40\%, 50\%, 60\%,$ and 70% as the case may be when viewed in a different perspective, and also $0.10, 0.30, 0.40, 0.10,$ and 0.10 are respective probabilities of the possible returns. Then, the expected which is $E(X) = \sum_{i=1}^{n} x_i \, p(x_i)$ will be evaluated is this way;

x_i	$p(x_i)$	$x_i \, p(x_i)$
30	0.1	3
40	0.3	12
50	0.4	20
60	0.1	6
70	0.1	7

Therefore, $E(X) = \sum_{i=1}^{n} x_i \, p(x_i)$ = 48%

$E(X) = 48\%$

Now we have seen the expected return; we can now talk about the risk calculation since the expected returns are not sufficient to determine the risk of an investment. In doing this, one may use several measures to do so, but the Mean-Variance approach is the widely accepted procedure for assessment of risk.

Therefore, we compute the Variance using the expected return $\bar{X}$ above. Let's denote variance as σ^2, thus,

$$\sigma^2 = \sum_{i=1}^{n} [(x_i - \bar{X})^2 \, p(x_i)], \ \ i = 1, 2, 3, \ldots, n.$$

The above equation is the summation of the products of the squares of the deviation with their respective probabilities.

x_i	$p(x_i)$	$(x_i - \bar{X})$	$(x_i - \bar{X})^2$	$(x_i - \bar{X})^2\, p(x_i)$
30	0.1	-18	324	32.4
40	0.3	-8	64	19.2
50	0.4	2	4	1.6
60	0.1	12	144	14.4
70	0.1	22	484	48.4

Therefore, $\sigma^2 = \sum_{i=1}^{n}[(x_i - \bar{X})^2\, p(x_i)] = 116.0.$

Finally, the square root of the variance σ^2 is the Standard deviation.

Remember, we said earlier that the risk associated with securities is made up of two components; systematic risk and unsystematic risk. Since the unsystematic risk is a type of risk that is peculiar to a company's security, it can be reduced by **diversification** of the investment, while systematic risk cannot be diversifiable, hence, needs to be looked into by the investor.

3.6 The Returns of an Investment

When we talk about Portfolio theory, we are talking about the theory that deals with the problems of constructing a collection of assets in an investment with the sole aim of making a profit.

While selecting a portfolio, we will also have the financial objectives and the risk tolerance of the investor in mind. The implication of the statement above shows that there are two basic features of the asset we will consider before selecting them. We must look at the average returns over some time and also, how risky it would be to stake on such asset over the proposed investment period.

Let A be our notation for an asset such that $A(0)$ and $A(t)$ be the values of the asset at the time equal to zero (t = 0 ie at the time of investing) and t respectively. Therefore, the rate of return (ρ) the asset is defined as

$$A(t) = (1 + \rho)A(0) \tag{3.1}$$

This rate of return can be seen in a layman's as an effective interest rate. Therefore the rate of the return is;

$$\rho = \frac{A(t) - A(0)}{A(0)} \tag{3.2}$$

Remember that the actual value of $A(t)$ is uncertain at the point $A(0)$ ie, at the point of investing, because of this uncertainty nature of the $A(t)$, we should consider it as a random variable in other to model it.

Therefore, the average return is

$$\mu = E(\rho) \tag{3.3}$$

where $E(\rho)$ is the expectation of a random variable which we called expected rate of return. This expected rate of return on its own can only show us how the return will be but will not bring out the riskiness of the assets.

Therefore, we make use of the variance to bring out the riskiness of the asset,

$$\sigma^2 = var(\rho) = E(|\rho - \mu|^2) \tag{3.4}$$

Due to the risky nature of the assets, it is vital to consider the coupling effect of the assets (which we are going to throw more light on it). If we have n^{th} collection of assets$\left(A_{1,}A_{2,\dots,}A_n\right)$, let us denote the rate of return and variance of any i^{th} asset by ρ_i and σ_i^2 respectively. Here we look at the coupling effect among the assets to be;

$$\sigma_{ij} = E\left[(\rho_i - \mu_i)\left(\rho_j - \mu_j\right)\right] \tag{3.5}$$

This coupling leads to the definition of covariance matrix V, knowing that$\sigma_{ij} = \sigma_{ji}$, also when $i = j$, we have that $\sigma_{ii} = \sigma_i^2$ or $\sigma_{jj} = \sigma_j^2$

$$V = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \cdots & \cdots & \cdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix} \tag{3.6}$$

Note that V is a symmetric matrix and positive definite.

3.7 The principal of portfolios

We have seen that a Portfolio is an investment made in several assets using a certain amount of money M . If we have n number of assets to invest on and we allocate a certain amount of money to each i^{th} asset, we can now denote the amount as M_i (which is otherwise called the weight). Remember that the total amount invested is M. Therefore,

$$\sum_{i=1}^{n} M_i = M \tag{3.7}$$

without much ambiguity, one can easily see that the fraction of the money invested on the i^{th} asset is $w_i = \frac{M_i}{M}$.

Therefore, we can see that,

$$\sum_{i=1}^{n} M_i = M$$

but

$$w_i = \frac{M_i}{M} \tag{3.8}$$

therefore,

$$\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} \frac{M_i}{M}$$

this implies that,

$$\sum_{i=1}^{n} w_i = \frac{\sum_{i=1}^{n} M_i}{M}$$

but remember that $\sum_{i=1}^{n} M_i = M$, therefore,

$$\sum_{i=1}^{n} w_i = \frac{\sum_{i=1}^{n} M_i}{M} = \frac{M}{M} = 1$$

therefore,

$$\sum_{i=1}^{n} w_i = 1 \tag{3.9}$$

3.8 The Value of a Portfolio at any Time

The value of the Portfolio at any time (t) can be verified as in Kevin S. (2013); let the value of the portfolio be denoted as Q_p and expressed as;

$$Q_p(t) = \sum_{i=1}^{n} \frac{M_i}{A_i(0)} A_i(t) \tag{3.10}$$

Remember that at the point of investing in the portfolio, the value of the portfolio $Q_p \; is \; equal \; to \; Q_p(0)$, then,

$$Q_p(0) = M = \sum_{i=1}^{n} M_i$$

from equ. (3. 2)

Recall that $\rho = \frac{A(t)-A(0)}{A(0)}$, therefore in that respect, we will have that $\rho_p(t)$ (which is the expected return of the portfolio) will be;

$$\rho_p(t) = \frac{Q_p(t) - Q_p(0)}{Q_p(0)} \tag{3.11}$$

this implies that,

$$\rho_p(t) = \frac{\sum_{i=1}^{n} \frac{M_i}{A_i(0)} A_i(t) - M}{M}$$

where $Q_p(t) = \sum_{i=1}^{n} \frac{M_i}{A_i(0)} A_i(t) \; and \; Q_p(0) = M = \sum_{i=1}^{n} M_i$

therefore,

$$\rho_p(t) = \frac{\sum_{i=1}^{n} \frac{M_i}{A_i(0)} A_i(t) - \sum_{i=1}^{n} M_i}{M}$$

$$= \sum_{i=1}^{n} \frac{M_i}{M} \frac{A_i(t)}{A_i(0)} - \sum_{i=1}^{n} \frac{M_i}{M}$$

$$= \sum_{i=1}^{n} \left(\frac{M_i}{M} \frac{A_i(t)}{A_i(0)} - \frac{M_i}{M} \right)$$

$$= \sum_{i=1}^{n} \frac{M_i}{M} \left(\frac{A_i(t)}{A_i(0)} - 1 \right)$$

we have seen that $m_i = \frac{M_i}{M}$ and $\frac{A_i(t)}{A_i(0)} - 1 = \frac{A_i(t) - A_i(0)}{A_i(0)}$ therefore,

$$\rho_p(t) = \sum_{i=1}^{n} m_i \left(\frac{A_i(t) - A_i(0)}{A_i(0)} \right)$$

but earlier we have seen that $\rho_i = \frac{A_i(t) - A_i(0)}{A_i(0)}$, therefore,

$$\rho_p(t) = \sum_{i=1}^{n} w_i \ \rho_i \qquad (3.12)$$

With this, we have seen that the rate of return of a portfolio is the weighted average of the rates of return of the assets. We should note that these weights are determined by w_i (ie the fraction of the money invested in each of the i^{th} asset).

3.9 Expected rate of return and Variance of the Portfolio

The expected rate of return of the portfolio will now be;

$$\mu_p = E\left(\sum_{i=1}^{n} w_i \, \rho_i \right) \qquad (3.13)$$

but we have seen that $\mu_i = E(\rho_i)$, therefore,

$$\mu_p = \sum_{i=1}^{n} w_i \, E(\rho_i)$$

$$= \sum_{i=1}^{n} w_i \, \mu_i \tag{3.14}$$

The variance of the rate of return of the portfolio is given by,

$$\sigma^2 = E\left(\left|\rho_p - \mu_p\right|^2\right) \tag{3.15}$$

but we have shown that $\rho_p = \sum_{i=1}^{n} m_i \, \rho_i$ and $\mu_p = \sum_{i=1}^{n} m_i \, \mu_i$ therefore,

$$\sigma^2 = E\left(\left|\sum_{i=1}^{n} w_i \, \rho_i - \sum_{i=1}^{n} w_i \, \mu_i\right|^2\right)$$

$$= E\left(\left|\sum_{i=1}^{n} w_i \, (\rho_i - \mu_i)\right|^2\right)$$

$$= E\left(\left[\sum_{i=1}^{n} w_i \, (\rho_i - \mu_i)\right]\left[\sum_{j=1}^{n} w_j (\rho_j - \mu_j)\right]\right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \, w_j E\left([\rho_i - \mu_i][\rho_j - \mu_j]\right)$$

$$= \sum_{i\,j=1}^{n} w_i \, w_j \sigma_{i\,j}$$

where $E\left([\rho_i - \mu_i][\rho_j - \mu_j]\right) = \sigma_{i\,j}$ *and* $\sum_{i\,j}^{n} \sigma_{i\,j} = V$ which is the Covariance Matrix. Therefore,

$$\sigma^2 = \sum_{i\,j=1}^{n} w_i \, w_j \sigma_{i\,j} = M^T V M \tag{3.16}$$

If we have two portfolios say x and y that contains various assets, we can quantitatively describe the coupling between the portfolios x and y by using the covariance of random rates of returns of the portfolios $\rho_p^{(x)}$ *and* $\rho_p^{(y)}$, it gives thus;

$$\sigma^{(x,y)} = E\left[\left(\rho_p^x - \mu_p^x\right)\left(\rho_p^y - \mu_p^y\right)\right] \tag{3.17}$$

$$= E\left[\left(\sum_{i=1}^{n} M_i^{(x)}(\rho_i - \mu_i)\right)\left(\sum_{j=1}^{n} M_j^{(y)}\left(\rho_j - \mu_j\right)\right)\right]$$

$$= \sum_{i\ j=1}^{n} M_i^{(x)} M_j^{(y)} E\left([\rho_i - \mu_i][\rho_j - \mu_j]\right)$$

$$= \sum_{i\ j=1}^{n} M_i^{(x)} M_j^{(y)} \sigma_{i\ j}$$

$$= \left(M_p^{(x)}\right)^T V M_p^{(y)}$$

3.10 Covariance and Correlation

Covariance is a statistical measure of how much two random variables interact. In our case, it is the measure of interaction of the returns of the assets in the portfolio.
Though the magnitude of this covariance is not easy to interpret, the sign spans from negative to positive, the sign of the covariance shows the tendency in the linear relationship between the assets in the portfolio. In other words, we can say that covariance is an absolute measure of the interaction of the risk between assets.
When the returns of two securities move in the same direction consistently, then the covariance is said to be positive, but when they move in the opposite direction consistently, it is said to be negative. It is zero if their movement is independent of each other. Therefore, if we have two securities x and y, their covariance is

$$Cov(x, y) = \sum_{i=1}^{n} \frac{\left[(\rho^x - \mu_x)\left(\rho^y - \mu_y\right)\right]}{N} \tag{3.18}$$

where N is the number of observations.
On the other hand, the coefficient of correlation is related to the covariance thus,

$$Cov(x, y) = \gamma_{x,y}\ \sigma_x \sigma_y$$

which implies that,

$$\gamma_{x,y} = \frac{Cov(x,y)}{\sigma_x \sigma_y} \qquad (3.19)$$

where σ_x, σ_y are the standard deviation of x and y respectively, while $\gamma_{x,y}$ is the coefficient of correlation of x and y.

3.11 The Case with two Risky assets.

In this case, we will consider the case of two Risky assets in a portfolio and see the effect of the interaction of the assets in terms of risk reduction. Remember that this interaction produces a coefficient which is called the coefficient of correlation $(\gamma_{x,y})$ and it takes its value from +1 to -1.

Let $M_x \; and \; M_y$ be the weight (ie the amount of money) invested in x and y assets respectively, $\gamma_{x,y}$ is the coefficient of correlation between x and y, while $\sigma_x \; and \; \sigma_y$ are the standard deviations of x and y. Then,

$$\left(\sigma^{(x,y)}\right)^2 = M^T V M$$

$$= \begin{bmatrix} M_x & M_y \end{bmatrix} \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} \begin{bmatrix} 1 & \gamma_{x,y} \\ \gamma_{x,y} & 1 \end{bmatrix} \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} \begin{bmatrix} M_x \\ M_y \end{bmatrix}$$

this implies that

$$\begin{bmatrix} M_x & M_y \end{bmatrix} \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} = \begin{bmatrix} M_x\sigma_x + 0 & 0 + M_y\sigma_y \end{bmatrix} = \begin{bmatrix} M_x\sigma_x & M_y\sigma_y \end{bmatrix}$$

also

$$\begin{bmatrix} M_x\sigma_x & M_y\sigma_y \end{bmatrix} \begin{bmatrix} 1 & \gamma_{x,y} \\ \gamma_{x,y} & 1 \end{bmatrix} = \begin{bmatrix} M_x\sigma_x + \gamma_{x,y}M_y\sigma_y & \gamma_{x,y}M_x\sigma_x + M_y\sigma_y \end{bmatrix}$$

then

$$\begin{bmatrix} M_x\sigma_x + \gamma_{x,y}M_y\sigma_y & \gamma_{x,y}M_x\sigma_x + M_y\sigma_y \end{bmatrix} \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} =$$

$$\begin{bmatrix} M_x\sigma_x^2 + \gamma_{x,y}M_y\sigma_x\sigma_y + 0 & 0 + \gamma_{x,y}M_x\sigma_x\sigma_y + M_y\sigma_y^2 \end{bmatrix} =$$

$$\begin{bmatrix} M_x\sigma_x^2 + \gamma_{x,y}M_y\sigma_x\sigma_y & \gamma_{x,y}M_x\sigma_x\sigma_y + M_y\sigma_y^2 \end{bmatrix}$$

Then finally,

$$\begin{bmatrix} M_x\sigma_x^2 + \gamma_{x,y}M_y\sigma_x\sigma_y & \gamma_{x,y}M_x\sigma_x\sigma_y + M_y\sigma_y^2 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \end{bmatrix} =$$

$$\begin{bmatrix} M_x^2\sigma_x^2 + \gamma_{x,y}M_y\sigma_x\sigma_y + \gamma_{x,y}M_x\sigma_x\sigma_y + M_y^2\sigma_y^2 \end{bmatrix} =$$

$$M_x^2\sigma_x^2 + 2\gamma_{x,y}M_xM_y\sigma_x\sigma_y + M_y^2\sigma_y^2$$

Therefore,

$$\left(\sigma^{(x,y)}\right)^2 = M_x^2 \tag{3.20}$$

Let us consider the special cases when $\gamma_{x,y}$ is +1, 0, and -1.

If $\gamma_{x,y} = +1$, then the equation of the variance;

$$\left(\sigma^{(x,y)}\right)^2 = M_x^2\sigma_x^2 + M_y^2\sigma_y^2 + 2\gamma_{x,y}M_xM_y\sigma_x\,\sigma_y$$

will become:

$$\left(\sigma^{(x,y)}\right)^2 = M_x^2\sigma_x^2 + M_y^2\sigma_y^2 + 2M_xM_y\sigma_x\,\sigma_y$$

This implies that,

$$\left(\sigma^{(x,y)}\right)^2 = (M_x\sigma_x + M_y\sigma_y)^2$$

Therefore,

$$\sigma^{(x,y)} = M_x\sigma_x + M_y\sigma_y \tag{3.21}$$

$M_x\sigma_x\, and\ M_y\sigma_y$ are both positive and we will have a straight line in the risk/return space

If $\gamma_{x,y} = 0$, then the equation of the variance;

$$\left(\sigma^{(x,y)}\right)^2 = M_x^2\sigma_x^2 + M_y^2\sigma_y^2 + 2\gamma_{x,y}M_xM_y\sigma_x\,\sigma_y$$

will become;

$$\left(\sigma^{(x,y)}\right)^2 = M_x^2\sigma_x^2 + M_y^2\sigma_y^2$$

This implies that,

$$\sigma^{(x,y)} = \sqrt{M_x^2\sigma_x^2 + M_y^2\sigma_y^2} \tag{3.22}$$

again, the standard deviation shrinks a bite, and we will have part of a hyperbola in the risk/return space.

If $\gamma_{x,y} = -1$, then the equation of the variance;

$$\left(\sigma^{(x,y)}\right)^2 = M_x^2\sigma_x^2 + M_y^2\sigma_y^2 + 2\gamma_{x,y}M_xM_y\sigma_x\,\sigma_y$$

will become:

$$\left(\sigma^{(x,y)}\right)^2 = M_x^2\sigma_x^2 + M_y^2\sigma_y^2 - 2M_xM_y\sigma_x\,\sigma_y$$

This implies that,

$$\left(\sigma^{(x,y)}\right)^2 = (M_x\sigma_x - M_y\sigma_y)^2$$

Therefore,

$$\sigma^{(x,y)} = M_x\sigma_x - M_y\sigma_y \tag{3.23}$$

here, we have two positive entities $M_x\sigma_x \; and \; M_y\sigma_y$, the difference will further bring down the valve of standard deviation; also, we will have a hooked line in risk/return space.

3.12. Analysis of coefficient correlation in portfolio diversification.

The coefficient of correlation plays a significant role in portfolio diversification. From the above computation, we can see that if well managed, the coefficient of correlation will reduce the risk to a bearable level.

When $\gamma_{x,y}$ is +1, you will see that the standard deviation $\sigma^{(x,y)} = M_x\sigma_x + M_y\sigma_y$ has two positive numbers $M_x\sigma_x \; and \; M_y\sigma_y$. Therefore, when computed, the standard deviation blows up (which shows that the risk becomes bigger than the individual risks). Thus, shows that if we have assets that the interactions of their returns are perfectly positively correlated, the risk reduction will not be achieved because the portfolio risk will not be reduced below the individual asset risk. Therefore, Portfolio diversification does little or nothing to risk averting.

When $\gamma_{x,y}$ is 0 (zero), we will see that $\left(\sigma^{(x,y)}\right)^2 = M_x^2\sigma_x^2 + M_y^2\sigma_y^2$ implies that;
$\sigma^{(x,y)} = \sqrt{M_x^2\sigma_x^2 + M_y^2\sigma_y^2}$.
Though the entities are positive numbers, the portfolio standard deviation will be less than the standard deviation of the individual assets. Therefore, portfolio diversification can reduce risk when the assets returns are uncorrelated.
When $\gamma_{x,y}$ is -1, again, you will see that the standard deviation $\sigma^{(x,y)} = M_x\sigma_x - M_y\sigma_y$ has two positive numbers $M_x\sigma_x \; and \; M_y\sigma_y$. But after the subtraction, we will notice that the difference will so reduce. Therefore, when the coefficient of correlation is - 1, so that the asset returns are perfectly negatively correlated, portfolio diversification reduces the risk of the portfolio to bearable minimum. Though this level of risk reduction might not be achieved in real life, we can reduce the risk by changing the weight of the investment (ie $M_x and \; M_y$) . In other words, the risk of a portfolio decreases as the coefficient of correlation of the assets moves from positive to negative.

3.13 Summary and conclusion

We have shown our research methodology, where we obtained the data used for the research and the detailed work plan together with our outputs and the deliverables. The remaining part of the thesis is presented as Chapters four, five, six and seven.

Chapter 4 Portfolio Optimization

4.0 Introduction

Portfolio optimisation is a vital part of the portfolio management, all the study around portfolio management geared towards building an optimal portfolio. This means the ability of the market practitioners, players or investors to take significant and essential decisions on the allocation of the available funds to different stocks, bonds and derivatives in other to get a maximum output (profit).

In 1952, Markowitz introduced the mean-variance (MV) portfolio model which has two intentions, first is to maximise the portfolio return which is measured by the mean of the expected return and secondly, to minimise the risks on the return which is measured the standard deviation (variance) of the portfolio return.

Now, we have seen that the objective of every investor, whether it is a simple portfolio hold by an individual or a big portfolio which is managed by a professional investment manager, is to arrange a set of assets. Also, assign a weight to each of the assets that will yield a maximum return with a risk of the minimum bearable level.

In this chapter, we will look at the evolution of portfolio optimisation in the literature, different techniques and method some authors used in their work. Finally, we will present our work where we optimised portfolio from the Nigerian stock exchange (NSE).

4.1 Literature review for portfolio optimisation and evolution.

The decision on which asset that will make the portfolio of investment is a practise done by the market players whether an individual investor or a big professional portfolio managing company. This is not an easy task since every investor hopes to make a profit in all his/her investment irrespective of the risks involved in that investment.

The nature of this investment problem has attracted a lot of researchers brainstorming to resolve the puzzle. This area of research was pioneered by Markowitz (1952) where he proposed and formulated a portfolio optimisation model

MV, which sees an investor as one who will invest in a portfolio that will guarantee a level of tolerance amount of risk (minimised risk) and with the best profit (maximised return). Therefore, having this in mind while formulating his MV, Markowitz defines his portfolio as a real-valued vector which contains weights of the available assets (which the weighted sum must be maximised and at the same time minimises the variance of the return). In the beginning, the model seems to work, but when compared with the realities of what is obtainable from the market, it shows that MV lacks some merits in its assumptions. This made so many researchers to seek ways to improve the model in different aspects which includes; increasing the robustness of the model to address the complexities associated with portfolio selection problems, the introduction of the use of classical techniques which provides for additional constraints and improved risk handling techniques.

An evolutional algorithm (EAs) published first in the late '80s where it is used to solve dynamic game (game theory) problems, as Chen (2002) indicates. Researchers have used EAs to solve problems associated with different areas like biology, ecology, engineering and social sciences see Sivanandam & Deepa (2008), Ahn (2006), Goldberg (1989), and Fogel (1999). In mathematical finance, EAs is not capable of delivering the desired result due to its complexity. This limitation only allows EAs to deal with optimisation problem that has only single objective functions. Some Economic and Mathematical finance problems have multiple conflicting objectives which no longer demand an optimal solution but a set of solutions that will guarantee the best trade-offs among the objectives of the problem Ponsich et al. (2013). Due to this task mentioned above, multi-objective evolutional algorithm (MOEAs) was formed to solve those problems which single-objective evolutional algorithm (SOEAs) and other techniques cannot solve.

Arnone et al. (1993) first proposed the use of MOEAs in solving the optimisation problems in the investment portfolios. This was meant to modify the initial Markowitz model so that it can adapt to the real-world requirements which are obtainable from the market. Ponsich et al. (2013) show that the modification on the initial MV concerning MOEAs was mainly in three ways and they are; bringing in of realistic constraints, the addition of new objectives that form the risk indicators, and the use of Sharpe ratio.

In the real-life market practises, floor-ceiling and cardinality constraints are obtainable, but the initial Markowitz considered only the sum of the weights on each asset to be exactly the available fund (ie $\sum_{i=1}^{n} w_i = 1$) as its constraints. We should note that the quadratic programme in Markowitz (1987) will no longer be useful if a new constraint added which transforms the optimisation problem into a nonconvex space.

Floor - ceiling constraints were first introduced by Streichert et al. (2004) as buy-in thresholds; this tends to impose bounds on each asset's weight. As the name implies, it imposes the minimum and maximum percentage of the invested capital to each asset. This ensures that the higher portion of the fund is not lumped in one asset (vice versa), so that risk can be minimised. The mathematical expression of floor-ceiling constraints is

$$l_i \leq w_i \leq \mu_i, \forall\ i = 1, 2, \dots, n \qquad (i)$$

Many researchers like Chiam et al. (2008), Krink and Paterlini (2008) and Chang et al. (2009) were among those who subsequently worked on floor - ceiling constraints, though some of them appear to do something slightly different from Streichert et al. (2004), but the basis and methodologies are the same.

Another interesting constraint is the total weight assigned to the asset class. This was introduced by Krink and Paterlini (2008) and Pai and Michel (2009), and it lays down bounds on the total capital which is allocated to a sector or class of assets in the portfolio. For an instant in section (5.4.1), NSE has 11 sectors; oil and gas, Conglomerates etc. Bounds are, therefore, assigned to the sectors meaning that the sum of the weight of the assets in the sector will have a limit. This type of constraint is similar to the floor - ceiling above, but the difference is that floor-ceiling has its bounds on the assets while this has its bounds on each sector. This allocation is done using the values of the market capitalisation of each sector. The sector with a higher value of market capitalisation gets a bigger weight compared to the sector with lower capitalisation.

Chang et al. (2000) are believed to have introduced the cardinality constraints. This will force the assets selected in a portfolio to obey some restrictions. One of the two cases they proposed in their work imposes a value K which is the desired number the selected assets must be equal to, while the other case provides lower and upper bounds on this value K. The first case is known as the exact version while the other is known as the soft version. It is observed that if no cardinality constraint is imposed, quadratic programming (QP) can be used to solve the problem but if it is imposed, QP will no longer be valid. In the above mentioned work, Chang et al (2000) introduce ε_i and δ_i which is the mini-buy and the limit exposure of the portfolio to asset i, and $z_i \in [0,1]$, $i = 1, 2, \ldots, n$. This gives

$$\varepsilon_i z_i \leq z_i \leq \delta_i z_i, i = 1, 2, \ldots, n \qquad (ii)$$

If asset i is held, $z_i = 1$ and w_i will lie between ε_i and δ_i, and if none is held, $z_i = 0$ and this will force $w_i = 0$.

So the number of assets in the portfolio lies between K_l and K_u for $K_l \neq K_u$ this implies that the mathematical expression is;

$$K_l \leq \sum_{i}^{n} z_i \leq K_u, \forall\, K = K_l, K_{l+1}. \ldots, K_u \qquad (iii)$$

Pia et al. (2009)

In application, the amount of money used to buy security runs in the multiple of the smallest transaction lot, which is the minimum volume of the asset that can be acquired. In this case, the weight of any asset i is computed through a lot of purchase Dastkhan et al (2011). This type of constraint is known as a round-lot constraint with purchasing price c_i and x_i, an integer, which is regarded as the number of purchase lot, thus;

$$w_i = \frac{x_i c_i}{\sum_{i=1}^{n} x_i c_i}, \forall\, i = 1, 2, \ldots, n \qquad (iv)$$

Krink and Paterlini (2008) considered that if there is a change in the asset's weight, the change in the current weight and the previous one must be greater than a particular mark. This restriction on the change in the asset's weight is known as a Turnover constraint. This is mostly considered in a multi-period investment. If the current weight is w and the previous weight is w', thus, the mathematical expression is

$$|w_i - w_i'| \geq \Delta_i \; or \; |w_i - w_i'| = 0, i = 1, 2, \dots, n \qquad (v)$$

Also, the summation of the absolute value of the change in the weights of the previous and current value must be less than the maximum turnover ratio (TR),

$$\sum_{i=1}^{n} |w_i - w_i'| \leq TR \qquad (vi)$$

Gomez et al. (2006) proposed a similar constraint which they called purchase or sale constraint. With $\bar{B}_i$and $\underline{B_i}$ as the maximum and minimum purchasing thresholds respectively, when the current weight is greater than the previous weight, that is $w_i > w_i'$ and $\bar{S}_i$and $\underline{S_i}$ as the maximum and minimum sale thresholds respectively, when the current weight is less than the previous weight, that is $w_i < w_i'$.

$$Max(w_i - w_i', 0) \leq \bar{B}_i \; and \; Max(w_i' - w_i, 0) \leq \bar{S}_i \qquad (vii)$$

for all values of $i = 1, 2, \dots, n$, as well as trading constraints

$$w_i = w_i' \; or \; w_i \geq w_i' + \underline{B_i} \; or \; w_i \geq w_i' - \underline{S_i} \qquad (viii)$$

4.2 Definitions of some concepts

Here we are going to give definitions of some of the standard concepts in mathematics used in this chapter.

Definition 4.2.1

The multi-objective optimisation problem is said to be linear if all the objective functions and constraint functions are linear, but nonlinear multi-objective optimisation problem if there exist any of the objective functions or constraint functions are nonlinear.

Definition 4.2.2

A function $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if for all $w^1, \; w^2 \in \mathbb{R}^n$ is valid that $f_i(\beta(w^1) + (1 - \beta)w^2) \leq \beta f_i(w^1) + (1 - \beta) f_i(w^2)$ for all values of $\beta \in [0, 1]$. If all the conditions hold, with the inequality $'' \leq ''$ change to $'' \geq ''$, the function is said to be concave.

Definition 4.2.3

A function $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at w^* if $f_i(w^* + d) - f_i(w^*) = \nabla f_i(w^*)^T d + ||d||\varepsilon(w^*, d)$, where $\nabla f_i(w^*)$ is the gradient of f_i at the point w^*, $d \in \mathbb{R}^n$ is a feasible direction emanating from $w \in \mathcal{C}$, and $\varepsilon(w^*, d) \rightarrow 0$ as $||d|| \rightarrow 0$. Furthermore, If all partial derivatives, that is $\frac{\partial f_i(w^*)}{\partial w_j}$ for $j = 1,2, \dots, n$, are continuous at the point w^*, then f_i is continuously differentiable at that point w^*

Definition 4.2.4

A function $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is twice differentiable at w^* if $f_i(w^* + d) - f_i(w^*) = \nabla f_i(w^*)^T d + \frac{1}{2} d^T \nabla^2 f_i(w^*) d + ||d||\varepsilon(w^*, d)$, where $\nabla f_i(w^*)$ is the gradient, the symmetric $n \times n$ matrix $\nabla^2 f_i(w^*)$ is a Hessian matrix of f_i at the point w^* and $\varepsilon(w^*, d) \rightarrow 0$ as $||d|| \rightarrow 0$. The Hessian matrix is a twice - differentiable function which consists of second order partial derivatives, that is $\frac{\partial^2 f_i(w^*)}{\partial w_i \partial w_j}$ for $j = 1, 2, \dots, n$. If all second-order partial derivatives are continuous at w^*, then f_i is twice differentiable at w^*.

Definition 4.2.5

A function $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is an increasing function if for all value w^1, $w^2 \in \mathbb{R}^n$ such that $w_j^1 < w_j^2 \rightarrow f_i(w^1) < f_i(w^2)$ for all $j = 1, 2, \dots, n$. If all other conditions hold with $w_j^1 < w_j^2 \rightarrow f_i(w^1) > f_i(w^2)$, f_i is said to be decreasing function. It is important to note that if the inequality is weaken, that is if $<$ is replaced with $\leq$ or $>$ with $\geq$, then the increasing function is known to be non-decreasing function while the decreasing function will be non-increasing function.

4.3 Portfolio models

1. w is a vector of continuous decision variables from the feasible set $\mathcal{C} \in \mathbb{R}^n$ defined by linear or non-linear (sometimes linear and non-linear combined) constraints, while $F(w) = [f_1(w), f_2(w), \dots, f_n(w)]'$ forms the objective vector.

2. A multi-objective optimisation problem is said to be convex if all the objective functions and the feasible region are convex.

3. A multi-objective optimisation problem is said to be non-differentiable if at least one of the objective functions or the constraint functions forming the region is non-differentiable.

4. Finally, a function is said to be monotone if the function is either decreasing or increasing function (non-decreasing or non-increasing).

4.3.1 Markowitz's portfolio basic model

Markowitz in 1952 developed a portfolio optimisation model known as mean-variance MV which is governed by four axioms which are,

1. Mean, and variance of different stocks form the basis from which decision is made for the returns and risks of every investment made.

2. All the market players are more concern with the dividend which they get at the end, and the end time for every investment is same to all involved.

3. The investors are homogenous, meaning that information on the decision-making process which includes meaning; variance and correlation (see equation 5.18) of different stocks are freely and equally available to all the market players or participants.

4. Finally, assets are fungible, by fungibility, we mean that they (assets) are capable of being substituted in place of one another (note that this is not bartering).

Since a standard Markowitz deals with mean and standard deviation as returns and risk. Therefore we will look at Markowitz's MV.

$$\mu = Expected\ [R] \tag{4.1}$$

If $Expected\ [R] = E[\sum_{i=1}^{n} w_i\, R_i]$ and w_i is the weight of asset i. This shows that the expected return is

$$\mu = \sum_{i=1}^{n} w_i \mu_i \qquad i = 1,2,\dots,n \ \ (n \in \mathbb{N}) \tag{4.2}$$

For the risk, it is measured as the combined standard deviation or variance of the expected return of the assets in the portfolio. This implies that

$$\sigma^2(R) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j} \tag{4.3}$$

4.3.2 Markowitz's portfolio MV with Objective model.

We have seen earlier that the MV model has some shortcomings; these shortcomings made a lot of researchers to be working on them (shortcomings) to improve the results gotten from the model. To provide an analytical solution for the market players who may want to maximise their returns or minimise the risks. As it was pointed out earlier that the risk is measured as the standard deviation and the gain or return is measured as the mean. Therefore, to optimise the portfolio, it is either minimise the risk or maximise the return depending on the available statistics and parameters. The quadratic objective function can be formulated using real-valued variables with linear constraints to optimise the Markowitz's MV as follows:

1. If we are interested in getting a better return over a particular risk, thus we maximise our return

$$maximise \sum_{i=1}^{n} w_i \mu_i \tag{4.4}$$

$$subject\ to \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j} = \sigma^2$$

$$\sum_{i=1}^{n} w_i = 1,$$

$where\ 0 \leq w_i \leq 1, and\ i = 1, 2, 3, \ldots, n$

2. If one is comfortable with the return but is concerned about the risk, the overall risk can be minimised by formulating the optimisation function as;

$$minimise \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j} \tag{4.5}$$

$$subject\ to\ \sum_{i=1}^{n} w_i \mu_i = \mu$$

$$\sum_{i=1}^{n} w_i = 1,$$

$where\ 0 \le w_i \le 1, and\ i = 1, 2, 3, \dots, n$

where σ^2 and μ are standard deviation and expected return of the portfolio respectively, while μ_i and $\sigma_{i,j}$ are expected return of the asset i and the covariance of any two assets i and j for $i, j = 1, 2, 3, \dots, n$ where n is the number of the assets in the portfolio. w_i is the weight invested on the asset i while the condition $0 \le w_i \le 1$, show that short selling is not allowed in the treading of the assets.

4.3.3 Markowitz's portfolio MV with a single objective model

The equations (4.4) and (4.5) can be merged to form a single objective by introducing a risk parameter which will combine the equations. Risk is a convex set see (1.7.6), therefore is we choose $\lambda \in [0,1]$ as the risk parameter, the objective function will be

$$maximise\ \ (1-\lambda)\left[\sum_{i=1}^{n} w_i \mu_i\right] + \lambda\left[-\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j}\right] \tag{4.6}$$

$$subject\ to\ \ \sum_{i=1}^{n} w_i = 1$$

$0 \le w_i \le 1, and\ i = 1, 2, 3, \dots, n$

σ^2, n, μ_i , $\sigma_{i,j}$, $0 \le w_i \le 1$, and w_i are as defined above and $i, j = 1, 2, 3, \dots, n$. The parameter λ takes it values from 0 to 1 irrespective of the value of the standard deviation (risk) of the portfolio. The parameter λ shows the degree of risk averseness of an investor, as the parameter increases from 0 to 1 the risk aversion increases while the interest on the return decreases. If more emphasis is on return, it takes from 1 to 0 which will in return, reduce the risk, averseness. Since $\lambda \in [0,1]$, therefore, it can assume a value 1 or 0. If $\lambda = 1$ or 0, equation (4.6) will automatically become equation (4.4) or (4.5) respectively.

4.3.4 Markowitz's portfolio MV with multi-objective model

Most people like the single objective optimisation model because the solutions are easy and well known, most often than not, only concepts from calculus are involved in getting the solution of the optimisation problem. The major aim of creating an investment portfolio is to maximise return and minimise risk simultaneously. At this instant, the equations (4.4) to (4.6) fail to provide the investors with their desired aim. Single objective model literally, cannot offer an optimal solution which will balance the inherent risk-return trade-off. To address this need, the multi-objective optimisation model is needed. Thus, the multi-objective optimisation model can be stated as;

$$\min f_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j} \leq \sigma_p \tag{4.7}$$

$$\max f_2 = \sum_{i=1}^{n} w_i \mu_i \geq \mu_p$$

$$subject\ to \sum_{i=1}^{n} w_i = 1,$$

$0 \leq w_i \leq 1, i = 1, \dots, n$, where σ_p and μ_p are the risk and return of the portfolio respectively.

The solution to equation (4.7) is feasible if it falls within the set of the feasible region and the image of that feasible region under F, for $F = [f_1, f_2, \dots, f_n]'$ forms the objective space.

4.3.5 Standard Multi-objective Optimisation problem

Since Markowitz's work, a lot of researchers have been doing a great job in studying and improving the model. Researchers like; Fonseca and Fleming (1995), Diosan (2005), Coello (2006), Castillo and Coello (2007), Ponsich et al. (2013) has worked on the various aspect of the multi-objective optimisation problem.

Our interest in this section is to present the standard general multi-objective optimisation problem. Mathematically, Miettinen (1999) defines a multi-objective optimisation problem which reflects some of the definitions above.

Let w be a vector of continuous decision variables from the feasible set $\boldsymbol{\mathcal{C}}$, which is a subset of $\mathbb{R}^n$ (where$\mathbb{R}$ is the set of real numbers), defined by linear or nonlinear constraints. Let $F(w) = [f_1(w), f_2(w), \dots, f_j(w)]'$ be the set of objective vector, then

$$\min_{w \in \mathcal{C}} F(w) = \begin{bmatrix} f_1(w) \\ \vdots \\ f_j(w) \end{bmatrix} \tag{4.8}$$

Subject to $\boldsymbol{\mathcal{C}}$

where $\mathcal{C} = \{w: g(w) = 0\,, h(w) \leq 0\}$ and $j \geq 2$, is called multi-objective constrained optimisation problem. $F: \mathbb{R}^n \rightarrow \mathbb{R}^J$ is required to be twice continuously differentiable where is the number of the variables and J is the number of the objectives. A solution w of the above equation; multi-objective optimisation problem will be feasible if w belongs to $\mathcal{C}$, the set of all such elements in $\mathbb{R}^n$ which forms the feasible region. The image of the function F of the feasible region forms the objective space which is the subset of $\mathbb{R}^j$.

4.4 Portfolio selection.

We considered building a portfolio that will contain only 24 risky assets of financial stocks from NSE that is, banks and the insurance company. Therefore, our portfolio will consist of assets that consistently traded in the market (NSM) within the time interval of the study.

The data used is the daily closing price of the market mention above running from 3rd of August 2009 to 4th August 2015 (Note that this excludes weekends and public holidays in Nigeria nationwide).

We assume that the total fund (capital) invested is equal to 1, and the allocation of the fund to different assets is a percentage of the fund (capital). We maximise the expected return and minimise the variance of the portfolio by using Markowitz's portfolio selection model and a three-objective linear programming model to allocate a different percentage of weight to different assets to obtain an optimal/feasible portfolio of the financial sector of the Nigerian stock exchange (NSE).

Finally, we assume that the entire fund is distributed among the 24 assets. The allocation vector of the fund, (that is the weight) is $w = (w_1, w_2, \ldots, w_n)$ where w_i denotes the allocation of the capital invested in the asset i for $i = 1, 2, \ldots, n$ where $n = 24$. It will be observed that the constraints is equation (2.9) that is, $\sum_{i=1}^{n} w_i = 1$ since we have allocated the entire available fund to the assets. We denote $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ where μ_i is the mean of the asset i over the period of time.

An equally weighted portfolio was constructed using the daily closing prices from the financial services sector of NSE, the mean and the standard deviation of the data from the sector of the market served as our constraints in the three-objective model used. Additionally, three portfolios were constructed with the aims of maximising the returns and the Sharpe ratio and minimising the standard deviation (variance), respectively. With the result of our simulation and analysis, we were able to select the assets that are considered to form the optimal portfolio and the weight allocation to each stock. An investor wishes to build a feasible portfolio w^* ; this feasible portfolio becomes the efficient one if it satisfies the following condition with at least one strict inequality;

1. $(w) \leq (w^*)\mu$,
2. $\sigma(w) \geq \sigma(w^*)$ and
3. $SR(w) \leq SR(w^*)$

where $\mu(w), \sigma(w)\ and\ SR(w)$ are the expected return, risk and the sharpe ratio of the portfolio respectively, $w = (w_1, w_2, \ldots, w_n)$.

Finally, we were able to provide advice to the investors and market practitioners on how best to invest in the sector of NSE.

4.4.1 Portfolio1: Equally weighted Portfolio

We first constructed a portfolio that is equally weighted using the daily closing prices of the market, we got a portfolio which the return is 0.00162%, and the standard deviation is 1.28% (see Table 4.3). Though the standard deviation of the portfolio seems to be better than what we have from the market (see Table 4.1), but the return is very poor. We can see that the single asset with the least risk is

CORNERST, which is 1.84% but unfortunately, with a very poor return (see Table 4.2). Now our objective is to maximise the portfolio's return with a portfolio standard deviation which should be less than or equal to the least risk, (in other words we want to construct a portfolio that the standard deviation will be less than or equal to that of CORNERST, but the return will be above its return).

Stocks	ACCESS	AIICO	CONTINSU	CORNERST	CUSTODYINS	DIAMONDBNK	FBNH	FCMB	FIDELITYBK	GUARANT	MANSARD	NEM
SD	2.62%	3.22%	2.96%	1.84%	3.12%	2.80%	2.34%	2.63%	2.79%	2.40%	2.93%	17.88%
Average	-0.02%	-0.02%	-0.03%	-0.03%	0.03%	-0.05%	-0.06%	-0.06%	-0.04%	0.03%	-0.01%	0.45%
Variance	0.069%	0.100%	0.088%	0.034%	0.097%	0.079%	0.055%	0.069%	0.078%	0.058%	0.086%	3.200%
Stocks	NIGERINS	PRESTIGE	ROYALEX	SKYEBANK	STERLNBANK	TRANSCORP	UAC-PROP	UBA	UBN	WAPIC	WEMABANK	ZENITHBANK
SD	2.04%	2.87%	2.60%	10.50%	3.10%	3.35%	2.89%	4.50%	7.16%	5.10%	4.63%	2.46%
Average	-0.07%	-0.16%	-0.04%	0.18%	0.02%	0.10%	-0.02%	0.01%	-0.17%	0.00%	0.02%	0.01%
Variance	0.042%	0.082%	0.068%	1.100%	0.096%	0.110%	0.084%	0.200%	0.510%	0.260%	0.210%	0.061%

Table 4.1. Table showing the mean, variance and standard deviation of the assets.

4.4.2 Portfolio2: Maximum return with risk less than or equal to 1.84%

In this case, we constructed the second portfolio (Portfolio2), where we want to maximise our return at risk less than or equal to 1.84%. Therefore, we set the objective function as thus,

$$maximize\ f(w_1, w_2, \dots, w_n) = \sum_{i=1}^{n} w_i \mu_i - \mu_p \qquad (4.9)$$

subject:

$$g_1(w_1, w_2, \dots, w_n) = \frac{1}{n-1} \sum_{i=1}^{n} w_i w_j \sigma_{i,j} \leq 1.84\%$$

$$g_2(w_1, w_2, \dots, w_n) = \sum_{i=1}^{n} w_i - 1 = 0$$

where w_i is the weight of individual assets, n is the number of the observations. After the simulation, the weights were distributed among the assets but assets like UBA, UBN, Diamond bank, ACCESS, FBNH, Fidelity bank, FCMB etc. were allocated with 0% of the weight while assets like Transcorp, Guaranty trust bank and Custody were given more percentage of the weight (see Table 4.4).

Individual	Assets			
	Average	Variance	Standard D	μ/σ
ACCESS	-0.022%	0.069%	2.620%	-0.830%
AIICO	-0.023%	0.100%	3.220%	-0.710%
CONTINSURE	-0.028%	0.088%	2.960%	-0.960%
CORNERST	-0.033%	0.034%	1.840%	-1.810%
CUSTODYINS	0.025%	0.097%	3.120%	0.810%
DIAMONDBNK	-0.051%	0.079%	2.800%	-1.820%
FBNH	-0.061%	0.055%	2.340%	-2.590%
FCMB	-0.064%	0.069%	2.630%	-2.430%
FIDELITYBK	-0.041%	0.078%	2.790%	-1.460%
GUARANTY	0.033%	0.058%	2.400%	1.390%
MANSARD	-0.014%	0.086%	2.930%	-0.480%
NEM	0.450%	3.200%	17.880%	2.520%
NIGERINS	-0.074%	0.042%	2.040%	-3.610%
PRESTIGE	-0.160%	0.082%	2.870%	-5.570%
ROYALEX	-0.045%	0.068%	2.600%	-1.720%
SKYEBANK	0.180%	1.100%	10.500%	1.710%
STERLNBANK	0.024%	0.096%	3.100%	0.760%
TRANSCORP	0.099%	0.110%	3.350%	2.950%
UAC-PROP	-0.023%	0.084%	2.890%	-0.810%
UBA	0.008%	0.200%	4.500%	0.180%
UBN	-0.170%	0.510%	7.160%	-2.370%
WAPIC	0.005%	0.260%	5.100%	0.092%
WEMABANK	0.018%	0.210%	4.630%	0.390%

ZENITHBANK	0.006%	0.061%	2.460%	0.240%

Table 4.2 Table showing the asset return, variance, standard deviation and sharp ratio

Though in this new portfolio, we got a standard deviation that is greater than that of the portfolio with equal weighted assets, the return is very encouraging. The return is about 52 times the return of the said portfolio (see Table 4.4). Again, if we look at the return of the asset with the least standard deviation (CORNERST with σ = 1.84%, see Table 4.2), you will notice that it cannot be compared to our new return. Finally, if we look at the Sharpe ration (SR) of the portfolios, SR of the equal-weighted portfolio and our new portfolio are 0.12% and 4.56% respectively (see Table 4.4), and the stock with the least standard deviation has its SR to be 1.81% (Table4. 2), this shows that 4.56% is best among all.

4.4.3 Portfolio 3: Minimization of Standard Deviation

In this case, we want to minimise the standard deviation of the single asset with Courville, T., & Thompson, B. (2001). Use of structure coefficients in published multiple regression articles: ß is not enough. Educational and Psychological Measurement, 61, 229-248. maximum SD (NEM) which is 17.88% (Table 4.2), to see if we will get a lower SD and an improved return (which may not necessarily be equal to the return of the said asset). Therefore, we apply

$$minimize\ f(w_1, w_2, \dots, w_n) = \frac{1}{n-1}\sum_{i=1}^{n} w_i w_j \sigma_{i,j} \tag{4.10}$$

Subject to

$$g_1(w_1, w_2, \dots, w_n) = \sum_{i=1}^{n} w_i \mu_i - \mu_p \geq 0.450\%$$

$$g_2(w_1, w_2, \dots, w_n) = \sum_{i=1}^{n} w_i - 1 = 0$$

After the simulation, we got a funny result where 100% of our weight is allocated to NEM, with return and SD equal to what we had abinitio and therefore this portfolio is not acceptable.

4.4.4 Portfolio 4: Maximization of Sharpe ratio

Finally, we maximise the Sharpe ratio SR. Here we have the equation as follows

$$maximize\ f(w_1, w_2, \ldots, w_n) = SR \tag{4.11}$$

$$g(w_1, w_2, \ldots, w_n) = \sum_{i=1}^{n} w_i - 1 = 0$$

Again, we have the return to be 0.095%, the SD to be 2.06% and SR 4.6%. The weights were loaded in Transcorp, Guaranty trust bank and Custody assets with very few distributed among Skye, Sterling and Wema Banks, others are Wapic and NEM.

Comparison of the results that are an equally weighted portfolio, Max. Return, Min. Standard deviation and Max SR as shown in Table 4.3

		Portfolios		
	Equal Wt	Max Return	Min St Dev	Max SR
μ_p	0.00162%	0.084%	0.45%	0.095%
σ_p	1.28%	1.84%	17.89%	2.06%
μ/σ	0.13%	4.56%	2.52%	4.60%

Table 4.3 A table showing the return, risk and Sharpe ratio of the four portfolios constructed.

If we take the equally weighted portfolio as our pivotal portfolio, with the return, standard deviation and Sharpe ratio as 0.00162%, 1.28% and 0.13% respectively, we notice that it returns was below expectations. Though the risk is very minimal, the return and the Sharpe ratio shows that it is not a good idea to invest in the sector with an equally weighted portfolio. The portfolio that minimises standard deviation has the highest return, but the risk is too much, and the Sharpe ratio is not encouraging. Also, the simulation allocated 100% of the weight to one stock (NEM), which does not encourage the diversification of funds. Therefore, these make it not healthy for investment. We are now left with two options which are, Max Return and Max SR which have their returns as multiples of 52 and 59 of the return of the equally weighted portfolio respectively. Though the risk value of both is higher than the value of the equally weighted portfolio, the Sharpe ratios are better, which again are multiples of 46 on approximate of the equally weighted portfolio.

		Portfolios		
	Equal Wt	Max Return	Min St Dev	Max SR
	None	at σ ≤	at μ =	None
Value of Constr	N/a	1.840%	0.450%	N/a
ACCESS	4.1666%	0.0000%	0.0000%	0.0000%
AIICO	4.1666%	0.0000%	0.0000%	0.0000%
CONTINSURE	4.1666%	0.0000%	0.0000%	0.0000%
CORNERST	4.1666%	0.0000%	0.0000%	0.0000%
CUSTODYINS	4.1666%	11.9229%	0.0000%	10.3272%
DIAMONDBNK	4.1666%	0.0000%	0.0000%	0.0000%
FBNH	4.1666%	0.0000%	0.0000%	0.0000%
FCMB	4.1666%	0.0000%	0.0000%	0.0000%
FIDELITYBK	4.1666%	0.0000%	0.0000%	0.0000%
GUARANTY	4.1666%	27.2599%	0.0000%	26.4968%
MANSARD	4.1666%	0.0000%	0.0000%	0.0000%
NEM	4.1666%	5.1779%	100.0000%	6.3839%
NIGERINS	4.1666%	0.0000%	0.0000%	0.0000%
PRESTIGE	4.1666%	0.0000%	0.0000%	0.0000%
ROYALEX	4.1666%	0.0000%	0.0000%	0.0000%
SKYEBANK	4.1666%	5.6436%	0.0000%	6.7855%
STERLNBANK	4.1666%	8.5136%	0.0000%	6.7090%
TRANSCORP	4.1666%	36.2113%	0.0000%	40.8552%
UAC-PROP	4.1666%	0.0000%	0.0000%	0.0000%
UBA	4.1666%	0.0000%	0.0000%	0.0000%
UBN	4.1666%	0.0000%	0.0000%	0.0000%
WAPIC	4.1666%	2.0102%	0.0000%	0.7009%
WEMABANK	4.1666%	2.9119%	0.0000%	1.7415%
ZENITHBANK	4.1666%	0.3488%	0.0000%	0.0000%
Σ w	100.00%	100.00%	100.00%	100.00%
μ_p	0.00162465%	0.0839671%	0.45000000%	0.09465438%
σ_p	1.2807793%	1.8400012%	17.8885438%	2.0596265%
μ/σ	0.1268485%	4.56342%	2.51557648%	4.59571%

		Portfolios		
	Equal Wt	Max Return	Min St Dev	Max SR
	None	at σ ≤	at μ =	None
Value of Constr	N/a	0.0184	0.0045	N/a
ACCESS	0.0417	0	3.66E-09	0
AIICO	0.0417	0	3.67E-09	0
CONTINSURE	0.041	0	3.67E	0

	7		-09	
CORNERST	0.0417	0	-3.80E-08	0
CUSTODYINS	0.0417	0.1192	3.67E-09	0.1033
DIAMONDBNK	0.0417	0	3.67E-09	0
FBNH	0.0417	0	3.67E-09	0
FCMB	0.0417	0	3.67E-09	0
FIDELITYBK	0.0417	0	3.67E-09	0
GUARANTY	0.0417	0.2726	3.67E-09	0.265
MANSARD	0.0417	0	-3.80E-08	0
NEM	0.0417	0.0518	1	0.0638
NIGERINS	0.0417	0	3.67E-09	0
PRESTIGE	0.0417	0	3.67E-09	0
ROYALEX	0.0417	0	3.67E-09	0

SKYEBANK	0.0417	0.0564	3.66E-09	0.0679
STERLNBANK	0.0417	0.0851	3.67E-09	0.0671
TRANSCORP	0.0417	0.3621	3.68E-09	0.4086
UAC-PROP	0.0417	0	3.68E-09	0
UBA	0.0417	0	3.64E-09	0
UBN	0.0417	0	3.67E-09	0
WAPIC	0.0417	0.0201	3.67E-09	0.007
WEMABANK	0.0417	0.0291	3.66E-09	0.0174
ZENITHBANK	0.0417	0.0035	3.67E-09	0
Σ w	1	1	1	1
μp	1.62E-05	8.40E-04	0.0045	9.47E-04
σp	0.0128	0.0184	0.1789	0.0206
μ/σ	0.0013	0.0456	0.0252	0.046

Table 4.4 Table of the four different portfolios constructed.

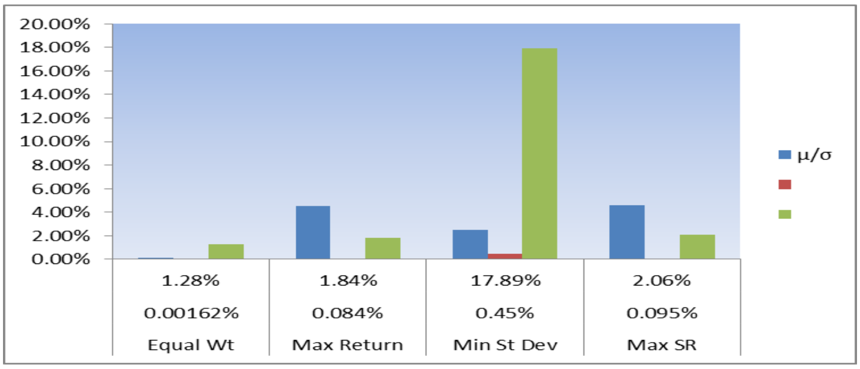

Fig 4.1 Histogram showing the return, risk and Sharpe ratio of the constructed portfolios.

Figure 4.1 above shows the histogram representation of the return, risk and Sharpe ratio of the portfolio we constructed. This has the four portfolios; the equally weighted portfolio, the maximised return portfolio, the minimised risk portfolio and the Sharpe ratio. While figure 4.2 below shows the interaction among different stocks in the portfolio. This is otherwise the cross-correlation matrix of the assets.

	ACCESS	AIICO	CONTINSU	CORNERST	CUSTODYI	DIAMOND	FBNH	FCMB	FIDELITYB	GUARANT	MANSARD	NEM	NIGERINS	PRESTIGE	ROYALEX	SKYEBANK	STERLNBA	TRANSCO	UACPRO	UBA	UBN	WAPIC	WEMABA	ZENITHBA
ACCESS	1	0.15076	0.056041	-0.01129	0.054676	0.239825	0.195369	0.127066	0.187493	0.165777	0.024933	0.022425	0.017909	0.013768	0.062961	-0.00181	0.12783	-0.02529	0.030219	0.067869	0.002778	0.053063	0.016861	0.223521
AIICO	0.15076	1	0.031982	0.009987	0.024212	0.067551	0.110694	0.061034	0.043642	0.132721	-0.02568	0.006563	0.015028	0.000717	0.006873	-0.00154	0.082536	-0.01036	0.002989	0.022636	-0.04742	0.018642	0.045983	0.120533
CONTINSU	0.056041	0.031982	1	0.061441	-0.07555	0.013918	0.044602	-0.00133	0.07853	0.016289	0.020515	0.010001	0.017147	-0.04602	-0.02966	0.011193	0.003815	0.050338	-0.00212	0.036778	0.007538	0.010738	-0.01125	0.031048
CORNERST	-0.01129	0.009987	0.061441	1	0.01274	0.005558	0.008376	0.057548	0.01488	-0.02078	-0.02035	0.006918	0.015995	-0.02565	-0.02834	0.013671	0.032636	0.006602	0.058526	-0.01141	0.001812	0.038141	0.024661	-0.02996
CUSTODYI	0.054676	0.024212	-0.07555	0.01274	1	0.049431	0.040057	0.045621	0.037637	0.046467	-0.04119	-0.00237	0.014387	0.007517	-0.00912	0.007332	0.072169	-0.00514	0.0397	0.029588	0.01162	-0.0102	0.018732	0.053487
DIAMOND	0.239825	0.067551	0.013918	0.005558	0.049431	1	0.174686	0.097734	0.290397	0.146793	-0.02485	0.000141	-0.0202	0.061032	0.031771	0.065686	0.142097	0.001916	0.01464	0.15934	-0.00886	0.102077	0.059745	0.145028
FBNH	0.195369	0.110694	0.044602	0.008376	0.040057	0.174686	1	0.117732	0.142378	0.372794	0.028788	0.005897	0.001805	-0.07033	0.013712	-0.01355	0.118883	-0.03332	0.080751	0.091059	-0.0229	0.020163	0.039201	0.385486
FCMB	0.127066	0.061034	-0.00133	0.057548	0.045621	0.097734	0.117732	1	0.106594	0.112151	0.029501	0.009275	-0.00342	-0.02713	0.006427	0.035481	0.164312	0.059917	0.072263	0.075343	-0.04152	0.0034	-0.00382	0.118524
FIDELITYB	0.187493	0.043642	0.07853	0.01488	0.037637	0.290397	0.142378	0.106594	1	0.140685	0.021668	-0.00039	0.013386	0.045131	0.088691	0.055757	0.117026	0.006328	0.015783	0.097524	0.013021	0.080512	0.073797	0.222817
GUARANT	0.165777	0.132721	0.016289	-0.02078	0.046467	0.146793	0.372794	0.112151	0.140685	1	0.008122	0.000976	-0.02638	-0.00331	0.01757	0.005144	0.108429	-0.04327	0.068898	0.075798	0.005193	0.053239	0.028267	0.357704
MANSARD	0.024933	-0.02568	0.020515	-0.02035	-0.04119	-0.02485	0.028788	0.029501	0.021668	0.008122	1	0.007896	0.042376	0.002973	0.051473	0.000686	-0.00373	0.016646	-0.02154	0.027622	0.043367	0.007491	0.042028	0.06349
NEM	0.022425	0.006563	0.010001	0.006918	-0.00237	0.000141	0.005897	0.009275	-0.00039	0.000976	0.007896	1	0.010726	-0.00539	0.008879	0.00444	0.004899	-0.01398	-0.00852	0.034025	-0.00015	-0.00128	0.007008	-0.00711
NIGERINS	0.017909	0.015028	0.017147	0.015995	0.014387	-0.0202	0.001805	-0.00342	0.013386	-0.02638	0.042376	0.010726	1	0.044131	-0.02136	0.001065	-0.00182	-0.01568	0.014461	0.015311	0.019744	0.049056	0.043319	0.018145
PRESTIGE	0.013768	0.000717	-0.04602	-0.02565	0.007517	0.061032	-0.07033	-0.02713	0.045131	-0.00331	0.002973	-0.00539	0.044131	1	0.021425	0.002879	-0.00308	-0.01809	-0.03582	-0.02624	-0.02189	0.014432	-0.01486	0.015711
ROYALEX	0.062961	0.006873	-0.02966	-0.02834	-0.00912	0.031771	0.013712	0.006427	0.088691	0.01757	0.051473	0.008879	-0.02136	0.021425	1	0.012932	0.067454	0.018112	-0.01278	0.002731	0.002697	0.011582	0.000495	0.038326
SKYEBANK	-0.00181	-0.00154	0.011193	0.013671	0.007332	0.065686	-0.01355	0.035481	0.055757	0.005144	0.000686	0.00444	0.001065	0.002879	0.012932	1	0.025861	0.029057	0.024996	0.753359	-0.01009	-0.0022	0.019219	0.019501
STERLNBA	0.12783	0.082536	0.003815	0.032636	0.072169	0.142097	0.118883	0.164312	0.117026	0.108429	-0.00373	0.004899	-0.00182	-0.00308	0.067454	0.025861	1	0.006659	0.070407	0.03845	0.008223	0.006269	0.074596	0.128894
TRANSCO	-0.02529	-0.01036	0.050338	0.006602	-0.00514	0.001916	-0.03332	0.059917	0.006328	-0.04327	0.016646	-0.01398	-0.01568	-0.01809	0.018112	0.029057	0.006659	1	0.022529	0.104056	0.000208	-0.01773	0.02563	-0.05526
UACPRO	0.030219	0.002989	-0.00212	0.058526	0.0397	0.01464	0.080751	0.072263	0.015783	0.068898	-0.02154	-0.00852	0.014461	-0.03582	-0.01278	0.024996	0.070407	0.022529	1	0.03557	-0.02204	0.057531	0.02092	0.02097
UBA	0.067869	0.022636	0.036778	-0.01141	0.029588	0.15934	0.091059	0.075343	0.097524	0.075798	0.027622	0.034025	0.015311	-0.02624	0.002731	0.753359	0.03845	0.104056	0.03557	1	-0.02388	0.059057	0.034955	0.09979
UBN	0.002778	-0.04742	0.007538	0.001812	0.01162	-0.00886	-0.0229	-0.04152	0.013021	0.005193	0.043367	-0.00015	0.019744	-0.02189	0.002697	-0.01009	0.008223	0.000208	-0.02204	-0.02388	1	0.004097	-0.00339	-0.00381
WAPIC	0.053063	0.018642	0.010738	0.038141	-0.0102	0.102077	0.020163	0.0034	0.080512	0.053239	0.007491	-0.00128	0.049056	0.014432	0.011582	-0.0022	0.006269	-0.01773	0.057531	0.059057	0.004097	1	0.012808	0.069573
WEMABA	0.016861	0.045983	-0.01125	0.024661	0.018732	0.059745	0.039201	-0.00382	0.073797	0.028267	0.042028	0.007008	0.043319	-0.01486	0.000495	0.019219	0.074596	0.02563	0.02092	0.034955	-0.00339	0.012808	1	0.073013
ZENITHBA	0.223521	0.120533	0.031048	-0.02996	0.053487	0.145028	0.385486	0.118524	0.222817	0.357704	0.06349	-0.00711	0.018145	0.015711	0.038326	0.019501	0.128894	-0.05526	0.02097	0.09979	-0.00381	0.069573	0.073013	1

Fig 4.2 the correlation matrix of the assets in the financial sector of NSE

4. 5 Efficient frontiers of our portfolio.

In this section, we wish to use the portfolio we formed to generate different allocations of weight to different stocks (see appendix). We have nine stocks from the financial sector of the market in the portfolio. We wish to get an efficient frontier.

Efficient frontier contains the efficient portfolio, that is, those portfolios that earn the highest return at a given level of risk. Sometimes this is done by locating those portfolios that have maximum Sharpe ratio These data are daily closing prices from the Nigerian Stock Exchange. From the return, we calculated the mean, variance and standard deviation of each asset in the portfolio. This is shown on the table below.

Normal Dist	CUSTODYINS	GUARANTY	NEM	SKYEBANK	STERLNBANK	TRANSCORP	WAPIC	WEMABANK	ZENITHBANK
Average	0.025%	0.033%	0.452%	0.184%	0.024%	0.099%	0.005%	0.018%	0.006%
Variance	0.097%	0.058%	3.196%	1.103%	0.096%	0.112%	0.260%	0.215%	0.061%
Standard D	3.116%	2.402%	17.878%	10.502%	3.098%	3.352%	5.100%	4.635%	2.461%

Table 4.5 the average return, variance and standard deviation of the selected assets

Now we try to get the expected return of the portfolio, having in mind that our weight allocation to different stocks is as follows;

CUSTODYINS	GUARANTY	NEM	SKYEBANK	STERLNBANK	TRANSCORP	WAPIC	WEMABANK	ZENITHBANK
11.92%	27.26%	5.18%	5.64%	8.51%	36.21%	2.01%	2.91%	0.36%

Table 4.6 weights of the assets in the portfolio.

With this, we computed the return of the portfolio for each using equation (2.14)

$$\mu_p = \sum_{i=1}^{n} w_i \mu_i$$

Where w_i and μ_i are the weights and daily return of the assets in appendix 1 respectively. After that, we need to compute the standard deviation of the portfolio. In order to that, we then use equation (2.16)

$$\sigma^2 = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

where $\sum_{i,j=1}^{n} \sigma_{ij}$ is the covariance matrix. And it is obtain using equation (2.6). Thus, recall that the covariance between asset x_i and y_i is $Cov(x,y) = \sum_{i=1}^{n} \frac{[(\rho^x - \mu_x)(\rho^y - \mu_y)]}{N}$, where N is the number of the observations (see equation 3.18). Before then, we need to find the excess return which is $x_i - \bar{x}$ which is the difference between the return and the average, and multiple it by its transpose. We denoted that as $X^T X$, remember that takes care of $\sum_{i=1}^{n}[(\rho^x - \mu_x)(\rho^y - \mu_y)]$ in the $Cov(x,y)$ equation.

X'X	CUSTODYINS	GUARANTY	NEM	SKYEBANK	STERLNBANK	TRANSCORP	WAPIC	WEMABANK	ZENITHBANK
CUSTODYINS	1.440	0.052	-0.018	0.036	0.103	-0.008	-0.024	0.040	0.061
GUARANTY	0.052	0.856	0.008	0.020	0.120	-0.051	0.097	0.047	0.314
NEM	-0.018	0.008	47.431	0.136	0.042	-0.118	-0.017	0.087	-0.046
SKYEBANK	0.036	0.020	0.136	16.361	0.125	0.154	-0.017	0.139	0.075
STERLNBANK	0.103	0.120	0.042	0.125	1.423	0.011	0.015	0.159	0.146
TRANSCORP	-0.008	-0.051	-0.118	0.154	0.011	1.668	-0.045	0.059	-0.068
WAPIC	-0.024	0.097	-0.017	-0.017	0.015	-0.045	3.857	0.045	0.129
WEMABANK	0.040	0.047	0.087	0.139	0.159	0.059	0.045	3.186	0.123
ZENITHBANK	0.061	0.314	-0.046	0.075	0.146	-0.068	0.129	0.123	0.898

Table 4.7 Matrix of excess return.

Finally, we have the covariance matrix by dividing each of the matrices above by n.

n= 1483									
Var-Cov Matrix	(1/n)X'X								
	CUSTODYINS	GUARANTY	NEM	SKYEBANK	STERLNBANK	TRANSCORP	WAPIC	WEMABANK	ZENITHBANK
CUSTODYINS	9.71E-04	3.49E-05	-1.21E-05	2.45E-05	6.97E-05	-5.116E-06	-1.62E-05	2.71E-05	4.10E-05
GUARANTY	3.49E-05	5.77E-04	5.70E-06	1.36E-05	8.08E-05	-3.452E-05	6.52E-05	3.15E-05	2.11E-04
NEM	-1.21E-05	5.70E-06	3.20E-02	9.17E-05	2.82E-05	-7.934E-05	-1.15E-05	5.89E-05	-3.10E-05
SKYEBANK	2.45E-05	1.36E-05	9.17E-05	1.10E-02	8.46E-05	1.041E-04	-1.17E-05	9.39E-05	5.05E-05
STERLNBANK	6.97E-05	8.08E-05	2.82E-05	8.46E-05	9.60E-04	7.146E-06	9.91E-06	1.07E-04	9.83E-05
TRANSCORP	-5.12E-06	-3.45E-05	-7.93E-05	1.04E-04	7.15E-06	1.125E-03	-3.03E-05	4.00E-05	-4.55E-05
WAPIC	-1.62E-05	6.52E-05	-1.15E-05	-1.17E-05	9.91E-06	-3.026E-05	2.60E-03	3.03E-05	8.73E-05
WEMABANK	2.71E-05	3.15E-05	5.89E-05	9.39E-05	1.07E-04	4.000E-05	3.03E-05	2.15E-03	8.33E-05
ZENITHBANK	4.10E-05	2.11E-04	-3.10E-05	5.05E-05	9.83E-05	-4.552E-05	8.73E-05	8.33E-05	6.05E-04

Table 4.8 Variance-Covariance Matrix.

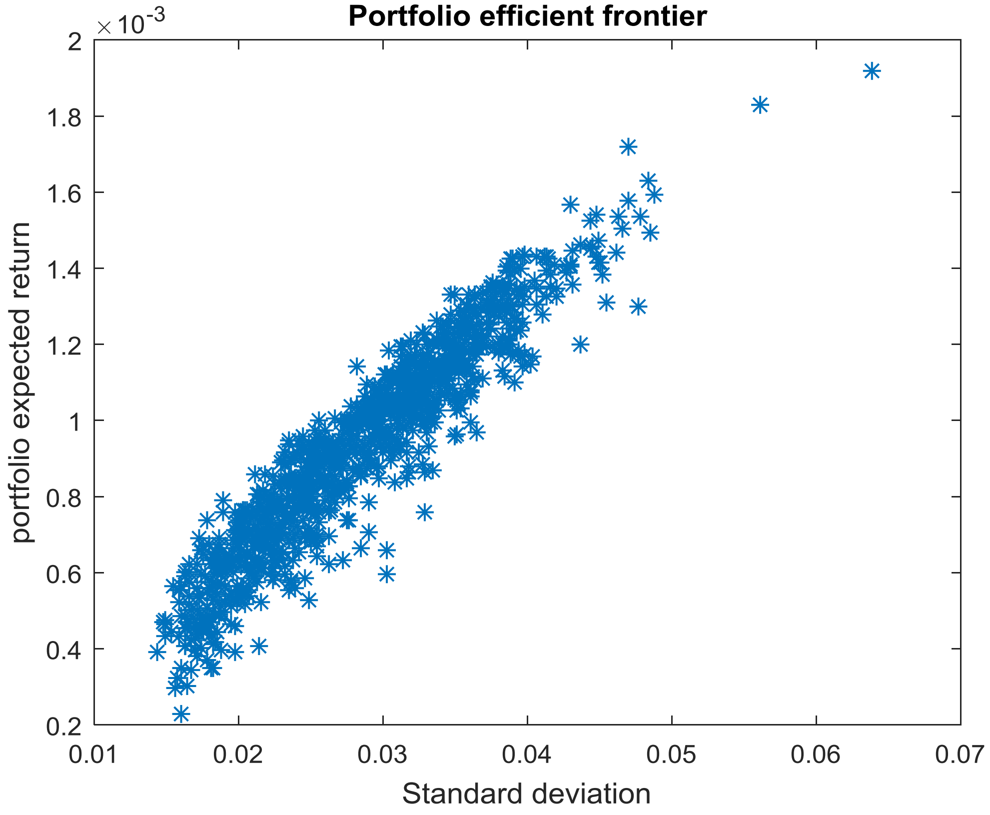

Fig 4.3 The portfolio efficient frontier.

4.6 Conclusion

We were able to construct four portfolios as we can see the summary in Table 4.3. The first portfolio is an equal-weighted portfolio, and the return is so small. Though the risk is low, the difference between the risk and the risks of the third and fourth portfolio is negligible. Therefore, an investor who is more interested in the number of returns (more significant return) to make will not be advised to invest in such a portfolio. Although Malladi, R. and Fabozzi, F.J. (2017), claimed that equal-weighted portfolio outperformed the value-weighted one, their theory was based on the data sample that ran from 1926 to 2014 which is a serious assumption that can only work in a developed market. Some emerging markets like NSE are so volatile that the turbulence in the market is nearly unpredictable. Banking sector of NSE was over one hundred stocks enlisted in the market before 2009 but was reduced to just 24 banks after 2009. When we completed our research, almost three or more banks couldn't survive, so they went off the market.

The second portfolio, which was formed with equation (4.9), gave the highest return with a very high risk. This is not encouraging; any risk-averse investor will never consider this portfolio as an option. Besides, the idea of diversification was not encouraged at all because the whole fund was allocated to one stock, which is NEM. Therefore we strongly advise the investors to disregard this portfolio.

Finally, the equations (4.8) and (4.10) gave us something closer to what we want, an appreciated return and a risk that can be tolerated and above all, their Sharpe ratios are quite commendable when compared with the former two. Investors who want to invest in this sector are advised to invest in the portfolio of equation (4.10), which we consider to be the optimal portfolio. Though the risk is slightly above the other, the return and the Sharpe ratio are very encouraging. Furthermore, we can see from the interaction of the stocks in the correlation matrix (fig 2), that the assets selected in the portfolio move in such a direction that will reduce risk. Investors are highly advised to invest in this portfolio.

Chapter 5 Random Matrix Theory and Applications

5.0 Introduction

In this Chapter, we discuss in detail the review of some standard and more recent techniques on Random Matrix Theory (RMT). These techniques can reduce the empirical noise and improve the standard Markowitz model's predictions. The analysis of eigenvalues using RMT helps to check if there is a presence of pertinent information by using Marcenko-Pastur distribution. And we look at cross-correlation among stocks of Nigerian Stock Exchange. We also examine the statistical properties of cross-correlation coefficients, the distribution of eigenvalues, the distribution of eigenvector components, and the inverse participation ratio. Finally, we present our results.

A random matrix is a matrix of a given type and size whose entries consist of random numbers from some specified distribution. The origin of RMT is a bit dicey, some school of taught claimed that it could be traced as far back as 1928 to the works of John Wishart, where he gave generalised distribution, and to calculate its moments up to the fourth order, Wishart (1928). They considered the case of three varieties, and after that a proved the general n-fold system.

5.1 Literature Review

Most literature gave the credit to Eugene Wigner, believing that he was the first to propose Random matrix theory (RMT) in Wigner (1951). However, RMT has become a popular technical tool for solving some more complicated problems in the areas of; number theory, quantum mechanics, condensed matter physics, Statistics, Mathematics, wireless communication etc. most importantly it is used for investigating the cross-correlation in financial markets.

At first, Wigner (1951) wished to describe the general properties of the energy levels of highly excited states of heavy nuclei as measured in the nuclear reactions Izenman (2008). He assumed that the interactions between the constituents comprising the nucleus are so complex and they should be modelled as random. He represented it by Hermitian operator H (called Hamiltonian), which behave like a large random matrix. The energy level of the system was approximated by the eigenvalues of a large random matrix and the spacing's between the energy levels of the nuclei could be modelled by the spacing of the eigenvalues of a random matrix. Dyson (1962a), (1962b) and (1962c) introduce and substantiate the works of Winger by giving the important symmetry classification of Hamiltonians which implies the existence of three major symmetry classes of random matrices - Orthogonal, Unitary and Symplectic, which cover the most relevant classical ensembles. They also develop a theory of their spectra and suggest a model of Brownian motion in random matrices ensembles. These types of random matrix ensemble are based upon the property of time-reversal invariance and have elements that are complex, real and self-dual quaternion. As the year's pass, RMT was receiving attention by a lot of researchers, Brody et al. (1981) brought it into the Nuclear Physics, Bohigas et al. (1984) introduced it to quantum chaos, while Beenaker (2007) brought in quantum transport.

The development of the financial market in recent years has provided a large amount of financial market data which needs more sophisticated tools and techniques to handle. RMT has shown that it can address these problems of the researchers, especially in the study of the cross-correlations among stocks of the market; removal/ elimination of the noisy eigenvalues from the correlation matrix. This has helped in the reduction and improved the forecast of the realised risk.

Laloux et al. (1999) and Plerou et al. (1999) introduced RMT to financial market research where it has been used in studying the statistical properties of cross-correlations in financial markets. They based their studies on noise filtering of particularly large dimensional systems like the stock market in the financial time serials.

Laloux et al. (1999) (2000) compared the statistics of the eigenvalues and eigenvectors of the S&P 500 data set, involving daily data over the period 1991 to 1996 market correlations, to those of a corresponding random matrix. They were able to cover 406 stocks in the entire time interval. In work mentioned above, they found out that the highest eigenvalue of the empirical correlation matrix was 25 times larger than the maximum eigenvalue predicted by RMT. Also, there is a good agreement between the distributions when compared the remaining eigenvalues with RMT. They determined that 94% of the eigenvalues were within the noise band predicted by RMT, while the highest 6% of eigenvalues were above the maximum random eigenvalue. This largest 6 % of eigenvalues were found to contain 26% of the total system volatility. They revealed that eigenvectors corresponding to the eigenvalues which were within the noise band, had components which were consistent with randomness, and the eigenvector for the largest eigenvalue was shown to have non-random elements. In conclusion, they said that this eigenvector represented the market, in the sense that it assigned a roughly equal weighting and the non-noisy eigenvectors were found to be more stable in time to each stock.

Laloux et al. (2000) try to obtain the optimised portfolio by using a cleaned correlation matrix, which is more reliable. They divided the total data set over some time into two equal sub-periods, and they cleaned up the correlation matrix, which was determined using the first sub-period. Again, the other data set was used to determine another correlation matrix, and the two were used to optimise portfolio using Markowitz optimisation to construct efficient frontiers. They observed that there is a better-realised risk of the filtered portfolio than the other. The difference in the risk of the two was seen to be reasonably constant, and the filtered one is always below the unfiltered one in every point the efficient frontier.

Plerou et al. (1999) and Laloux et al. (2000), analyse the cross-correlation matrix of returns of a database containing the price of the stock of 1000 publicly-traded companies of US stocks for a period of 2-year, from 1994 to 1995. They found that 20 of the largest eigenvalues (2%) show deviations from the predictions of RMT while (98%) of the eigenvalues were found within the RMT bonds. Though the two groups were working independently, there are points of agreement in the outcome of their research. They agree at these points; the eigenvalues of the correlation matrix of the returns were consistent with the ones obtained from random returns. They also agree that the higher percentage of the eigenvalues which are not consistent with random returns had eigenvectors that are more stable over time and both groups agree that filtering reduces the realised risk of the optimised portfolio and also improves the forecasting of the realised risk.

Sharifi et al. (2004) applied RMT to an empirically measured financial correlation matrix and showed that the matrix contains a large amount of noise. Finally, they attempted to separate the noisy part from the non-noisy part of the matrix. Rak et al.

(2006) (2008) applied RMT in the Warsaw stock market, and they used the correlation matrix formation to study the temporal aspects of the Warsaw stock market evolution as represented by the WIG20 index in 2006. In 2006, they studied the inter-stock correlation for the largest component listed on the Warsaw Stock Exchange and WIG20 index. Their result inferred that a one-factor model could well describe the Warsaw stock market; this inference was due to their correlation matrix analysis.

Others like; Wilcox and Gebbie (2007) worked on the South African stock market, where they constructed correlation matrix from ten (10) years daily data from Johannesburg Stock Exchange and applied RMT to compare correlation matrix estimator obtained from the date from the market.

Kwlkarni and Deo (2007) and Pan and Sinha (2007) worked on the Indian stock, while Curkura et al. 2007) worked on the Istanbul stock market. Wang et al. (2013) found some new results of the cross-correlation in the US stock market. They examined the statistical properties of cross-correlation in the US market and found out that the detrended cross-correlation analysis (DCCA) coefficient method has similar results and properties with Pearson's correlation coefficient (PCC).

El Alaoui (2015) studied cross-correlation among stocks of Casablanca Stock Exchange using RMT. He tries to observe if the difference between predicted risk and realised risk will be reduced by cleaning the noisy element of the correlation matrix.

In this work, we studied Nigerian market and used Random Matrix Theory (RMT) to analyse the eigen structure of the empirical correlations of 82 stocks which are consistently traded in the Nigerian Stock Exchange (NSE) over a 4-year study period 3 August 2009 to 26 August 2013. We applied the Marcenko-Pastur distribution of eigenvalues of a purely random matrix to investigate the presence of investment-pertinent information contained in the empirical correlation matrix of the selected stocks. We used hypothesised standard normal distribution of eigenvector components from RMT to assess deviations of the empirical eigenvectors to this distribution for different eigenvalues. We also use the Inverse Participation Ratio (IPR) to measure the deviation of eigenvectors of the empirical correlation matrix from RMT results. These preliminary results on the dynamics of asset price correlations in the NSE are essential for improving risk-return trade-offs associated with Markowitz's portfolio optimisation in the stock exchange, which we achieve by cleaning up the correlation matrix. In this work, we propose to measure the extent of closeness or otherwise in selected sectors of the NSE and the Johannesburg Stock Exchange (JSE) in subsequent work.

5.2 Definitions of some concepts

In this section, we gave the definitions of the common terms we frequently used in Random matrix theory.

Definition 5.2.1

Let C be any square matrix, μ_c is the probability distribution. If μ_c puts equal the mass of each eigenvalue of C, then μ_c is called the empirical spectral distribution (*ESD*) of C.

This implies that if λ_i, $i = 1,2,\dots,n$ are eigenvalues of the matrix C (where C is $n\,x\,n$ matrix), then

$$\mu_c = \frac{1}{n}(\delta\lambda_1 + \delta\lambda_2 +, \dots, +\delta\lambda_n) \tag{5.1}$$

Where δ is the Kronecker delta. Note that if C is $n\,x\,n$ matrix of multiplicity m and λ is it's eigenvalues, then *ESD* μ_c puts mass $m/_n$ at λ.

Equation (5.1) above can be put into a general form thus,

$$\mu_c = \frac{1}{n}\sum_{1}^{n} \delta\lambda_i \tag{5.2}$$

Definition 5.2.2

Let X_{ij}, $1 \le i < j < \infty$ be independent identically distributed (i.i.d) (real) random variables with mean 0 and variance 1 and set $X_{ji} = X_{ii}$. Let X_{ii} be i.i.d (real) random variables (with possibility of a different distribution) with mean 0 and variance 1, then, $C_n = [X_{ij}]_{ij=1}^{n}$ will be a random $n\,x\,n$ symmetric matrix. This is called the (real) Wigner matrix.

In general, let $\{X_{ij}\}_{ij=1}^{n}$ be a sequence of independent random variables with the same distribution with all first moments 0 and all second moments 1. Let $\{X_{ii}\}_{i=1}^{n}$ be a sequence of independent random variables with the same distribution with all first moments 0. Assume all $\{X_{ij}\}_{ij=1}^{n}$and $\{X_{ii}\}_{i=1}^{n}$ are independent of each other. And, assume all moments of both sets of random variables are finite. A Wigner matrix is a symmetric matrix C_n of size n such that

$$\{X_{ij}\}_{ij=1}^{n} = \{X_{ji}\}_{ji=1}^{n} = \begin{cases} X_{ij}\Big/\sqrt{n} \; if\ i < j \\ X_{ii}\Big/\sqrt{n} \; if\ i = j \end{cases} \tag{5.3}$$

Note: If $X_{ij} \in \mathbb{C}$**,** that is the set of the complex number and $X_{ij} = \bar{X}_{ji}$, $\{X_{ij}\}_{ij=1}^{n}$ is called a random $n\,x\,n$ Hermitian matrix. But in this work, our interest is in the Real case.

Definition 5.2.3
A matrix R is called a Wishart matrix if $x_{ij} \in X$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, L,$ is a double array of independent identically distributed real random variables with mean zero and variance one, if such that $R = \frac{1}{L}X'X$ where $\mathbb{x}_j = \left(x_{1,j}, x_{2,j}, \dots, x_{n,j}\right)'$ and $X^t = [\mathbb{x}_1, \mathbb{x}_2, \dots, \mathbb{x}_L]$.

5.3 The background and Methodology.

5.3.1 Equally weighted covariance and RMT

Let X be a $L\,x\,N$ matrix that contains the return of the market (hourly, daily, weekly, monthly, quarterly, whichever be the case), where n is the number of assets we are considering. Therefore, the price entering of the returns of these assets will be n, this gives us the n number of columns while L is rows that indicate the time series of our observations. Thus,

$$X = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & r_{ij} & \vdots \\ r_{L1} & \cdots & r_{Ln} \end{bmatrix} \tag{5.4}$$

where $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, L$. (J.P. Morgan 1996) when the standard deviation and covariance around a zero mean is computed and weigh each observation with a probability of $\frac{1}{L}$, we will have our covariance matrix to be

$$\sum = \frac{X'X}{L} \tag{5.5}$$

where X' is the transpose of X and this can be presented in a general as;

$$\frac{X'X}{L} = \begin{bmatrix} \frac{1}{L}\sum_{i=1}^{L} r_{i1}^2 & \cdots & \frac{1}{L}\sum_{i=1}^{L} r_{i1}r_{in} \\ \vdots & \ddots & \vdots \\ \frac{1}{L}\sum_{i=1}^{L} r_{i1}r_{in} & \cdots & \frac{1}{L}\sum_{i=1}^{L} r_{in}^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1n}^2 \\ \vdots & \ddots & \vdots \\ \sigma_{L1}^2 & \cdots & \sigma_n^2 \end{bmatrix} \tag{5.6}$$

If we divide each entering of X with a corresponding standard deviation which will help us generate a correlation matrix. This means normalisation of X.

$$A = \begin{bmatrix} \frac{r_{11}}{\sigma_1} & \cdots & \frac{r_{1n}}{\sigma_n} \\ \vdots & \ddots & \vdots \\ \frac{r_{L1}}{\sigma_1} & \cdots & \frac{r_{Ln}}{\sigma_n} \end{bmatrix} \tag{5.7}$$

remember that

$$\sigma_{ij} = \frac{1}{L}\sqrt{\sum_{i=1}^{L} r_{ij}^2} \qquad j = 1,2,3, \ldots, n \tag{5.8}$$

5.3.2 Exponentially weighted covariance and RMT

Pafka et al. (2004) introduced a covariance matrix estimator that uses exponentially weighted moving average to account for the heteroscedasticity of financial returns, and reduce the effect of noise through a technique browed from RMT

if $C = \{c_{ij}\}_{ij=1}^{N}$ with

$$c_{ij} = \sum_{k=0}^{\infty}(1-\alpha)\alpha^k x_{ik}x_{jk} \tag{5.9}$$

where $\{x_{ij}\}_{k=0,\ldots,\infty}^{i=1,\ldots,N}$ are assume to be normally distributed with mean 0 and standard deviation σ^2 (that is, N.I.D. (μ, σ^2)) and α is called a decay factor.

According to Pafka et al. 2004, in a situation when $N \to \infty$ and $\alpha \to 1$ with $Q \equiv \frac{1}{(N(1-\alpha))}$ fixed, $\rho(\lambda)$ which is the probability density of the eigenvalue of C is given by

$$\rho(\lambda) = \frac{Qv}{\pi} \tag{5.10}$$

where v is the root of the equation

$$F(v) = \frac{\lambda}{\sigma^2} - \frac{v\lambda}{\tan(v\lambda)} + \ln(v\sigma^2) - \ln(\sin(v\lambda)) - \frac{1}{Q} \tag{5.11}$$

Daly et al. (2008) shows that $F(v)$ is a function that is well define on the interval $\left(0, \frac{\pi}{\lambda}\right)$. For any given value of λ whose root does not exist within the open interval $\left(0, \frac{\pi}{\lambda}\right)$, the probability density of that λ is equal to zero, thus, $\rho(\lambda) = 0$. Then the exponentially weighted covariance matrix $V^* = \{\sigma_{ij}^*\}_{ij=1}^{n}$ is defined as

$$\sigma_{i,j}^* = \frac{1-\alpha}{1-\alpha^L}\sum_{l=0}^{L-1}\alpha^l\left(X_{i,L-t} - \bar{X}_{il}\right)\left(X_{j,L-1} - \bar{X}_{jl}\right) \tag{5.12}$$

where the corresponding exponentially weighted correlation matrix is $C^* = \{\rho_{ij}^*\}_{ij=1}^{n}$ is

$$\rho_{ij}^* = \frac{\sigma_{ij}^*}{\sqrt{\sigma_{ii}^*\sigma_{jj}^*}} \tag{5.13}$$

where α is the decay factor.

Definition 5.5.1

Let $P_i(t)$ be the closing price of the index on the day (t) of stock i and we define the natural logarithmic returns of the index (i.e. the log-difference of $P_i(t+1)\ and\ P_i(t)$) is

$$r_i(t) = \ln P_i(t+1) - \ln P_i(t) \tag{5.14}$$

$r_i(t)$ has L number of observations.

Before establishing the portfolio selection process, we will compute the mean return and standard deviation of each i

We first calculate the price change ('return') of the stocks to quantify correlations, see Plerou et al. (2001) and Sharifi et al. (2004). Therefore, the price change of stock $i = 1, \ldots, N$, over a time scale Δt.

Definition 5.5.2

Let's denote the price of i^{th} asset at time t as $S_i(t)$, therefore, we define its price change as

$$G_i(t) = \ln S_i(t+\Delta t) - \ln S_i\,(t) \tag{5.15}$$

Since there is variation in the levels of volatility (standard deviation) of different stocks, we, therefore, define a normalised return concerning its standard deviation σ_i as follows:

$$g_i(t) = \frac{G_i(t) - \langle G_i \rangle}{\sigma_i} \tag{5.16}$$

where $\sigma_i = \sqrt{\langle G_i^2 \rangle - \langle G_i \rangle^2}$ is the standard deviation of G_i for the assets $i = 1, \ldots, N$ and $\langle G_i \rangle$ is denoted as the time average of G_i over the period studied which can be computed as follows, $\langle G_i \rangle = \frac{1}{L} \sum_{l=0}^{L-1} G_i$.

Note 5.5.3

(5.14) and (5.15) may seem to be the same but in the real sense, they are not.

$P_i(t)$ is the daily closing price of the market, $r_i(t)$ gives us the log difference (natural log) of the closing price of one day compared with the previous day in (5.14).

$S_i(t)$ is the price of the stock at a time therefore, $G_i(t)$ gives us the change in price within the time interval of the sampling. $G_i(t)$ in (5.15) is a more general formula because some studies are done within the interval of 30 minutes (see Plerou et al. (1999) equation 1), in some cases the intervals may be one day, weekly, fortnight, monthly, quarterly etc. see Wilcox & Gebbie (2007). Therefore, if the interval of sampling is one day equations (5.14) and (5.15) coincide.

$$C_{ij} \equiv \langle g_i(t), g_j(t) \rangle \tag{5.17}$$

Being correlation coefficients, the elements of C_{ij} are restricted to the domain $C_{ij} \in [-1,1]$ i.e. $-1 \leq C_{ij} \leq 1$ during the construction of the C_{ij}, where $C_{ij} = 1$ corresponds to perfect positive correlation, $C_{ij} = 0$ corresponds to uncorrelated pairs of stocks and $C_{ij} = -1$ corresponds to anti-correlation , that is a perfect negative correlation .

There are difficulties in the analysis of the correlation between any two stocks i, j , Plerou et al. (2001), (2000) and El Alaoui (2015) note that two main difficulties arise in the analysis of the correlations between any two stocks or more generally, the correlation structure of a portfolio of assets in a financial market. Firstly, market conditions change with time. Hence, the correlation C_{ij} between any two pairs i, j of the stocks may not be stationary. Secondly, time averaging over a finite time series introduces 'measurement noise'. These facts have implications for portfolio selection and optimisation, given the centrality of cross-correlations among assets to Markowitz portfolio optimization. Recall that the basic tenets of Markowitz's theory of optimal portfolios are a) to determine the optimal weights of assets with given average returns and risks which maximizes the overall returns for a fixed level of risk, or b) minimises the risk for a given level of overall return. For this purpose, if $\{R_i\}$ are expected returns of the portfolio assets, $\{p_i\}$are the relative amounts of capital invested in the assets, and $C = (C_{ij})$ is the matrix of covariances of asset returns, then Markowitz's optimization uses Langrangian multiplier approach to minimize the overall portfolio variance $\sigma_P^2 = \sum_{i,j=1}^{N} p_i p_j C_{ij}$ for a given value of overall return $r_P = \sum_{i=1}^{N} p_i R_i$.The results of this scheme are summarised graphically in mean-variance Efficient Frontiers which shows the range of optimal risk-return combinations possible.

The point, therefore, of Random Matrix Theory (RMT) is to compare the structure of the empirical cross-correlations (in effect covariance) among portfolio assets with the behaviour of a purely random matrix in which the assets are independent. For this, both Eigen-structure and time dependence (or stability) of the matrix C are of interest and constitute what is known in RMT as dynamics of the correlation matrix. The following notes summarise what is known about these dynamics. For a portfolio of N assets, the matrix C has $N(N-1)/2$ entries to be determined from N time series of length L. Here N and L correspond to the numbers of the stocks listed and the days studied respectively. In one of our results, $N = 82$ and $L = 1018$ correspond respectively to the numbers of the stocks listed in NSE and the days studied.

Although NSE has about 188 stocks listed within the period of the time series, so many stocks were not consistently traded over the period studied. Hence, we choose 82 stocks that were consistent in the market during the sample period. We then compute the equal-time cross-correlation matrix C with elements:

If L is not very large relative to N, we are basically estimating too many model parameters from sparse information, which introduces the above-mentioned 'measurement noise' in empirical correlation matrices. This makes the use of such matrices in applications less accurate in portfolio optimization than alternative matrices filtered for pertinent information using key results of RMT summarised below. Indeed, in RMT this information is cleaned from the behaviour of eigenvalues and eigenvectors of C, compared to a 'null hypothesis' purely random matrix such as represented by a finite time series of strictly independent and uncorrelated assets.

5.3.3 Eigenvalue distribution of correlation matrix.

In matrix notation, the correlation matrix can be expressed as

$$C = \frac{1}{L} GG' \tag{5.18}$$

where G is an $N \times L$ matrix with elements $\{g_{im} \equiv g_i(m\Delta t): i = 1, \dots, N; m = 0, \dots, L-1\}$, and $G'$is the transpose of G.

The RMT method used here is to compare the empirical cross-correlation matrix C against the null hypothesis of a random matrix of the same type which is R (see Laloux et al. 1999).

So, we consider a random correlation matrix

$$R = \frac{1}{L} AA' \tag{5.19}$$

where A is an $N \times L$ matrix containing N time series of L random element a_{im}, with zero mean and unit variance that are mutually uncorrelated and A' is the transpose of A.

By construction, Muirhead (1982) referred to R as Wishart matrices in multivariate statistics. Kim et al. (2010) shows that if the eigenvalues from the empirical matrix C is like the result of the random matrix R from the component analysis, this shows that the empirical data of the market is noisy. But when it is larger than the random matrix R, then the empirical data is meaningful and, it contains useful information about the market.

By diagonalisation of matrix C, we obtain

$$Cu_k = \lambda_k u_k \tag{5.20}$$

where λ_k are the eigenvalues and u_k are the eigenvectors, and $k = 1, \ldots, N$ are arranged in order of increasing eigenvalues. Statistical properties of random matrices such as R are known in the limit of large dimensions, In particularly, the limit $N \to \infty$, $L \to \infty$ such that $Q \equiv \frac{L}{N}$ (> 1) is fixed, Sengupta & Mitra (1999) shown analytically that the probability density function $P_{rm}(\lambda)$ of eigenvalues λ of the random correlation matrix, R is given by

$$P_{rm}(\lambda) = \frac{Q}{2\pi\sigma^2} \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{\lambda} \tag{5.21}$$

For λ within the bounds $\lambda_- \leq \lambda_i \leq \lambda_+$, where λ_- and $\lambda_+$are the minimum and maximum eigenvalues of R, respectively, given by

$$\lambda_{\pm} = \sigma^2 \left(1 + \frac{1}{Q} \pm 2\sqrt{\frac{1}{Q}} \right) \tag{5.22}$$

where σ^2 is equal to the variance of the elements R, Sengupta & Mitra (1999). The eigenvalues of R falls within the range $[\lambda_-, \lambda_+]$ as predicted by Random matrix theory, Rosenow et al. (2003). σ^2 is equal to unity in the case of a normalised matrix A. Also, the maximum and minimum (theoretical) eigenvalues determine the (theoretical) bounds of the eigenvalue's distribution and if, the eigenvalues of the matrix C are beyond the bounds, they are said to deviate from the random bound which suggests that they contain pertinent investment information. Being able to detect assets with such real information as opposed to random market noise will enable investors to include them in portfolio constructions for more optimal risk control.

It is also known that the first three eigenvalues represent the overall market information based on the random behaviour of investment returns (random walk hypothesis), which the random matrix represents. Hence, deviations between realised values of the first three eigenvalues (especially the first one) and the predicted maximum eigenvalue indicate the extent to which a stock market is consistent with the RMT assumptions.

In line with the assumption of pure randomness and independence, the distribution of the components $\{\boldsymbol{u_k}(l) | l = 1, 2, \ldots, N\}$ of an eigenvector $\boldsymbol{u_k}$ of a random correlation matrix, R should obey the standard normal distribution with zero mean and unit variance given by

$$P_R(u) = \frac{1}{\sqrt{2\pi}} \, exp\left(-\frac{u^2}{2} \right) \tag{5.23}$$

5.4 Data Analysed.

We analyse the sample data from two emerging markets, namely the Nigerian Stock Exchange (NSE) and the Johannesburg Stock Exchange (JSE).

5.4.1 Nigerian Stock Exchange Data

The Nigerian Stock Exchange (NSE) has 188 stocks listed under eleven (11) sectors, namely: Agriculture with 5 stocks, Conglomerates with 6 stocks, Construction/Real estate with 9 stocks, Consumer goods with 28 stocks, Financial services with 57 stocks, ICT with 9 stocks, Health care with 11, Industrial goods with 21, Natural resources, Oil and Gas, and Services with 5, 14 and 23 stocks, respectively.

The data set used was the daily closing price of 4 years of stock data listed in the NSE. We have 1019 every day closing prices running from 3rd August 2009 to 26th August 2013, excluding weekends and public holidays in Nigeria (Nationwide). These stock price data were converted into 1018 logarithmic returns see equ (5.14). These data were screened to remove stocks that were delisted, infrequently traded or not traded at all during the sample period. This reduced the data to 82 securities only.

5.4.2 Johannesburg Stock Exchange Data

Johannesburg Stock Exchange (JSE) is claimed to be the biggest stock exchange in Africa; in 2003, the JSE had an estimated 472 listed companies. These companies are listed under three categories called issuers, namely Equity, Structural product, and Hybrids issuers.

In the course of this work, we analyse a total of 35 selected securities from various sectors which cut across Banking, Insurance, Health care, Telecommunications, oil and gas, food and drugs, Tobacco, Pharmaceuticals and Beverages, etc. The data set used the daily closing price of the stock data listed above of JSE. We have 1485 every day closing prices running from 2nd January 2009 to 1st August 2013, excluding weekends and public holidays in South Africa. These stock price data were converted into 1484 logarithmic returns and was used in our analysis.

5.4.3 Eigenvalue analysis

In one of our Analysis with the Nigerian market Nnanwa et al. (2017), we analysed $N = 82$ stocks from NSE which a total of $L = 1019$ daily closing prices. We found out that the theoretical (Random) eigenvalue bounds of the correlation matrix form equ. (5.22) , are $\lambda_{max} = 1.6484$ and $\lambda_{min} = 0.5128$ as maximum and minimum eigenvalues respectively. The value of our Q is 12.4146 (Note that $Q = {}^{L}/_{N}$).

We compute the eigenvalues λ_i of the empirical correlation matrix and ranked them in an ordered form that is, $\lambda_{i+1} > \lambda_i$, and the compare the probability distribution $P(\lambda)$ with that of the random matrix $P_{rm}(\lambda)$ from equ. (5.21) for $= 12.4146$.

$$\lambda_{\pm} = \sigma^2 \left(1 + \frac{1}{Q} \pm 2\sqrt{\frac{1}{Q}} \right)$$

where $\frac{1}{Q} = 0.08055$, $\sqrt{\frac{1}{Q}} = 0.2839{,}2\sqrt{\frac{1}{Q}} = 0.5678$ and $\sigma^2 = 1$, we have that equ. (5.22) is

$$\lambda_{\pm} = \left(1 + 0.0806 \ \pm 2\sqrt{0.0806}\right)$$

and we have $\lambda_{max} = (1 + 0.0806 + 0.5678)$ and $\lambda_{min} = (1 + 0.0806 - 0.5678)$, therefore $\lambda_{max} = 1.6484$ and $\lambda_{min} = 0.05128$, this gives our eigenvalue bound to be $[\, \lambda_{min}, \, \lambda_{max}] = [0.05128, 1.6484]$.

We observe from the results that the bulk of the eigenvalues fall within the bounds of the random spectrum $[\lambda_{min}, \lambda_{max}]$ for $P_{rm}(\lambda)$, it is also observed that the largest eigenvalue = 4.87 which is 2.95 (approximately 3) times bigger than the predicted RMT value above. Also, 6 of out our eigenvalues deviated from the above RMT eigenvalue spectrum, which accounts for 10.98% of the total eigenvalues (see fig. 5a). Therefore, this suggests the presence of true information about the stock market is in about 11% of the selected stocks and purely random information in the remaining 89% associated with the purely random matrix, (Laloux et al. 1999) and (Plerou et al. 1999).

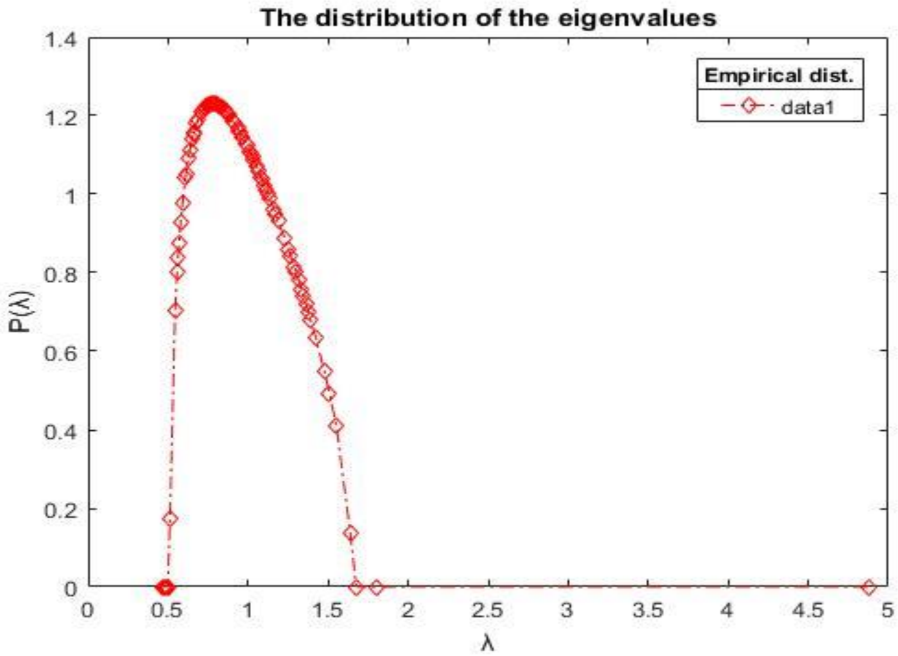

Fig 5a: Empirical distr. of eigenvalues

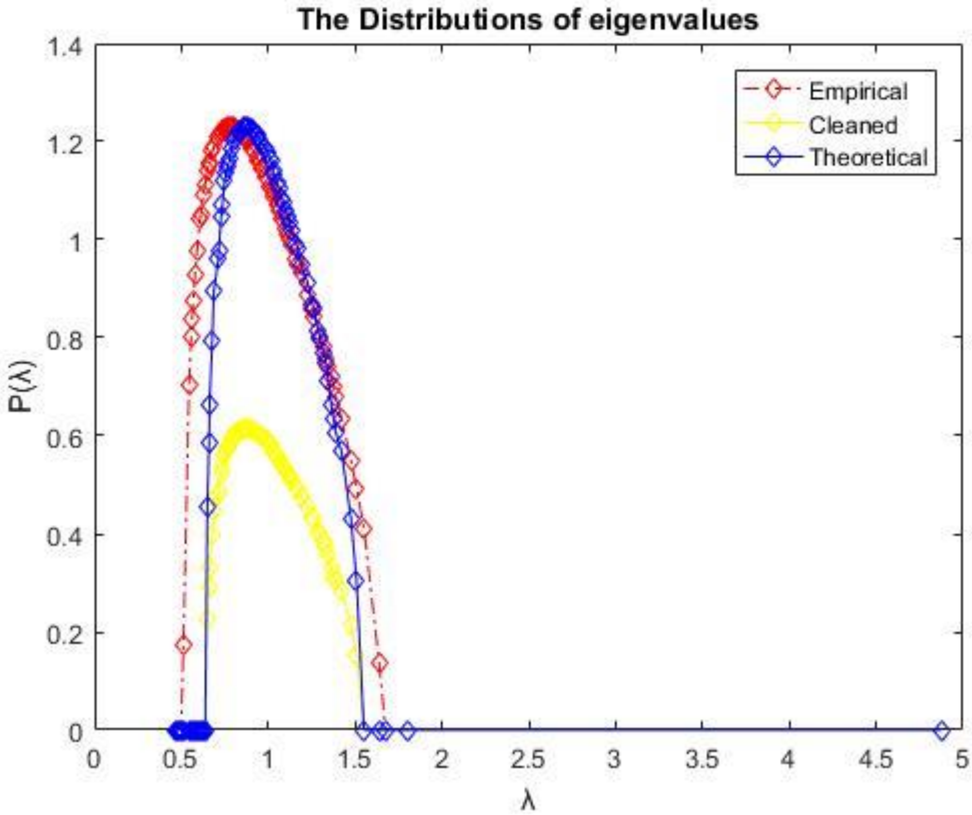

Fig 5b: Theoretical (Marcenko-Pastur), empirical and Cleaned up distr. of eigenvalues

In the other development, Urama et al., (2017) try to compare the results of Nnanwa et al. (2017) on NSE and JSE. They analysed 1149 daily closing price of 35 stocks for Joburg exchange, therefore have N and L as 35 and 1148 respectively. From the analysis, we have that the theoretical bounds of the correlation matrix are $[0.21, 2.37]$, having λ_{max}=2.37 and λ_{min}= 0.21 as maximum and minimum eigenvalues respectively. The value of Q is 32.77 while the largest eigenvalue is 11.86 which 5 times the value of the largest eigenvalue from the predicted RMT that is λ_{max}=2.37. In this case, 8.57% of the total eigenvalue lies outside the theoretical bounds of the eigenvalue thus have the penitent information about the market.

The corresponding eigenvector is the 'market' itself, which has approximately equal components on all the 82 stocks. The pure noise RMT hypothesis may, therefore, not be consistent with the NSE stocks.

5.4.4 Distribution of eigenvectors component analysis

Using Figures Fig 5c, Fig 5d, Fig 5e, Fig 5f and Fig 5g below we analyse the distribution of the eigenvectors by comparing the distribution of the eigenvector components that are inside the boundaries of the RMT prediction (i.e. $[\lambda_{min}, \lambda_{max}]$) with those outside the bound. The normal distribution of the eigenvectors u_1 of the first eigenvalue (which represents the market on the selected stocks) is clearly right-tail asymmetric with a positive mean. The data are also non-normal and therefore inconsistent with RMT predictions. Fig 5d shows a similar non-normal asymmetric distribution of eigenvectors associated with the second eigenvalue, as with the first, but with a reversed left-tailed skew. Normality begins to occur with much higher eigenvalue values such as the eigenvectors associated with the twentieth eigenvalue (see Fig 5e). Thus, potentially portfolio-enhancing stocks lay outside but not too far away from predicted RMT the bands. Particularly for the first eigenvalue which represents the market, we can say that the Nigerian Stock Market, as revealed by the 82 stocks, reacts more to positive variations than negative variations.

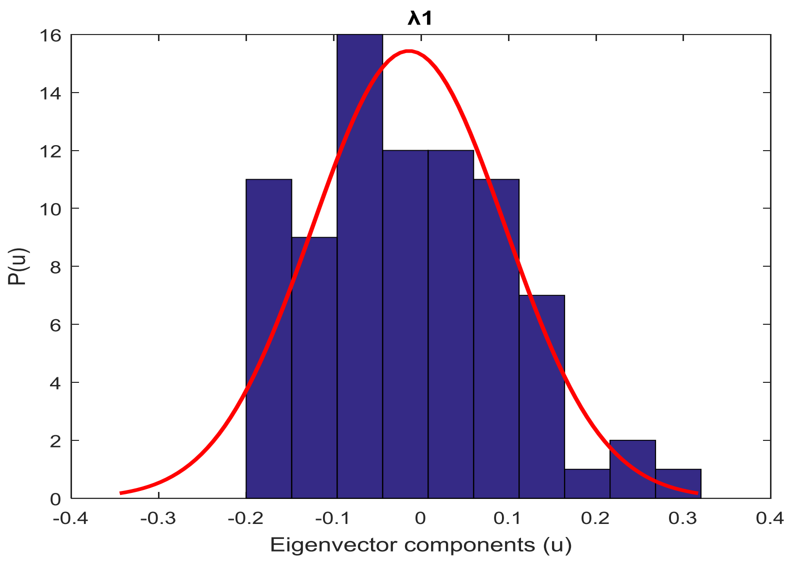
λ1
16
14
12
10
8
6
4
2
0
P(u)
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
Eigenvector components (u)

Fig 5c

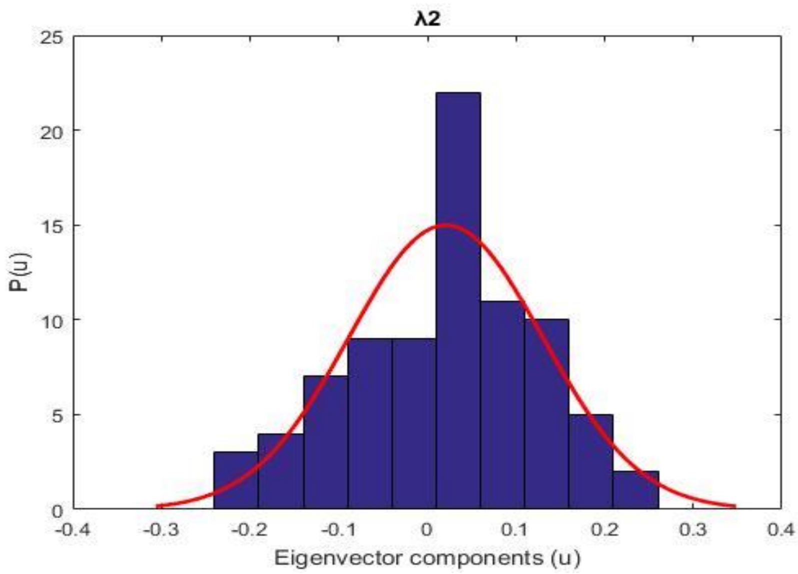
λ2
25
20
15
10
5
0
P(u)
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
Eigenvector components (u)

Fig 5d

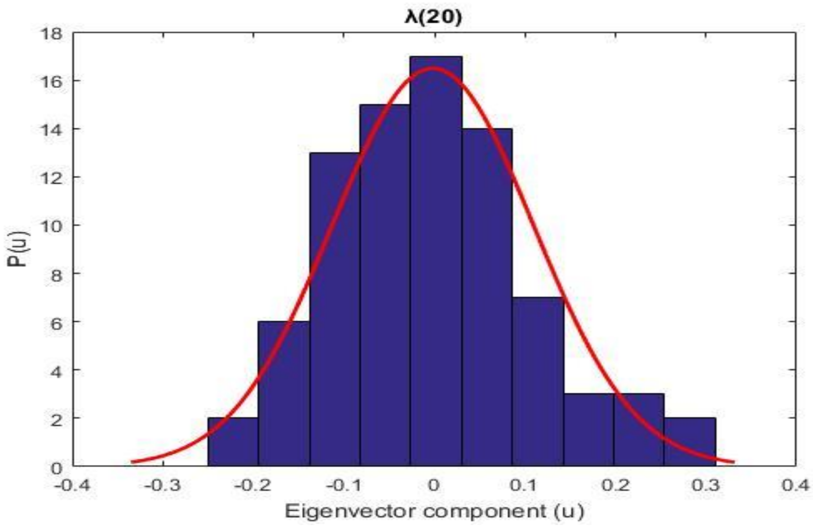
λ(20)
P(u)
Eigenvector component (u)
18
16
14
12
10
8
6
4
2
0
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4

Fig 5e

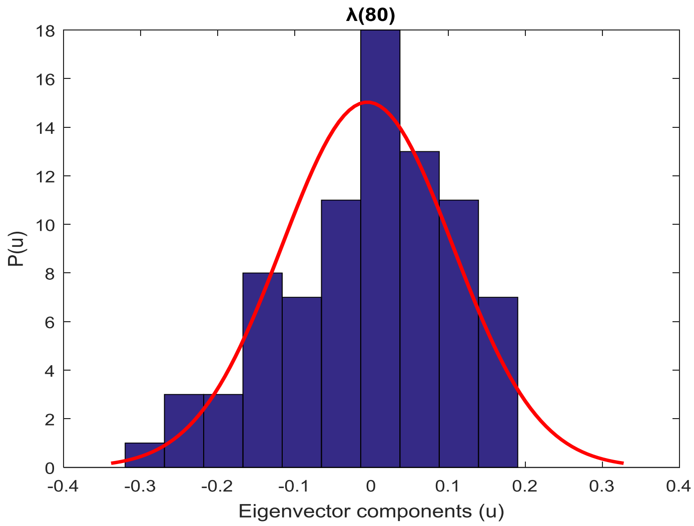
λ(80)
P(u)
Eigenvector components (u)
18
16
14
12
10
8
6
4
2
0
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4

Fig 5f

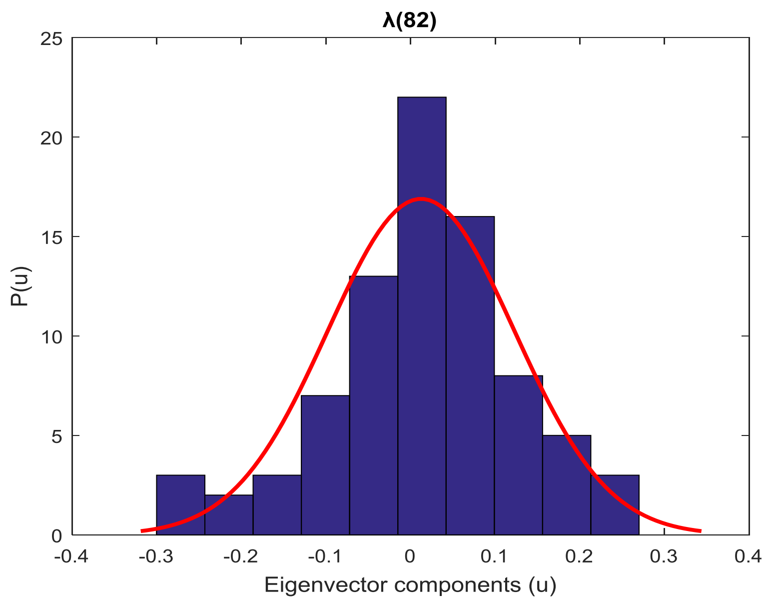

Fig 5g

5.4.5 Inverse Participation Ratio (IPR)

The Inverse participation ratio (IPR) is used to analyse the structure of the eigenvectors whose eigenvalues are lying outside the noisy band of RMT predicted eigenvalues (see Plerou et al. 1999, 2000). It measures the number of components that participate significantly in each eigenvector (see Gurh et al. 1998). It also indicates the degree to which the distribution of eigenvectors of the empirical correlation matrix deviate from RMT results, particularly distinguishing an eigenvector with roughly equal components and another with a small number of large components.

Definition 5.4.5

Let u_l^k denote the l^{th} component of the eigenvector u^k . The IPR is defined as

$$I^k = \sum_{l=1}^{N}\left(u_l^k\right)^4 , l = 1, 2, 3, \dots, N \qquad (5.24)$$

where $N = 82$ is the number of assets.

The IPR is the reciprocal of the number of eigenvector components significantly different from zero El Alaoui, (2015). This can be illustrated by two limiting cases: (i) if the components of the eigenvector are identical $u_1^k \equiv 1/\sqrt{N}$ has $I^k = 1/N$, and (ii) a vector with one component $u_i^k \equiv 1$ and the remainder zero has $I^k = 1$. Practically, inverting an observed IPR estimates the number of active elements in the time series of a financial asset, that is, the number of eigenvector components that contribute more than random noise to the portfolio risk-return characteristics.

5.4.6 Analysis of Inverse participation ratio

The theoretical mean IPR is approximately equal to $3/N = 3/82 = 0.0366$. A look at Fig.5h shows that the IPR values are close to a mean level of noise (0.04) indicating that a few stocks do not dominate the dynamics and most stocks participate in the correlation dynamics of asset returns. However, the deviations in IPR values are strong for a few initial elements (17 times more for the highest score of 0.17 compared with the lowest non-zero score of about 0.01 between elements 1-6). Overall, there does not appear to be a localisation effect in the spread of IPR values, with very large values for a few elements and very small values for the rest El Alaoui, (2015). It would be interesting to compare this overall market behaviour against sectorial results as mentioned in the introduction to this chapter.

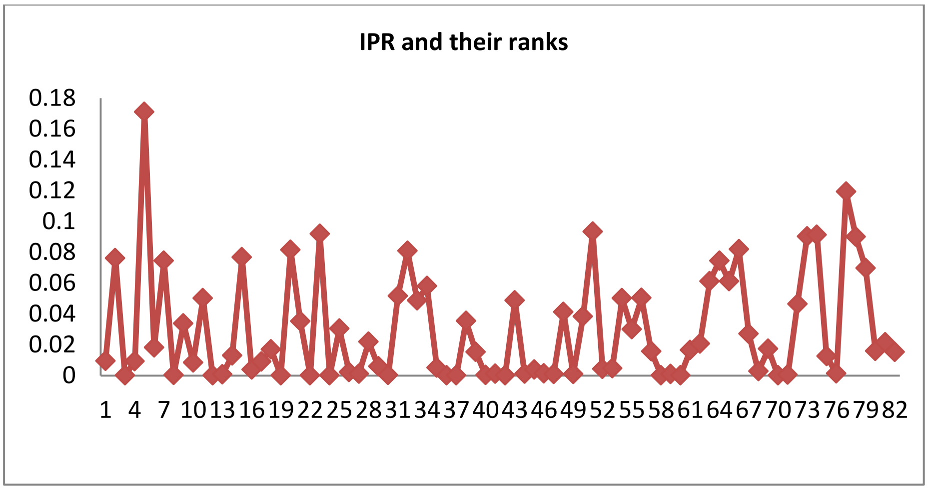

Fig 5h: Inverse participation ratio and their rank

5.5 Noise reduction

The covariance matrix C of the returns of the assets in a portfolio is estimated. If a portfolio of N assets, the matrix C has $N(N-1)/2$ entries to be determined from N time series of length L, the entries will be NL data. To realise a reasonable result the length of L needs to be large, but in the real-life experiment, one is not expected to use data that is more than four years. It is believed that four years is a lot of time; therefore, a lot of policies which might affect the data may have changed (example economic policies see Kondor et al. (2005). Due to these reasons, the covariance matrix C determined from the empirical financial data where L is approximately 1000 appear to contain a lot of noise.

5.5.1 The Clean-up Technique.

Several scholars have studied noise in financial analysis, authors like Lalour et al. (1999) (2000), Plerou et al. (1999) (2000) and Sharifi et al. (2004) have shown that the correlation matrix contains a lot of noise. They argue that using the empirical correlation matrix to optimise a portfolio will result in the new portfolio containing a lot of underestimated risk. Since the risk and return of this portfolio is not well controlled and estimated, there is a need for the system to be denoised.

Lalour et al. (2000) and Plerou et al. (2002) introduced the cleaning up (otherwise called filtering technique) of the correlation matrix to improve the estimated risk of the optimal portfolio by separating errors from the right and real correlations to the matrix, in order to extract the useful (real) information from the market. This is done by replacing the noisy eigenvalues of the correlation matrix by at the identity matrix with a coefficient such that the trace of the matrix is conserved. After cleaning the correlation matrix, the cleaned up one is used to compute the corresponding covariance matrix and then the optimal portfolio is constructed.

Some of the authors mentioned above have their peculiar way or method of filtering the noisy eigenvalues; Lalour et al. (2000) did their filtering by replacing the noisy eigenvalues by their average, while Plerou et al. (2002) did theirs by replacing the noisy eigenvalue with zeros after which the original main diagonal is restored. Sharifi et al. (2004) did theirs by replacing them with ones that are maximally and equally spaced and ensures that the sum of the eigenvalues is maintained.

5.5.2 Cleaning empirical correlation matrices

We noted earlier in the introduction to this chapter that estimating empirical correlation matrices introduces measure noise because of the large number of correlation parameters required. Using such matrices further in portfolio optimisation as depicted by Markowitz's mean-variance Efficient Frontiers is therefore subject to inaccuracies in predicting portfolio risks. We examine this in more detail when constructing suitable investment portfolios from overall and sector based RMT results. For this, we explain briefly how the process works. The first step is to divide the observed series of stock prices and returns into two equal sub-periods. The first period analyses predicted risk and the second realised risk. The second step filters RMT eigenvalues into noisy and non-noisy elements, with the later situated outside the RMT eigenvalue spectrum. These eigenvalues are maintained, and those in RMT bound $[\lambda_{min}, \lambda_{max}]$ are cleaned by replacing them with their average values while maintaining the same matrix trace. The third step uses known methods to construct the cleaned correlation matrix from the noisy elements El Alaoui, (2015); Laloux et al. (1999), (2000). After cleaning up the correlation matrix, the return increases and the risk is reduced by 13.7% (see Fig 4 below).

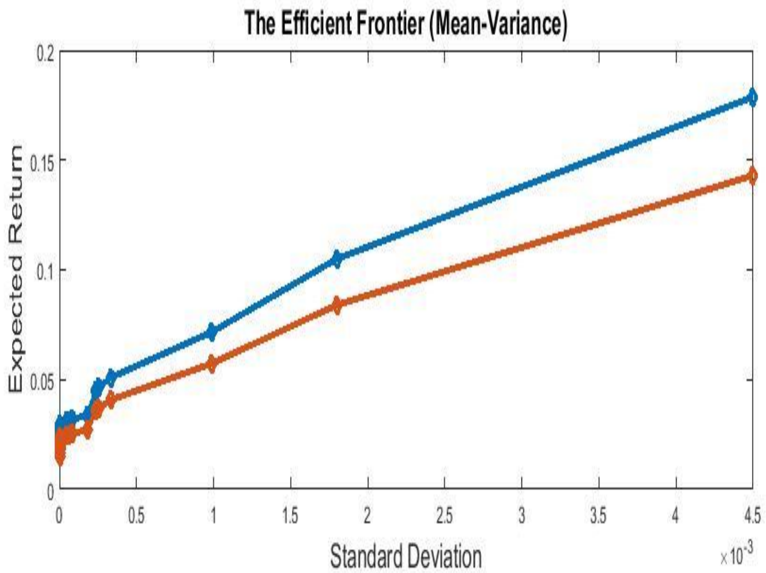

Fig 5i

5.6 Conclusion

RMT enabled us to analyse in some detail the correlation structure of stock returns in the Nigerian Stock Exchange. Marcenko-Pastur distribution predicted a theoretical eigenvalue range of between 0.52 and 1.65 approximately. About 6 out of 82 eigenvalues of the selected stocks were outside this eigenvalue spectrum, indicating that about 11% of the stocks have important information that can be used in constructing portfolios with more stable returns and risk characteristics than a null hypothesis purely random market allows in the remaining 89% cases.

The eigenvectors associated with the essential eigenvalues$[\lambda_{min}, \lambda_{max}]$are non-normal and highly asymmetric which is inconsistent with RMT. This fact suggests the presence of market signals such as depicted also in the information-carrying stocks. Given that the first eigenvalue represents the market, it also suggests that the Nigerian Stock Market (NSM) is inefficient, a fact that is consistent with known results in Ezepue and Omar (2012), for example. Importantly, this positive asymmetry in the distribution of eigenvectors of λ_1 shows that the NSM reacts more strongly to positive variations (and news) than negative variations.

While the RMT bounds imply that most stocks analysed (89%) will suitably follow a zero-mean normal distribution in their returns, the normal distribution is not a good enough fit for the materially informative stocks outside the RMT eigenvalue bounds. Finally, the inverse participation ratio gives additional insights about the spread or localisation of eigenvectors by their ranks. It shows an even spread which suggest that there are no dominant stocks in among the 82 stocks.

These insights are useful for constructing more optimal portfolios. For example, RMT Eigen-structure results are used to clean the empirical correlation matrices and thereby improve the realised and predicted risks associated with Markowitz mean-variance Efficient Frontier. Also, detailed risk analysis of individual stock and portfolio returns outside RMT bounds should use suitable non-normal distributions.

Future lines of work along these lines also include the development of financial derivatives in the NSM using the information on NSM-JSE market affinities

Chapter 6 Risk Management

6.1 Introduction

Risk is a word that is pervasive to all that come across it, but the truth is that one cannot do without risk. Life on its own is risky; it is unimaginable to think about the world without risk. Risk if appropriately managed, turns out to be a force that induces the growth of an establishment by making good returns from the uncertain profit.

Over the years, the financial market has grown tremendously and become more robust. The market has developed to the level where the practitioners can combine assets from different exchanges in a portfolio. With this development, the market is becoming more complex as the day goes by, thereby becoming riskier to invest in the market.

Due to this expansion of the market and the risk involved, the market practitioners and the investors became more sceptical about their investment. This gave rise to the study of risks associated with the financial markets and portfolios. Since researchers began to study the risks associated with the financial market and portfolio management, so many people have developed a lot of models to help and manage the risk of the market.

This risk can be calculated using three methods; therefore, people who try to develop models that will manage risk try to adopt any of the three methods or combination of them. These include;

1) Measuring the risky through the interactions of the stocks in the portfolio; otherwise called the covariance method.

2) Another one is calculating the risk using the historical data of the shares. And lastly

3) Monte Carlo. This is a method where the random market scenarios are generated; while the risk factors are assumed to be a multivariate normal distribution. The return of the portfolio is computed based on the scenarios.

The risk calculation proposed by Markowitz in 1952 before now used to be the easiest way of working out the risk of a portfolio due to its assumption. His work was the foremost work of the risk optimisation of portfolio; he introduced Mean-Variance (MV) in portfolio optimisation. It is assumed to have multi-normal distribution, but the results from the asset returns indicated that this assumption is not correct; instead, they have fat tails and high kurtosis. The result gotten from the correlation of the different stocks in the market has shown that there is no linear dependence between the asset returns as assumed earlier. Due to these problems mentioned, Markowitz classical approach to the risk management of a portfolio was criticised by a lot of market practitioners which inspired many researchers to develop some models to take care of those deficiencies which MV has shown while estimating the risk of a portfolio.

Sharpe (1970) came up with what he called the Capital asset pricing model (CAPM), which investigates the relationship between the expected return and risk of assets. Sharpe designed the CAPM to handle some of the deficiencies discover from MV; it tends to manage the systemic risk which MV and its diversification could not handle. Ross (1976), in his model, considered the relationship of the asset and many risk factors; he called his model Arbitrage pricing theory (APT). This model tries to correct the mispricing from the theoretical predictions of the price of the securities. APT uses the expected return and factors of several risk premiums of the asset, which is flexible; therefore, it is seen as an alternative model to CAPM.

As the year goes by, the financial market was growing and becoming more robust and volatile; therefore, the risks associated with the market were also increasing. This situation demands that sophisticated risk measures need to be developed to meet up with the challenges imposed by the rapid growth of the market. A good number of researchers worked and developed different risk measure, which we have discussed in section 2.2.2 from subsection (a) to (g).

Some of the models developed to tackle this problem used dependency structure without bringing into consideration the market and asset interdependency structure, that is, the joint tail realisations. Some authors have criticised the use of the linear correlation between securities; it is believed that its feasible values depend on the marginal distribution, according to Embrechts et al., (2002). Also, Stulajter (2009) pointed out that when assets have positive dependence, it does not mean correlation of one and zero correlation does not always imply independence. In his analysis, he demonstrated with scattered plots where he showed the stronger tail dependence, which underestimates risk and thereby makes the portfolio to be less diversified.

These problems persisted until researchers brought the results of Sklar (1959), called copulas into financial market studies. The origin of this area of research is traced back to Frechet (1951) where he studied distribution functions defined on a probability space with given marginals. Though Frechet introduced the concept, it was not clear at that moment; this made Abe Sklar work and improve the idea he Frechet (1951) introduced. This gave rise to Sklar (1959). It was in Sklar's work where copulas were mentioned first. Copulas help to determine the distribution of the return of a portfolio which depends on the univariate distributions and the dependence between assets in the portfolio Andrew (2007).

Though copulas can be applied in so many areas of research, our interest will be in finance and economics. It has been widely used in the areas of decision making in finance and economics like; risk management, option pricing, credit risk, and the relationship between different financial markets Patton (2006).

In risk management, it is used to evaluate the tail probabilities and market trade-offs. This was difficult to evaluate using other risk measures. Cherubini and Luciano (2001) dropped the joint normality assumption on returns; they used value-at-risk (VaR) and copulas to recover the marginal probability distribution and then calibrate copula function and recover the joint distribution. Embrechts et al. (2003) with a host of other researchers also used copulas method to study VaR.

Joe (1997) and Nelsen (2006) were believed to have introduced statistical methods into copulas and used it in derivatives and options pricing while Cherubini et al. (2004) gave a detailed introduction of copulas to option pricing which serves as an alternative to the statistical methods of Joe above.

Therefore, the copula is a multivariate distribution function defined on the unit cube $[0,1]^n$ that joins to its one-dimensional marginal distribution function. Since the calculation of the correlation coefficient works perfectly with normal distributions, and some of the market calculations are meanly skewed, copula is applied to deal with the skewness.

Definition 6.1.1

Let X be a vector random variables where $X = [x_1, x_2, \dots, x_n]'$ and F as joint distribution with marginathe l distribution $F_1, F_2, \dots, F_n$, Sklar (1959) proves that there exists a function C such that

$$F(x_{1,}x_{2,\dots,}x_n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)), \quad \forall\, x \in \mathbb{R}^n \tag{6.1}$$

This shows that from the multivariate distribution F we can get the marginal distributions and the copula, F_i and C respectively. From Sklar's theorem, the copula function C is $C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)) = C(U_1, U_{2,\dots,}U_n)$ is the distribution function of random variables. This means that C meets the following conditions in the case of n- dimension Nelsen (2006);

For every $U \in [0,1]^n$, $C(U) = 0$, if at least one element of U is 0.

For every $U \in [0,1]^n$, $C(U) = U$, if all coordinates of U are 1 except U_j,

C is an increasing function (see definition 1.7.9)

Therefore, equation (6.1) shows that C is to determine the dependence between X_i since each F_i contains all the univariate information of each X_i and the joint distribution F has univariate and multivariate information.

If the margins are continuous, the n-times partial derivative of equation (6.1) with respect to all the variables gives the multivariate density for the data.

$$f(x_{1,}x_{2,\dots,}x_n) = c(F_1(x_1), F_2(x_2), \dots, F_n(x_n)) f_1(x_1), \dots, f_n(x_n) \quad \forall\, x \in \mathbb{R}^n, \tag{6.2}$$

where $c(F_1(x_1), F_2(x_2), \dots, F_n(x_n))$ is called the copula density, which is derived from the partial derivative.

6.1.1 Copula Density

Let X be a multivariate random data set defined above, and C is a cumulative distribution function or that vector $X = [x_1, x_2, \dots, x_n]'$ defined on a unit square $[0,1]^n$ with uniform marginal distribution as $U_i = F_i(x_i)$ for $i = 1, 2, \dots, n$. If C is continuous, the probability density function (pdf) of C will be

$$c(U_1, U_2, \dots, U_.) = \frac{\partial^n}{\partial U_1, \partial U_2, \dots, \partial U_n} C(U_1, U_2, \dots, U_.), \tag{6.3}$$

Where c is the copula density. Without loss of generality, equation (6.3) can be expressed as

$$f(x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f_i(x_i) \, . \, c\big(F_1(x_1), F_2(x_2), \dots, F_n(x_n)\big) \qquad (6.5)$$

The above equation shows that copula density controls the dependence between the x_i, therefore, f is the product of the univariate marginal. Patton (2009) pointed out that the joint log-likelihood is the summation of the univariate log-likelihood and the copula log-likelihood. This is possible because, the joint density is the product of the marginal densities and the copula density.

$$\log f(x) = \sum_{i=1}^{n} \log f_i(x_i) + \log c\big(F_1(x_1), F_2(x_2), \dots, F_n(x_n)\big). \qquad (6.6)$$

The tail dependence and bounds for dependence form the central concept of copula functions theory. The conditional probability of copula function is represented by the lower or upper tail dependence coefficient; this implies that x_1 takes a value in its lower or upper tail given that x_2 takes a value in its lower or upper tail for all $x_1, x_1 \in X$.

When lower and upper tail dependence coefficients of copula functions are equal, it is said to have symmetric tail dependence. But when their lower and upper tail dependence coefficients are different, it said to have asymmetric tail dependence.

In the discussion of copulas, there are three fundamental concepts: they are copulas representing independence, copulas with perfect positive dependence and copulas with perfect negative dependence. Hence, the perfect positive and perfect negative dependence of copulas is defined as Frechet upper and lower bound copulas.

Alexander (2008) and Stulajter (2009) said that no copula could take a value that is greater than the value of Frechet upper bound copula or a value less than Frechet lower bound copula.

6.1.2 Dependence

There are three dependence natures of copula functions; these include independence, copulas with perfect positive dependence and copulas with perfect negative dependence.

Let $U = (u_1, u_2, \dots, u_n)$ be a random vector with $u_i, i = 1, .., n$ as independence uniform random variables, if the distribution function of U is a copula function C which is

$$C(u_1, u_2, \dots, u_n) = \prod_{i=1}^{n} u_i, \tag{6.7}$$

Then C is known as the independence copula. This shows that random variables are independent if the associate copula is equal to the independence copula.

Perfect positive dependence is a term used in probability theory, which refers to the dependence between the components of a random vector. Let X be a random vector that is $X = (x_1, x_2, \dots, x_n)'$ such that

$$P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) = \min\{X_i \leq x_i\}_{i=1}^{n} \tag{6.8}$$

For $x_i \in \mathbb{R}^n, i = 1, 2, \dots, n$. Then the dependence of X is called perfect positive dependence. It is sometimes called comonotone random variable and seen as the upper bound of the Frechet - Hoeffind bound when the inequalities are equality.
Perfect negative dependence is an expression that shows countercomonotonicity copula which is of the form

$$W(u_1, u_1, \dots, u_1) = max\{u_1 + u_2 + \dots + u_n + 1 - \mathrm{d}, 0\} \tag{6.9}$$

6.1.3 Frechet - Hoeffding bounds

Frechet - Hoeffding bounds is a maximal and minimal bivariate copula which every other take values in between the bounds. The Frechet - Hoeffding upper and lower bounds correspond to the perfect positive and perfect negative dependence, respectively. Kort (2007) shows that for any copula C with domain $X_1 \times X_2$,

$$C^-(u, v) = \max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v) = C^+ \tag{6.10}$$

for every $(u, v) \in X_1 \times X_2$. C^+ and C^- are called the Frechet - Hoeffding upper and lower bounds respectively.

Equation (6.10) can be given in a generalised form. This is Frechet - Hoeffding theorem and states thus, for any copula $C: [0,1]^n \to [0,1]$ and any $(u_1, u_2, \dots, u_n) \in [0,1]^n$, then the following bounds hold $C^-(u_1, u_2, \dots, u_n) \leq C(u_1, u_2, \dots, u_n) \leq C^-(u_1, u_2, \dots, u_n)$. This implies that,

$$C^-(u_1, \dots, u_n) = max\left\{\sum_{i=1}^{n} u_i + 1 - n, 0\right\} \leq C(u_1, \dots, u_n) \leq \min(u_1, \dots, u_n) = C^-(u_1, \dots, u_n) \tag{6.11}$$

C^+ and C^- are upper and lower bounds and they correspond to comontone and counter-monotonic random variables respectively.

Note: As the parameter values of the copula functions change, it tends to one of the Frechet - Hoeffding bounds either positive or negative dependence between the variables. If it does not tend to upper or lower bound as correlation approaches 1 or -1 respectively, we then say that the copula is a Gaussian copula.

6.2 Copula Family

In this section, we will present different families of the copula. Copula families refer to the parameters that control the strength of the dependence of the copula functions, while the copula class refers to a collection of copula families that have similar properties.

Nelsen (2006) states that there are two main methods used to derive copula function; these include; inversion method and generator functions method. The normal or student t distribution is a type of multivariate distribution copula that is derived from the inversion method. A good example is the elliptical copulas. The Archimedean copula is an example of copula functions constructed by a generator function.

6.2.1 Elliptical copulas

Elliptical copulas are a type of copula that is derived from elliptical distributions. These include multivariate normal (Gaussian) distribution and student t distribution, and this is as a result of Sklar's theorem.

Definition 6.2.2

Let $X = (X_1, X_2, \dots, X_n)$ be a random vector with $X_i, i = 1, .., n$, for some $\mu \in \mathbb{R}^n$ the mean vector, $\sum \; = AA'$ is the covariance matrix (see 5.3.3) and $\phi : [0, +\infty[\rightarrow [0, +\infty[$ be the generator, the characteristic function $\varphi X - \mu(t)$ of $X - \mu$ is a function of the quadratic form $t^T \sum t, \varphi X - \mu(t) = \phi(t^T \sum t)$. Then X has an elliptical distribution with parameters $\mu, \sum$ and ϕ, and one can write $X \sim \mathcal{E}(\mu, \sum, \phi)$, if it can be expressed in the form

$$X = \mu + RAU, \tag{6.12}$$

Example 6.2.3

An elliptical copula with multivariate normal distribution is called multivariate Gaussian copula. This can be written as

$$C(u_1, u_2, \dots, u_n) = \phi\big(\phi^{-1}(u_1), \phi^{-1}(u_2), \dots, \phi^{-1}(u_n)\big), \qquad (6.13)$$

where ϕ^{-1} is the inverse of the standard multivariate normal distribution function ϕ. The density of the function is

$$C(u_1, u_2, \dots, u_n) = \frac{1}{|R|^{\frac{1}{2}}} exp\left(-\frac{1}{2}\omega^T(R^{-1} - 1)\omega\right), \qquad (6.14)$$

where $\omega = (\phi^{-1}(u_1), \dots, \phi^{-1}(u_n))^T$ and R is the correlation matrix (see 5.3.3). And $\omega = (\omega_1, \omega_2, \dots, \omega_n)'$, where ω_i is the u_i quantile of the standard normal random variable X_i which is

$$u_i = P(X_i < \omega_i), \qquad X_i \sim N(0,1), i = 1, 2, \dots, n. \qquad (6.15)$$

Example 6.2.4
An elliptical copula with multivariate Student's t –distribution, correlation matrix R and v degree of freedom is called multivariate Student's t –copula. This can be written as

$$C(u_1, u_2, \dots, u_n) = t_{v,R}\big(t_v^{-1}(u_1), t_v^{-1}(u_2), \dots, t_v^{-1}(u_n)\big), \qquad (6.16)$$

where $t_{v,R}$ is a standardised multivariate Student's t –distribution and t_v^{-1} is the inverse of the univariate cumulative distribution function of Student's t with v degree of freedom

The density of the Student's t- copula is

$$C(u_1, u_2, \dots, u_n) = \frac{\Gamma((v+d)/2)\big[\Gamma(\frac{v}{2})\big]^d (1 + \omega^T R^T \omega)^{-(v+d)/2}}{|R|^{\frac{1}{2}}\Gamma(v/2)[\Gamma((v+1)/2)]^d \prod_{i=1}^d \left(1 + \frac{\omega_i^2}{v}\right)^{-\frac{v+1}{2}}} \qquad (6.17)$$

where $\omega = \big(t_v^{-1}(u_1), t_v^{-1}(u_2), \dots, t_v^{-1}(u_n)\big)^T$, and $\Gamma(*)$ is the gamma function.

As we have seen above, the elliptical family of copula functions has two main examples; Gaussian and Student's t –copula. Though, the Gaussian copula seems to be more popular than the Student's t –copula but the Gaussian copula does not possess tail dependence like the Student's t –copula. This allows an increase in the probability of joint extreme events, and serves as an advantage over Gaussian copula, see He and Gong (2009). Secondly, Student's t- copula has a parameter v

called the degree of freedom, a higher value for v decreases the probability of tail events and as v tends to infinite; $(v \to \infty)$ the Student's t –copula converges to the Gaussian copula.

Finally, Kole et al. (2007) show that the Student's t –copula matches both the dependence in the centre and the dependence in the tails than Gaussian copula which fails to capture tail dependence.

6.2.2 Archimedean copulas

Archimedean copulas are a class of copulas that are easily derived, and they are the type of copulas that allow a wide range of dependence. Unlike the elliptical copulas (Gaussian and Student - t copulas, which is implicit copulas and are built using an inversion method from multivariate distribution), Archimedean copulas are cumulative distribution functions without integral, but they have regular form and an alternative method of building which is depends on the generators. Therefore, given any generator, one can easily define a corresponding Archimedean copula.

Definition 6.2.4

Let $\emptyset: [0,1] \to [0,\infty)$ be a continuous decreasing convex function such that $\emptyset(1) = 0$ and $\emptyset(0) = \infty$ which is strictly decreasing is called an Archimedean generator if the pseudo-inverse of $\emptyset$ is;

$$\emptyset^{[-1]} = \begin{cases} \emptyset^{-1}(u), & 0 \le u \le \emptyset(0) \\ 0, & \emptyset(0) \le u \le \infty \end{cases} \tag{6.18}$$

If $\emptyset$ is a strict generator when $\emptyset(0) = \infty, \emptyset^{[-1]} = \emptyset^{-1}$. Se Demarta and

Definition 6.2.5

Let $\emptyset$ in definition 6.2.4 be a strict generator and its inverse $\emptyset^{-1}: [0,\infty) \to [0,1]$ is completely monotone, the n-dimensional function generates by $\emptyset$ is called a multivariate Archimedean copula if

$C(u_1, u_2, \ldots, u_n) = \emptyset^{-1}(\emptyset(u_1) + \emptyset(u_2) + \ldots + \emptyset(u_n)),$

which is

$$C(u_1, u_2, \ldots, u_n) = \emptyset^{-1}\left(\sum_{i=1}^{n} \emptyset(u_i)\right) \tag{6.19}$$

In the literature, there are groups of copula families generated by an Archimedean generator which has one or more parameters. Different types of Archimedean copula are as a result of the generator used; any parameter used gives a different result. For example, one parameter families are Clayton copula, Gumbel copula and Frank copula.

Example 6.2.6

Let $\emptyset(u) = (-ln(u))^{\theta}$, for $\theta \geq 1$. If $\emptyset(u)$ is continuous and $\emptyset(1) = 0$, then derivative of $\emptyset(u)$ is $\emptyset'(u) = -\theta(-\ln(u))^{\theta-1}\frac{1}{u}$ and $\emptyset$ is a strictly decreasing function from the domain $[0,1]$ to the range$[0,\infty)$. If the second derivative exists and $\emptyset''(u) \geq 0$ on $[0,1]$, so $\emptyset(u)$ is convex. With $\emptyset(u) = \infty$, $\emptyset$ is a strict generator? Then from equation (6.19), the copula will be

$$C_{\theta}(u_1, u_2, \dots, u_n) = \emptyset^{-1}\left(\sum_{i=1}^{n} \emptyset(u_i)\right) = exp\left(-\left[(-lnu_1)^{\theta} + \cdots + (-lnu_n)^{\theta}\right]^{1/\theta}\right)$$

this implies that

$$C_{\theta}(u_1, u_2, \dots, u_n) = exp\left(-\left[\sum_{i=1}^{n}(-\ln(u_i))^{\theta}\right]^{1/\theta}\right) \tag{6.20}$$

This type of copula family is called the Gumbel family. The Gumbel copulas cannot represent negative dependence and can model symmetric dependence in the data due to its stronger upper dependence and with a weaker lower dependence.

Example 6.2.7

Let $\emptyset(u) = (u^{-\theta} - 1)/\theta$, for $\theta \in [-1, \infty)\backslash\{0\}$, this will generate a copula of the form

$$C_{\theta}(u_1, u_2, \dots, u_n) = \max\left(\left[1 + \sum_{i=1}^{n}\left(u_i^{-\theta} - 1\right)\right]^{-1/\theta}, 0\right) \tag{6.21}$$

but for $\theta > 0$, and copulas are strict, equation (6.21) will be simplified to

$$C_{\theta}(u_1, u_2, \dots, u_n) = \left(1 + \sum_{i=1}^{n}\left(u_i^{-\theta} - 1\right)\right)^{-1/\theta} \tag{6.22}$$

This type of copula is called the family of Clayton copulas. The Clayton copulas are used to study correlated risk in mathematical finance because of its capacity to capture lower tail dependence. If the parameter $\theta = 1$, this generates an independence copula, while the copula coincides with the comonotonicity when $\theta = \infty$. For any other value of the parameter, the copula moves between the independence and comonotonicity and can never be negative dependence.

Example 6.2.8

Let $\emptyset(u) = -ln\frac{e^{-\theta u}-1}{e^{-\theta}-1}$, for $\theta \in \mathbb{R}\backslash\{\theta\}$ this parameter will generate a copula in the form

$$C_\theta(u_1, u_2, \dots, u_n) = -\frac{1}{\theta}\, ln\left(1 + \frac{\prod_{i=1}^{n}\left(e^{-\theta u_i} - 1\right)}{(e^{-\theta} - 1)^{n-1}}\right) \tag{6.23}$$

This is a strict Archimedean copula called the Frank copulas. In n-dimension, the parameter is restricted to the range $(0, \infty)$ and the copula is reduced to positive while the bivariate case, the parameter is allowed to take both positive and negative values, and the bounds of Frechet - Hoeffding are both obtained at $\theta = -\infty$ or $\theta = +\infty$. This shows that Frank copula describes symmetric dependence.

In Nelsen (1997), it shows that if $\emptyset$ is a strict generator for an Archimedean copula with $\emptyset(t^\alpha)$ and $[\emptyset(t)]^\beta$ which are considered as two parameter family of strict generators for all $\alpha \in (0,1]$ and $\beta \geq 1$, and $\emptyset_{\alpha,\beta}(t) = \emptyset_\beta \circ \emptyset_\alpha(t) = [\emptyset(t^\alpha)]^\beta$. If $\emptyset_\alpha^{-1}$ and $\emptyset_{\alpha,\beta}^{-1}$ are completely monotonic in $(0, \infty)$, then $\emptyset_{\alpha,\beta}$generates an n-copula.

Example 6.2.9

Let$\emptyset(t) = t^{-1} - 1$, for $\alpha > 0$ it is obvious that the inverse $\emptyset_\alpha^{-1}(t) = (1 + t)^{-\frac{1}{\alpha}}$ is completely monotonic in the interval $(0, \infty)$, then $\emptyset_\beta \circ \emptyset_\alpha(t) = (t^{-\alpha} - 1)^\beta$ for $\beta \geq 1$ will generates the family of copula

$$C_{\alpha,\beta}(u_1, u_2, \dots, u_n) = \left\{\left[\sum_{i=1}^{n}(u_i^{-\alpha} - 1)^\beta\right]^{\frac{1}{\beta}} + 1\right\}^{-\frac{1}{\alpha}} \tag{6.24}$$

for $u_i \in [0,1]$ and $n \geq 2$. When $\beta = 1$,we can easily see that equation (6.19) will be the Clayton family with one parameter.

One method to estimate the parameters is to calibrate with Kendall's tau. The relation between the parameter θ and Kendall's tau, τ is summarized in the following table below for the three Archimedean copulas.

Copula Type	τ	Formula for θ
Clayton	$\theta/(\theta+2)$	$2\tau/(1-\tau)$
Gumbel	$1-1/\theta$	$1/(1-\tau)$
Frank	$1-4\theta^{-1}(1-D_1(\theta))$	No closed form

Table 6.1. The summary of the parameters and their tau.

6.3 Measuring the coupled risk

Value - at - Risk (VaR) is widely used as a risk measure; it measures the potential loss in value of risky assets in a portfolio within a given interval of confidence over some time. It is often used by commercial and investment banks to capture the probability of the most they can lose on their investment. VaR took centre stage until Artzner et al. (1997) and others criticised it. They claimed that among its problem was its inability to satisfy subadditivity; and therefore, not a coherent risk measure. Secondly, it does not measure adequately; the possible loses in the tail of the distribution. It is accepted that the alternative risk measure to VaR is conditional Value - at - Risk (CVaR). CVaR manages and has a better description of the distribution of loses on the tail, Rockafeller and Uryasev (2000a)

6.3.1 Estimation of VaR and CVaR

Value - at - Risk (VaR) is a concept developed to help and manage risk in financial studies. It is a measure that defines the maximum amount a portfolio of assets is likely to lose over some time at a specific confidence level.

Definition 6.3.3

Given $\alpha \in (0,1)$ some confidence level threshold, which is normally 5% and 1% are taken in practise, and then VaR is defined as;

$$VaR_{\alpha}(X) = inf\,\{m: P[X+m<0] \leq \alpha\} \qquad (6.25)$$

So, if we chose $\alpha = 5\%$, this shows that we are 95% confident that our lost will not exceed the value of the VaR. If we let the lose to be $-X$, then the VaR is defined as;

$$VaR_{\alpha}(X) = inf\ \{m: P[-X < m]\alpha\} \geq 1 - \alpha \tag{6.26}$$

If we assume X to be normal distributed with the mean and variance as μ and σ^2 respectively, then VaR will be

$$VaR_{\alpha}(X) = \ -[\mu + N^{-1}(\alpha).\sigma] \tag{6.27}$$

This shows that VaR is the $(1 - \alpha)$ quantile of the return distribution which is specified in most cases. Also, we have seen that VaR depends on some concerts like confidence level, holding period, volatility and correlation of the stocks.

Some authors criticised VaR because of its inability to satisfy sub-additivity axiom and therefore, lacks the coherent property as a risk measure Artzner et al. (1997). Also, it was claimed that VaR does not put into consideration the tail of the distribution well; that is, it does not capture the losses at the tail of a distribution.

To address these challenges mentioned above and many more problems with VaR, Rockafellar and Uryasev (2000a) introduced Conditional Value at Risk (CVaR). CVaR propose a minimisation formulation that gives a convex or linear problem; this helps in providing a better description of the loss on the tail of the distribution Rockafellar and Uryasev (2000a) (2000b).

Definition 6.3.4
Let X be a continuous random variable representing loss, given a parameter$\alpha \in (0, 1)$, and then Conditional Value at Risk (CVaR) of X is defined as

$$CVaR_{\alpha} = E[X/X \geq VaR_{\alpha}(X)] \tag{6.28}$$

Where E is seen as a conditional expectation of losses above the threshold value of the loss. If one looks at the definition of $CVaR_{\alpha}$ above and equ (6.28), one can easily see that $CVaR_{\alpha}$ is always not less than VaR_{α}.

In some cases, authors refer to $CVaR_{\alpha}$ as mean excess loss, tail VaR_{α} and mean shortfall. $CVaR_{\alpha}$ can be directly derived from VaR_{α} if the cut off level periodicity of the data and the assumption of the stochastic volatility appear the same, thus

$$CVaR_{\alpha} = \frac{1}{1 - \alpha} \int_{-1}^{VaR_{\alpha}} x\, p(x) dx \tag{6.29}$$

where $p(x)dx$ is the probability density of a return X.

Having defined VaR_α and $CVaR_\alpha$, we will now define the two in terms of the copula. Remember, that in the definition (6.1.1), we stated X is a vector of random variables and F is a set of joint distribution with the marginal distribution $f_1, f_2, \ldots, f_n$ where $u = F(X)$, is assumed to be a continuous distribution with the function $C(.)$ called copula function.

If $g(w,u)$ is a cost function which does not exceed a threshold β, and $\breve{g}(w,u) = g(w, F^{-1}(u))$, where $F^{-1}(u) = (F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_n^{-1}(u_n))'$. This transforms the domain of $g(.,.)$ from $\mathbb{R}^n \rightarrow \mathrm{I}^n$ which was implied by the transformation of $u_i = f_i(x_i)$ in (6.1.2). Therefore, we can now define the copula version of VaR_α as thus

Definition 6.3.4

Let X be a random vector in $\mathbb{R}^n$ with $n-$ copula function $\emptyset$, $\mathrm{c}: \mathrm{I}^n \rightarrow [0, \infty)$ where c is the copula density define in (6.1.2), with α as the confidence level, VaR_α is defined as thus;

$$VaR_\alpha = min\,\{\beta \in \mathbb{R} : \emptyset(w, \beta) \geq \alpha\} \tag{6.30}$$

this implies that

$$VaR_\alpha = min\,\{\beta \in \mathbb{R}: C(u/\breve{g}(w,u) \leq \beta\} \geq \alpha \tag{6.31}$$

where $w \in \mathbb{R}^n$. We can now give the definition of $CVaR_\alpha$ in terms of a copula.

Definition 6.3.5

Given $w, u, F(X)$ and $\breve{g}(w,u)$ above, and for a confidence level α $CVaR_\alpha$ can be defined as

$$CVaR_\alpha = \frac{1}{1-\alpha} \int\limits_{\breve{g}(w,u) \geq VaR_\alpha} \breve{g}(w,u) C(u) du \tag{6.32}$$

6.4 Numerical Application

In this section, we give numerical examples and applications of modelling the risk of a portfolio using copula, $CVaR_\alpha$ and VaR_α.

In the examples, we use the following stocks: Custodyins, Guaranty Trust Bank, Nem, Skye Bank, Sterling Bank, Transcorp, Wapic, Wema Bank and Zenith Bank

Plc. These represent nine stocks from the financial sector of NSE, and the data used cover the period between August 2009 and August 2015, which we look at the daily returns of the stocks. This is shown in figure (6.1) below.

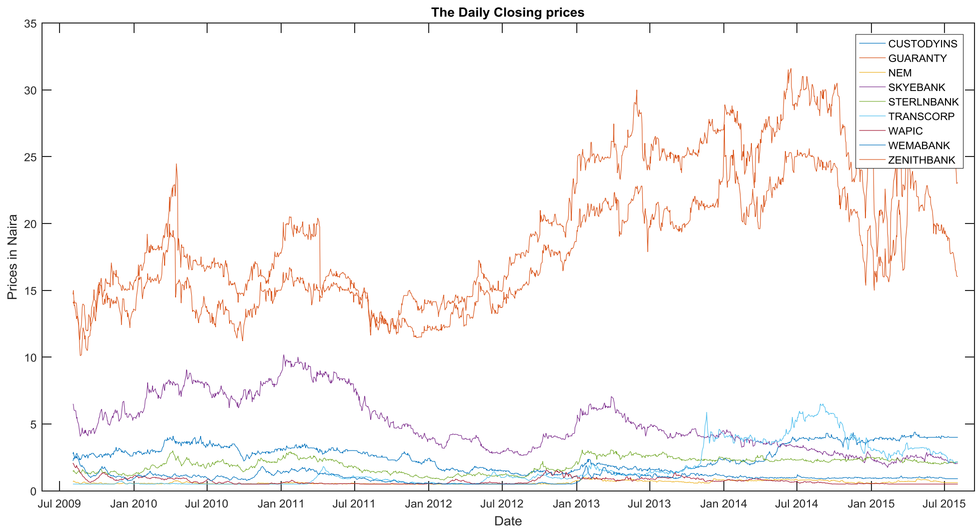

Fig 6.1: shows the closing price from 3rd August 2009 to 3rd August 2015.

6.4.1 Multiple regression analyses of Variance.

In our analysis, multiple regression analyses were conducted to examine the relationship between risk (which is our response variable) and weights (which is our explanatory variables, i.e. the various banks).

The number of the parameters used is $9\ (n)$, while the number of the observation is 1483. The degree of freedom is $n - 1$ equal to 8 (which is, number of the parameters minus 1) and in other to accept or reject the null hypothesis, we have to look at the tabulated F and compare it with our calculated one. This gives us 1.95 and when compared with the calculated F from the analysis of variance table displayed in table 6.2, the F value of 1901.21 (with an associated p-value that is 0), we found out that the calculate F is greater than the tabulated F, indicates a significant relationship between the dependent variable, risk, and at least one of the explanatory variables see Courville and Thompson (2001), LeBreton et al (2004) and Nathans et al (2012). Therefore, we have to reject the null hypothesis because the implication of the above statement is that since the tabulated F is less than the calculated one, indicates that the model as a whole has the statistically significant predictive capability.

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95.0%	Upper 95.0%
Intercept	2.69E-02	4.80E-04	5.61E+01	0.00E+00	2.60E-02	2.79E-02	2.60E-02	2.79E-02
GUARANTY	-5.08E-02	2.15E-02	-2.37E+00	1.80E-02	-9.29E-02	-8.72E-03	-9.29E-02	-8.72E-03
ZENITHBANK	-5.73E-02	2.11E-02	-2.72E+00	6.65E-03	-9.86E-02	-1.59E-02	-9.86E-02	-1.59E-02
NEM	3.03E-01	2.68E-03	1.13E+02	0.00E+00	2.97E-01	3.08E-01	2.97E-01	3.08E-01
SKYEBANK	2.84E-01	4.57E-03	6.20E+01	0.00E+00	2.75E-01	2.93E-01	2.75E-01	2.93E-01
STERLNBANK	-6.72E-02	1.57E-02	-4.27E+00	2.06E-05	-9.80E-02	-3.63E-02	-9.80E-02	-3.63E-02
TRANSCORP	-5.26E-03	1.43E-02	-3.67E-01	7.14E-01	-3.34E-02	2.29E-02	-3.34E-02	2.29E-02
CUSTODYINS	4.46E-03	1.55E-02	2.89E-01	7.73E-01	-2.58E-02	3.48E-02	-2.58E-02	3.48E-02
WAPIC	1.75E-01	9.43E-03	1.85E+01	5.56E-69	1.56E-01	1.93E-01	1.56E-01	1.93E-01
WEMABANK	1.01E-01	1.04E-02	9.67E+00	1.80E-21	8.02E-02	1.21E-01	8.02E-02	1.21E-01

Table 6.2. Regression 1

The t –statistic is employed for making inferences about the regression coefficients. The hypothesis test on coefficients of the explanatory variables, tests the null hypothesis that it is equal to zero – meaning the corresponding term is not significant – versus the alternate hypothesis that the coefficient is different from zero. The coefficients' p –values are used to determine which terms to keep in the regression model. The coefficient's p –value for TRANSCORP is 0.71 and CUSTODYINS is 0.77. Both are considered not statistically significant at 5% a significance level see LeBreton et al (2004) and Nathans et al (2012). Hence, their removal from the model is considered to optimize the predictive capability of the model. A better model in terms of having more predictive capability were obtained after removing the variables TRANSCORP and CUSTODYINS, as shown in the results presented in the table 6.3 below.

The R –square value shows that approximately 92% of the variability in the risk (which is our response variable) can be accounted for or explained by differences between weights (which is our specific explanatory variables, i.e. the various allocations of weights to the banks stocks). Hence, up to 92% of the variability of the response data around its mean could be explained by this model.

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95.0%	Upper 95.0%
Intercept	2.51E-02	5.31E-04	4.74E+01	1.95E-298	2.41E-02	2.62E-02	2.41E-02	2.62E-02
GUARANTY	-5.55E-02	2.37E-02	-2.34E+00	1.94E-02	-1.02E-01	-8.99E-03	-1.02E-01	-8.99E-03
ZENITHBANK	-6.00E-02	2.33E-02	-2.58E+00	1.00E-02	-1.06E-01	-1.44E-02	-1.06E-01	-1.44E-02
NEM	3.38E-01	2.97E-03	1.14E+02	0.00E+00	3.32E-01	3.44E-01	3.32E-01	3.44E-01
SKYEBANK	3.17E-01	5.05E-03	6.27E+01	0.00E+00	3.07E-01	3.27E-01	3.07E-01	3.27E-01
STERLNBANK	-7.43E-02	1.74E-02	-4.28E+00	1.95E-05	-1.08E-01	-4.03E-02	-1.08E-01	-4.03E-02
WAPIC	1.94E-01	1.04E-02	1.86E+01	2.43E-69	1.73E-01	2.14E-01	1.73E-01	2.14E-01
WEMABANK	1.18E-01	1.15E-02	1.03E+01	5.91E-24	9.56E-02	1.41E-01	9.56E-02	1.41E-01

Table 6.3. Regression 2

The regression equation is thus: $\hat{Y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6 + \beta_7 x_7$ where $\hat{Y}$ is a risk that is, the response variable, β_0 is the intercept in table 6.3 and $\beta_i\,; i = 1,2,\dots,7$ are the coefficients of the various allocations of weights to the banks stocks on the table 6.3 above while $x_i\,; i = 1,2,\dots,7$ are the specific explanatory variables which is the various allocations of weights to the banks stocks on the table 6.3 .

Therefore, the regression model will be;

$$Risk \;=\; 0.0251 - 0.0555x_1 - 0.0600x_2 \;+\; 0.3377x_3 + \;0.3168x_4 - 0.0743x_5 \\ + \;0.1937x_6 \;+\; 0.1181x_7$$

Which implies that the predicted risk is;

$$Risk \;=\; 0.0251 - 0.0555(Guaranty) - 0.0600(Zenithbank) \;+\; 0.3377(Nem) \\ + \;0.3168(Skybank) - 0.0743(Sterlnbank) \;+\; 0.1937(Wapic) \\ + \;0.1181(Wemabank)$$

6.4.2 Analysis of the portfolio using VaRα and CVaRα.

To study the coupled risk, we need to compute the VaR_α and $CVaR_\alpha$ also the para, meters are estimated so as to select the type of copula we need.

From the definition of VaR_α above, we have that $P\,(X \leq VaR_\alpha | \Omega_{t-1}) = \alpha$ which implies that there is $(1-\alpha)\%$ confidence that loss in our chosen period will not be greater than VaR_α. We will note that while $CVaR_\alpha$ gives us an average expected loss, VaR_α gives a range of potential losses there by making $CVaR_\alpha$ to be more presided than VaR_α.

In this case, we calculated the VaR_α and $CVaR_\alpha$ of the nine (9) portfolios using the variance-covariance and historical simulation methods at $90\%, 95\%,$ and 99%. In the table 6.4 below, we have the variance, the average and standard deviation of each of the nine stocks in the portfolio while we show some of the calculations we did to get the different values of α for the variance-covariance and historical simulation methods at $90\%, 95\%,$ and 99% in the appendix.

Normal Dist VaR	CUSTODYINS	GUARANTY	NEM	SKYEBANK	STERLNBANK	TRANSCORP	WAPIC	WEMABANK	ZENITHBANK
Average	0.025%	0.033%	0.452%	0.184%	0.024%	0.099%	0.005%	0.018%	0.006%
Variance	0.097%	0.058%	3.196%	1.103%	0.096%	0.112%	0.260%	0.215%	0.061%
Standard D	3.116%	2.402%	17.878%	10.502%	3.098%	3.352%	5.100%	4.635%	2.461%

Table 6.4. The average, variance, and standard deviation of the portfolio.

Table 6.5 below shows the explicit figures of the variance-covariance and historical simulation of VaR_α and that of $CVaR_\alpha$ with the valves of α as 1%, 5%, and 10%. While the detailed calculation is in the appendix

		VaR			Historical	VaR		CVaR	
Stocks	1% VAR	5% VAR	10% VAR	1% VAR	5% VAR	10% VAR	1% CVAR	5% CVAR	10% CVAR
CUSTODYINS	-0.072	-0.051	-0.040	-0.084	-0.050	-0.047	-0.094	-0.062	-0.055
GUARANTY	-0.056	-0.039	-0.030	-0.051	-0.038	-0.024	-0.093	-0.056	-0.043
NEM	-0.411	-0.290	-0.225	-0.059	-0.044	-0.038	-0.078	-0.055	-0.047
SKYEBANK	-0.243	-0.171	-0.133	-0.052	-0.050	-0.042	-0.075	-0.055	-0.051
STERLNBANK	-0.072	-0.051	-0.042	-0.054	-0.049	-0.045	-0.083	-0.057	-0.052
TRANSCORP	-0.077	-0.054	-0.065	-0.099	-0.049	-0.042	-0.103	-0.068	-0.057
WAPIC	-0.119	-0.084	-0.059	-0.095	-0.050	-0.046	-0.101	-0.068	-0.058
WEMABANK	-0.108	-0.076	-0.059	-0.095	-0.050	-0.047	-0.101	-0.065	-0.056
ZENITHBANK	-0.057	-0.040	-0.031	-0.053	-0.043	-0.029	-0.086	-0.056	-0.045

Table 6.5. The variance-covariance and historical simulation of VaR_α and that of $CVaR_\alpha$

	The Student's t-copula		The Gaussian copula		The Student's t-copula		The Gaussian copula		The Student's t-copula		The Gaussian copula	
Stocks	CVaR 1%	VaR 1%	CVaR 1%	VaR 1%	CVaR 5%	VaR 5%	CVaR 5%	VaR 5%	CVaR 10%	VaR 10%	CVaR 10%	VaR 10%
CUSTODYINS	0.773	0.683	0.776	0.687	0.746	0.644	0.739	0.643	0.740	0.647	0.708	0.641
GUARANTY	0.642	0.546	0.645	0.549	0.609	0.508	0.607	0.513	0.610	0.514	0.576	0.511
NEM	0.773	0.680	0.778	0.687	0.741	0.641	0.732	0.641	0.738	0.646	0.701	0.639
SKYEBANK	0.656	0.558	0.655	0.558	0.628	0.525	0.626	0.527	0.626	0.529	0.595	0.525
STERLNBANK	0.751	0.655	0.759	0.665	0.734	0.629	0.724	0.627	0.735	0.637	0.693	0.625
TRANSCORP	0.883	0.796	0.884	0.800	0.894	0.806	0.874	0.785	0.879	0.792	0.843	0.783
WAPIC	0.864	0.779	0.866	0.781	0.857	0.763	0.847	0.757	0.850	0.761	0.816	0.755
WEMABANK	0.782	0.684	0.790	0.696	0.776	0.675	0.764	0.668	0.767	0.675	0.733	0.666
ZENITHBANK	0.712	0.608	0.714	0.613	0.664	0.554	0.654	0.550	0.663	0.559	0.623	0.548

Table 6.6. the Student's t copula and Gaussian copula of VaR and CVaR

In table 6.6, we displayed the result we got from the portfolio return of VaR and CVaR with $99\%, 95\%$ and 90% confidence levels. The result shows that a 99% confidence level is higher than 95%, and 95% is bigger than the 90% confidence level in each of the VaR and CVaR.

Secondly, it is observed that CVaR is bigger than the corresponding VaR in all confidence level. This shows that CVaR captures more information that VaR about the tail of the distribution. Finally, the Student's t copula gives us better approximation than that of Gaussian copula.

6.4.2 Normal, Historical VaR and Expected Shortfall

Here we analysed the risk using the normal and historical VaR and expected shortfall. Using the closing prices of the stock we studied, we first calculated the return, (which we have gotten in the previous calculations). After this, we calculate the mean and standard deviation of each stock return see the table below.

Normal Dist	CUSTODYINS	GUARANTY	NEM	SKYEBANK	STERLNBANK	TRANSCORP	WAPIC	WEMABANK	ZENITHBANK
Average	0.025%	0.033%	0.452%	0.184%	0.024%	0.099%	0.005%	0.018%	0.006%
Variance	0.097%	0.058%	3.196%	1.103%	0.096%	0.112%	0.260%	0.215%	0.061%
Standard D	3.116%	2.402%	17.878%	10.502%	3.098%	3.352%	5.100%	4.635%	2.461%

Table 6.6 the mean and standard deviation

In our analysis, we looked at the 5% and 10% VaR from the bottom cases and the results are displayed in the table below.

VAR(%)	CUSTODYINS	GUARANTY	NEM	SKYEBANK	STERLNBANK	TRANSCORP	WAPIC	WEMABANK	ZENITHBANK
Bottom 10%	-3.97%	-3.05%	-22.47%	-13.28%	-3.95%	-4.20%	-6.53%	-5.92%	-3.15%
Bottom 5%	-5.10%	-3.92%	-28.97%	-17.10%	-5.07%	-5.42%	-8.39%	-7.61%	-4.04%

Table 6.7 the 5% and 10% VaR from the bottom

The table above shows that we have 90% and 95% confidence that the loss on each stock in our portfolio will not exceed the percentages displayed on the table respectively. We need to compare the result with the historical VaR at 5% and 10% as we did earlier.

Again, we look at $1\%, 5\% \, and \, 10\%$ bottom cases; this will give $72.2nd$ and $148.4th$ respectively from the bottom. Remember that the number of our observation is 1484; therefore the 1%, 5% and 10% is gotten through interpolation of the numbers $14th$ and $15th$, $72nd$ and $73rd$ and also $148th$ and $149th$ respectively. This is shown on the tables below.

VAR(%)	CUSTODYINS	GUARANTY	NEM	SKYEBANK	STERLNBANK	TRANSCORP	WAPIC	WEMABANK	ZENITHBANK
Bottom 10%	-0.0470	-0.0239	-0.0377	-0.0424	-0.0451	-0.0419	-0.0458	-0.0465	-0.0290
Bottom 5%	-0.0510	-0.0392	-0.0445	-0.0499	-0.0490	-0.0494	-0.0504	-0.0496	-0.0426
Bottom 1%	-0.0836	-0.0513	-0.0590	-0.0523	-0.0544	-0.0994	-0.0953	-0.0946	-0.0530

Table 6.8 The Historical VaR

The table above gives us 1%, 5% and 10% historical data while the table below gives the corresponding CVaR

CVAR	CUSTODYINS	GUARANTY	NEM	SKYEBANK	STERLNBANK	TRANSCORP	WAPIC	WEMABANK	ZENITHBANK
CVAR 10%	-0.055480896	-0.043293711	-0.047166861	-0.05121148	-0.051992646	-0.056684287	-0.057956594	-0.056223692	-0.045458107
CVAR 5%	-0.062177852	-0.055877842	-0.054659744	-0.055243022	-0.05667699	-0.067868344	-0.068285723	-0.064573281	-0.056000174
CVAR 1%	-0.094164487	-0.093250599	-0.078011283	-0.074843702	-0.082775929	-0.103236006	-0.101395322	-0.101228439	-0.085569261

Table 6.8 the corresponding CVaR

We look at the values from the 1%, 5% and 10% historical data on the table above and compare them with the corresponding 1%, 5% and 10% values we got from the returns. We notice that the values are different, which means that the distribution may not be a normal distribution as thought. So, we look at the statistical distribution of the return. This is shown on the tables below for each stock.

	GUARANTY	ZENITHBANK	NEM	SKYEBANK	STERLNBANK	TRANSCORP	CUSTODYINS	WAPIC	WEMABANK
Mean	3.35E-04	5.84E-05	4.52E-03	1.84E-03	2.35E-04	9.87E-04	2.53E-04	4.70E-05	1.82E-04
Standard Error	6.24E-04	6.39E-04	4.64E-03	2.73E-03	8.05E-04	8.71E-04	8.10E-04	1.32E-03	1.20E-03
Median	0	0	0	0	0	0	0	0	0
Mode	0	0	0	0	0	0	0	0	0
Standard Deviation	2.40E-02	2.46E-02	1.79E-01	1.05E-01	3.10E-02	3.35E-02	3.12E-02	5.10E-02	4.64E-02
Sample Variance	5.77E-04	6.06E-04	3.20E-02	1.10E-02	9.60E-04	1.12E-03	9.72E-04	2.60E-03	2.15E-03
Kurtosis	1.92E+01	7.94E+00	1.42E+03	1.27E+03	-1.79E-01	1.12E+00	1.41E-01	4.97E+02	2.79E+02
Skewness	-1.52E+00	-6.05E-01	3.72E+01	3.44E+01	-1.31E-02	1.63E-01	-1.55E-01	1.70E+01	1.10E+01
Range	3.59E-01	3.45E-01	6.90E+00	3.99E+00	2.01E-01	2.03E-01	2.01E-01	1.60E+00	1.28E+00
Minimum	-2.62E-01	-2.48E-01	-9.66E-02	-1.00E-01	-1.05E-01	-1.05E-01	-1.05E-01	-1.05E-01	-1.05E-01
Maximum	9.74E-02	9.72E-02	6.81E+00	3.89E+00	9.53E-02	9.76E-02	9.67E-02	1.50E+00	1.18E+00
Sum	4.96E-01	8.67E-02	6.70E+00	2.74E+00	3.49E-01	1.46E+00	3.75E-01	6.96E-02	2.70E-01
Count	1483	1483	1483	1483	1483	1483	1483	1483	1483
P-value	1.80E-02	6.65E-03	0.00E+00	0.00E+00	2.06E-05	7.14E-01	7.73E-01	5.56E-69	1.80E-21
t Stat	-2.37E+00	-2.72E+00	1.13E+02	6.20E+01	-4.27E+00	-3.67E-01	2.89E-01	1.85E+01	9.67E+00
Coefficients	-5.08E-02	-5.73E-02	3.03E-01	2.84E-01	-6.72E-02	-5.26E-03	4.46E-03	1.75E-01	1.01E-01

Table 6.9 the statistical distribution of the return

6.5 Conclusion

It is always good to estimate the risk of a portfolio correctly so that the investor will do a proper allocation of the funds to yield an optimal return. The relationship among the prices of the stocks in the portfolio, otherwise called the correlation plays an important role in determining the movement of the stock and therefore helps in the computation of risk.

Chapter 7 Summary and Conclusion

7.0 Introduction

In this chapter, we present a summary of our work and also bring out explicitly the contributions to knowledge. We try to harmonise chapters 4, 5 and 6.

7.1 Correlation of implied volatility of the portfolio.

As we have seen in chapter 5, correlations among different assets in the market are beneficial not only in portfolio selection but also, in the option pricing and certain multivariate econometric models for pricing forecasting and volatility estimate see Engle and Figlewski (2014). Therefore, a perfect correlation among stocks in a portfolio is significant in quantifying the risk of the portfolio, option pricing and forecasting.

The variance ρ of a portfolio of options exposed to Vega risk only is defined by Black-Scholes (1973) as

$$Var(\rho) = \sum_{i,j,k,l} \frac{w_i w_l \Lambda_{ij} \Lambda_{lk} C_{jk}}{v_j v_k \sigma_j \sigma_k} \tag{7.1}$$

where w_i are the weights of the portfolio, C_{ij} is correlation coefficient between assets i and j in the implied volatility matrix and the Vega matrix has $ij-th$ elements Λ_{ij} which is defined as

$$\Lambda_{ij} = \frac{\partial p_i}{\partial v_j} \tag{7.2}$$

Where p_i the price of the option is i, v_j is the implied volatility of asset underlying option j and σ_i is the standard deviation of the implied volatility v_i.

According to the Chicago Board options exchange (2009), the variance of a portfolio consisting of n assets can be calculated as

$$\sigma_{port}^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} C_{ij} w_i w_j \, \sigma_i \sigma_j \tag{7.3}$$

where σ_{port} is standard deviation or volatility of the portfolio, σ_i, σ_j are the standard deviations or volatility of the i and j assets, w_i, w_j are weights of i and j assets respectively, and C_{ij} is the correlation coefficient between i and j assets.

Kawee and Nattachai Numpacharoen (2013) has it that if the relationship among assets in the portfolio is described using equi-correlation, then the variance of the portfolio will be written as

$$\sigma_{port}^2 = \sum_{1=1}^{N} w_i^2 \sigma_i^2 + 2r \sum_{i=1}^{N-1} \sum_{j>1}^{N} w_i w_j \sigma_i \sigma_j \qquad (7.4)$$

This implies that

$$r = \frac{\sigma_{port}^2 - \sum_{i=1}^{n} w_i^2 \sigma_i^2}{2 \sum_{i=1}^{N-1} \sum_{j>1}^{N} w_i w_j \sigma_i \sigma_j} \qquad (7.5)$$

If we denote the implied volatility of the portfolio as σ_{port}^Q and σ_i^Q, σ_j^Q as the implied volatility of the i and j assets respectively, then equation (7.5) can be rewritten as

$$r^Q = \frac{(\sigma_{port}^Q)^2 - \sum_{i=1}^{n} w_i^2 (\sigma_i^Q)^2}{2 \sum_{i=1}^{N-1} \sum_{j>1}^{N} w_i w_j \sigma_i^Q \sigma_j^Q} \qquad (7.6)$$

r^Q is a real constant such that $-1 \leq r \leq 1$.

Furthermore, it was evident that $w_i, w_j, \sigma_i, \sigma_j$ and σ_{port} are all non-negative numbers, then the terms $\sum_{i=1}^{n} w_i^2 (\sigma_i^Q)^2$ and $\sum_{i=1}^{N-1} \sum_{j>1}^{N} w_i w_j \sigma_i^Q \sigma_j^Q$ in the equation (7.6) is always positive. Therefore, if $w_i, w_j, \sigma_i,$ and σ_j remain constant, σ_{port} will be directly proportional to r^Q, higher σ_{port} gives higher r and lower σ_{port} gives lower r^Q. r^Q is called the implied correlation coefficient of the equi-correlation matrix.

Kawee and Nattachai Numpacharoen (2013) described R^Q as the realistic implied correlation matrix while R^p is a valid correlation matrix. Then the implied volatility of the portfolio is

$$(\sigma_{port}^Q)^2 = W * V^Q * R^Q * V^Q * W' \qquad (7.7)$$

for a case where correlation among the assets is not identical. W is the weight of the individual asset in the portfolio.

$$W = [w_1 \quad \cdots \quad w_n]'$$

V^Q is the diagonal matrix got from the implied standard deviation of the individual assets being considered.

$$V^Q = \begin{bmatrix} \sigma_1^Q & 0 \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n^Q \end{bmatrix}$$

R^Q is the desired realistic implied correlation matrix which will be obtained from R^p.

$$R^Q = \begin{bmatrix} 1 & R_{2,1}^Q \cdots R_{n-1,1}^Q & R_{n,1}^Q \\ \vdots & \ddots & \vdots \\ R_{n,1}^Q & R_{n,2}^Q \cdots R_{n-1,n}^Q & 1 \end{bmatrix}$$

R^p is a valid correlation matrix (empirical) see Buss and Vilkov (2012).

Kawee and Nattachai Numpacharoen (2013) set the realistic correlation coefficient to be $r^Q = r^p - \varphi(1 = r^p)$ for $\varphi \in (-1, 0]$. Furthermore, they let I_{nxn} be an nxn matrix whose entries are 1, this implies that the realistic matrix can be written as

$$R^Q = R^p - \varphi(I_{nxn} - R^p) \tag{7.8}$$

where R^Q is a function of R^p with φ as a parameter to be identified, note that I_{nxn} is not an identity matrix. To obtain φ, we substitute equation (8) into equation (7) and this becomes;

$$(\sigma_{port}^Q)^2 = W * V^Q * [R^p - \varphi(I_{nxn} - R^p)] * V^Q * W'$$

$$(\sigma_{port}^Q)^2 = W * V^Q * R^p * V^Q * W' - \varphi(I_{nxn} - R^p)(W * V^Q * V^Q * W')$$

$$(\sigma_{port}^Q)^2 - W * V^Q * R^p * V^Q * W' = -\varphi(W * V^Q * (I_{nxn} - R^p) * V^Q * W')$$

This implies that φ is

$$\varphi = -\frac{(\sigma_{port}^Q)^2 - W * V^Q * R^p * V^Q * W'}{W * V^Q * (I_{nxn} - R^p) * V^Q * W'} \tag{7.9}$$

We can find R^Q from equation (8) once we have obtained φ. Also, the implied volatility of the portfolio σ_{port}^{p} can be from R^p thus,

$$(\sigma_{port}^{p})^2 = W * V^Q * R^p * V^Q * W' \quad (7.10)$$

Therefore,

$$\sigma_{port}^{p} = \sqrt{W * V^Q * R^p * V^Q * W'} \quad (7.11)$$

When $\varphi$is positive, that is, consider the cases when $\varphi > 0$, this occurs when $\sigma_{port}^{P} > \sigma_{port}^{Q}$ since$(I_{nxn} - R^P) \geq 0$. This could make the realistic correlation matrix R^Q to be invalid when $\sigma_{port}^{P} < \sigma_{port}^{Q}$.

They propose a formula for a valid correlation matrix that will take care of this shortcoming, as stated below. Given any two valid correlation matrices B and D of dimensions nxn then, there exists a convex valid correlation matrix F of the same dimension such that $F = wD + (1-w)B$ with $0 \leq w \leq 1$as the weight. If I_{nxn} and L_{nxn} are corresponding equi-correlation matrices entries are 1 and $-\frac{1}{n-1}$ for $i \neq j$ and1 for $i = j$ respectively, I_{nxn} and L_{nxn} upper equicorrelation matrix and lower equi-correlation matrix respectively. Replacing F and B in equation F with R^Q and by R^p, we will obtain;

$$R^Q = wD + (1-w)R^P \quad (7.12)$$

This implies that

$$R^Q = wD + (1-w)R^P$$

$$= wD + R^P - wR^P$$

$$= R^P + wD - wR^P$$

$$= R^P + w(D - R^P)$$

Therefore,

$$R^Q = R^P + w(D - R^P) \quad (7.13)$$

But finding w implies;

$$R^Q - R^P = +w(D - R^P)$$

so;

$$w = \frac{R^Q - R^P}{D - R^P}$$

Therefore, substituting for the weight in equation (7) is;

$$w = \frac{(\sigma_{port}^{Q})^2 - (\sigma_{port}^{P})^2}{WV^Q(D - R^P)V^QW'} \tag{7.14}$$

This is useful in assigning different weights to different assets in our portfolio as seen in the example below, which will help in the maximisation and minimisation of the returns and the risk of the portfolio, respectively.

We, therefore, use a practical demonstration to obtain the correlation matrix from some of the assets we studied in the Nigerian stock market.

7.2 Empirical Example

In this section, we use the correlation matrix obtained from 20 of the 82 NSM stocks considered in chapter 5. These assets are7UP, ABC transport, Access Bank, AgLevent, AIICO Insurance, Air service, Ashaka Cement, Julius Berger, Cadbury Nigeria Plc, CAP, CCNN, Cileasing, Conoil, Contisure, Cornerstone, Costain Construction, Courtvile, Custodian, Cutix Cables and Dangote Cement.

We, therefore, want to compute the realistic empirical correlation matrix for some assets already considered in the RMT as below (see Table 7.1 below):

	A7UP	ABCTRANS	ACCESS	AGLEVENT	AIICO	AIRSERVIC	ASHAKACI	BERGER	CADBURY	CAP	CCNN	CILEASING	CONOIL	CONTINSU	CORNERST	COSTAIN	COURTVIL	CUSTODYI	CUTIX	DANGCEM
A7UP	1	-0.05084	-0.00262	0.00322	-0.00143	0.01566	-0.0029	0.041205	0.035481	0.014103	0.015787	0.010815	0.009635	0.028702	0.033092	-0.01957	-0.01251	-0.02231	0.008773	0.040708
ABCTRANS	-0.05084	1	0.056511	0.107324	0.026388	-0.00063	0.057519	-0.03212	0.016378	0.046174	0.054952	-0.00091	-0.02445	0.040527	0.027983	-0.01186	-0.03699	-0.0094	-0.05143	-0.01032
ACCESS	-0.00262	0.056511	1	0.041798	0.165593	0.005204	0.096497	0.054031	0.134552	0.03997	0.175307	0.055145	-0.04035	0.062478	-0.01304	0.062428	0.016785	0.01893	-0.04865	0.037644
AGLEVENT	0.00322	0.107324	0.041798	1	0.026699	0.001187	0.061421	0.009062	0.040987	0.019843	0.04584	0.009799	0.000529	0.005	0.031901	0.003182	0.015911	0.052039	0.00793	0.009541
AIICO	-0.00143	0.026388	0.165593	0.026699	1	-0.0073	0.079441	-0.06684	0.034305	0.035806	0.127068	0.084493	0.013289	0.052223	0.011167	0.079949	-0.03631	0.016342	0.039694	0.000341
AIRSERVIC	0.01566	-0.00063	0.005204	0.001187	-0.0073	1	0.013793	0.01008	0.019304	0.027947	0.014157	0.008397	-0.00943	-0.01882	0.037871	0.03219	0.024177	0.0165	-0.02333	-0.02294
ASHAKACI	-0.0029	0.057519	0.096497	0.061421	0.079441	0.013793	1	0.040604	0.131813	-0.01865	0.136209	0.023381	0.068711	0.041732	0.024245	0.033759	-0.05434	0.062872	-0.00385	0.051222
BERGER	0.041205	-0.03212	0.054031	0.009062	-0.06684	0.01008	0.040604	1	0.004316	-0.05637	-0.01496	-0.00019	0.003384	-0.02062	0.031925	0.001533	0.027304	0.002867	0.01409	0.045171
CADBURY	0.035481	0.016378	0.134552	0.040987	0.034305	0.019304	0.131813	0.004316	1	0.039896	0.06141	-0.02738	0.044002	-0.05896	-0.01341	0.078438	-0.00591	0.003203	0.006094	-0.00317
CAP	0.014103	0.046174	0.03997	0.019843	0.035806	0.027947	-0.01865	-0.05637	0.039896	1	0.032908	0.040034	-0.02318	-0.01104	0.011431	0.03451	0.021587	-0.00672	-0.02004	0.084764
CCNN	0.015787	0.054952	0.175307	0.04584	0.127068	0.014157	0.136209	-0.01496	0.06141	0.032908	1	0.049229	-0.06661	0.024031	-0.00524	0.040984	0.007728	0.065341	0.001566	0.030894
CILEASING	0.010815	-0.00091	0.055145	0.009799	0.084493	0.008397	0.023381	-0.00019	-0.02738	0.040034	0.049229	1	0.042932	0.032846	0.030753	0.00305	-0.01163	0.075287	-0.00421	0.037988
CONOIL	0.009635	-0.02445	-0.04035	0.000529	0.013289	-0.00943	0.068711	0.003384	0.044002	-0.02318	-0.06661	0.042932	1	0.017264	0.018679	-0.01089	-0.07688	0.059827	0.00496	0.041872
CONTINSU	0.028702	0.040527	0.062478	0.005	0.052223	-0.01882	0.041732	-0.02062	-0.05896	-0.01104	0.024031	0.032846	0.017264	1	0.07067	0.022625	0.008551	-0.08108	0.007992	0.008142
CORNERST	0.033092	0.027983	-0.01304	0.031901	0.011167	0.037871	0.024245	0.031925	-0.01341	0.011431	-0.00524	0.030753	0.018679	0.07067	1	0.016227	-0.01527	0.013627	-0.04751	0.011117
COSTAIN	-0.01957	-0.01186	0.062428	0.003182	0.079949	0.03219	0.033759	0.001533	0.078438	0.03451	0.040984	0.00305	-0.01089	0.022625	0.016227	1	-0.03288	0.030068	-0.02884	-0.0262
COURTVIL	-0.01251	-0.03699	0.016785	0.015911	-0.03631	0.024177	-0.05434	0.027304	-0.00591	0.021587	0.007728	-0.01163	-0.07688	0.008551	-0.01527	-0.03288	1	-0.00295	0.009347	0.006857
CUSTODYI	-0.02231	-0.0094	0.01893	0.052039	0.016342	0.0165	0.062872	0.002867	0.003203	-0.00672	0.065341	0.075287	0.059827	-0.08108	0.013627	0.030068	-0.00295	1	-0.00166	0.027586
CUTIX	0.008773	-0.05143	-0.04865	0.00793	0.039694	-0.02333	-0.00385	0.01409	0.006094	-0.02004	0.001566	-0.00421	0.00496	0.007992	-0.04751	-0.02884	0.009347	-0.00166	1	0.079732
DANGCEM	0.040708	-0.01032	0.037644	0.009541	0.000341	-0.02294	0.051222	0.045171	-0.00317	0.084764	0.030894	0.037988	0.041872	0.008142	0.011117	-0.0262	0.006857	0.027586	0.079732	1

Table7.1: Empirical correlation matrix from NSM price return

The weights W and implied volatility V^Q computed from option prices for the listed assets which we studied from the NSM. To carry out this assignment, we create hypothetical or assumed weights and implied volatilities are represented as

$$W = \begin{bmatrix} .05;\ .08;\ .01;\ .04;\ .03;\ .06;\ .01;\ .03;\ .05;\ .07;\ .02;\ .04;\ .02; \\ .07;\ .09;\ .04;\ .02;\ .07;\ .12;\ .08 \end{bmatrix}$$

and their corresponding implied volatility represented by

$$V^Q = \begin{matrix} [.36; .26; .30; .10; .15; .20; .25; .40; .19; .24; .38; \\ .27; .10; .22; .21; .40; .28; .30; .16; .29] \end{matrix}$$

The empirical correlation matrix has its eigenvalues as [0.71; 0.79; 0.81; 0.82; 0.83; 0.86; 0.89; 0.92; 0.95; 0.95; 0.98; 0.99; 1.06; 1.094; 1.11; 1.12; 1.15; 1.16; 1.121; 1.60].

Thus, the minimum eigenvalue of $R^p = 0.071$ which shows that R^p is a valid correlation matrix.

Therefore, to estimate the realistic implied correlation matrix R^Q from the assumed, implied volatility for the given portfolio consisting of twenty assets, we assume that the implied volatility of the portfolio $\sigma^{Q}_{port} = 0.04$. If the weight W, then the following values, $R^{\,p}$and V^{Q} will be used to solve for $(\sigma^{p}_{port})^2$ in the equation (10). But

$$\sigma^{p}_{port} = \sqrt{(W * V^{Q} * R^{p} * V^{Q} * W')}$$

$$\sigma^{p}_{port} = 0.0361$$

Remember that our V^{Q} is;

0.26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0.16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0.3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0.21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0.15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0.23	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0.36	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0.24	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0.19	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0.24	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0.18	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0.17	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0.21	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0.29	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.32	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.25	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.16	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.23	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.18	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.14

To this effect we will replace D in equation (14) with the nxn matrix whose entries are1; I_{nxn}. Since 0.04 is greater than 0.0361, this implies that $\sigma^{Q}_{port} > \sigma^{p}_{port}$.

$$w = \frac{(\sigma^{Q}_{port})^2 - (\sigma^{P}_{port})^2}{W * V^{Q} * (I_{20x20} - R^{P}) * V^{Q} * W'}$$

$$= \frac{0.0016 - 0.001303}{W * V^{Q} * (I_{20x20} - R^{P}) * V^{Q} * W'}$$

$$= \frac{0.0003}{-6.5737e^{-04}}$$

$$= -0.4564$$

The eigenvalues of $R^{Q} = [.57; .69; .72; .74; .76;\ .80;\ .84;\ .89;\ .93;\ .93;\ .96;$

$.98; 1.09; 1.14; 1.16; 1.18; 1.22; 1.23; 1.30; 1.98]$ from where we obtain the minimum eigenvalue to be 0.57 showing that R^{Q} is also a positive semi-definite. We now verify our solution from equation (8) to compute the variance of the portfolio using the obtained realistic implied correlation matrix R^Q gotten above:

The implied correlation matrix is applicable in hedging risks associated with foreign exchange. Large corporations are always very interested in hedging their currency exposures by using a basket of options instead of taking separate put options for the respective countries where they have their investments Bensman (1997). This will help guard against the unnecessary losses they might incur in the event of the rising value of the domestic currency where they have their investments. Companies that are therefore exposed to a variety of currency fluctuations find it profitable to directly hedge their aggregate risk by using a basket of options made possible through the use of estimated implied correlation matrix in the form of a basket of options H Krishnan and I Nelken (2001). For a manufacturing firm in the United States that sources its raw materials in Nigeria, Ghana, and South Africa and pays for its operation in those countries in local currencies will be exposed to exchange rate risk. To hedge against the risk of falling United States dollars against Naira, Cedi, and Rand, the manufacturing firm has to use a basket of option in its risk management strategy. The company could, therefore, directly buy an option on a basket of currencies at a lower price than it can purchase through separate options on the individual currencies. This is possible through the use of historical return time series correlation, as we have done with some stocks in the NSM, and the major concern will then be the amount of weight to be assigned to the individual stocks (or currencies). For optimal portfolio on the investment, we need to predict the future correlation of the respective option values correctly by observing the correlation throughout the life span of the option.

7.3 Kurtosis and Skewness

Looking at data on the tables, we find out that the Kurtosis and the Skewness both are not zero in each of the stocks in our portfolio. We now look at the frequency distribution of the individual stock.

The histogram can give you a general idea of the shape, but two numerical measures of shape (skewness and kurtosis) provide a more precise evaluation. A frequency distribution is said to be skewed when its mean and median are different, or more generally when it is asymmetric. The kurtosis of a frequency distribution is the concentration of scores at the mean, or how the distribution peaked appears if depicted graphically.

Skewness tells you the amount and direction of skew (departure from horizontal symmetry), and kurtosis tells you how tall and sharp the central peak is, relative to a standard bell curve. Many statistics inferences require that a distribution be normal or nearly normal. A normal distribution has skewness and excess kurtosis of 0, so if your distribution is close to those values, then it is probably close to normal.

If skewness is positive, the data are positively skewed or skewed right, meaning that the right tail of the distribution is longer than the left. If skewness is negative, the data are negatively skewed or skewed left, meaning that the left tail is longer.

Variance is a measurement of the spread between numbers in a data set. The variance measures how far each number in the set is from the mean. Variance is used in statistics for a probability distribution. Since variance measures the variability (volatility) from an average or means and volatility is a measure of risk, the variance statistic can help determine the risk an investor might assume when purchasing a specific security. A variance value of zero indicates that all values within a set of numbers are identical; all non-zero variances will be positive numbers. A significant deviation indicates that numbers in the set are far from the mean and each other, while a small variance indicates the opposite.

The standard error is an indication of the reliability of mean. A small value of the standard error indicates that the mean is a more accurate reflection of the actual population mean – implying that it could be used as an accurate representation of the population.

The median is the value of the 50% or half population (observation).

7. 4 Conclusion and hints on future work

The analysis of the correlation and structure of stock market returns has provided the necessary data for a hypothetical analysis of implied correlation in the NSM, carried out in this research using the concept of RMT. This foregrounds research on comparative derivative pricing in the NSM, especially on foreign exchange futures. Marcenko-Pastur eigenvalue distribution predicted that the theoretical eigenvalues should be in the range of 0.52 and 1.65 for NSM. It was observed that 6 out of 82 stocks considered, have their corresponding eigenvalues lie outside this theoretical bound of eigenvalues. Therefore, 89% of the information from the return distributions is purely random thereby leaving us with the alternative hypothesis of the RMT which states that the information on the market lies on the deviating eigenvalues which imply then that for NSM the actual market characteristic lies with only 11% of the stocks considered.

As stated earlier, these correlation matrices contain some relevant information for options pricing and hedging, J. Hull (1997). The realistic implied correlation matrix R^Q has positive coefficients meaning that the respective stock move in the same direction hence the diversification method in the portfolio is not an optimal portfolio strategy. It is; therefore, better to invest in some derivative products like call and or put option to hedge against the risk on the portfolio for the hypothetical weight and implied volatility used in the estimated implied correlation matrix.

The eigenvectors associated with the essential eigenvalues$[\lambda_{min}, \lambda_{max}]$are non-normal and highly asymmetric; this suggests the presence of market signals such as depicted also in the information-carrying stocks. The fact that the first eigenvalue represents the market (that is, it contains the most information of the market), it also suggests that the Nigerian Stock Market (NSM) is inefficient and contains a lot of noise. The positive asymmetry in the distribution of eigenvectors of λ_1 shows that, the NSM reacts more strongly to positive variations than negative variations. It shows a fairly even spread which suggest that there are no dominant stocks in among the 82 stocks we studied. Therefore, we advised the investors and the market players who may want to control the risk of their portfolios to select most of their stocks in the portfolio from the same sector of the market due to the nature of NSM. This is what we demonstrated in chapter six.

We also noted that the concept of implied correlation could be used in options trading and hedging the risks associated with the portfolio of investment, including the use of a basket of options in hedging foreign exchange risk. In this regard, it is, therefore, open to further work for the investigation of the use of the basket of an option to hedge against exchange rate risk especially in emerging markets like Nigeria and South Africa. As Nigeria has already started trading on foreign exchange futures, it is pertinent therefore to note that soon market data will be available for the empirical application of the concept of the implied correlation matrix for the hedging of exchange risk in Nigeria.

The risks of any portfolio should be calculated accurately so that the investor can manage his investment adequately. We intend to help the investor maximise the expected returns for the targeted values of VaR/CVaR. The correlation among the prices of the stocks in the portfolio plays a vital role in the proper estimation of VaR/CVaR. When there is a non-linear arrangement, it becomes difficult to capture the risk very well with the correlation alone. Therefore, the use of copula functions becomes necessary to capture those non-linear and the fat-tailed dependencies. Our analysis shows that the Student t copula provides a better approximation to describe the joint distribution of pairs of stocks in a portfolio better than Gaussian copula in portfolio optimisation.

Future works

In our proposals for future work, we want to study in details some of the African emerging markets like South Africa and Morocco markets, look at the similarities and find the best portfolio selection/combination of assets from those markets studied.

Nigeria has not included the financial derivatives in her market, so we include the development of financial derivatives in the NSM using the information from NSM-JSE market affinities, which was foreshadowed in this work.

We will apply and verify Fernandez et al. (2016) claims on multivariate lower and upper dependence coefficient to analyse the relationship among pairwise, mutual and extremal tail independence, using a selected number of stocks from NSM and other African emerging markets hopefully.

Finally, we will use the principles of optimality for uncertain optimal control to solve our portfolio selection problem.

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Appendix

5.0 Distribution of Eigenvalues

Remember that the probability density function $P_{rm}(\lambda)$ of eigenvalues λ of the random correlation matrix, R is given by equation (5.21) in chapter 5 and its stated thus,

$$P_{rm}(\lambda) = \frac{Q}{2\pi\sigma^2}\frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{\lambda}$$

For λ within the bounds $\lambda_- \leq \lambda_i \leq \lambda_+$, that is $[\lambda_+, \lambda_-]$ where λ_- and $\lambda_+$are the minimum and maximum eigenvalues of R respectively, and this given by equation (5.22) in chapter 5 and its stated thus,

$$\lambda_\pm = \sigma^2\left(1 + \frac{1}{Q} \pm 2\sqrt{\frac{1}{Q}}\right)$$

To solve for the $\lambda_\pm$ in the above equation, we need to first solve for Q in that equation.

But $Q = \frac{L}{N}$, where $N = 82$ stocks we analysed from NSE and $L = 1019$ is the total of daily closing prices of those stocks. Therefore,

$$Q = \frac{1019}{82} = 12.426$$

This implies $Q = 12.4$, then $\frac{1}{Q} = 0.0806$ which is the reciprocal of Q. To get the eigenvalue bound $[\lambda_+, \lambda_-]$, we substitute for Q in the equation above having in mind that σ^2 is unitary. Thus,

$$\lambda_\pm = \sigma^2\left(1 + \frac{1}{Q} \pm 2\sqrt{\frac{1}{Q}}\right)$$

$$= 1^2\left(1 + \frac{1}{12.4} \pm 2\sqrt{\frac{1}{12.4}}\right)$$

$$= \left(1 + 0.0806 \pm 2\sqrt{0.0806}\right)$$

$$= (1.0806 \pm 2(0.28398))$$

Therefore, $\lambda_+ = 1.0806 + 0.56796$ or $\lambda_- = 1.0806 - 0.56796$. Thus $\lambda_+ = 1.64856$ and $\lambda_+ = 0.51264$.

Now, with the eigenvalue bound $[\lambda_+, \lambda_-]$, we can get the eigenvalue distribution by ploting the density $P(\lambda_i)$ against its value, where $i = 1, 2, \ldots, 82$ and λ_i is the eigenvalues from Empirical, theoretical and the cleaned up matrices.

The distribution of the eigenvalues is figure $5a$ and $5b$ in chapter 5. Tables below are the collection of the eigenvalues of Empirical, theoretical and the cleaned-up matrices.

	The Empirical eignevalue s from the empirical Matrix with the 82 stocks studied								
1	2	3	4	5	6	7	8	9	10
0.462222	0.473459	0.488408	0.49809	0.511803	0.543933	0.55661	0.562031	0.568181	0.578575
11	12	13	14	15	16	17	18	19	20
0.590499	0.609337	0.611057	0.627254	0.636702	0.649487	0.659342	0.66186	0.677746	0.686433
21	22	23	24	25	26	27	28	29	30
0.712838	0.715223	0.72883	0.736066	0.748083	0.756257	0.768498	0.771428	0.779155	0.788935
31	32	33	34	35	36	37	38	39	40
0.79658	0.807143	0.825354	0.831951	0.842357	0.855395	0.861913	0.867174	0.879861	0.904011
41	42	43	44	45	46	47	48	49	50
0.910028	0.914347	0.933889	0.941113	0.951644	0.967685	0.987744	0.993686	1.008948	1.016445
51	52	53	54	55	56	57	58	59	60
1.026461	1.03542	1.050241	1.064123	1.082887	1.088031	1.10157	1.116517	1.122921	1.135287
61	62	63	64	65	66	67	68	69	70
1.160098	1.173476	1.18679	1.229139	1.252345	1.264717	1.289052	1.29836	1.314655	1.333823
71	72	73	74	75	76	77	78	79	80
1.346487	1.362685	1.375556	1.389884	1.420827	1.476302	1.508283	1.551156	1.638806	1.675504
81	82								
1.797996	4.87499								

Table 5a

	The theorrtical eigenvalues from the Random Matrix with the 82 stocks studied								
1	2	3	4	5	6	7	8	9	10
0.513932	0.532032	0.567023	0.568798	0.590938	0.605732	0.623991	0.636007	0.640739	0.648842
11	12	13	14	15	16	17	18	19	20
0.655356	0.674591	0.679524	0.692426	0.710797	0.717694	0.722189	0.728217	0.736619	0.755268
21	22	23	24	25	26	27	28	29	30
0.758787	0.765712	0.775475	0.776365	0.798904	0.801509	0.818156	0.828238	0.847754	0.850238
31	32	33	34	35	36	37	38	39	40
0.856627	0.862422	0.877382	0.88091	0.891943	0.9054	0.92724	0.934375	0.939671	0.945271
41	42	43	44	45	46	47	48	49	50
0.959132	0.975637	0.997079	1.005844	1.010911	1.022984	1.033933	1.042242	1.061527	1.070929
51	52	53	54	55	56	57	58	59	60
1.076723	1.087992	1.108933	1.115786	1.127053	1.135099	1.148131	1.155036	1.16007	1.179377
61	62	63	64	65	66	67	68	69	70
1.191439	1.203998	1.2104	1.231302	1.243953	1.268141	1.276897	1.294013	1.309118	1.329654
71	72	73	74	75	76	77	78	79	80
1.350229	1.365713	1.392018	1.394322	1.410243	1.428856	1.449078	1.460471	1.502005	1.530906
81	82								
1.581872	1.655286								

Table 5b

		The eigenvalues from the Cleaned up Matrix							
1	2	3	4	5	6	7	8	9	10
0.395332	0.409255	0.436172	0.437537	0.454568	0.465948	0.479993	0.489237	0.492876	0.499109
11	12	13	14	15	16	17	18	19	20
0.50412	0.518916	0.522711	0.532636	0.546767	0.552072	0.55553	0.560167	0.56663	0.580976
21	22	23	24	25	26	27	28	29	30
0.583683	0.58901	0.596519	0.597204	0.614542	0.534339	0.545437	0.552159	0.565169	0.566826
31	32	33	34	35	36	37	38	39	40
0.571085	0.574948	0.584922	0.587273	0.594629	0.6036	0.545435	0.549632	0.552748	0.556042
41	42	43	44	45	46	47	48	49	50
0.564195	0.573904	0.586517	0.591673	0.594653	0.601755	0.608196	0.613083	0.624428	0.629958
51	52	53	54	55	56	57	58	59	60
0.633367	0.639995	0.652314	0.656344	0.662973	0.667705	0.765421	0.770024	0.77338	0.786251
61	62	63	64	65	66	67	68	69	70
0.794293	0.802666	0.806934	0.820868	0.829302	0.845428	0.851265	0.995395	1.007014	1.022811
71	72	73	74	75	76	77	78	79	80
1.038637	1.050548	1.070783	1.072556	1.084802	1.09912	1.114675	1.123439	1.155389	1.17762
81	82								
1.216825	1.273297								

5c

$$p(\lambda_i) = \frac{Q}{2\pi\sigma^2}\frac{\sqrt{(1.6484\ - \lambda_i) * (\lambda_i\ - 0.5128)}}{\lambda_i}$$

where $i = 1, 2, \dots, 82$ and λ_i is the eigenvalues from Empirical, theoretical and the cleaned-up matrices. This value is plotted against the eigenvalues in the tables above to get the distribution of such eigenvalue.