Frequency domain criteria for lp-robust stability of systems with fuzzy parameters

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Frequency Domain Criteria for l^p-Robust Stability of Systems with Fuzzy Parameters

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***Abstract*—The paper deals with the problem of determining stability margin of linear continuous-time system with fuzzy parametric uncertainty. Non-symmetric multivariate membership functions with l^p-constraints describing the uncertainty of characteristic polynomial parameters are considered. An elegant solution, graphical in nature, based on generation of Tsypkin-Polyak plot is presented.**

Keywords: Robust Stability, Fuzzy Parameters

I. INTRODUCTION

Uncertainty plays a key role in control and when dealing with real systems some uncertainty has to be always considered. If the structure of a system is supposed to be given but the parameters are not known precisely we speak about parametric uncertainty. In engineering practice it is of fundamental importance that the system preserves stable behaviour for a whole admissible parameter variations. In this view it could be also appropriate to know, if a system is stable for some nominal values of its parameters, within what boundary the stability remains preserved. Such a problem is called stability margin determination.

The problems mentioned above and solved by classical robust analysis approach [1,2,3] assume that the uncertainty remains the same independently on the working conditions. It means that the worst case has to be considered and conservative results are obtained. However, in many practical situations the uncertainty varies, e.g. depending on operation conditions. In such a case the uncertainty interval can be often parameterized by confidence level. This parameter is usually difficult to measure but it can be estimated by a human operator. If each coefficient of a system is described in this way the system corresponds to a family of interval linear time-invariant systems parameterized by the confidence level.

To handle this type of uncertain systems it is desired to have a mathematical framework. Such a framework was proposed by Bondia and Pico in [4]. They adopted the concept of fuzzy numbers and fuzzy functions [5]. The approach interprets a set of intervals parameterized by a confidence level as a fuzzy number with its membership degree given by this confidence level. It means that all the parameters a_i are characterized by means of fuzzy numbers with the membership functions $\alpha_i = \mu_{\tilde{a}_i}(a_i)$.

When a confidence level α_i is specified then the parameter interval is determined by α_i −cut $[\tilde{a}_i]_{\alpha_i}$. If $\alpha_i = 1$ (the maximum confidence level – the system works in normal operating conditions) the parameter a_i can take any value (crisp or interval) within the cores of $\tilde{a}_i$'s: $a_i = core\left(\tilde{a}_i\right)$. If $\alpha_i = 0$ (the minimum confidence level) the parameter a_i is the interval equal to the support of $\tilde{a}_i$'s ($a_i \in supp\left[\tilde{a}_i\right]$).

Let us consider a characteristic polynomial

$$\tilde{p}(s) = \tilde{a}_0 + \tilde{a}_1 s + \ldots + \tilde{a}_n s^n \quad (1)$$

The parameter $\tilde{a}_i$, i = 0,…, n are supposed to be described by fuzzy numbers characterized by arbitrary convex, normal, continuous membership functions $\alpha = \mu_{\tilde{a}_i}(a_i)$ sharing common confidence level α with one-element cores $core\left\{\tilde{a}_i\right\} = a_i^0$ and bounded support supp$\left\{\tilde{a}_i\right\} = \left[a_i^-, a_i^+\right]$. Then there exist the monotonic increasing and monotonic decreasing functions $a_i^-(\alpha_i)$ and $a_i^+(\alpha_i)$, respectively, defined for $\alpha \in [0,1]$ as

$$a_i^-(\alpha) = \mu_{a_i}^{-1}(\alpha) \text{ for } a_i^- \le a_i^-(\alpha) \le a_i^0 \quad (2)$$

$$a_i^+(\alpha) = \mu_{a_i}^{-1}(\alpha) \text{ for } a_i^0 \le a_i^+(\alpha) \le a_i^+$$

that constitute the α-cut representation of parameter $\tilde{a}_i$, $\left[\tilde{a}_i\right]_\alpha = a_i(\alpha) = \left[a_i^-(\alpha),\, a_i^+(\alpha)\right]$. The functions $a_i^-(\alpha)$ and $a_i^+(\alpha)$ satisfy $a_i^-(0) = a_i^-$, $a_i^+(0) = a_i^+$, $a^-(1) = a^+(1) = a_i^0$,

$a_i^-(\alpha) \le a_i^+(\alpha)$. So the α-cut representation of polynomial (1) correspondence to an uncertain polynomial

$$\left[\tilde{P}(s,\tilde{a})\right]_\alpha = P^\alpha(s,a) = a_0 + a_1 s + \ldots + a_n s^n \quad (3)$$

where $a \in Q \subset R^n$ is vector of interval parameters $a_i = \left[\tilde{a}_i\right]_\alpha$, $i = 0, \ldots, n$ with nominal value a_i^0 corresponding to $\alpha = 1$ and bounded values a_i^-, a_i^+ corresponding to $\alpha = 0$.

The intervals $a_i(\alpha) = \left[a_i^-(\alpha), a_i^+(\alpha)\right]$ are non-symmetric in general and can be described in the following useful form:

$$a_i(\alpha) = a_i^0 + \delta a_i(\alpha), \quad -\delta a_i^-(\alpha) \le \delta a_i(\alpha) \le \delta a_i^+(\alpha), \quad (4)$$

where in the case of triangular membership functions we have

$$\delta a_i^-(\alpha) = (1-\alpha)\Delta a_i^-, \ \delta a_i^+(\alpha) = (1-\alpha)\Delta a_i^+, \quad \Delta a_i^- = \Delta a_i^-(0) = a_i^0 - a_i^-, \quad \Delta a_i^+ = \Delta a_i^+(0) = a_i^+ - a_i^0 \quad (5)$$

In this case the uncertain polynomial (3) can be rewritten as follows:

$$P^\alpha(s) = P^0(s) + (1-\alpha)\delta P(s), \quad P^0(s) = a_0^0 + a_1^0 s + \ldots + a_n^0 s^n, \quad \delta P(s) = \Delta a_0 + \Delta a_1 s + \ldots + \Delta a_n s^n, \quad (6)$$

where

$$\Delta a_i = \Delta a_i^- \ \text{ if } \ a_i < a_i^0, \qquad \Delta a_i = \Delta a_i^+ \ \text{ if } \ a_i \ge a_i^0.$$

The problem is to check robust stability of linear uncertain polynomial (3) for a fixed confidence level α and to find minimum confidence level (or maximal margin) $\alpha_{\min} \in [0, 1]$ preserving robust stability. Note that while the polynomial (3) is robust stability for $\alpha_{\min}$ the fuzzy polynomial (1) is robustly stable for all $\alpha > \alpha_{\min}$ and unstable for $\alpha \le \alpha_{\min}$.

When the coefficients $\tilde{a}_i, i = 0,1, \ldots, n$ are independent fuzzy numbers, the problem has been solved using Argoun stability test in [5] or with the help of Tsypkin-Polyak locus in [6]. In this concept the author [7, 8] derived a method for stability analysis of closed-loop control systems with fuzzy parameters.

Nevertheless, the parameters of the system are very often identified using measured input-output data. In such case it is more realistic to characterize the set of parameters by a multidimensional membership function rather than employing fuzzy numbers. For example, when utilizing well-known prediction error identification algorithm the parameters $a = \left[a_0, a_1, \ldots, a_n\right]^T$ lie in an ellipsoidal set

$$\left(a - a^0\right)^T W(a - a_0) \le 1 \quad (7)$$

where W is a positive definite matrix.

This constrain is mathematically called a ball in l^2 norm $\|\delta a\|_{2,W} \le 1$. Then it is reasonable to consider fuzzy set with membership function

$$\mu_{\tilde{a}}(a) = -\left(a - a^0\right)^T W(a - a^0) + 1 \quad (8)$$

where the confidence level $\alpha = \mu_{\tilde{a}}(a)$ indicates the belief in the experiment the measured data were obtained by. For $\alpha = 0$ the parameters a_i can take any value inside hyperellipsoid $\left(a - a^0\right)^T W(a - a^0) \le 1$, for $\alpha = 1$ equals to the nominal value, $a = a^0$. This case is solved in [9].

In this paper a generalized case is considered, when relative perturbation form a ball in l^p norm, $1 \le p \le \infty$:

$$\left[\sum_{i=1}^{n} \left|\frac{a_i - a_i^0}{\Delta a_i}\right|^p\right]^{\frac{1}{p}} \le 1 - \alpha, \quad (9)$$

where

$$\Delta a_i = \Delta a_i^- \ \text{ if } \ a_i < a_i^0, \qquad \Delta a_i = \Delta a_i^+ \ \text{ if } \ a_i \ge a_i^0. \quad (10)$$

In classical robust analysis the problem for verifying stability of uncertain polynomial (3), (9) has been solved elegantly by using Tsypkin-Polyak plot [2,3], where the ball is symmetrical $\Delta a_i^- = \Delta a_i^+$, $i = 0,\ldots,n$. In the case that $p = 2$, the solution of this problem was derived earlier in [1, 10]. It will be shown in this paper that the Tsypkin-Polyak plot can be extended for the case of non-symmetric ball to solve the problem of finding stability margin of the polynomial (3),(9),(10).

II. Generalized Tsypkin-Polyak plot

Let us introduce the conjugate index q, $1 \le q \le \infty : 1/p + 1/q = 1$ and the functions

$$U^0(\omega) = \operatorname{Re} P^0(j\omega) = \sum_{k_{even}} (-1)^{k/2} a_k \omega^k = a_0^0 - a_2^0\omega^2 + a_4^0\omega^4 \ldots,$$

$$V^0(\omega) = \operatorname{Im} P^0(j\omega) = \sum_{k_{odd}} (-1)^{k-1/2} a_k \omega^{k-1} = a_1^0 - a_3^0\omega^2 + a_5^0\omega^4 \ldots,$$

$$S(\omega) = 0{,}5\{[S^-(\omega) + S^+(\omega)] + [(S^-(\omega) - S^+(\omega)] sign U^0(\omega)\},$$

$$S_{even}(\omega)=\left(\sum_{k/2_{even}}\left(\Delta a_k^{+}\omega^k\right)^q+\sum_{k/2_{odd}}\left(\Delta a_k^{-}\omega^k\right)^q\right)^{1/q},$$

$$S_{odd}(\omega)=\left(\sum_{k/2_{even}}\left(\Delta a_k^{-}\omega^k\right)^q+\sum_{k/2_{odd}}\left(\Delta a_k^{+}\omega^k\right)^q\right)^{1/q},$$

$$S^{+}(\omega)=\begin{cases}S_{even}(\omega), & if\ q\ is\ even\\ S_{odd}(\omega), & if\ q\ is\ odd\end{cases},$$

$$S^{-}(\omega)=\begin{cases}S_{odd}(\omega), & if\ q\ is\ even\\ S_{even}(\omega), & if\ q\ is\ odd\end{cases}, \tag{11}$$

$$T(\omega)=0{,}5\{[T^{-}(\omega)+T^{+}(\omega)]+$$
$$+[(T^{-}(\omega)-T^{+}(\omega)]signV^0(\omega)\},$$

$$T_{even}(\omega)=$$
$$=\left(\sum_{k-1/2_{even}}\left(\Delta a_k^{+}\omega^{k-1}\right)^q+\sum_{k-1/2_{odd}}\left(\Delta a_k^{-}\omega^{k-1}\right)^q\right)^{1/q},$$

$$T_{odd}(\omega)=$$
$$=\left(\sum_{k-1/2_{even}}\left(\Delta a_k^{-}\omega^{k-1}\right)^q+\sum_{k-1/2_{odd}}\left(\Delta a_k^{+}\omega^{k-1}\right)^q\right)^{1/q},$$

$$T^{+}(\omega)=\begin{cases}T_{even}(\omega), & if\ q\ is\ even\\ T_{odd}(\omega), & if\ q\ is\ odd\end{cases}$$

$$T^{-}(\omega)=\begin{cases}T_{odd}(\omega), & if\ q\ is\ even\\ T_{even}(\omega), & if\ q\ is\ odd\end{cases}$$

$$x(\omega)=U^0(\omega)/S(\omega),$$
$$y(\omega)=V^0(\omega)/T(\omega),$$
$$z(\omega)=x(\omega)+jy(\omega).$$

Then the theorem can be stated as:

Theorem. The uncertain polynomial (3),(9),(10) is stable if and only if frequency plot $z(\omega)$

a) goes through n quadrants in counterclockwise direction,
b) does not intersect l^p circle of radius $(1-\alpha)$ (i.e. the set $\left(|x|^p+|y|^p\right)^{1/p}\le(1-\alpha)$),
c) $a_0^0/\Delta a_0^{-}>(1-\alpha)$, if q is even
   $a_0^0/\Delta a_0^{+}>(1-\alpha)$, if q is even
d) $a_n^0/\Delta a_n^{-}>(1-\alpha)$, if q is even
   $a_n^0/\Delta a_n^{+}>(1-\alpha)$, if q is even

Proof. As $x(\omega)$ and $y(\omega)$ have the same sign as $U^0(\omega),\ V^0(\omega)$, the plot $z(\omega)$ goes through the same quadrants as the plot $P^0(j\omega)$. So the condition a) is obviously from necessary stability condition of nominal polynomial $P^0(s)$ (i.e. the Mikhailop's test).

Let us denote the value set $P(\omega)=P^{\alpha}(j\omega)$ for all uncertain parameters a_i satisfied constraint (9) at certain ω. The perturbed real and imaginary parts of $P^{\alpha}(j\omega)-P^0(j\omega)$ can be expressed as

$$\mathrm{Re}\{P^{\alpha}(j\omega)-P^0(j\omega)\}=\sum_{k_{even}}(-1)^{k/2}\delta a_k\omega^k,$$
$$\mathrm{Im}\{P^{\alpha}(j\omega)-P^0(j\omega)\}=\sum_{k_{odd}}(-1)^{k-1/2}\delta a_k\omega^{k-1},$$

respectively.

So in this case, when $\Delta a_i^{-}=\Delta a_i^{+},\ i=0,\dots,n$ we have

$$-(1-\alpha)\sum_{k_{even}}(-1)^{k/2}\Delta a_k\omega^k\le$$
$$\le\mathrm{Re}\{P^{\alpha}(j\omega)-P^0(j\omega)\}\le$$
$$\le(1-\alpha)\sum_{k_{even}}(-1)^{k/2}\Delta a_k\omega^k,$$
$$-(1-\alpha)\sum_{k_{odd}}(-1)^{k-1/2}\Delta a_k\omega^{k-1}\le$$
$$\le\mathrm{Im}\{P^{\alpha}(j\omega)-P^0(j\omega)\}\le$$
$$\le(1-\alpha)\sum_{k_{odd}}(-1)^{k-1/2}\Delta a_k\omega^{k-1}$$

and it can be easily proved [2,3] that the set $P(\omega)$ consists of the points $x+jy$ subject to

$$\left[\left|\left(\frac{x-U^0(\omega)}{S(\omega)}\right)\right|^p+\left|\left(\frac{y-V^0(\omega)}{T(\omega)}\right)\right|^p\right]^{1/p}\le$$
$$\le(1-\alpha)$$

with

$$S(\omega)=S^{-}(\omega)=S^{+}(\omega),$$
$$T(\omega)=T^{-}(\omega)=T^{+}(\omega).$$

In particular case $p=2$ it is an ellipse with semi-axes $S(\omega),T(\omega)$ [1].

Now in the generalized case, when $\Delta a_i^{-}\ne\Delta a_i^{+},\ i=1,\dots,n$ it is not difficult to proved that

$$-(1-\alpha)S^{-}(\omega)\le\mathrm{Re}\{P^{\alpha}(j\omega)-P^0(j\omega)\}\le$$
$$\le(1-\alpha)S^{+}(\omega),$$
$$-(1-\alpha)T^{-}(\omega)\le\mathrm{Im}\{P^{\alpha}(j\omega)-P^0(j\omega)\}\le$$
$$\le(1-\alpha)T^{+}(\omega)$$

and the set $P(\omega)$ consists of the points $x+jy$ subject to

$$\left[\left|\left(\frac{x-U^0(\omega)}{S(\omega)}\right)\right|^p + \left|\left(\frac{y-V^0(\omega)}{T(\omega)}\right)\right|^p\right]^{1/p} \leq$$

$$\leq (1-\alpha) \tag{12}$$

with $S(\omega), T(\omega)$ are defined in (11).

For $p = 2$ the value set $P(\omega)$ is non-symmetric ellipse (Fig.1).

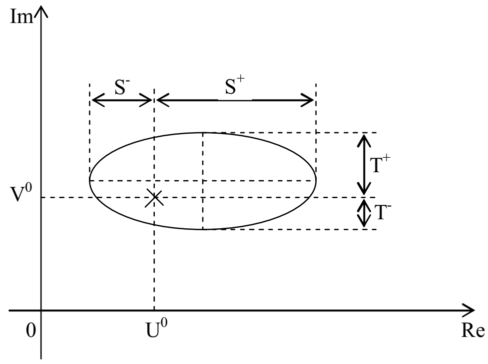

Figure 1. The non-symmetric value set

Using Zero Exclusion Principle [1,3] the condition $0 \notin P(\omega)$ transforms to

$$\left(|x|^p + |y|^p\right)^{1/p} > (1-\alpha)).$$

This formula is equivalent to condition b). The conditions c) and d) in theorem are derived from boundary conditions $|x(0)| > (1-\alpha)$ and $|x(\infty)| > (1-\alpha)$, $|y(\infty)| > (1-\alpha)$

The case $p = \infty$ corresponds to interval constraints studied in [6] and in the case $p = 2$ the derived theorem is equivalent to the key theorem 4 in [9].

For illustrating purpose we consider the case $p = 2$ with the following example.

Example [9]

Let the coefficients of a 6-th order polynomial $\tilde{P}(s) = \sum_{k=0}^{6} \tilde{a}_k s^k$ be characterized by membership function (7) with the following parameters:

$a^0 = [a_0^0, a_1^0, a_2^0, a_3^0, a_4^0, a_5^0, a_6^0]^T =$

$= [433.5, 667.25, 502.25, 251.25, 80.25, 14, 1]^T$

$\Delta a^+ = [\Delta_0^+, \Delta_1^+, \Delta_2^+, \Delta_3^+, \Delta_4^+, \Delta_5^+, \Delta_6^+]^T =$

$[196.98, 143.48, 117.42, 45.95, 12.98, 8.56, 0.6]^T$

$\Delta a^- = [\Delta_0^-, \Delta_1^-, \Delta_2^-, \Delta_3^-, \Delta_4^-, \Delta_5^-, \Delta_6^-]^T =$

$= [173.4, 133.44, 100.548, 60.3, 22.47, 5.6, 0.4]^T$

The frequency plot $z(\omega)$ is depicted in Fig.2. Maximum radius of circle satisfying condition b) of the theorem is $r_{\max} = 0.66$ and the minimum confidence level preserving stability of $\tilde{P}(s)$, $\alpha_{\min} = 1 - r_{\max} = 0.34$.

This result is the same as in [9].

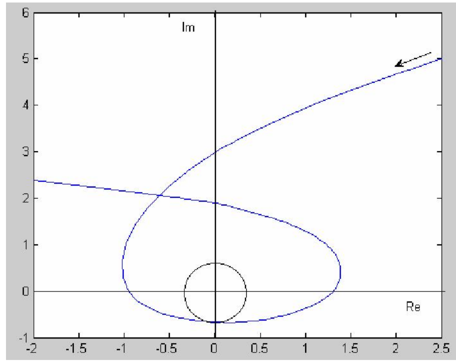

Figure 2. The generalized Tsypkin-Polyak plot for *p*=2

III. CONCLUSIONS

In this paper, an unified approach to stability analysis of continuous-time linear systems with fuzzy parametric uncertainty satisfying l^p-constraints was presented. An elegant, graphical in nature, algorithm of determining minimum confidence level preserving system stability was derived based on generalization of Tsypkin-Polyak plot [2,3]. The main advantage of the proposed method is its general character at high level. Many earlier works in classical robust stability analysis [2,3], and fuzzy control [6,9] are typical particular cases of the proposed method. Furthermore, the proof of the proposed method is much simpler (in comparison with [9], for example).

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