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The potential of recreational mathematics to support the development of mathematical learning

A literature review establishes a working definition of recreational mathematics: a type of play which is enjoyable and requires mathematical thinking or skills to engage with. Typically, it is accessible to a wide range of people and can be effectively used to motivate engagement with and develop understanding of mathematical ideas or concepts. Recreational mathematics can be used in education for engagement and to develop mathematical skills, to maintain interest during procedural practise and to challenge and stretch students. It can also make cross-curricular links, including to history of mathematics. In undergraduate study, recreational mathematics can be used for engagement within standard curricula and for extra-curricular interest. Beyond this, there are opportunities to develop important graduate-level skills in problem-solving and communication. The development of a module 'Game Theory and Recreational Mathematics' is discussed. This provides an opportunity for fun and play, while developing graduate skills. It teaches some combinatorics, graph theory, game theory and algorithms/complexity, as well as scaffolding a Pólya-style problem-solving process. The assessment of problem-solving as a process via examination is outlined. Student feedback gives some indication that students appreciate the aims of the module, benefit from the explicit focus on problem-solving and understand the active nature of the learning.

Keywords: recreational mathematics, skills development, problem-solving, undergraduate mathematics, history of mathematics

Introduction

The primary focus of this paper is to establish what recreational mathematics is and how it can be used in education. A precise, unambiguous definition of recreational mathematics is surprisingly elusive, but a systematic literature search leads to a working definition. We then argue that recreational mathematics is useful in education in various ways, including that it is well suited for developing authentic skills expected of a graduate mathematician around communicating and especially problem-solving. Finally, the design and initial delivery of a new module 'Game Theory and Recreational

Mathematics' is briefly described. As this module is new, involves a single small cohort and, at the time of writing, has not completed its first year of delivery, a detailed evaluation is not attempted. However, some reflections and small-scale student feedback are presented as indicative findings. The main contribution of this paper is to establish what recreational mathematics is for an educational context and to theorise that it can be used to develop skills needed by graduate mathematicians, including around communication and problem-solving.

What is recreational mathematics?

Hankin says recreational mathematics is 'easier to recognize than define' [1,p.48]. Singmaster says 'an essay on recreational mathematics should begin with a discussion of what recreational mathematics is, or at least what its author thinks it is' [2,p.579], recognising the problematic definition of the term. With this in mind, we begin with a systematic literature review which attempts to answer the question 'what is recreational mathematics?' This results in a working definition, which is used to discuss the use of recreational mathematics in education.

Approach

Definitions of recreational mathematics were collected by conducting a literature search via our university library search facility and Google Scholar, using key words 'recreational mathematics' and 'definition'. In addition, where sources used for other aspects of this paper contained definitions, these were included in the collection.

Note that a general definition is sought, recognising that 'mathematical tastes are highly individualized, so no classification of particular mathematical topics as recreational or not is likely to gain universal acceptance' [3,p.21].

Extremely broad definitions

In an interview, the modern recreational mathematics pioneer Martin Gardner defined recreational mathematics as ‘anything that has a spirit of play about it’ [4,p.238]. Some refer to mathematics that is ‘fun’ [5,6]. Fun is a subjective measure and potentially very broad as ‘many professional mathematicians consider all mathematics to be pleasurable’ [3,p.21]. MathsJam, an international network of recreational mathematics meetings, involves ‘maths enthusiasts’ sharing puzzles, games, problems or ‘anything else they thought was cool or interesting’ [7,p.115].

It is an attractive feature of such definitions that they desire to be inclusive. However, they are too broad and could easily incorporate all of mathematics and much else besides. The remainder of the findings from this review are grouped into thematic headings that emerged from analysis of literature collected.

Fun, play and popularity

Singmaster observes that recreational mathematics is ‘fun and popular’ [6,p.13]. Play in recreational mathematics may involve a puzzle, game, magic trick, paradox, fallacy or curious piece of mathematics [8], which is linked to positive feeling and enjoyment [9,10]. Such feelings are key aspects in making a topic recreational, since engagement is ‘based on positive motivation’ [11,p.123]. Play can be argued as a fundamental aspect of mathematics [12], though the satisfaction at completing an intellectual pursuit may go beyond being simply fun [9,12].

Some regard as a key aspect of recreational mathematics that it can be communicated to the general public [1]. The prevalence of puzzles in the popular press shows a broad interest in recreational mathematics exists [13,14]. Both MathsJam and Recreational Mathematics Magazine make a point of highlighting that recreational mathematics is not just for mathematicians [7,10].

Prior knowledge requirements

Some argue that recreational mathematics does not require knowledge of advanced mathematical notation or terminology [1,14]. Trigg says 'many' consider mathematics to be recreational if it is sufficiently elementary [3,p.21].

Some impose a further restriction, that recreational mathematics topics must be readily accessible to all while allowing meaningful exploration that might interest a professional mathematician [16,3]. Some make the accessibility requirement for the statement of puzzles only, allowing that solutions might not be fully understood by a general reader [5,6].

However, there are suggestions that recreational mathematics can easily stray beyond that understandable by a general reader. Moscovich has to make 'a great effort' to ensure his puzzles are 'understandable to everyone' [5,p.5]. *Recreational Mathematics Magazine* publishes only exceptionally: 'some papers that do not require any mathematical background' [10,p.3]. Remembering that recreational mathematics is fun, and fun is subjective, Trigg recognises that [3,p.20]:

> for many individuals, as they approach the limit of their abilities, mathematics loses its fun aspect. When a topic is undeveloped, it is recreational to many. As the theory is developed and becomes more abstract, fewer persons find it recreational.

The discussion of prior, non-recreational mathematical knowledge raises questions about the interaction between recreational and other mathematics.

Interaction with history of mathematics

Recreational mathematics has a history as old as formal mathematics, with examples found on the oldest mathematical texts such as the Rhind papyrus and Babylonian tablets [11]. We do not know whether these were playful diversions or an interaction between recreational and 'serious' mathematics, but there are certainly more recent examples of recreational problems acting as stimulus for non-recreational topics [17]. Two illustrative examples are found in graph theory, which has origins in Euler's and Hamilton's work on the Königsberg bridges problem and Icosian Game, respectively, and modern probability theory, which has origins in games of chance analysed by Cardano, Tartaglia, Pascal and Fermat, including de Méré's dice game and the problem of points. 'Serious' mathematics has also led to new insights in recreations, for example graph theory cast light on old puzzles such as the Knight's Tour and Guarini's problem.

Interaction with non-recreational mathematics

Some believe recreational mathematics can be chosen to lead into more 'significant' [4,p.234] or 'serious' mathematics [16,p.13], but the distinction between 'recreational' and 'serious' mathematics is not universally accepted [5,10].

Some have recreational mathematics as a personal activity carried out for entertainment or self-education [16], seeing this as distinct from formal education or research activity [3]. Such restrictions seem unnecessarily self-limiting. Certainly they considerably undermine the mission of publications such as *Recreational Mathematics Magazine* [10] or work attempting to use recreational mathematics in education.

Educational value

A 1982 UK Government report into mathematics teaching said 'carefully planned use of mathematical puzzles and "games" can clarify the ideas in a syllabus and assist the development of logical thinking' [13,p.67]. Indeed, recreational mathematics is widespread in school curricula of some countries, especially India where the emphasis is on the development of a positive attitude towards mathematics; by comparison, for example, the United States curriculum focuses on mathematics as a tool without considering the personal experience of using it [11].

Links to educational value may be less formal, with recreational mathematics being used to provoke curiosity [5] and develop concepts and skills through play [18]. The use of recreational mathematics might also be motivational [4], which is explored further in the next section. Some authors write passionately about the role of puzzles in fostering their own early mathematical development [14,19].

A definition of 'recreational mathematics'

Recreational mathematics cannot be everything mathematicians find fun or interesting, as this is too broad. There is general agreement that recreational mathematics is fun, in the sense of being playful and enjoyable. There is agreement, too, that recreational mathematics can be popular and widely accessible, but not that this is a necessary condition. We do not accept those definitions that stop topics being recreational as soon as professional interest is directed towards them, as these undermine the practice of engaging in recreational mathematics and particularly its use in education. Instead, we accept that those with specialist mathematical knowledge can engage in play, and that an interface between recreational and non-recreational mathematics is personal and hard to define. There is strong agreement that recreational mathematics can encourage engagement in and develop skills associated with learning and doing mathematics, and that recreational topics can be used to develop a deeper understanding of mathematical ideas or concepts, but not that it must do so.

All that said, we take a deep breath and propose the following definition:

Recreational mathematics is a type of play which is enjoyable and requires mathematical thinking or skills to engage with; typically, it is accessible to a wide range of people and can be effectively used to motivate engagement with and develop understanding of mathematical ideas or concepts.

This is, of course, not perfect. Subjectivity is injected in particular via the adjective ‘enjoyable’. This should be taken to mean ‘some people find it enjoyable’, rather than necessarily ‘I find it enjoyable’. Though broad, our definition is designed to match the use of the term in practice, and therefore it is offered as a working definition.

Use of recreational mathematics in education

In general

As discussed, recreational mathematics can be used to enhance engagement and interest [4,9,11,14,19]. Teaching using games has been shown to improve engagement and attitudes [20–22]. Recreational mathematics has the potential to develop and expand mathematical skills, including problem-solving, and deepen understanding [23,24].

The use of recreational mathematics in teaching is not new. Vankúš [25] provides a brief history starting with Plato and Aristotle being in favour of the use of games in educating children. There is evidence of extensive use of recreational mathematics in school curricula [11]. Where recreational mathematics topics are not explicitly mentioned in curricula, they could still be used for relevant topics at the teacher’s discretion [13].

Introducing a topic via a recreational problem can help with student interest and engagement [18]. For example, it may be considered ‘natural’ to introduce probability using ‘simple experiments by tossing coins, rolling dice, and so forth’ [26,p.9], mirroring historical developments. Once introduced, repetition and practise is a way in which mathematical knowledge and ideas are reinforced [27]. However, students may lose interest and become unmotivated, indeed students who do not continue mathematics to university commonly report they do not enjoy it and find it boring [28]. Recreational problems may counteract this by ‘taking the drudgery out of the practice of skills' [20,p.3], potentially developing creative, problem-solving skills alongside procedural fluency [29,30]. Recreational problems can also be used to challenge and stretch students and encourage them to go beyond what they have learned [18,20].

Sometimes recreational mathematics might be included within other aspects, for example history of mathematics is emphasised in various countries' curricula [31]. History of mathematics has potential to motivate, encourage links between mathematics and other subjects and, via study of the problems experienced by those originally developing topics, to develop perseverance [32,33].

In undergraduate mathematics

There are significant opportunities to introduce and illuminate undergraduate topics through recreations [34]. The Maths Arcade initiative [35] is an informal, extra-curricular games-based activity. It operates as an informal drop-in session where students can play strategy games with other students and staff, and can help to develop strategic thinking and problem-solving skills [23]. Recreational mathematics can make contributions within the curriculum beyond engagement, however.

In the UK, design, delivery and review of degree courses is guided by the Quality Assurance Agency's Subject Benchmark Statement [36]. This document provides limited guidance on content that must be included [36,p.14], but does clearly outline expectations for skills development, including [36,p.16-18]:

- problem-solving: to 'formulate [problems] mathematically'; to 'select and apply appropriate mathematical processes'; to 'assess problems logically and approach them analytically'; to 'construct and develop logical mathematical arguments'; to 'work independently with patience and persistence, pursuing the solution of a problem to its conclusion'; to ‘transfer knowledge from one context to another’; ‘adaptability’, including ‘readiness to address new problems from new areas’; and,

- communication: to present mathematical arguments and conclusions ‘with accuracy and clarity’; to clearly identify assumptions and conclusions.

A discussion by mathematicians on graduate expectations reported by Good found limited agreement on specific content, but a similar list of skills, adding ‘some knowledge of the culture of mathematics’ [37,p.15]. Some other countries have a similar statement by a panel of experts describing disciplinary norms. For example, the Australian Mathematical Threshold Learning Outcomes document [38], which also says relatively little about topics that are expected in a mathematics degree, contains detail on graduate skills around problem-solving and communicating clearly.

It seems there is generally clearer agreement on the skills that should be developed in undergraduate mathematics than on the topics that must be covered. As seen above, recreational mathematics can promote communication and clarity of explanation, and provides links to the history and culture of the discipline. It also offers considerable opportunity and motivation to practise problem-solving.

Problem-solving is central to the practice of being a mathematician [39]. Foster has problem-solving built on fluency and reasoning skills, and makes an important distinction between ‘an exercise in following the teacher’s method’ and problem-solving, ‘what you do when you do not know what to do’ [39,p.8]. He argues that instead of ‘scaffolding a problem’, we need to be ‘scaffolding the problem solving’ [39,p.9], and favours problem-solving practise built on ‘mathematical content that was learned some time ago and is quite robustly known’ [39,p.10].

Recreational mathematics is also linked to effective teaching methods. Su recognises that play is an important aspect of mathematics (and life), and says [12,p.486]:

> Play is part of what makes inquiry-based learning and other forms of active learning so effective. There's overwhelming evidence that students learn better with active learning.

Games and recreational mathematics are fun, and an opportunity for play. They also can make use of various interesting mathematical topics, including game theory, combinatorics, graph theory and algorithms/complexity. For undergraduates in the final year, there is an opportunity to teach new topics that could be applied in a puzzles and games context, while practising problem-solving skills based on mathematical content learned earlier in the degree. There is also an opportunity to link to the history and culture of the discipline. Therefore, games and recreations give the opportunity to focus on developing key graduate skills around communication and problem-solving.

A new module: Game Theory and Recreational Mathematics

Our new module began with a section exploring what recreational mathematics is and its link to the historical development of mathematics. Following this was a section on the use of recreational mathematics in education and problem-solving. Students were encouraged to use a Pólya-style problem-solving process. Pólya outlined a problem-solving process that highlighted understanding the problem and planning before solving the problem, and reflection once a solution is reached. In the understanding and planning phases, he encourages activities such as restating the problem, drawing a picture and trying to solve a simpler problem [40].

Following this was the development of specific topics in a recreational context: combinatorics, graph theory, game theory (combinatorial and classic) and the use of algorithms, especially for searching game trees. A final section focused again on problem-solving practise, using mathematics that is relatively straightforward (for final year undergraduates) or was first developed much earlier in the module (though, alas, not two years earlier, as Foster suggests [39]).

A first coursework (40% of the marks) asked students to: write about the historical development of a topic; present a puzzle or game in a magazine article aimed at school leavers; or, write about how a puzzle or game could be used to teach a topic on the school curriculum. A second coursework (30%) focused on the use of algorithms to analyse games and puzzles.

The remaining 30% of module marks were for an exam. This contained two sections: section A (40%) contained basic questions on topics taught in the module around combinatorics, graph theory and game theory; section B (60%) was based around unseen problem-solving. Students were warned that there were no mathematical topics to revise for this and they were not expected to obtain a full solution, but the exam paper states: ‘Full marks will be obtained for a correct solution that is very clearly communicated. For an incorrect or partially-correct solution, marks will be awarded for clear evidence of applying a systematic problem-solving approach.’

The idea is to give the students the opportunity to demonstrate their ability in problem-solving, using the scaffolded approach delivered in the taught content. Students were encouraged to write down their attempts at various aspects of the Pólya-style process: rewrite the question in your own words; state the crucial features of the problem; draw a diagram; write an explanation of what you need to do and why this is difficult; try variants of the problem; make a conjecture and test it. Ultimately, if a solution cannot be found, write a clear explanation of what has been tried and why it didn't work and/or a critique of the problem and why it is hard to solve. Some guidance given to students, designed to illustrate the problem-solving process and emphasise that getting the answer is not the most important aspect, is presented in Figure 1. An example question and student attempts at its solution are reviewed later in this article.

Exam preparation sessions focused on practice questions, which are also unseen problems, which students attempted to solve. Teaching delivery here was focused on the problem-solving process, with most class time devoted to students trying to solve unseen problems. Staff would join in discussions, but focus on how to apply the problem-solving advice and what students could write during the exam to demonstrate that they were following this, rather than provide advice on solving particular problems.

Student feedback

Degree-level staff-student feedback meetings aggregate feedback from students on all modules. Minutes of three meetings this year indicate that no issues have been raised with this module and students find it 'very interesting' and 'really fun'. An institutional module evaluation questionnaire, completed before the exam, received a low response (7 students from 24). This is not enough for robust conclusions, but some responses are discussed here as indicative findings.

All respondents agreed 'This module is intellectually stimulating', 'This module has provided me with opportunities to apply what I have learnt' and 'It is clear to me how this module forms an integral part of my course'. Six out of seven respondents agreed, and none disagreed, with the statement 'I feel confident in tackling unfamiliar problems as a result of this module'.

Students were asked for free-text responses to 'What one thing did you particularly enjoy about this module?'. Students were positive about the idea of the module, including that it is ‘fun’, ‘engaging’, ‘interesting’ and ‘very different to other modules’. Students liked ‘apply[ing] maths we had previously learnt to games’ and ‘applying familiar mathematics to problems you would not initially associate it with’, which supports the idea of problem-solving based on content that is ‘quite robustly known’ [39,p.10] and students’ ability to ‘transfer knowledge from one context to another’ [36,p.17]. One student wrote ‘All the problem solving skills were useful in questions for the assignments as well as other modules. So it is practical for in university and in real-life.’

In response to a free-text question ‘What advice would you give to students studying this module next year?’, students indicated awareness of the need to actively participate in the learning, advising future students to ‘attend lectures as looking at the [lecture slides] is not sufficient to know what to do’ and remarking that ‘if you attend all lectures/tutorials and actively participate in the problem-solving techniques you will be able to achieve a reasonable grade overall and feel comfortable enough when it comes tackling new problems and scenarios’.

Negative feedback (responding to questions about aspects that could be improved) focused exclusively on organisational matters, especially about the administration of one assignment, that are not intrinsic issues with the module.

Final year mathematics students at Sheffield Hallam University choose three elective modules from a list of ten. In the 2018/19 academic year, the new module was taken by 24 students out of 75 in the cohort (32%). For 2019/20, a cohort of 113 students were recently asked to select modules and 59 chose Game Theory and Recreational Mathematics (52%). This level of selection shows that students are interested in this module, and the increase in proportion may be based on us being able to present more details about the module and students hearing about the positive experience of students in the current year.

Student work in the examination

After the examination had taken place and been marked, students were asked via online questionnaire to consent to their written scripts being used as research data and quoted in research outputs. This process was approved by the Sheffield Hallam University Research Ethics Committee. Of the 24 students who took the exam, eleven responded to the request and all consented to their scripts being used.

To illustrate our practice of marking the problem-solving process rather than the correctness of the solution, an example problem is discussed. Figure 2 shows a question from the unseen problems part (section B) of the 2019 exam.

One method of solution is to consider moving from the bottom left to upper right spaces as a sequence of moves up (*U*), right (*R*) and diagonal (*D*); for example, a successful sequence could be *DUURRUUURR* or *DUDURDD*. The maximum number of *U* moves that can be made is 6, of *R* moves is 5, and of *D* moves is 5. Each *D* move decreases the number of *U* and *R* moves by one each, so if we make k diagonal moves then we must make $6-k$ up moves and $5-k$ right moves, altogether giving $k+6-k+5-k=11-k$ total moves made. There are $(11-k)!$ sequences of length $11-k$ when all of the symbols are different. Since the *U*, *R* and *D* moves are repeated then we must divide out by the appropriate factorials. The total number of paths from bottom left to top right is then given by the sum over all values of k:

$$\sum_{k=0}^{5}\frac{(11-k)!}{k!\,(6-k)!\,(5-k!)}.$$

Working this out for the problem in figure 2 gives the sum

$$462 + 1260 + 1260 + 560 + 105 + 6 = 3563.$$

Alternative methods include directly counting different paths, which the grid size is designed to dissuade (but which might work for smaller cases), and noticing a pattern and extrapolating. From the bottom left space, there is one way to get to the space immediately above and one to get immediately to the right. The diagonally-adjacent square can be reached either directly via *D* or in two steps via either *UR* or *RU*. The number of ways of reaching a square is thus the sum of the number of ways of reaching the three squares to the left, below and diagonal-left. Extrapolating this leads to the table shown in figure 3.

We did not prefer any particular method when marking. Rather, we marked for problem-solving process and quality of communication, using the mark scheme outlined in figure 4. Notice in particular the different criteria for whether the answer was correct or incorrect, so that a student might get the correct answer and still a low (3^{rd} class) mark, or a student without the correct answer might nevertheless access the full range of marks. Students had approximately half an hour within a two-hour exam for this question.

To illustrate that a student without the correct answer might obtain a 1^{st} class mark, we will now review one student's response. First, the student defines a notation, using 'r', 'u' and 'd' for right, up and diagonal moves. They use this notation to count the number of routes for smaller grid sizes 3x3 and 2x3, drawing diagrams of each. Apparently not finding a pattern, they write: 'Solving for a simplified problem: Assume the King cannot travel diagonally. ∴ we have: rrrrruuuuuu ∴ no. of possibilities is 11C5 = 11C6 = 462. If the King cannot travel diagonally then there are 462 ways.'

The student then writes 'What if he only travels diagonally?'. A diagram is then used to indicate that the target space cannot be reached using only diagonal moves.

Through writing out sequences, the student attempts to investigate the cases where the diagonal moves are taken consecutively, for example they observe there are ‘4 going 4 diagonally: uuddddr, uurdddd, urddddu and rdddduu’. They claim to have found ‘70 extra combinations’ via this route.

The student then concludes their answer writing ‘In my own words the puzzle has asked me to find how many routes there are from the bottom left to the top right such that we travel up, right or diagonal. This is different from my solution of 462 as this figure does not include for diagonals. This is difficult as there are many variants that allow for diagonals, however I have worked out a minimum number of 532. This is not the total number as it only counts each diagonal once for 5, 4, 3, 2, 1 diagonals (if possible) and does not consider multiple routes.’

Although it does not obtain the correct answer, there is much that is positive about this attempt. There is clear use of the problem-solving approach scaffolded in class, with the student defining notation, drawing diagrams, trying simpler related problems (both smaller grids and rule tweaks), writing the problem in their own words and, in the end, reflecting on the progress made, recognising that their solution is incorrect and giving a reason why. The mark given is 22/30 (1st class).

By contrast, another student wrote out the equivalent of the table in figure 3 for grid sizes 2x2, 2x3 and 3x3 with the comment ‘The number in the square represent number of ways to that space’. Then they drew out the full table in figure 3 and stated the answer in words. Although the correct answer is obtained and there is some evidence of trying smaller cases, there is limited attempt to explain what method is being used or justify why the student believes this is correct. The mark given is 15/30 (2.2).

Discussion

Recreational mathematics is hard to precisely define, but a study of the literature led to a working definition based on how the term is used in practice. This highlights the playful and mathematical aspects and includes that it might be widely accessible and used to motivate engagement with and develop mathematical understanding.

It is clear that recreational mathematics can be useful in education in several ways, including for engagement, to develop mathematical skills, to maintain interest during procedural practise, to challenge and stretch students and to make cross-curricular links. In undergraduate mathematics, recreational mathematics can be used for engagement with standard topics and for extra-curricular interest such as via the Maths Arcade. There are also opportunities to develop important skills that we expect of graduate mathematicians, including around problem-solving and communication. The explicit link between the benefits of recreational mathematics in education and the skills expected of graduate mathematicians is not made in the literature surveyed.

With this link in mind, we developed a module 'Game Theory and Recreational Mathematics' for final year undergraduate study. This provided an opportunity for fun and play, while developing key skills. This module allowed us to teach a little combinatorics, graph theory and algorithms/complexity and provided an opportunity to scaffold a Pólya-style problem-solving process via direct teaching and assessment.

Limited student feedback is positive about the principles and aims of the module, and indicates that students benefit from the explicit focus on problem-solving and understand the active nature of the learning. The scaffolded problem-solving process can be seen in evidence in student answers to exam questions. Further research is needed to investigate the process of setting unseen problems for exam papers and the responses and experiences of the students answering such questions, which will be attempted with the next cohort.

Reviewing the module, we feel it has gone very well for a first time and are looking forward to increased numbers next year. We suspect our students may be more open to this kind of teaching due to two aspects of the wider degree course. First, the availability of a Maths Arcade means, even if they don't regularly attend, students are aware of games making up a part of the activity of our community, normalising the use of recreations as a mathematical activity. Second, the widespread use of student-centred and active teaching methods in other modules in the degree, meaning students are not used to traditional 'chalk and talk' delivery in all modules. That said, we can imagine a place for something like our module in many degree courses. Certainly, there is value in not bombarding students with new ideas but making better use of existing ones, and allowing students greater ownership of their learning through flexible teaching and assessment methods.

Though we went quite far, the first experience teaching this module has taught us that more can be done to explicitly scaffold a problem-solving process and provide plenty of opportunity to practise. To this end, we are considering making the second coursework a series of tasks as continuous assessment, so that students are actively encouraged to exercise their problem-solving skills throughout the module. We are also considering introducing formal group assessment (rather than just delivery via group learning) because discussing problems with other people is part of the problem-solving process (especially around being stuck), and this has potential to contribute to the skill of communicating mathematical ideas 'with accuracy and clarity' [36,p.17].

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An excellent exam answer might be:

Some messy working.
Here is the problem in my own words: “…”.
However, I have noticed this is not quite the same problem because X is not Y.
So the crucial feature of the problem is Y. This is difficult because…
(Diagram illustrating the situation.)
The question asks for n=10. Here I solve the problem for n=0,1,2,3.
My approach does not work for n=4 because…
I could solve n=4 if condition k were relaxed.
The main trouble I am having is not being able to account for k.

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A quite poor answer might be:

Some messy working.
L=71,213.

Notes:

Say 71,213 is correct, but the problem does not define L.
This does not explain:

how the solution came about;
what L represents;
how you know this is correct;
how you knew to stop here.

The module is about problem-solving, and this is what we are trying to assess here. So this is what we will be looking for.

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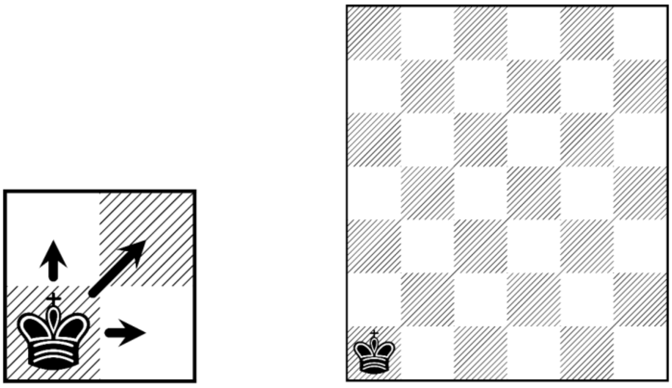
Figure 1: Guidance given to students on answering questions in exam section B.

On a 7×6 chessboard, how many ways are there for the king to move from the lower left square to the top right square? The only permitted moves are to adjacent squares moving up, right and diagonal-right up.



Permitted moves (left) and 7×6 board (right).

Figure 2: Sample question from exam section B.

1	13	85	377	1289	3653
1	11	61	231	681	1289
1	9	41	129	321	681
1	7	25	63	129	231
1	5	13	25	41	61
1	3	5	7	9	11
1	1	1	1	1	1

Figure 3: The number of ways of moving to each position on the board for the problem shown in figure 2.

If answer is correct:

Class	Indicative description	Marks range
1st class	Shows evidence of problem-solving process. Very well explained.	21-30
2.1	Some evidence of problem-solving, fairly well-explained OR just the answer but well explained.	18-20
2.2	Some evidence of problem-solving but not well-explained OR Just the answer with minimal explanation.	15-17
3rd class	Just states the answer with very little meaningful explanation.	12-14
Fail	Not meeting 3rd class.	0-11

If answer is not correct:

1st class	Clear evidence of problem-solving process. Well-explained. Making progress. Sound reasoning.	21-30
2.1	Evidence of problem-solving process, but with more flaws or not well-explained.	18-20
2.2	Limited evidence of problem-solving. More flaws and not well-explained.	15-17
3rd class	Incoherent, with little explanation.	12-14
Fail	Not meeting 3rd class.	0-11

Figure 4: Mark scheme for exam section B (unseen problems).