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On the derivatives of composite functions

J.K. Langley and E.F. Lingham

Abstract

Let g be a non-constant polynomial and let f be transcendental and meromorphic of sub-exponential growth in the plane. Then if $k \geq 2$ and Q is a polynomial the function $(f \circ g)^{(k)} - Q$ has infinitely many zeros. The same conclusion holds for $k \geq 0$ and with Q a rational function if f has finitely many poles. We also show by example that this result is sharp.

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1 Introduction

This paper will use standard notation of value distribution theory [7], including $\rho(f)$ for the order of growth of a meromorphic function f in the plane. In [4], the second author proved the following result, for application in a theorem concerning normal families.

Theorem A Let $k \in \mathbb{N}$. Let f be a transcendental entire function with $\rho(f) < 1/2$. Let g and Q be polynomials, with g non-constant. Let F and H be defined by

$$F = f \circ g, \quad H = F^{(k)} - Q. \tag{1}$$

Then H has infinitely many zeros.

The hypothesis $\rho(f) < 1/2$ was needed for the proof of Theorem A in [4], which made use of the celebrated $\cos \pi \rho$ minimum modulus theorem [8, Chapter 6]. In this paper, we show that Theorem A can be extended to transcendental functions of sub-exponential growth, that is, functions of at most order 1, minimal type. We state the result as follows.

Theorem 1.1 Let k be a non-negative integer. Let g be a non-constant polynomial and let f be a transcendental meromorphic function in the plane with

$$\lim_{r \to \infty} \frac{T(r, f)}{r} = 0. \tag{2}$$

Let F and H be defined by (1), with Q a rational function. Then the following conclusions hold. (a) If f has finitely many poles then the exponent of convergence of the zeros of H is equal to the order $\rho(F)$ of F.

(b) If k > 2 and Q is a polynomial, or if k = 1 and $Q \equiv 0$, then H has infinitely many zeros.

Theorem 1.1 is sharp in the following sense. If $f(w) = e^w + P(w)$ and P and g are polynomials with g non-constant, then $F = f \circ g$ satisfies $F^{(k)} = Re^g + S$ with R and S polynomials so that the equation $F^{(k)}(z) = S(z)$ has finitely many solutions in the plane.

The case k=0 of Theorem 1.1 may be compared with results on the frequency of fixpoints of $f\circ g$. Bergweiler proved that if g is a transcendental entire function and f is transcendental and meromorphic in the plane then f(g)-Q has infinitely many zeros, for any non-constant rational function Q [3] (see also [1, 2]). It seems plausible that if f is a transcendental entire function satisfying (2) and g is any non-constant entire function then the function H defined by (1) has infinitely many zeros, for any f and any rational function f some partial results are proved in [4] and [11, Theorem 6].

2 Proof of Theorem 1.1

We need the following result by the first author [10].

Lemma 2.1 Suppose that G is meromorphic of finite order in the plane, and that $G^{(k)}$ has finitely many zeros, for some $k \geq 2$. Then G has finitely many poles.

We now prove Theorem 1.1. We first note that since f has finite order and g is a polynomial it follows that

$$\rho(H) = \rho(F^{(k)}) = \rho(F) < \infty. \tag{3}$$

Next, we observe that it suffices to prove part (a). To see this, note first that the case where k=1 and $Q\equiv 0$ is handled by the argument of [11, p.137], and is based on the fact that (2) implies that f' has infinitely many zeros, by a result of Eremenko, Langley and Rossi [5], from which it follows since g is a non-constant polynomial that so has F'=f'(g)g'. Suppose next that $k\geq 2$, that Q is a polynomial, and that H has finitely many zeros. Choose a polynomial Q_1 with $Q_1^{(k)}=Q$ and set $G=F-Q_1$. Then $H=G^{(k)}$ and it follows from Lemma 2.1 that G has finitely many poles and, again since g is a polynomial, so has f. Hence H is a transcendental meromorphic function of finite order with finitely many zeros and poles so that $\rho(F)=\rho(H)\geq 1$, using (3), and a contradiction arises from part (a).

To prove part (a), assume that f has finitely many poles but that the exponent of convergence of the zeros of H is less than $\rho(F)$. It follows using (3) that there exist a positive integer n, a meromorphic function Π with finitely many poles and a polynomial P, such that

$$H = F^{(k)} - Q = \Pi e^P, \quad \rho(\Pi) < n, \quad \deg P = \rho(H) = n.$$
 (4)

Then we may write

$$g(z) = a_m z^m + \ldots + a_0, \quad P(z) = b_n z^n + \ldots + b_0, \quad b_n \in (0, \infty).$$
 (5)

Here the assumption that b_n is real and positive involves no loss of generality, since otherwise we may apply a rotation of the independent variable z.

Since f has finitely many poles and satisfies (2), a standard application of the Poisson-Jensen formula [7, p.1] gives (compare [7, Theorem 1.6, p.18])

$$\log M(r, f) \le 3m(2r, f) + O(\log r) = o(r) \quad \text{as} \quad r \to \infty. \tag{6}$$

Denote positive constants by c, M, not necessarily the same at each occurrence. Combining (3), (4), (5) and (6) and using the fact that f and F have finitely many poles, it follows that

$$cr^n \leq T(r, e^P)$$

$$\leq (1 + o(1))T(r, F^{(k)})$$

$$\leq (1 + o(1))T(r, F)$$

$$\leq (1 + o(1))\log M(r, F)$$

$$\leq (1 + o(1))\log M(M(r, g), f)$$

$$= o(M(r, g))$$

as $r \to \infty$. Using (5) again we deduce at once that

$$m > n.$$
 (7)

Let δ and η be positive constants, with δ and η/δ small. If r_1 is large then (4) and (5) give

$$F^{(k)}(z) = O(|z|^M)$$
 for $|z| \ge r_1$, $\frac{\pi}{2n} + \delta \le \arg z \le \frac{3\pi}{2n} - \delta$.

Hence repeated integration, starting from the point $r_1e^{i\pi/n}$, leads to

$$f(g(z)) = F(z) = O(|z|^{M+k}) \quad \text{for} \quad |z| \ge r_1, \quad \frac{\pi}{2n} + \delta \le \arg z \le \frac{3\pi}{2n} - \delta. \tag{8}$$

Let u_1, u_2, \ldots be the zeros of Π , repeated according to multiplicity. Since $\rho(\Pi) < n$ it follows that

$$\sum_{u_j \neq 0} |u_j|^{-n} < \infty$$

and a standard application of the Poisson-Jensen formula [7, p.1] gives

$$\log |\Pi(z)| = o(|z|^n) \quad \text{as} \quad z \to \infty, \quad z \notin E = \bigcup_{u_j \neq 0} B(u_j, |u_j|^{-n}). \tag{9}$$

Moreover, standard estimates based on the differentiated Poisson-Jensen formula [7, p.22] (see [6] and [9, p.89]) give

$$\frac{F^{(k)}(z)}{F(z)} = O(|z|^M) \tag{10}$$

for large z outside a union E' of discs, having finite sum of radii. Let r be large and positive, such that the circle |z|=r does not meet the exceptional sets E,E' corresponding to (9) and (10). Then (4), (5), (9) and (10) yield

$$\log|f(g(z))| = \log|F(z)| > cr^n \quad \text{for} \quad |z| = r, \quad |\arg z| \le \frac{\pi}{2n} - \delta. \tag{11}$$

With this same value of r let

$$\Omega_1 = \left\{ re^{i\theta} : -\frac{\pi}{2n} + 2\delta \le \theta \le \frac{\pi}{2n} - 2\delta \right\}, \quad \Omega_2 = \left\{ re^{i\theta} : \frac{\pi}{2n} + 2\delta \le \theta \le \frac{3\pi}{2n} - 2\delta \right\}. \tag{12}$$

Then (12) implies that for j=1,2 the image of the arc Ω_j under the mapping

$$\zeta = h(z) = a_m z^m \tag{13}$$

is the set $\{\zeta=|a_m|r^me^{i\phi}:\phi\in I_j\}$, where I_j is an interval of length

$$m\left(\frac{\pi}{n} - 4\delta\right) > \pi + \delta,$$

since δ is small by assumption and m > n by (7). Hence there exist

$$z_1 \in \Omega_1, \quad z_2 \in \Omega_2 \quad \text{such that} \quad w = h(z_1) = h(z_2).$$
 (14)

For $|z-z_2|=\eta r$ we have using (5), (13) and (14), since η is small and r is large,

$$|h(z) - h(z_1)| = |h(z) - h(z_2)| \ge cr^m$$
, $g(z) - h(z) = o(r^m)$, $h(z_1) - g(z_1) = o(r^m)$.

We then write

$$g(z) - g(z_1) = h(z) - h(z_1) + g(z) - h(z) + h(z_1) - g(z_1)$$

and apply Rouche's theorem, which shows using (14) again that g takes the value $g(z_1)$ at some $z_3 \in B(z_2, \eta r)$, provided r is large enough. Moreover,

$$\frac{\pi}{2n} + \delta \le \arg z_3 \le \frac{3\pi}{2n} - \delta,$$

using (12) and (14), since η is small compared to δ .

But now (8) gives

$$|f(g(z_1))| = |f(g(z_3))| \le c|z_3|^{M+k} \le c(|z_2| + \eta r)^{M+k} = O(r^{M+k}),$$

whereas (11), (12) and (14) give

$$\log|f(g(z_1))| > cr^n.$$

These estimates are obviously incompatible since r is large, and this contradiction completes the proof of part (a) and of Theorem 1.1.

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