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LINGHAM, Eleanor and LANGLEY, James K

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On the derivatives of composite functions

J.K. Langley and E.F. Lingham

Abstract

Let g be a non-constant polynomial and let f be transcendental and meromorphic of sub-exponential growth in the plane. Then if $k \geq 2$ and Q is a polynomial the function $(f \circ g)^{(k)} - Q$ has infinitely many zeros. The same conclusion holds for $k \geq 0$ and with Q a rational function if f has finitely many poles. We also show by example that this result is sharp.

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1 Introduction

This paper will use standard notation of value distribution theory [7], including $\rho(f)$ for the order of growth of a meromorphic function f in the plane. In [4], the second author proved the following result, for application in a theorem concerning normal families.

Theorem A *Let $k \in \mathbb{N}$. Let f be a transcendental entire function with $\rho(f) < 1/2$. Let g and Q be polynomials, with g non-constant. Let F and H be defined by*

$$F = f \circ g, \quad H = F^{(k)} - Q. \tag{1}$$

Then H has infinitely many zeros.

The hypothesis $\rho(f) < 1/2$ was needed for the proof of Theorem A in [4], which made use of the celebrated $\cos \pi\rho$ minimum modulus theorem [8, Chapter 6]. In this paper, we show that Theorem A can be extended to transcendental functions of sub-exponential growth, that is, functions of at most order 1, minimal type. We state the result as follows.

Theorem 1.1 *Let k be a non-negative integer. Let g be a non-constant polynomial and let f be a transcendental meromorphic function in the plane with*

$$\lim_{r \to \infty} \frac{T(r, f)}{r} = 0. \tag{2}$$

Let F and H be defined by (1), with Q a rational function. Then the following conclusions hold.
(a) If f has finitely many poles then the exponent of convergence of the zeros of H is equal to the order $\rho(F)$ of F.
(b) If $k \geq 2$ and Q is a polynomial, or if $k = 1$ and $Q \equiv 0$, then H has infinitely many zeros.

Theorem 1.1 is sharp in the following sense. If $f(w) = e^w + P(w)$ and P and g are polynomials with g non-constant, then $F = f \circ g$ satisfies $F^{(k)} = Re^g + S$ with R and S polynomials so that the equation $F^{(k)}(z) = S(z)$ has finitely many solutions in the plane.

The case $k = 0$ of Theorem 1.1 may be compared with results on the frequency of fixpoints of $f \circ g$. Bergweiler proved that if g is a transcendental entire function and f is transcendental and meromorphic in the plane then $f(g) - Q$ has infinitely many zeros, for any non-constant rational function Q [3] (see also [1, 2]). It seems plausible that if f is a transcendental entire function satisfying (2) and g is any non-constant entire function then the function H defined by (1) has infinitely many zeros, for any $k \geq 1$ and any rational function Q. Some partial results are proved in [4] and [11, Theorem 6].

2 Proof of Theorem 1.1

We need the following result by the first author [10].

Lemma 2.1 *Suppose that G is meromorphic of finite order in the plane, and that $G^{(k)}$ has finitely many zeros, for some $k \geq 2$. Then G has finitely many poles.*

We now prove Theorem 1.1. We first note that since f has finite order and g is a polynomial it follows that

$$\rho(H) = \rho(F^{(k)}) = \rho(F) < \infty. \tag{3}$$

Next, we observe that it suffices to prove part (a). To see this, note first that the case where $k = 1$ and $Q \equiv 0$ is handled by the argument of [11, p.137], and is based on the fact that (2) implies that f' has infinitely many zeros, by a result of Eremenko, Langley and Rossi [5], from which it follows since g is a non-constant polynomial that so has $F' = f'(g)g'$. Suppose next that $k \geq 2$, that Q is a polynomial, and that H has finitely many zeros. Choose a polynomial Q_1 with $Q_1^{(k)} = Q$ and set $G = F - Q_1$. Then $H = G^{(k)}$ and it follows from Lemma 2.1 that G has finitely many poles and, again since g is a polynomial, so has f. Hence H is a transcendental meromorphic function of finite order with finitely many zeros and poles so that $\rho(F) = \rho(H) \geq 1$, using (3), and a contradiction arises from part (a).

To prove part (a), assume that f has finitely many poles but that the exponent of convergence of the zeros of H is less than $\rho(F)$. It follows using (3) that there exist a positive integer n, a meromorphic function Π with finitely many poles and a polynomial P, such that

$$H = F^{(k)} - Q = \Pi e^P, \quad \rho(\Pi) < n, \quad \deg P = \rho(H) = n. \tag{4}$$

Then we may write

$$g(z) = a_m z^m + \ldots + a_0, \quad P(z) = b_n z^n + \ldots + b_0, \quad b_n \in (0, \infty). \tag{5}$$

Here the assumption that b_n is real and positive involves no loss of generality, since otherwise we may apply a rotation of the independent variable z.

Since f has finitely many poles and satisfies (2), a standard application of the Poisson-Jensen formula [7, p.1] gives (compare [7, Theorem 1.6, p.18])

$$\log M(r, f) \leq 3m(2r, f) + O(\log r) = o(r) \quad \text{as} \quad r \to \infty. \tag{6}$$

Denote positive constants by c, M, not necessarily the same at each occurrence. Combining (3), (4), (5) and (6) and using the fact that f and F have finitely many poles, it follows that

$$cr^n \leq T(r, e^P)$$

$$
\begin{aligned}
&\leq (1+o(1))T(r,F^{(k)}) \\
&\leq (1+o(1))T(r,F) \\
&\leq (1+o(1))\log M(r,F) \\
&\leq (1+o(1))\log M(M(r,g),f) \\
&= o(M(r,g))
\end{aligned}
$$

as $r \to \infty$. Using (5) again we deduce at once that

$$
m > n. \tag{7}
$$

Let δ and η be positive constants, with δ and η/δ small. If r_1 is large then (4) and (5) give

$$
F^{(k)}(z) = O(|z|^M) \quad \text{for} \quad |z| \geq r_1, \quad \frac{\pi}{2n} + \delta \leq \arg z \leq \frac{3\pi}{2n} - \delta.
$$

Hence repeated integration, starting from the point $r_1 e^{i\pi/n}$, leads to

$$
f(g(z)) = F(z) = O(|z|^{M+k}) \quad \text{for} \quad |z| \geq r_1, \quad \frac{\pi}{2n} + \delta \leq \arg z \leq \frac{3\pi}{2n} - \delta. \tag{8}
$$

Let $u_1, u_2, \ldots$ be the zeros of Π, repeated according to multiplicity. Since $\rho(\Pi) < n$ it follows that

$$
\sum_{u_j \neq 0} |u_j|^{-n} < \infty
$$

and a standard application of the Poisson-Jensen formula [7, p.1] gives

$$
\log |\Pi(z)| = o(|z|^n) \quad \text{as} \quad z \to \infty, \quad z \notin E = \bigcup_{u_j \neq 0} B(u_j, |u_j|^{-n}). \tag{9}
$$

Moreover, standard estimates based on the differentiated Poisson-Jensen formula [7, p.22] (see [6] and [9, p.89]) give

$$
\frac{F^{(k)}(z)}{F(z)} = O(|z|^M) \tag{10}
$$

for large z outside a union E' of discs, having finite sum of radii. Let r be large and positive, such that the circle $|z| = r$ does not meet the exceptional sets E, E' corresponding to (9) and (10). Then (4), (5), (9) and (10) yield

$$
\log |f(g(z))| = \log |F(z)| > cr^n \quad \text{for} \quad |z| = r, \quad |\arg z| \leq \frac{\pi}{2n} - \delta. \tag{11}
$$

With this same value of r let

$$
\Omega_1 = \left\{ re^{i\theta} : -\frac{\pi}{2n} + 2\delta \leq \theta \leq \frac{\pi}{2n} - 2\delta \right\}, \quad \Omega_2 = \left\{ re^{i\theta} : \frac{\pi}{2n} + 2\delta \leq \theta \leq \frac{3\pi}{2n} - 2\delta \right\}. \tag{12}
$$

Then (12) implies that for $j = 1, 2$ the image of the arc Ω_j under the mapping

$$
\zeta = h(z) = a_m z^m \tag{13}
$$

is the set $\{\zeta = |a_m| r^m e^{i\phi} : \phi \in I_j\}$, where I_j is an interval of length

$$
m\left(\frac{\pi}{n} - 4\delta\right) > \pi + \delta,
$$

since δ is small by assumption and $m > n$ by (7). Hence there exist

$$z_1 \in \Omega_1, \quad z_2 \in \Omega_2 \quad \text{such that} \quad w = h(z_1) = h(z_2). \tag{14}$$

For $|z - z_2| = \eta r$ we have using (5), (13) and (14), since η is small and r is large,

$$|h(z) - h(z_1)| = |h(z) - h(z_2)| \geq cr^m, \quad g(z) - h(z) = o(r^m), \quad h(z_1) - g(z_1) = o(r^m).$$

We then write

$$g(z) - g(z_1) = h(z) - h(z_1) + g(z) - h(z) + h(z_1) - g(z_1)$$

and apply Rouché's theorem, which shows using (14) again that g takes the value $g(z_1)$ at some $z_3 \in B(z_2, \eta r)$, provided r is large enough. Moreover,

$$\frac{\pi}{2n} + \delta \leq \arg z_3 \leq \frac{3\pi}{2n} - \delta,$$

using (12) and (14), since η is small compared to δ.

But now (8) gives

$$|f(g(z_1))| = |f(g(z_3))| \leq c|z_3|^{M+k} \leq c(|z_2| + \eta r)^{M+k} = O(r^{M+k}),$$

whereas (11), (12) and (14) give

$$\log |f(g(z_1))| > cr^n.$$

These estimates are obviously incompatible since r is large, and this contradiction completes the proof of part (a) and of Theorem 1.1.

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School of Mathematical Sciences, University of Nottingham, NG7 2RD

Department of Mathematical Sciences, Loughborough University, LE11 3TU