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FENTON, P.C. and LINGHAM, Eleanor

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The size of Wiman-Valiron discs for subharmonic functions of a certain type

P.C. Fenton
University of Otago, Dunedin, New Zealand
pfenton@maths.otago.ac.nz

E.F. Lingham
De Montfort University, Leicester, United Kingdom
elingham@dmu.ac.uk

Abstract

Wiman-Valiron theory and the results of Macintyre about “flat regions” describe the asymptotic behaviour of entire functions in certain discs around maximum points. We use a technique of Bergweiler, Rippon and Stallard to describe the asymptotic behaviour of a certain type of subharmonic function, and a technique of Bergweiler to estimate the size of its Wiman-Valiron discs from above and below. The results are extended to $\delta$-subharmonic functions.

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1 Introduction

Given $r > 0$ and an upper semi-continuous function $u : \mathbb{C} \to [-\infty, \infty)$, let $B(r, u) := \max_{|z|=r} u(z)$ be its maximum function. We recall that if $u$ is subharmonic then $B(r, u)$ is a convex, increasing function of $\log r$. The left and right derivatives of $B(r, u)$ thus exist, and $B(r, u)$ is differentiable outside a countable set. We write

$$a(r, u) := \frac{dB(r, u)}{d \log r},$$

(1)

taking the right derivative at points where $B(r, u)$ is not differentiable.

We say that $z_r$ is a maximum point of $u$ if $|z_r| = r$ and $u(z_r) = B(r, u)$. Slightly abusing this notation, we call $z_r$ a maximum point for a holomorphic function $f$ if it is a maximum point for the subharmonic function $\log |f|$. We denote by $D(a, r) = \{ z : |z - a| < r \}$ the disc centred at $a$ of radius $r$, and say that a set $F \subseteq [0, \infty)$ has logarithmic measure $\int_F \frac{dt}{t}$. For a subharmonic function $u$, we denote by $\mu_u$ the Riesz measure of $u$.

A result of Macintyre ([13], Theorem 3) says that an entire function $f$ can be estimated on a disc near its maximum point. That is, given $\epsilon > 0$, there exists
a set $F$ of finite logarithmic measure such that
\[ f(z) \sim \left( \frac{z}{z_r} \right)^{a(r, \log |f|)} f(z_r) \] (2)
as $r \to \infty$, $r \notin F$, uniformly for $z \in D(z_r, r/(\log M(r, f))^{\frac{1}{2} + \epsilon})$. As Bergweiler, Rippon and Stallard point out [2, p. 372], it can be deduced from (2) that, for each $k \in \mathbb{N} \cup \{0\}$,
\[ f^{(k)}(z) \sim \left( \frac{a(r, \log |f|)}{z} \right)^k \left( \frac{z}{z_r} \right)^{a(r, \log |f|)} f(z_r) \]
as $r \to \infty$, $r \notin F$, uniformly for $z \in D(z_r, r/(\log M(r, f))^{\frac{1}{2} + \epsilon})$. This work followed results from Wiman-Valiron theory which found that sufficiently close to their maximum points, entire functions act like monomials, namely the dominant term of their Taylor power series (see [15], [14], [13]).

More recently, Bergweiler, Rippon and Stallard [2] proved a result similar to (2) where it is not required that $f$ is entire, but only that $f$ has a direct tract. To explain this, let $D$ be an unbounded domain in $\mathbb{C}$ whose boundary consists of piecewise smooth curves, and whose complement is unbounded. Let $f$ be a complex-valued function whose domain of definition contains the closure $\overline{D}$ of $D$. Then $D$ is called a direct tract of $f$ if $f$ is holomorphic in $D$ and continuous in $\overline{D}$, and there exists $R > 0$ such that $|f(z)| = R$ for $z$ on the boundary of $D$, while $|f(z)| > R$ for $z \in D$.

The main result of [2] says that for every $\tau > \frac{1}{2}$, there exists a set $F$ of finite logarithmic measure such that, for $r \notin F$, the disc $D(z_r, r/a(r, \log |f|)^{\tau})$ is contained in $D$ and (2) holds for $z \in D(z_r, r/a(r, \log |f|)^{\tau})$ as $r \to \infty$. In [1], Bergweiler investigated the size of the disc around $z_r$ in which (2) holds (which can be described as a Wiman-Valiron disc), and proved results from below and above as follows. Let $\phi : [t_0, \infty) \to (0, \infty)$ be a differentiable function satisfying
\[ \int_{t_0}^{\infty} \frac{1}{\phi(t)} dt < \infty \] (3)
and
\[ K \leq \frac{t \phi'(t)}{\phi(t)} \leq L \] (4)
for certain constants $K$ and $L$ satisfying $0 \leq K \leq 1 < L < 2$. Let $f$ be a function with a direct tract $D$ and let $z_r \in D$ be a maximum point of $f$ in $D$. Then there exists a set $F$ of finite logarithmic measure such that the disc $D(z_r, r/\sqrt{\phi(a(r, \log |f|))}) \subseteq D$ and (2) holds on $D(z_r, r/\sqrt{\phi(a(r, \log |f|))})$ uniformly as $r \to \infty$, $r \notin F$. On the other hand, if
\[ \int_{t_0}^{\infty} \frac{1}{\phi(t)} dt = \infty \]
and (4) holds for $K = 1$ and some $L < \frac{6}{5}$, then there exists an entire function $f$ which has exactly one direct tract $D$ and is such that if $r$ is sufficiently large and $|z| = r$, then the disc $D(z, r/\sqrt{\phi(a(r, \log |f|))})$ contains a zero of $f$, and thus (2) cannot hold.

With these results in mind, we seek analogous results for subharmonic functions of the form

$$u(z) = \sum_{j=1}^{\infty} c_j \log \left| 1 - \frac{z}{z_j} \right|,$$

(5)

where $c_j \in \mathbb{R}^+$, $z_j \in \mathbb{C}$ and $\sum_{j=1}^{\infty} \left| \frac{c_j}{z_j} \right| < \infty$. In any disc that contains no $z_j$, $u$ can also be written in the form

$$u(z) = \log |f(z)|,$$

where $f(z) = \prod_{j=1}^{\infty} \left( 1 - \frac{z}{z_j} \right)^{c_j}$. The $c_j$ are the Riesz masses of $u$ at the points $z_j$; at all other points $u$ is harmonic and the Riesz mass is zero. Since the $z_j$ are the only “problem” points of $u$ in $\mathbb{C}$, we do not need to assume the existence of a direct tract as in [2] and [1].

To achieve results for $u$, it is necessary to impose some lower growth condition on the masses $c_j$. For if, for example, the sequence $z_j$ consisted of all the rational points in the plane, then every open disc would contain a $z_j$ and we could not have (2). We will assume that

$$\lim_{j \to \infty} \frac{c_j}{I(a(|z_j|, u))^2} > 0$$

(6)

where

$$I(r) := \int_r^{\infty} \frac{1}{\phi(t)} \, dt$$

(7)

and $\beta$ is a positive constant.

We are now ready to state our first theorem.

**Theorem 1** Let $u$ be as in (5) with the growth condition (6) on the masses $c_j$ of $u$, for some constant $\beta > 0$. Let $t_0 > 0$ and let $\phi : [t_0, \infty) \to (0, \infty)$ be a differentiable function satisfying (3) and (4) for constants $K = \min\{\beta, 1\}$ and $L$ satisfying $1 < L < 3/2$. Let $r > 0$. If $\beta < 1$ there exists a set $F \subseteq [0, \infty)$ of finite logarithmic measure such that the disc $D(z_r, r/\sqrt{\phi(a(r, u))})$ contains no $z_j$, and (2) holds uniformly for $z \in D(z_r, r/\sqrt{\phi(a(r, u))})$, as $r \to \infty$, $r \notin F$.

Remark: Theorem 1 can be extended to $\delta$-subharmonic functions, that is, functions

$$v(z) = v_1(z) - v_2(z),$$

3
where \( v_1, v_2 \) are subharmonic functions of the form (5). We assume that the set \( D = \{ z \in \mathbb{C} : v(z) > 0 \} \) has an unbounded component \( D_0 \) on which \( v \) is harmonic. The function which is \( v \) on \( D_0 \) and 0 elsewhere, which we denote by \( v^+ \), is then subharmonic. The complement of \( D_0 \) may contain islands \( I \), that is, closed bounded components of the form \( \{ z : v^+(z) = 0 \} \), and the growth condition we impose on \( v_1 \) and \( v_2 \) involves islands, rather than points, as follows. Either there are finitely many islands, or

\[
\lim_{r \to \infty} \frac{\mu_{v_1}(I) - \mu_{v_2}(I)}{I(a(r^+, u))^\beta} > 0,
\]

where \( 0 < \beta < 1 \) and \( r^+ = \sup \{ |z| : z \in I \} \). The proof relies on the fact that \( \mu_{v^+}(I) = \mu_{v_1}(I) - \mu_{v_2}(I) \), but is otherwise similar to the proof of Theorem 1.

We state our second theorem as follows.

**Theorem 2** Let \( t_0 > 0 \) and let \( \phi : [t_0, \infty) \to (0, \infty) \) be a differentiable function that satisfies (3). Let \( I \) be as defined in (7) and suppose that, given \( \kappa > 1 \) and \( \lambda > 0 \),

\[
1 \leq \frac{r \phi'(r)}{\phi(r)} - \frac{(1 + (\kappa - 1)/\lambda)r}{\phi(r)I(r)} \leq \frac{r \phi'(r)}{\phi(r)} \leq L,
\]

for \( r \geq t_0 \), where \( 1 < L < 6/5 \). Then there exists a subharmonic function of the form (5), with

\[
c_j = (1 + o(1))(I(a(|z_j|, u)))^{\kappa/\lambda} \quad (j \to \infty),
\]

for which, for all large \( r \), there are no Wiman-Valiron discs of radius greater than

\[
r \sqrt{\phi(a(r, u))I(a(r, u))}^{1-1/\lambda}.
\]

**Corollary 1** Theorem 1 fails if \( \beta > 1 \).

Assuming Theorem 2 for the moment, let us prove Corollary 1. The function

\[
\phi(r) = \eta^{-1} r \cdot \log r \cdot \log_2 r \cdots \log_l r \cdot (\log_{l+1} r)^{1+\eta},
\]

where \( \eta > 0 \), \( l \) is a positive integer and \( \log_t \) is the \( t \)-times iterated logarithm, satisfies (3) and, for any \( \lambda > 0 \) and \( \kappa > 1 \), satisfies (8) for all large \( r \). For in that case \( I(r) = (\log_{l+1} r) ^{-\eta} \) and

\[
\frac{r \phi'(r)}{\phi(r)} - \frac{(1 + (\kappa - 1)/\lambda)r}{\phi(r)I(r)}
= 1 + \frac{1}{\log r} + \cdots + \frac{1}{\log r \log_2 r \cdots \log_l r} + \frac{1 - (\kappa - 1)\eta/\lambda}{\log r \log_2 r \cdots \log_{l+1} r}.
\]
Thus, for this $\phi$, the result of Theorem 2 holds for any $\kappa > 1$ and $\lambda > 0$. If $\beta > 1$ we may choose $\kappa$ and $\lambda$ satisfying $\kappa/\lambda = \beta$ with $\kappa > 1$ and $\lambda < 1$, and then (10) contradicts the conclusion of Theorem 1.

Remarks: 1. The case $\beta = 1$ is open.
2. Suppose that $0 < \beta < 1$ and that in Theorem 2 we choose $\kappa > 1$ arbitrarily and $\lambda = \kappa/\beta$. The difference between the radii in Theorem 1 and Theorem 2 is $\sqrt{I(a(r,u))}^{1-\beta/\kappa}$. To get some idea of the significance of this factor, consider $\phi$ of (11). The radius of the disc in Theorem 1 is

$$\eta^{-1} t \cdot \log t \cdot \log_2 t \cdot \log_3 t \cdots \log_{l+1} t \cdot (\log_{l+1} t)^{1+\eta} t = a(r,u)$$

and in Theorem 2 is

$$\eta^{-1} t \cdot \log t \cdot \log_2 t \cdot \log_3 t \cdots \log_{l+1} t \cdot (\log_{l+1} t)^{1+\beta \eta/\kappa} t = a(r,u).$$

Numerous applications of the theories of Wiman-Valiron and Macintyre exist, in areas including complex dynamics ([2], [5], [9]), complex differential equations ([6], [10], [11], [16]) and the zero distribution of derivatives ([3], [12]). We note also that functions of the form (5) are connected to the electrostatic fields generated by positively charged wires which meet the complex plane at the points $z_j$ [4].

2 Proof of Theorem 1

First, with $u$ given by (5), we have $a(r,u) \to \infty$ as $r \to \infty$. For otherwise $B(r,u) = O(\log r)$ and therefore $\mu_u(D(0,r)) = O(1)$ as $r \to \infty$, which gives that $\sum_{j \in \mathbb{N}} c_j < \infty$. By (6) the $c_j$ would be bounded below, and this would mean that there are only finitely many $c_j$, which is a contradiction.

We need two lemmas. The first is based on a well-known result about real functions and can be found in ([2], Lemma 6.10).

**Lemma 3** Let $v : \mathbb{C} \to [-\infty, \infty)$ be subharmonic, and let $\epsilon > 0$. Then there exists a set $F \subseteq [1, \infty)$ of finite logarithmic measure such that

$$a(r,v) \leq B(r,v)^{1+\epsilon}$$

for $r \geq 1$, $r \notin F$.

The second lemma summarises, with a small change, the lemmas and discussion in ([1], Section 2). The change we make concerns two functions, $\sigma_1$ and $\sigma_2$, that occur in the proof of Lemma 2.2 of [1]. Rather than (in the notation of [1]) $\sigma_1 = V^{R/2}\sqrt{\psi}$, we take $\sigma_1 = \sqrt{\psi}$ and $\sigma_2 = V^R \sqrt{\psi}$; the proof is unchanged except that the restriction on the constant $L$ needs to be strengthened (from $L < 2$ to $L < 3/2$).
Lemma 4 Let $t_0 > 0$, let $v : \mathbb{C} \to [-\infty, \infty)$ be a subharmonic function, and let $\phi : [t_0, \infty) \to (0, \infty)$ be a differentiable function satisfying (3) and (4) for certain constants $0 < K < 1 < L < 3/2$. Then there exists a set $F \subseteq [t_0, \infty)$ of finite logarithmic measure such that

$$B(s, v) \leq B(r, v) + a(r, v) \log \frac{s}{r} + I(a(r, v))^K$$

for

$$|\log \frac{s}{r}| \leq \frac{1}{\sqrt{\phi(a(r, v))}},$$

uniformly as $r \to \infty$, $r \notin F$.

We apply Lemma 4 with $K = \beta$ and $L$ satisfying $1 < L < 3/2$, and we apply Lemma 3 for $\epsilon = \beta$. Let $F$ be the union of the exceptional sets of these lemmas. We put $\rho = 2r/\phi(a(r, u))^{\beta/2}$ whenever $r$ is so large that $a(r, u) \neq 0$. Let $C$ be a positive constant sufficiently large that for all large $r$ and any $z_j \in D(z_r, C \rho)$ we have

$$c_j > (\log C)^{-1} I(a(|z_j|, u))^\beta,$$

which is possible by (6). We consider the function

$$v(z) = u(z) - a(r, u) \log \frac{|z|}{r} = u(z) - B(r, u) - a(r, u) \log \frac{|z|}{r}. \quad (14)$$

For $z \in \overline{D(z_r, C \rho)}$ we have

$$\left| \frac{z - z_r}{z_r} \right| \leq \frac{C \rho}{r} = \frac{2C}{\phi(a(r, u))^{3/2}} = o(1)$$

as $r \to \infty$, by (4) and since $a(r, u) \to \infty$. Thus since $\beta < 1$,

$$\left| \log \frac{|z|}{r} \right| = \left| \log \left( 1 + \frac{z - z_r}{z_r} \right) \right| \leq 2 \left| \frac{z - z_r}{z_r} \right| \leq \frac{4C}{\phi(a(r, u))^{3/2}} \leq \frac{1}{\sqrt{\phi(a(r, f))}}$$

for large $r$, and the hypotheses of Lemma 4 are satisfied. Since

$$v(z) \leq B(|z|, u) - B(r, u) - a(r, u) \log \frac{|z|}{r}$$

by (14), we conclude from Lemma 4 applied to $u$ that

$$v(z) \leq I(a(r, u))^\beta$$

as $r \to \infty$, $r \notin F$.

For large $r$, we have $0 \notin D(z_r, \rho)$ so that the difference of $v$ and $u$ as defined in (14) is harmonic in $D(z_r, \rho)$, and hence their Riesz measures in the disc coincide. Thus

$$\mu_v(D(z_r, \rho)) = \sum_{z_j \in D(z_r, \rho)} c_j$$

(16)
where the $z_j$ are matched with the $c_j$ in (5).

On the other hand, we have $\mu_v(D(z_r, \rho)) \log C \leq \int_0^{C \rho} \frac{\mu_v(D(z_r, t))}{t} \, dt$, and by (15) and the fact that $v(z_r) = 0$,

$$
\int_0^{C \rho} \frac{\mu_v(D(z_r, t))}{t} \, dt = \frac{1}{2\pi} \int_0^{2\pi} v(z_r + C\rho e^{i\zeta}) \, d\zeta - v(z_r)
\leq \frac{1}{2\pi} \int_0^{2\pi} I(a(r, u))^\beta \, d\zeta
= I(a(r, u))^\beta.
$$

Combining this with (16) gives

$$
\sum_{z_j \in D(z_r, C\rho)} c_j \log C \leq I(a(r, u))^\beta.
$$

In view of (13), we deduce that there do not exist any $z_j \in D(z_r, C\rho)$, and thus in $D(z_r, r/\sqrt{\phi(a(r, u))})$.

The remainder of the proof, including the derivation of (2), reproduces the arguments in ([2], Theorem 2.2).

3 Proof of Theorem 2, preliminaries

We follow Bergweiler’s intricate construction with slight changes, first introducing certain auxiliary functions in terms of which the example of Theorem 2 is defined. The main concern in this section is to show that certain key calculations that Bergweiler makes carry over in their modified form. Wherever possible, results from [1] have been simply quoted, with appropriate references.

Let $\phi$ be as in the statement of Theorem 2 and define $\Psi, \chi : [t_0, \infty) \to (0, \infty)$ by

$$
\Psi(t) = \lambda \phi(t) I(t)^{1+\kappa-1}/\lambda
$$

(17)
and

$$
\chi(t) = \lambda \phi(t) I(t)^{1-1/\lambda}
$$

(18)
where $\lambda$ and $\kappa$ are constants satisfying $\lambda > 2$ and $\lambda/2 \geq \kappa > 1$, and $I(t)$ is given by (7). Note that

$$
\int_{t_0}^t \frac{ds}{\Psi(s)} = \frac{1}{\kappa - 1} \left( I(t)^{-(\kappa-1)/\lambda} - I(t_0)^{-(\kappa-1)/\lambda} \right), \quad \int_{t_0}^{\infty} \frac{ds}{\chi(s)} = I(t)^{1/\lambda}.
$$

(19)

From the first of these we have

$$
I'(t) = -\frac{\lambda I(t)^{1+\kappa-1}/\lambda}{\Psi(t)}.
$$

(20)
In what follows it is helpful to know that $I(t) \leq 1$ for $t \geq t_0$, which can always be achieved by taking a somewhat larger value of $t_0$. Let us suppose that this has been done.

As in [1], we define $A_1 : [1, \infty) \to [t_0, \infty)$ by

$$
\log t = \int_{t_0}^{A_1(t)} \frac{1}{\Psi(s)} ds.
$$

From the first part of (19) we have

$$
I(A_1(t)) = ((\kappa - 1) \log(C_1 t))^{-\lambda/(\kappa - 1)},
$$

where $C_1 = \exp\{((\kappa - 1)^{-1} I(t_0)^{-(\kappa - 1)/\lambda})$. We also introduce

$$
A_2(t) := t A_1'(t), \quad A_3(t) := t A_2'(t), \quad A_0(t) := \int_1^t A_1(s) \frac{ds}{s}.
$$

Differentiating (21), we have

$$
A_2(t) = \Psi(A_1(t));
$$

also (see [1], (3.4))

$$
A_3(t) \geq A_2(t) \geq A_1(t) \geq t, \quad t \geq 1,
$$

and (see [1], (3.10))

$$
A_2(t) = o\left(A_0(t)^{\frac{1}{\lambda}}\right)
$$

as $t \to \infty$, where $L$ is the number of (8). Define $G : [1, \infty) \to [0, \infty)$ by

$$
G(t) = \int_1^t \sqrt{\chi(A_1(s)) \frac{ds}{s}},
$$

and let $H : [0, \infty) \to [1, \infty)$ be the inverse function. We have, using (17), (18), (24), (25) and the fact that $I \leq 1$,

$$
\frac{H(t)}{H'(t)} = \sqrt{\chi(A_1(H(t)))} = \sqrt{\frac{\Psi(A_1(H(t)))}{I(A_1(H(t)))}^{\kappa/\lambda}} = \sqrt{\frac{A_2(H(t))}{I(A_1(H(t)))}^{\kappa/\lambda}} \geq 1. \tag{28}
$$

Also, observing that from (20) and (24),

$$
I'(A_1(t)) = \frac{\chi(A_1(H(t)))^{1-(\kappa - 1)/\lambda}}{A_2(t)}
$$

we obtain after some calculation, and using (23),

$$
\frac{d}{dt} \left( \frac{H(t)}{H'(t)} \right) = \frac{d}{dt} \left( \sqrt{\frac{A_2(H(t))}{I(A_1(H(t)))}^{\kappa/\lambda}} \right)
= \frac{A_3(H(t))}{2A_2(H(t))} + \frac{\kappa}{2} I(A_1(H(t)))^{(\kappa - 1)/\lambda}. \tag{29}
$$
Since \((H/H')' = 1 - HH''/(H')^2\), we have from (29),
\[
\frac{H''(t)}{H'(t)} = \sqrt{\frac{I(A_1(H(t)))^{\kappa/\lambda}}{A_2(H(t))} \left( 1 - \frac{A_3(H(t))}{2A_2(H(t))} - \frac{\kappa}{2} I(A_1(H(t)))^{(\kappa-1)/\lambda} \right)}.
\]
(30)

From (29), (25) and the fact that \(I \leq 1\) we have
\[
\frac{d}{dt} \left( H(t) \frac{H'(t)}{H'(t)} \right) \leq (\kappa + 1) \frac{A_3(H(t))}{2A_2(H(t))}.
\]
(31)

We need a related estimate, for \(\frac{d}{dt} \left( I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left( \frac{r}{H(t)} \right) \right)\). Using (20), (24) and (31) we have, for \(r\) fixed and \(t\) such that \(H(t) \leq r\),
\[
\left| \frac{d}{dt} \left( I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left( \frac{r}{H(t)} \right) \right) \right| \leq \left( \kappa + (\kappa + 1) \frac{A_3(H(t))}{2A_2(H(t))} \right) \log r + 1.
\]

Since, as Bergweiler has shown (see [1], (3.15) and (3.2)), \(A_3/A_2 \leq Le^{1/L}A_2^{1-1/L}\), where \(c = \Psi(h_0)\) and \(L\) is the number of (8), the preceding estimate gives
\[
\left| \frac{d}{dt} \left( I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left( \frac{r}{H(t)} \right) \right) \right| \leq \Lambda A_2(r)^{1-1/L} \log r + 1
\]
(32)

for \(H(t) \leq r\), where \(\Lambda = \kappa + L(\kappa + 1)c^{1/L}/2\).

Finally, from (30) we have (as in [1] between formulas (3.13) and (3.14))
\[
\frac{H''(t)}{H'(t)} \leq \frac{1}{2} (\kappa + 3) \frac{A_3(H(t))}{A_2(H(t))^{3/2}} = o(1)
\]
as \(t \to \infty\), and we deduce from this (cf [1], (3.14)) that
\[
H'(t + s) = (1 + o(1))H'(t)
\]
as \(t \to \infty\), uniformly for \(0 \leq s \leq 1\).

4 Proof of Theorem 2

Consider
\[
u(z) = \sum_{j=1}^{\infty} c_j \log \left| 1 + \left( \frac{z}{H(j)} \right)^{\left[ \frac{r(j)}{H(j)} \right]} \right|,
\]
(34)
where \( \left\lfloor \frac{H(j)}{H'(j)} \right\rfloor \) means the integer part of \( \frac{H(j)}{H'(j)} \), and with the notation of the preceding section,

\[
c_j = I(A_1(H(j)))^{\kappa/\lambda}.
\]

Since the terms of the series (34) are subharmonic and the series converges locally uniformly (as will be evident from our subsequent calculations), \( u \) is subharmonic.

As in [1] we write

\[
A_j = \log \left( 1 + \left( \frac{r}{H(j)} \right) \left\lfloor \frac{H(j)}{H'(j)} \right\rfloor \right),
\]

for \( r \geq 0 \), and

\[
S_1 = \sum_{j=1}^{[G(r)]} c_j A_j, \quad S_2 = \sum_{j=[G(r)]+1}^{[G(\rho r)]} c_j A_j, \quad S_3 = \sum_{j=[G(\rho r)]+1}^{\infty} c_j A_j,
\]

where \( G \) is as in (27) and \( \rho = 1 + A_1(r)/(2A_2(r)) \). Then

\[
B(r, u) \leq S_1 + S_2 + S_3,
\]

and we estimate \( S_1, S_2 \) and \( S_3 \) in turn.

For \( S_1 \), since \( H(j) \leq r \) for \( j \leq G(r) \) we have, using (28), (32) and Bergweiler’s Lemma 3.1 of [1],

\[
S_1 \leq \sum_{j=1}^{[G(r)]} I(A_1(H(j)))^{\kappa/\lambda} \left( \frac{H(j)}{H'(j)} \right) \log \left( \frac{r}{H(j)} \right) + \log 2
\]

\[
\leq \sum_{j=1}^{[G(r)]} I(A_1(H(j)))^{\kappa/\lambda} \frac{H(j)}{H'(j)} \log \left( \frac{r}{H(j)} \right) + G(r) \log 2
\]

\[
\leq \int_{0}^{G(r)} I(A_1(H(j)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left( \frac{r}{H(t)} \right) \, dt
\]

\[
+ (\Lambda A_2(r)^{1-1/L} \log r + 1)G(r) + G(r) \log 2
\]

\[
= \int_{0}^{G(r)} I(A_1(H(j)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left( \frac{r}{H(t)} \right) \, dt
\]

\[
+ O \left( G(r)A_2(r)^{1-1/L} \log r \right)
\]

(37)
as \( r \to \infty \). With the change of variable \( s = H(t) \) we obtain

\[
\int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left( \frac{r}{H(t)} \right) dt
\]

\[
= \int_1^r I(A_1(s))^{\kappa/\lambda} s G'(s) \log \left( \frac{r}{s} \right) ds
\]

\[
= \int_1^r I(A_1(s))^{\kappa/\lambda} \chi(A_1(s)) \log \left( \frac{r}{s} \right) \frac{ds}{s}
\]

\[
= \int_1^r \frac{\Psi(A_1(s)) \log \left( \frac{r}{s} \right)}{s} ds
\]

\[
= \int_1^r \frac{A_2(s) \log \left( \frac{r}{s} \right)}{s} ds
\]

\[
= \int_1^r A_1'(s) \log \left( \frac{r}{s} \right) ds
\]

\[
= A_0(r) - t_0 \log r,
\]

(38)

in view of (23), the last step following after integrating by parts. Concerning the error term in (37) we have, from (22) and (24), and using \( \chi = \Psi/I^{\kappa/\lambda} \),

\[
G(r) \leq \sqrt{\chi(A_1(r))} \log r = O \left( \sqrt{A_2(r)} \left( \log r \right)^{\frac{3\kappa-2}{2\kappa-2}} \right)
\]

(39)

as \( r \to \infty \), and \( A_0(r) = \int_1^r (A_1(s)/s) ds \geq r - 1 \) from (25), so that

\[
\log r \leq (1 + o(1)) \log A_0(r)
\]

(40)

as \( r \to \infty \). Thus, using (26),

\[
G(r) A_2(r)^{1-{1/L}} \log r = O \left( A_2(r)^{(3L-2)/(2L)} \left( \log r \right)^{(3\kappa-2)/(2\kappa-2)} \right)
\]

\[
= O \left( A_0(r)^{(3L-2)/(4-2L)} \left( \log r \right)^{(3\kappa-2)/(2\kappa-2)} \right)
\]

\[
= o(A_0(r))
\]

(41)

as \( r \to \infty \). Combining (37), (38) and (41), and using the fact that \( L < 6/5 \), we conclude that

\[
S_1 \leq (1 + o(1)) A_0(r) \quad (r \to \infty).
\]

(42)

Turning to \( S_2 \), we have, using (27) and (24), and since \( H(j) > r \),

\[
S_2 \leq I(A_1(r))^{\kappa/\lambda} G(pr) \log 2
\]

\[
\leq I(A_1(r))^{\kappa/\lambda} \sqrt{\chi(A_1(pr))} \log(pr) \log 2
\]

\[
= I(A_1(r))^{\kappa/(2\lambda)} \sqrt{\Psi(A_1(pr))} \log(pr) \log 2
\]

\[
\leq \sqrt{A_2(pr)} \log(pr) \log 2.
\]
As Bergweiler shows ([1], p. 28), \( \sqrt{A_2(\rho r)} \log(\rho r) = o(A_0(r)) \) as \( r \to \infty \), and thus
\[
S_2 = o(A_0(r)) \quad (r \to \infty). 
\]
(43)

Finally, for \( S_3 \), since \( H(j) \geq \rho r \) for \( j \geq G(\rho r) \),
\[
S_3 \leq \sum_{j=[G(\rho r)]+1}^{\infty} c_j \left( \frac{r}{H(j)} \right) \left[ \frac{H(j)}{H'(j)} \right]
\]
\[
\leq \rho \sum_{j=[G(\rho r)]+1}^{\infty} I(A_1(H(j)))^{\kappa/\lambda} \exp \left( -\frac{\tau H(j)}{H'(j)} \right),
\]
where \( \tau = \log \rho \geq 0 \). Thus, since, from (31), \( H/H' \) is increasing,
\[
S_3 \leq \rho \left( 1 + \int_{G(\rho r)}^\infty I(A_1(H(t)))^{\kappa/\lambda} \exp \left( -\frac{\tau H(t)}{H'(t)} \right) \, dt \right).
\]
(44)

As in [1], p. 29, we deduce from (44) that \( S_3 = o(A_0(r)) \), and combining this, (42), (43) and (36), we obtain
\[
B(r,u) \leq (1 + o(1))A_0(r) \quad (r \to \infty). 
\]
(45)

To establish the reverse inequality, we use Jensen’s inequality and Lemma 3.1 of [1], and obtain
\[
B(r,u) \geq \sum_{j=1}^{G(r)} c_j \left[ \frac{H(j)}{H'(j)} \right] \log \left( \frac{r}{H(j)} \right)
\]
\[
\geq \int_{G(r)}^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \left( \frac{H(t)}{H'(t)} - 1 \right) \log \left( \frac{r}{H(t)} \right) \, dt
\]
\[
+ O \left( G(r) \sup_{0 < t < G(r)} |F'(t)| \right),
\]
(46)

where
\[
F(t) = I(A_1(H(t)))^{\kappa/\lambda} \left( \frac{H(t)}{H'(t)} - 1 \right) \log \left( \frac{r}{H(t)} \right).
\]
Since, for \( H(t) \leq r \),
\[
\left| \frac{d}{dt} \left( I(A_1(H(t)))^{\kappa/\lambda} \log \left( \frac{r}{H(t)} \right) \right) \right| = \left( \kappa I(A_1(H(t)))^{(2\kappa-1)/\lambda} \log \left( \frac{r}{H(t)} \right) + I(A_1(H(t)))^{\kappa/\lambda} \right) \frac{H'(t)}{H(t)} \\
\leq \kappa \log r + 1,
\]
from (20) and (28), we have, taking account of (32) and (26),
\[
|F'(t)| \leq (\Lambda + \kappa) A_2(r)^{1-1/L} \log r + 2 = o \left( A_0(r)^{(L-1)/(2-L)} \log r \right)
\]
as \( r \to \infty \). We deduce from (41) that
\[
G(r) \sup_{0 < t < G(r)} |F'(t)| = o(A_0(r)) \quad (47)
\]
as \( r \to \infty \). Also, from (40),
\[
\int_0^1 I(A_1(H(t)))^{\kappa/\lambda} \left( \frac{H(t)}{H'(t)} - 1 \right) \log \left( \frac{r}{H(t)} \right) dt = O(\log r) = O(\log A_0(r)).
\]
From this, (46) and (47),
\[
B(r, u) \geq \int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \left( \frac{H(t)}{H'(t)} - 1 \right) \log \left( \frac{r}{H(t)} \right) dt + o(A_0(r))
\]
as \( r \to \infty \). We have
\[
\int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left( \frac{r}{H(t)} \right) dt = (1 + o(1)) A_0(r)
\]
as \( r \to \infty \), from (38) and (40), and with the change of variable \( s = H(t) \),
\[
\int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \log \left( \frac{r}{H(t)} \right) dt \\
= \int_1^r I(A_1(s))^{\kappa/\lambda} \log \left( \frac{r}{s} \right) G'(s) ds \\
\leq \int_1^r \log \left( \frac{r}{s} \right) G'(s) ds = \int_1^r \frac{G(s)}{s} ds \leq G(r) \log r = o(A_0(r))
\]
as \( r \to \infty \), from (39) and (40). It follows that \( B(r, u) \geq (1 + o(1)) A_0(r) \), and combining this with (45) we conclude that
\[
B(r, u) = (1 + o(1)) A_0(r)
\]
and, as in ([1], p. 30), that
\[
a(r, u) = (1 + o(1)) A_1(r) \quad (48)
\]
as \( r \to \infty \).

In the context of Theorem 2, \( H(j) = |z_j| \) and so, from (35) and (48), \( c_j = I((1 + o(1))a(|z_j|, u))^{e/\lambda} \) as \( j \to \infty \). From (8) we have \( r_2/r_1 \leq \phi(r_2)/\phi(r_1) \leq (r_2/r_1)^{\lambda} \) for \( r_2 \geq r_1 \geq t_0 \), and it follows from this that \( \int_r^{(1 + o(1))r} \phi(t)^{-1}dt = o(\int_r^{2r} \phi(t)^{-1}dt) \) as \( r \to \infty \). Thus \( I((1 + o(1))r) = (1 + o(1))I(r) \) as \( r \to \infty \), and we have (9).

Bergweiler’s argument ([1], section 3.3), which requires (33), shows that for all large \( z \), a disc centred at \( z \) with radius \( d(|z|) = 9|z|/\sqrt{\chi(a(r, u)/2)} \) contains one of the singularities of \( u \), and thus no Wiman-Valiron disc can have radius greater that \( d(|z|) \). Since \( \chi = \lambda \phi I^{1-1/\lambda} \), and since, arguing as in the preceding paragraph, \( \phi(r/2) \geq \text{const} \cdot \phi(r) \), and also \( I(r/2) > I(r) \), we have \( d(|z|) \leq \text{const} \cdot |z|/\sqrt{\phi(a(r, u))I(a(r, u))^{1-1/\lambda}} \). If we first obtain this result for a value slightly smaller than \( \lambda \), and with \( \kappa \) adjusted so that the ratio \( \kappa/\lambda \) remains constant, the conclusion of Theorem 2 follows.

References


