The size of Wiman–Valiron discs for subharmonic functions of a certain type

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Abstract

Wiman-Valiron theory and the results of Macintyre about "flat regions" describe the asymptotic behaviour of entire functions in certain discs around maximum points. We use a technique of Bergweiler, Rippon and Stallard to describe the asymptotic behaviour of a certain type of subharmonic function, and a technique of Bergweiler to estimate the size of its Wiman-Valiron discs from above and below. The results are extended to δ-subharmonic functions.

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1 Introduction

Given $r > 0$ and an upper semi-continuous function $u : \mathbb{C} \to [-\infty, \infty)$, let $B(r, u) := \max_{|z|=r} u(z)$ be its *maximum* function. We recall that if u is subharmonic then $B(r, u)$ is a convex, increasing function of $\log r$. The left and right derivatives of $B(r, u)$ thus exist, and $B(r, u)$ is differentiable outside a countable set. We write

$$a(r, u) := \frac{dB(r, u)}{d \log r}, \tag{1}$$

taking the right derivative at points where $B(r, u)$ is not differentiable.

We say that z_r is a *maximum point* of u if $|z_r| = r$ and $u(z_r) = B(r, u)$. Slightly abusing this notation, we call z_r a maximum point for a holomorphic function f if it is a maximum point for the subharmonic function $\log |f|$. We denote by $D(a, r) = \{z : |z - a| < r\}$ the disc centred at a of radius r, and say that a set $F \subseteq [0, \infty)$ has *logarithmic measure* $\int_F \frac{dt}{t}$. For a subharmonic function u, we denote by μ_u the *Riesz measure* of u.

A result of Macintyre ([13], Theorem 3) says that an entire function f can be estimated on a disc near its maximum point. That is, given $\epsilon > 0$, there exists a set F of finite logarithmic measure such that

$$f(z) \sim \left(\frac{z}{z_r}\right)^{a(r,\log|f|)} f(z_r) \tag{2}$$

as $r \to \infty$, $r \notin F$, uniformly for $z \in D(z_r, r/(\log M(r,f))^{\frac{1}{2}+\epsilon})$. As Bergweiler, Rippon and Stallard point out [2, p. 372], it can be deduced from (2) that, for each $k \in \mathbb{N} \cup \{0\}$,

$$f^{(k)}(z) \sim \left(\frac{a(r,\log|f|)}{z}\right)^k \left(\frac{z}{z_r}\right)^{a(r,\log|f|)} f(z_r)$$

as $r \to \infty$, $r \notin F$, uniformly for $z \in D(z_r, r/(\log M(r,f))^{\frac{1}{2}+\epsilon})$. This work followed results from Wiman-Valiron theory which found that sufficiently close to their maximum points, entire functions act like monomials, namely the dominant term of their Taylor power series (see [15], [14], [13]).

More recently, Bergweiler, Rippon and Stallard [2] proved a result similar to (2) where it is not required that f is entire, but only that f has a *direct tract*. To explain this, let D be an unbounded domain in $\mathbb{C}$ whose boundary consists of piecewise smooth curves, and whose complement is unbounded. Let f be a complex-valued function whose domain of definition contains the closure $\overline{D}$ of D. Then D is called a *direct tract* of f if f is holomorphic in D and continuous in $\overline{D}$, and there exists $R > 0$ such that $|f(z)| = R$ for z on the boundary of D, while $|f(z)| > R$ for $z \in D$.

The main result of [2] says that for every $\tau > \frac{1}{2}$, there exists a set F of finite logarithmic measure such that, for $r \notin F$, the disc $D(z_r, r/a(r,\log|f|)^\tau)$ is contained in D and (2) holds for $z \in D(z_r, r/a(r,\log|f|)^\tau)$ as $r \to \infty$. In [1], Bergweiler investigated the size of the disc around z_r in which (2) holds (which can be described as a *Wiman-Valiron disc*), and proved results from below and above as follows. Let $\phi : [t_0, \infty) \to (0, \infty)$ be a differentiable function satisfying

$$\int_{t_0}^{\infty} \frac{1}{\phi(t)} dt < \infty \tag{3}$$

and

$$K \leq \frac{t\phi'(t)}{\phi(t)} \leq L \tag{4}$$

for certain constants K and L satisfying $0 \leq K \leq 1 < L < 2$. Let f be a function with a direct tract D and let $z_r \in D$ be a maximum point of f in D. Then there exists a set F of finite logarithmic measure such that the disc $D(z_r, r/\sqrt{\phi(a(r,\log|f|))}) \subseteq D$ and (2) holds on $D(z_r, r/\sqrt{\phi(a(r,\log|f|))})$ uniformly as $r \in \infty$, $r \notin F$. On the other hand, if

$$\int_{t_o}^{\infty} \frac{1}{\phi(t)} dt = \infty$$

and (4) holds for $K = 1$ and some $L < \frac{6}{5}$, then there exists an entire function f which has exactly one direct tract D and is such that if r is sufficiently large and $|z| = r$, then the disc $D(z, r/\sqrt{\phi(a(r, \log|f|))})$ contains a zero of f, and thus (2) cannot hold.

With these results in mind, we seek analogous results for subharmonic functions of the form

$$u(z) = \sum_{j=1}^{\infty} c_j \log\left|1 - \frac{z}{z_j}\right|, \tag{5}$$

where $c_j \in \mathbb{R}^+$, $z_j \in \mathbb{C}$ and $\sum_{j=1}^{\infty}\left|\frac{c_j}{z_j}\right| < \infty$. In any disc that contains no z_j, u can also be written in the form

$$u(z) = \log|f(z)|,$$

where $f(z) = \prod_{j=1}^{\infty}\left(1 - \frac{z}{z_j}\right)^{c_j}$. The c_j are the Riesz masses of u at the points z_j; at all other points u is harmonic and the Riesz mass is zero. Since the z_j are the only "problem" points of u in $\mathbb{C}$, we do not need to assume the existence of a direct tract as in [2] and [1].

To achieve results for u, it is necessary to impose some lower growth condition on the masses c_j. For if, for example, the sequence z_j consisted of all the rational points in the plane, then every open disc would contain a z_j and we could not have (2). We will assume that

$$\varliminf_{j\to\infty} \frac{c_j}{I(a(|z_j|, u))^{\beta}} > 0 \tag{6}$$

where

$$I(r) := \int_r^{\infty} \frac{1}{\phi(t)} dt \tag{7}$$

and β is a positive constant.

We are now ready to state our first theorem.

Theorem 1 *Let u be as in (5) with the growth condition (6) on the masses c_j of u, for some constant $\beta > 0$. Let $t_0 > 0$ and let $\phi : [t_0, \infty) \to (0, \infty)$ be a differentiable function satisfying (3) and (4) for constants $K = \min\{\beta, 1\}$ and L satisfying $1 < L < 3/2$. Let $r > 0$. If $\beta < 1$ there exists a set $F \subseteq [0, \infty)$ of finite logarithmic measure such that the disc $D(z_r, r/\sqrt{\phi(a(r, u))})$ contains no z_j, and (2) holds uniformly for $z \in D(z_r, r/\sqrt{\phi(a(r, u))})$, as $r \to \infty$, $r \notin F$.*

Remark: Theorem 1 can be extended to δ-subharmonic functions, that is, functions

$$v(z) = v_1(z) - v_2(z),$$

where v_1, v_2 are subharmonic functions of the form (5). We assume that the set $D = \{z \in \mathbb{C} : v(z) > 0\}$ has an unbounded component D_0 on which v is harmonic. The function which is v on D_0 and 0 elsewhere, which we denote by v^+, is then subharmonic. The complement of D_0 may contain *islands* $\mathcal{I}$, that is, closed bounded components of the form $\{z : v^+(z) = 0\}$, and the growth condition we impose on v_1 and v_2 involves islands, rather than points, as follows. Either there are finitely many islands, or

$$\varliminf_{r_{\mathcal{I}} \to \infty} \frac{\mu_{v_1}(\mathcal{I}) - \mu_{v_2}(\mathcal{I})}{I(a(r_{\mathcal{I}}, v^+))^{\beta}} > 0,$$

where $0 < \beta < 1$ and $r_{\mathcal{I}} = \sup\{|z| : z \in \mathcal{I}\}$. The proof relies on the fact that

$$\mu_{v^+}(\mathcal{I}) = \mu_{v_1}(\mathcal{I}) - \mu_{v_2}(\mathcal{I}),$$

but is otherwise similar to the proof of Theorem 1.

We state our second theorem as follows.

Theorem 2 *Let $t_0 > 0$ and let $\phi : [t_0, \infty) \to (0, \infty)$ be a differentiable function that satisfies* (3)*. Let I be as defined in* (7) *and suppose that, given $\kappa > 1$ and $\lambda > 0$,*

$$1 \le \frac{r\phi'(r)}{\phi(r)} - \frac{(1 + (\kappa - 1)/\lambda)r}{\phi(r)I(r)} \le \frac{r\phi'(r)}{\phi(r)} \le L, \tag{8}$$

for $r \ge t_0$, where $1 < L < 6/5$. Then there exists a subharmonic function of the form (5)*, with*

$$c_j = (1 + o(1))(I(a(|z_j|, u)))^{\kappa/\lambda} \qquad (j \to \infty), \tag{9}$$

for which, for all large r, there are no Wiman-Valiron discs of radius greater than

$$\frac{r}{\sqrt{\phi(a(r, u))I(a(r, u))^{1-1/\lambda}}}. \tag{10}$$

Corollary 1 *Theorem 1 fails if $\beta > 1$.*

Assuming Theorem 2 for the moment, let us prove Corollary 1. The function

$$\phi(r) = \eta^{-1} r \cdot \log r \cdot \log_2 r \cdots \log_l r \cdot (\log_{l+1} r)^{1+\eta}, \tag{11}$$

where $\eta > 0$, l is a positive integer and $\log_l$ is the l-times iterated logarithm, satisfies (3) and, for any $\lambda > 0$ and $\kappa > 1$, satisfies (8) for all large t. For in that case $I(r) = (\log_{l+1} r)^{-\eta}$ and

$$\begin{aligned} &\frac{r\phi'(r)}{\phi(r)} - \frac{(1 + (\kappa - 1)/\lambda)r}{\phi(r)I(r)} \\ &\qquad = 1 + \frac{1}{\log r} + ... + \frac{1}{\log r \log_2 r \cdots \log_l r} + \frac{1 - (\kappa - 1)\eta/\lambda}{\log r \log_2 r \cdots \log_{l+1} r}. \end{aligned} \tag{12}$$

Thus, for this ϕ, the result of Theorem 2 holds for any $\kappa > 1$ and $\lambda > 0$. If $\beta > 1$ we may choose κ and λ satisfying $\kappa/\lambda = \beta$ with $\kappa > 1$ and $\lambda < 1$, and then (10) contradicts the conclusion of Theorem 1.

Remarks: 1. The case $\beta = 1$ is open.

2. Suppose that $0 < \beta < 1$ and that in Theorem 2 we choose $\kappa > 1$ arbitrarily and $\lambda = \kappa/\beta$. The difference between the radii in Theorem 1 and Theorem 2 is $\sqrt{I(a(r,u))^{1-\beta/\kappa}}$. To get some idea of the significance of this factor, consider ϕ of (11). The radius of the disc in Theorem 1 is

$$\eta^{-1} t \cdot \log t \cdot \log_2 t \cdots \log_l t \cdot (\log_{l+1} t)^{1+\eta}|_{t=a(r,u)}$$

and in Theorem 2 is

$$\eta^{-1} t \cdot \log t \cdot \log_2 t \cdots \log_l t \cdot (\log_{l+1} t)^{1+\beta\eta/\kappa}|_{t=a(r,u)}.$$

Numerous applications of the theories of Wiman-Valiron and Macintyre exist, in areas including complex dynamics ([2], [5], [9]), complex differential equations ([6], [10], [11], [16]) and the zero distribution of derivatives ([3], [12]). We note also that functions of the form (5) are connected to the electrostatic fields generated by positively charged wires which meet the complex plane at the points z_j [4].

2 Proof of Theorem 1

First, with u given by (5), we have $a(r,u) \to \infty$ as $r \to \infty$. For otherwise $B(r,u) = O(\log r)$ and therefore $\mu_u(D(0,r)) = O(1)$ as $r \to \infty$, which gives that $\sum_{j\in\mathbb{N}} c_j < \infty$. By (6) the c_j would be bounded below, and this would mean that there are only finitely many c_j, which is a contradiction.

We need two lemmas. The first is based on a well-known result about real functions and can be found in ([2], Lemma 6.10).

Lemma 3 *Let $v : \mathbb{C} \to [-\infty, \infty)$ be subharmonic, and let $\epsilon > 0$. Then there exists a set $F \subseteq [1,\infty)$ of finite logarithmic measure such that*

$$a(r,v) \le B(r,v)^{1+\epsilon}$$

for $r \ge 1$, $r \notin F$.

The second lemma summarises, with a small change, the lemmas and discussion in ([1], Section 2). The change we make concerns two functions, σ_1 and σ_2, that occur in the proof of Lemma 2.2 of [1]. Rather than (in the notation of [1]) $\sigma_1 = \sigma_2 = V^{K/2}\sqrt{\psi}$, we take $\sigma_1 = \sqrt{\psi}$ and $\sigma_2 = V^K\sqrt{\psi}$; the proof is unchanged except that the restriction on the constant L needs to be strengthened (from $L < 2$ to $L < 3/2$).

Lemma 4 *Let $t_0 > 0$, let $v : \mathbb{C} \to [-\infty, \infty)$ be a subharmonic function, and let $\phi : [t_0, \infty) \to (0, \infty)$ be a differentiable function satisfying* (3) *and* (4) *for certain constants $0 < K < 1 < L < 3/2$. Then there exists a set $F \subseteq [t_0, \infty)$ of finite logarithmic measure such that*

$$B(s,v) \leq B(r,v) + a(r,v) \log \frac{s}{r} + I(a(r,v))^K$$

for

$$\left|\log \frac{s}{r}\right| \leq \frac{1}{\sqrt{\phi(a(r,v))}},$$

uniformly as $r \to \infty$, $r \notin F$.

We apply Lemma 4 with $K = \beta$ and L satisfying $1 < L < 3/2$, and we apply Lemma 3 for $\epsilon = \beta$. Let F be the union of the exceptional sets of these lemmas. We put $\rho = 2r/\phi(a(r,u))^{\beta/2}$ whenever r is so large that $a(r,u) \neq 0$. Let C be a positive constant sufficiently large that for all large r and any $z_j \in D(z_r, C\rho)$ we have

$$c_j > (\log C)^{-1} I(a(|z_j|, u))^{\beta}, \tag{13}$$

which is possible by (6). We consider the function

$$v(z) = u(z) - u(z_r) - a(r,u) \log \frac{|z|}{r} = u(z) - B(r,u) - a(r,u) \log \frac{|z|}{r}. \tag{14}$$

For $z \in \overline{D}(z_r, C\rho)$ we have

$$\left|\frac{z - z_r}{z_r}\right| \leq \left|\frac{C\rho}{r}\right| = \frac{2C}{\phi(a(r,u))^{\beta/2}} = o(1)$$

as $r \to \infty$, by (4) and since $a(r,u) \to \infty$. Thus since $\beta < 1$,

$$\left|\log \frac{|z|}{r}\right| = \left|\log \left|1 + \frac{z - z_r}{z_r}\right|\right| \leq 2 \left|\frac{z - z_r}{z_r}\right| \leq \frac{4C}{\phi(a(r,u))^{\beta/2}} \leq \frac{1}{\sqrt{\phi(a(r,f))}}$$

for large r, and the hypotheses of Lemma 4 are satisfied. Since

$$v(z) \leq B(|z|, u) - B(r,u) - a(r,u) \log \frac{|z|}{r}$$

by (14), we conclude from Lemma 4 applied to u that

$$v(z) \leq I(a(r,u))^{\beta} \tag{15}$$

as $r \to \infty$, $r \notin F$.

For large r, we have $0 \notin D(z_r, \rho)$ so that the difference of v and u as defined in (14) is harmonic in $D(z_r, \rho)$, and hence their Riesz measures in the disc coincide. Thus

$$\mu_v(D(z_r, \rho)) = \sum_{z_j \in D(z_r, \rho)} c_j \tag{16}$$

where the z_j are matched with the c_j in (5).

On the other hand, we have $\mu_v(D(z_r,\rho))\log C \le \int_0^{C\rho} \frac{\mu_v(D(z_r,t))}{t}dt$, and by (15) and the fact that $v(z_r)=0$,

$$\begin{aligned}\int_0^{C\rho} \frac{\mu_v(D(z_r,t))}{t}dt &= \frac{1}{2\pi}\int_0^{2\pi} v(z_r + C\rho e^{i\zeta})d\zeta - v(z_r)\\ &\le \frac{1}{2\pi}\int_0^{2\pi} I(a(r,u))^{\beta}d\zeta\\ &= I(a(r,u))^{\beta}.\end{aligned}$$

Combining this with (16) gives

$$\sum_{z_j\in D(z_r,C\rho)} c_j \log C \le I(a(r,u))^{\beta}.$$

In view of (13), we deduce that there do not exist any $z_j \in D(z_r, C\rho)$, and thus in $D(z_r, r/\sqrt{\phi(a(r,u))})$.

The remainder of the proof, including the derivation of (2), reproduces the arguments in ([2], Theorem 2.2).

3 Proof of Theorem 2, preliminaries

We follow Bergweiler's intricate construction with slight changes, first introducing certain auxiliary functions in terms of which the example of Theorem 2 is defined. The main concern in this section is to show that certain key calculations that Bergweiler makes carry over in their modified form. Wherever possible, results from [1] have been simply quoted, with appropriate references.

Let ϕ be as in the statement of Theorem 2 and define Ψ, $\chi : [t_0,\infty) \to (0,\infty)$ by

$$\Psi(t) = \lambda\phi(t)I(t)^{1+(\kappa-1)/\lambda} \tag{17}$$

and

$$\chi(t) = \lambda\phi(t)I(t)^{1-1/\lambda} \tag{18}$$

where λ and κ are constants satisfying $\lambda > 2$ and $\lambda/2 \ge \kappa > 1$, and $I(t)$ is given by (7). Note that

$$\int_{t_0}^{t} \frac{ds}{\Psi(s)} = \frac{1}{\kappa-1}\left(I(t)^{-(\kappa-1)/\lambda} - I(t_0)^{-(\kappa-1)/\lambda}\right), \quad \int_t^{\infty} \frac{ds}{\chi(s)} = I(t)^{1/\lambda}. \tag{19}$$

From the first of these we have

$$I'(t) = -\frac{\lambda I(t)^{1+(\kappa-1)/\lambda}}{\Psi(t)}. \tag{20}$$

In what follows it is helpful to know that $I(t) \le 1$ for $t \ge t_0$, which can always be achieved by taking a somewhat larger value of t_0. Let us suppose that this has been done.

As in [1], we define $A_1 : [1, \infty) \to [t_0, \infty)$ by

$$\log t = \int_{t_0}^{A_1(t)} \frac{1}{\Psi(s)} ds. \tag{21}$$

From the first part of (19) we have

$$I(A_1(t)) = ((\kappa - 1)\log(C_1 t))^{-\lambda/(\kappa-1)}, \tag{22}$$

where $C_1 = \exp\{(\kappa-1)^{-1} I(t_0)^{-(\kappa-1)/\lambda}\}$. We also introduce

$$A_2(t) := tA_1'(t),\ A_3(t) := tA_2'(t),\ A_0(t) := \int_1^t A_1(s)\frac{ds}{s}. \tag{23}$$

Differentiating (21), we have

$$A_2(t) = \Psi(A_1(t)); \tag{24}$$

also (see [1], (3.4))

$$A_3(t) \ge A_2(t) \ge A_1(t) \ge t, \quad t \ge 1, \tag{25}$$

and (see [1], (3.10))

$$A_2(t) = o\left(A_0(t)^{\frac{L}{2-L}}\right) \tag{26}$$

as $t \to \infty$, where L is the number of (8). Define $G : [1, \infty) \to [0, \infty)$ by

$$G(t) = \int_1^t \sqrt{\chi(A_1(s))}\,\frac{ds}{s}, \tag{27}$$

and let $H : [0, \infty) \to [1, \infty)$ be the inverse function. We have, using (17), (18), (24), (25) and the fact that $I \le 1$,

$$\frac{H(t)}{H'(t)} = \sqrt{\chi(A_1(H(t)))} = \sqrt{\frac{\Psi(A_1(H(t)))}{I(A_1(H(t)))^{\kappa/\lambda}}} = \sqrt{\frac{A_2(H(t))}{I(A_1(H(t)))^{\kappa/\lambda}}} \ge 1. \tag{28}$$

Also, observing that from (20) and (24),

$$I'(A_1(t)) = -\frac{\lambda I(A_1(t))^{1+(\kappa-1)/\lambda}}{A_2(t)},$$

we obtain after some calculation, and using (23),

$$\begin{aligned}\frac{d}{dt}\left(\frac{H(t)}{H'(t)}\right) &= \frac{d}{dt}\left(\sqrt{\frac{A_2(H(t))}{I(A_1(H(t)))^{\kappa/\lambda}}}\right)\\ &= \frac{A_3(H(t))}{2A_2(H(t))} + \frac{\kappa}{2} I(A_1(H(t)))^{(\kappa-1)/\lambda}. \end{aligned}\tag{29}$$

Since $(H/H')' = 1 - HH''/(H')^2$, we have from (29),

$$\frac{H''(t)}{H'(t)} = \sqrt{\frac{I(A_1(H(t)))^{\kappa/\lambda}}{A_2(H(t))}} \left(1 - \frac{A_3(H(t))}{2A_2(H(t))} - \frac{\kappa}{2} I(A_1(H(t)))^{(\kappa-1)/\lambda}\right). \quad (30)$$

From (29), (25) and the fact that $I \leq 1$ we have

$$\frac{d}{dt}\left(\frac{H(t)}{H'(t)}\right) \leq (\kappa+1)\frac{A_3(H(t))}{2A_2(H(t))}. \quad (31)$$

We need a related estimate, for $\frac{d}{dt}\left(I(A_1(H(t)))^{\kappa/\lambda}\frac{H(t)}{H'(t)}\log\left(\frac{r}{H(t)}\right)\right)$. Using (20), (24) and (31) we have, for r fixed and t such that $H(t) \leq r$,

$$\begin{aligned}
&\left|\frac{d}{dt}\left(I(A_1(H(t)))^{\kappa/\lambda}\frac{H(t)}{H'(t)}\log\left(\frac{r}{H(t)}\right)\right)\right| \\
&\quad = \Bigg| -\kappa(I(A_1(H(t))))^{(2\kappa-1)/\lambda}\log\left(\frac{r}{H(t)}\right) \\
&\qquad + I(A_1(H(t)))^{\kappa/\lambda}\left(\log\left(\frac{r}{H(t)}\right)\frac{d}{dt}\left(\frac{H(t)}{H'(t)}\right) - 1\right)\Bigg| \\
&\quad \leq \left(\kappa + (\kappa+1)\frac{A_3(H(t))}{2A_2(H(t))}\right)\log r + 1.
\end{aligned}$$

Since, as Bergweiler has shown (see [1], (3.15) and (3.2)), $A_3/A_2 \leq Lc^{1/L}A_2^{1-1/L}$, where $c = \Psi(t_0)t_0^{-L}$ and L is the number of (8), the preceding estimate gives

$$\left|\frac{d}{dt}\left(I(A_1(H(t)))^{\kappa/\lambda}\frac{H(t)}{H'(t)}\log\left(\frac{r}{H(t)}\right)\right)\right| \leq \Lambda A_2(r)^{1-1/L}\log r + 1 \quad (32)$$

for $H(t) \leq r$, where $\Lambda = \kappa + L(\kappa+1)c^{1/L}/2$.

Finally, from (30) we have (as in [1] between formulas (3.13) and (3.14))

$$\left|\frac{H''(t)}{H'(t)}\right| \leq \frac{1}{2}(\kappa+3)\frac{A_3(H(t))}{A_2(H(t))^{3/2}} = o(1)$$

as $t \to \infty$, and we deduce from this (cf [1], (3.14)) that

$$H'(t+s) = (1+o(1))H'(t) \quad (33)$$

as $t \to \infty$, uniformly for $0 \leq s \leq 1$.

4 Proof of Theorem 2

Consider

$$u(z) = \sum_{j=1}^{\infty} c_j \log\left|1 + \left(\frac{z}{H(j)}\right)^{\left[\frac{H(j)}{H'(j)}\right]}\right|, \quad (34)$$

where $\left[\frac{H(j)}{H'(j)}\right]$ means the integer part of $\frac{H(j)}{H'(j)}$, and with the notation of the preceding section,

$$c_j = I(A_1(H(j)))^{\kappa/\lambda}. \tag{35}$$

Since the terms of the series (34) are subharmonic and the series converges locally uniformly (as will be evident from our subsequent calculations), u is subharmonic.

As in [1] we write

$$A_j = \log\left(1 + \left(\frac{r}{H(j)}\right)^{\left[\frac{H(j)}{H'(j)}\right]}\right),$$

for $r \geq 0$, and

$$S_1 = \sum_{j=1}^{[G(r)]} c_j A_j, \quad S_2 = \sum_{j=[G(r)]+1}^{[G(\rho r)]} c_j A_j, \quad S_3 = \sum_{j=[G(\rho r)]+1}^{\infty} c_j A_j,$$

where G is as in (27) and $\rho = 1 + A_1(r)/(2A_2(r))$. Then

$$B(r,u) \leq S_1 + S_2 + S_3, \tag{36}$$

and we estimate S_1, S_2 and S_3 in turn.

For S_1, since $H(j) \leq r$ for $j \leq G(r)$ we have, using (28), (32) and Bergweiler's Lemma 3.1 of [1],

$$\begin{aligned}
S_1 &\leq \sum_{j=1}^{[G(r)]} I(A_1(H(j)))^{\kappa/\lambda} \left(\left[\frac{H(j)}{H'(j)}\right] \log\left(\frac{r}{H(j)}\right) + \log 2\right) \\
&\leq \sum_{j=1}^{[G(r)]} I(A_1(H(j)))^{\kappa/\lambda} \frac{H(j)}{H'(j)} \log\left(\frac{r}{H(j)}\right) + G(r)\log 2 \\
&\leq \int_0^{G(r)} I(A_1(H(j)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log\left(\frac{r}{H(t)}\right)\, dt \\
&\quad + (\Lambda A_2(r)^{1-1/L}\log r + 1)G(r) + G(r)\log 2 \\
&= \int_0^{G(r)} I(A_1(H(j)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log\left(\frac{r}{H(t)}\right)\, dt \\
&\quad + O\left(G(r)A_2(r)^{1-1/L}\log r\right)
\end{aligned} \tag{37}$$

as $r \to \infty$. With the change of variable $s = H(t)$ we obtain

$$\begin{aligned}
&\int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log\left(\frac{r}{H(t)}\right) dt \\
&\quad = \int_1^r I(A_1(s))^{\kappa/\lambda} s G'(s)^2 \log\left(\frac{r}{s}\right) ds \\
&\quad = \int_1^r I(A_1(s))^{\kappa/\lambda} \chi(A_1(s)) \log\left(\frac{r}{s}\right) \frac{ds}{s} \\
&\quad = \int_1^r \Psi(A_1(s)) \log\left(\frac{r}{s}\right) \frac{ds}{s} \\
&\quad = \int_1^r A_2(s) \log\left(\frac{r}{s}\right) \frac{ds}{s} \\
&\quad = \int_1^r A_1'(s) \log\left(\frac{r}{s}\right) ds \\
&\quad = A_0(r) - t_0 \log r, \qquad (38)
\end{aligned}$$

in view of (23), the last step following after integrating by parts. Concerning the error term in (37) we have, from (22) and (24), and using $\chi = \Psi/I^{\kappa/\lambda}$,

$$G(r) \le \sqrt{\chi(A_1(r))} \log r = O\left(\sqrt{A_2(r)}\, (\log r)^{\frac{3\kappa-2}{2\kappa-2}}\right) \qquad (39)$$

as $r \to \infty$, and $A_0(r) = \int_1^r (A_1(s)/s)\, ds \ge r - 1$ from (25), so that

$$\log r \le (1 + o(1)) \log A_0(r) \qquad (40)$$

as $r \to \infty$. Thus, using (26),

$$\begin{aligned}
G(r) A_2(r)^{1-1/L} \log r &= O\left(A_2(r)^{(3L-2)/(2L)} (\log r)^{(3\kappa-2)/(2\kappa-2)}\right) \\
&= O\left(A_0(r)^{(3L-2)/(4-2L)} (\log r)^{(3\kappa-2)/(2\kappa-2)}\right) \\
&= o(A_0(r)) \qquad (41)
\end{aligned}$$

as $r \to \infty$. Combining (37), (38) and (41), and using the fact that $L < 6/5$, we conclude that

$$S_1 \le (1 + o(1)) A_0(r) \qquad (r \to \infty). \qquad (42)$$

Turning to S_2, we have, using (27) and (24), and since $H(j) > r$,

$$\begin{aligned}
S_2 &\le I(A_1(r))^{\kappa/\lambda} G(\rho r) \log 2 \\
&\le I(A_1(r))^{\kappa/\lambda} \sqrt{\chi(A_1(\rho r))} \log(\rho r) \log 2 \\
&= I(A_1(r))^{\kappa/(2\lambda)} \sqrt{\Psi(A_1(\rho r))} \log(\rho r) \log 2 \\
&\le \sqrt{A_2(\rho r)} \log(\rho r) \log 2.
\end{aligned}$$

As Bergweiler shows ([1], p. 28), $\sqrt{A_2(\rho r)}\log(\rho r) = o(A_0(r))$ as $r \to \infty$, and thus

$$S_2 = o(A_0(r)) \qquad (r \to \infty). \tag{43}$$

Finally, for S_3, since $H(j) \geq \rho r$ for $j \geq G(\rho r)$,

$$\begin{aligned} S_3 &\leq \sum_{j=[G(\rho r)]+1}^{\infty} c_j \left(\frac{r}{H(j)}\right)^{\left[\frac{H(j)}{H'(j)}\right]} \\ &\leq \rho \sum_{j=[G(\rho r)]+1}^{\infty} I(A_1(H(j)))^{\kappa/\lambda} \exp\left(-\tau \frac{H(j)}{H'(j)}\right), \end{aligned}$$

where $\tau = \log \rho \geq 0$. Thus, since, from (31), H/H' is increasing,

$$S_3 \leq \rho \left(1 + \int_{G(\rho r)}^{\infty} I(A_1(H(t)))^{\kappa/\lambda} \exp\left(-\tau \frac{H(t)}{H'(t)}\right) dt\right).$$

Making the change of variable $s = H(t)$, and using (28) and the fact that $\chi(A_1) \geq \Psi(A_1) = A_2$ and $I \leq 1$, we have

$$\begin{aligned} S_3 &\leq \rho \left(1 + \int_{\rho r}^{\infty} I(A_1(s))^{\kappa/\lambda} \sqrt{\chi(A_1(s))} \exp\left(-\tau \sqrt{\chi(A_1(s))}\right) \frac{ds}{s}\right) \\ &= \rho \left(1 + \int_{\rho r}^{\infty} I(A_1(s))^{\kappa/(2\lambda)} \sqrt{A_2(s)} \exp\left(-\tau \sqrt{\chi(A_1(s))}\right) \frac{ds}{s}\right) \\ &\leq \rho \left(1 + \int_{\rho r}^{\infty} \sqrt{A_2(s)} \exp\left(-\tau \sqrt{A_2(s)}\right) \frac{ds}{s}\right). \end{aligned} \tag{44}$$

As in [1], p. 29, we deduce from (44) that $S_3 = o(A_0(r))$, and combining this, (42), (43) and (36), we obtain

$$B(r,u) \leq (1 + o(1))A_0(r) \qquad (r \to \infty). \tag{45}$$

To establish the reverse inequality, we use Jensen's inequality and Lemma 3.1 of [1], and obtain

$$\begin{aligned} B(r,u) &\geq \sum_{j=1}^{[G(r)]} c_j \left[\frac{H(j)}{H'(j)}\right] \log\left(\frac{r}{H(j)}\right) \\ &\geq \int_{G(r)-[G(r)]}^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \left(\frac{H(t)}{H'(t)} - 1\right) \log\left(\frac{r}{H(t)}\right) dt \\ &\quad + O\left(G(r) \sup_{0<t<G(r)} |F'(t)|\right), \end{aligned} \tag{46}$$

where

$$F(t) = I(A_1(H(t)))^{\kappa/\lambda} \left(\frac{H(t)}{H'(t)} - 1\right) \log\left(\frac{r}{H(t)}\right).$$

Since, for $H(t) \le r$,

$$\begin{aligned}
&\left| \frac{d}{dt} \left(I(A_1(H(t)))^{\kappa/\lambda} \log \left(\frac{r}{H(t)} \right) \right) \right| \\
&\quad = \left(\kappa I(A_1(H(t)))^{(2\kappa-1)/\lambda} \log \left(\frac{r}{H(t)} \right) + I(A_1(H(t)))^{\kappa/\lambda} \right) \frac{H'(t)}{H(t)} \\
&\quad \le \kappa \log r + 1,
\end{aligned}$$

from (20) and (28), we have, taking account of (32) and (26),

$$|F'(t)| \le (\Lambda + \kappa) A_2(r)^{1-1/L} \log r + 2 = o\left(A_0(r)^{(L-1)/(2-L)} \log r \right)$$

as $r \to \infty$. We deduce from (41) that

$$G(r) \sup_{0<t<G(r)} |F'(t)| = o(A_0(r)) \tag{47}$$

as $r \to \infty$. Also, from (40),

$$\int_0^1 I(A_1(H(t)))^{\kappa/\lambda} \left(\frac{H(t)}{H'(t)} - 1 \right) \log \left(\frac{r}{H(t)} \right) dt = O(\log r) = O(\log A_0(r)).$$

From this, (46) and (47),

$$B(r,u) \ge \int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \left(\frac{H(t)}{H'(t)} - 1 \right) \log \left(\frac{r}{H(t)} \right) dt + o(A_0(r))$$

as $r \to \infty$. We have

$$\int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left(\frac{r}{H(t)} \right) dt = (1 + o(1)) A_0(r)$$

as $r \to \infty$, from (38) and (40), and with the change of variable $s = H(t)$,

$$\begin{aligned}
&\int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \log \left(\frac{r}{H(t)} \right) dt \\
&\quad = \int_1^r I(A_1(s))^{\kappa/\lambda} \log \left(\frac{r}{s} \right) G'(s) ds \\
&\quad \le \int_1^r \log \left(\frac{r}{s} \right) G'(s) ds = \int_1^r \frac{G(s)}{s} ds \le G(r) \log r = o(A_0(r))
\end{aligned}$$

as $r \to \infty$, from (39) and (40). It follows that $B(r,u) \ge (1 + o(1)) A_0(r)$, and combining this with (45) we conclude that

$$B(r,u) = (1 + o(1)) A_0(r)$$

and, as in ([1], p. 30), that

$$a(r,u) = (1 + o(1)) A_1(r) \tag{48}$$

as $r \to \infty$.

In the context of Theorem 2, $H(j) = |z_j|$ and so, from (35) and (48), $c_j = I((1+o(1))a(|z_j|, u))^{\kappa/\lambda}$ as $j \to \infty$. From (8) we have $r_2/r_1 \leq \phi(r_2)/\phi(r_1) \leq (r_2/r_1)^L$ for $r_2 \geq r_1 \geq t_0$, and it follows from this that $\int_r^{(1+o(1))r} \phi(t)^{-1} dt = o(\int_r^{2r} \phi(t)^{-1} dt$ as $r \to \infty$. Thus $I((1+o(1))r) = (1+o(1))I(r)$ as $r \to \infty$, and we have (9).

Bergweiler's argument ([1], section 3.3), which requires (33), shows that for all large z, a disc centred at z with radius $d(|z|) = 9|z|/\sqrt{\chi(a(r,u)/2)}$ contains one of the singularities of u, and thus no Wiman-Valiron disc can have radius greater that $d(|z|)$. Since $\chi = \lambda \phi I^{1-1/\lambda}$, and since, arguing as in the preceding paragraph, $\phi(r/2) \geq \text{const} \cdot \phi(r)$, and also $I(r/2) > I(r)$, we have $d(|z|) \leq \text{const} \cdot |z|/\sqrt{\phi(a(r,u))I(a(r,u))^{1-1/\lambda}}$. If we first obtain this result for a value slightly smaller than λ, and with κ adjusted so that the ratio κ/λ remains constant, the conclusion of Theorem 2 follows.

References

[1] W. Bergweiler, The size of Wiman-Valiron disks, *Complex Var. Elliptic Equ.* 56 (2011), no. 1-4, 13-33.

[2] W. Bergweiler, P.J. Rippon, G.M. Stallard, Dynamics of meromorphic functions with direct or logarithmic singularities, *Proc. London Math. Soc.* (3) 97 (2008), no. 2, 368-400.

[3] J.G. Clunie, On a hypothetical theorem of Pólya, *Complex Variables Theory Appl.* 12 (1989), 87-94.

[4] J. Clunie, A. Eremenko and J. Rossi, On equilibrium points of logarithmic and Newtonian potentials, *J. London Math. Soc.* (2) 47 (1993), 309-320.

[5] A.E. Eremenko, On the iteration of entire functions, in *Dynamical systems and ergodic theory* (Warsaw, 1986), Banach Center Publ. 23, PWN, Warsaw, 1989, pp. 339-345.

[6] A. Eremenko, L.W. Liao, T.W. Ng, Meromorphic solutions of higher order Briot-Bouquet differential equations, *Math. Proc. Cambridge Philos. Soc.* 146 (2009), 197-206.

[7] W.H.J. Fuchs, A Phragmén-Lindeløf theorem conjectured by D.J. Newman, *Trans. Amer. Math. Soc.* 267 (1981), 285-293.

[8] W.K. Hayman and P.B. Kennedy, Subharmonic Functions Vol. 1, *Academic Press London etc.* (1976), xvii + 284 pp.

[9] X.H. Hua and C.C. Yang, Dynamics of transcendental functions, *Asian Math. Ser. 1* Gordon and Breach Science Publishers, Amsterdam, 1998.

[10] G. Jank and L. Volkmann, Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen, *UTB fd́dotur Wissenschaft: Grosse Reihe. Birkhauser Verlag* Basel, 1985.

[11] I. Laine, Nevanlinna theory and complex differential equations, *de Gruyter Studies in Mathematics* 15, Walter de Gruyter & Co., Berlin, 1993.

[12] J.K. Langley, Proof of a conjecture of Hayman concerning f and f'', *J. London Math. Soc.* (2) 48 (1993), no. 3, 500-514.

[13] A.J. Macintyre, Wiman's method and the "flat regions" of integral functions, *Q. J. Math.,* Oxf. Ser. 9 (1938), 81-88.

[14] G. Valiron, Lectures on the general theory of integral functions, *Edouard Privat* Toulouse, 1923.

[15] A. Wiman, Über den Zusammenhang zwischen dem Maximalbetrage einer analytischen Funktion und dem größten Betrage bei gegebenem Argumente der Funktion, *Acta Math.* 41 (1916), 1-28.

[16] H. Wittich, Neuere Untersuchungen über eindeutige analytische Funktionen, *Ergebnisse der Mathematik und ihrer Grenzgebiete* 8, Springer-Verlag, Berlin, second edition, 1968.