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FENTON, P.C. and LINGHAM, Eleanor

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The size of Wiman-Valiron discs for subharmonic functions of a certain type

P.C. Fenton
University of Otago, Dunedin, New Zealand
pfenton@maths.otago.ac.nz

E.F. Lingham
De Montfort University, Leicester, United Kingdom
elingham@dmu.ac.uk

Abstract

Wiman-Valiron theory and the results of Macintyre about “flat regions” describe the asymptotic behaviour of entire functions in certain discs around maximum points. We use a technique of Bergweiler, Rippon and Stallard to describe the asymptotic behaviour of a certain type of subharmonic function, and a technique of Bergweiler to estimate the size of its Wiman-Valiron discs from above and below. The results are extended to δ -subharmonic functions.

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1 Introduction

Given $r > 0$ and an upper semi-continuous function $u : \mathbb{C} \rightarrow [-\infty, \infty)$, let $B(r, u) := \max_{|z|=r} u(z)$ be its *maximum function*. We recall that if u is subharmonic then $B(r, u)$ is a convex, increasing function of $\log r$. The left and right derivatives of $B(r, u)$ thus exist, and $B(r, u)$ is differentiable outside a countable set. We write

$$a(r, u) := \frac{dB(r, u)}{d \log r}, \quad (1)$$

taking the right derivative at points where $B(r, u)$ is not differentiable.

We say that z_r is a *maximum point* of u if $|z_r| = r$ and $u(z_r) = B(r, u)$. Slightly abusing this notation, we call z_r a maximum point for a holomorphic function f if it is a maximum point for the subharmonic function $\log |f|$. We denote by $D(a, r) = \{z : |z - a| < r\}$ the disc centred at a of radius r , and say that a set $F \subseteq [0, \infty)$ has *logarithmic measure* $\int_F \frac{dt}{t}$. For a subharmonic function u , we denote by μ_u the *Riesz measure* of u .

A result of Macintyre ([13], Theorem 3) says that an entire function f can be estimated on a disc near its maximum point. That is, given $\epsilon > 0$, there exists

a set F of finite logarithmic measure such that

$$f(z) \sim \left(\frac{z}{z_r}\right)^{a(r, \log |f|)} f(z_r) \quad (2)$$

as $r \rightarrow \infty$, $r \notin F$, uniformly for $z \in D(z_r, r/(\log M(r, f))^{\frac{1}{2}+\epsilon})$. As Bergweiler, Rippon and Stallard point out [2, p. 372], it can be deduced from (2) that, for each $k \in \mathbb{N} \cup \{0\}$,

$$f^{(k)}(z) \sim \left(\frac{a(r, \log |f|)}{z}\right)^k \left(\frac{z}{z_r}\right)^{a(r, \log |f|)} f(z_r)$$

as $r \rightarrow \infty$, $r \notin F$, uniformly for $z \in D(z_r, r/(\log M(r, f))^{\frac{1}{2}+\epsilon})$. This work followed results from Wiman-Valiron theory which found that sufficiently close to their maximum points, entire functions act like monomials, namely the dominant term of their Taylor power series (see [15], [14], [13]).

More recently, Bergweiler, Rippon and Stallard [2] proved a result similar to (2) where it is not required that f is entire, but only that f has a *direct tract*. To explain this, let D be an unbounded domain in \mathbb{C} whose boundary consists of piecewise smooth curves, and whose complement is unbounded. Let f be a complex-valued function whose domain of definition contains the closure \overline{D} of D . Then D is called a *direct tract* of f if f is holomorphic in D and continuous in \overline{D} , and there exists $R > 0$ such that $|f(z)| = R$ for z on the boundary of D , while $|f(z)| > R$ for $z \in D$.

The main result of [2] says that for every $\tau > \frac{1}{2}$, there exists a set F of finite logarithmic measure such that, for $r \notin F$, the disc $D(z_r, r/a(r, \log |f|)^\tau)$ is contained in D and (2) holds for $z \in D(z_r, r/a(r, \log |f|)^\tau)$ as $r \rightarrow \infty$. In [1], Bergweiler investigated the size of the disc around z_r in which (2) holds (which can be described as a *Wiman-Valiron disc*), and proved results from below and above as follows. Let $\phi : [t_0, \infty) \rightarrow (0, \infty)$ be a differentiable function satisfying

$$\int_{t_0}^{\infty} \frac{1}{\phi(t)} dt < \infty \quad (3)$$

and

$$K \leq \frac{t\phi'(t)}{\phi(t)} \leq L \quad (4)$$

for certain constants K and L satisfying $0 \leq K \leq 1 < L < 2$. Let f be a function with a direct tract D and let $z_r \in D$ be a maximum point of f in D . Then there exists a set F of finite logarithmic measure such that the disc $D(z_r, r/\sqrt{\phi(a(r, \log |f|))}) \subseteq D$ and (2) holds on $D(z_r, r/\sqrt{\phi(a(r, \log |f|))})$ uniformly as $r \rightarrow \infty$, $r \notin F$. On the other hand, if

$$\int_{t_0}^{\infty} \frac{1}{\phi(t)} dt = \infty$$

and (4) holds for $K = 1$ and some $L < \frac{6}{5}$, then there exists an entire function f which has exactly one direct tract D and is such that if r is sufficiently large and $|z| = r$, then the disc $D(z, r/\sqrt{\phi(a(r, \log |f|))})$ contains a zero of f , and thus (2) cannot hold.

With these results in mind, we seek analogous results for subharmonic functions of the form

$$u(z) = \sum_{j=1}^{\infty} c_j \log \left| 1 - \frac{z}{z_j} \right|, \quad (5)$$

where $c_j \in \mathbb{R}^+$, $z_j \in \mathbb{C}$ and $\sum_{j=1}^{\infty} \left| \frac{c_j}{z_j} \right| < \infty$. In any disc that contains no z_j , u can also be written in the form

$$u(z) = \log |f(z)|,$$

where $f(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j} \right)^{c_j}$. The c_j are the Riesz masses of u at the points z_j ; at all other points u is harmonic and the Riesz mass is zero. Since the z_j are the only “problem” points of u in \mathbb{C} , we do not need to assume the existence of a direct tract as in [2] and [1].

To achieve results for u , it is necessary to impose some lower growth condition on the masses c_j . For if, for example, the sequence z_j consisted of all the rational points in the plane, then every open disc would contain a z_j and we could not have (2). We will assume that

$$\liminf_{j \rightarrow \infty} \frac{c_j}{I(a(|z_j|, u))^\beta} > 0 \quad (6)$$

where

$$I(r) := \int_r^\infty \frac{1}{\phi(t)} dt \quad (7)$$

and β is a positive constant.

We are now ready to state our first theorem.

Theorem 1 *Let u be as in (5) with the growth condition (6) on the masses c_j of u , for some constant $\beta > 0$. Let $t_0 > 0$ and let $\phi : [t_0, \infty) \rightarrow (0, \infty)$ be a differentiable function satisfying (3) and (4) for constants $K = \min\{\beta, 1\}$ and L satisfying $1 < L < 3/2$. Let $r > 0$. If $\beta < 1$ there exists a set $F \subseteq [0, \infty)$ of finite logarithmic measure such that the disc $D(z_r, r/\sqrt{\phi(a(r, u))})$ contains no z_j , and (2) holds uniformly for $z \in D(z_r, r/\sqrt{\phi(a(r, u))})$, as $r \rightarrow \infty$, $r \notin F$.*

Remark: Theorem 1 can be extended to δ -subharmonic functions, that is, functions

$$v(z) = v_1(z) - v_2(z),$$

where v_1, v_2 are subharmonic functions of the form (5). We assume that the set $D = \{z \in \mathbb{C} : v(z) > 0\}$ has an unbounded component D_0 on which v is harmonic. The function which is v on D_0 and 0 elsewhere, which we denote by v^+ , is then subharmonic. The complement of D_0 may contain *islands* \mathcal{I} , that is, closed bounded components of the form $\{z : v^+(z) = 0\}$, and the growth condition we impose on v_1 and v_2 involves islands, rather than points, as follows. Either there are finitely many islands, or

$$\liminf_{r_{\mathcal{I}} \rightarrow \infty} \frac{\mu_{v_1}(\mathcal{I}) - \mu_{v_2}(\mathcal{I})}{I(a(r_{\mathcal{I}}, v^+))^\beta} > 0,$$

where $0 < \beta < 1$ and $r_{\mathcal{I}} = \sup\{|z| : z \in \mathcal{I}\}$. The proof relies on the fact that

$$\mu_{v^+}(\mathcal{I}) = \mu_{v_1}(\mathcal{I}) - \mu_{v_2}(\mathcal{I}),$$

but is otherwise similar to the proof of Theorem 1.

We state our second theorem as follows.

Theorem 2 *Let $t_0 > 0$ and let $\phi : [t_0, \infty) \rightarrow (0, \infty)$ be a differentiable function that satisfies (3). Let I be as defined in (7) and suppose that, given $\kappa > 1$ and $\lambda > 0$,*

$$1 \leq \frac{r\phi'(r)}{\phi(r)} - \frac{(1 + (\kappa - 1)/\lambda)r}{\phi(r)I(r)} \leq \frac{r\phi'(r)}{\phi(r)} \leq L, \quad (8)$$

for $r \geq t_0$, where $1 < L < 6/5$. Then there exists a subharmonic function of the form (5), with

$$c_j = (1 + o(1))(I(a(|z_j|, u))^\kappa)^{\lambda} \quad (j \rightarrow \infty), \quad (9)$$

for which, for all large r , there are no Wiman-Valiron discs of radius greater than

$$\frac{r}{\sqrt{\phi(a(r, u))I(a(r, u))^{1-1/\lambda}}}. \quad (10)$$

Corollary 1 *Theorem 1 fails if $\beta > 1$.*

Assuming Theorem 2 for the moment, let us prove Corollary 1. The function

$$\phi(r) = \eta^{-1} r \cdot \log r \cdot \log_2 r \cdots \log_l r \cdot (\log_{l+1} r)^{1+\eta}, \quad (11)$$

where $\eta > 0$, l is a positive integer and \log_l is the l -times iterated logarithm, satisfies (3) and, for any $\lambda > 0$ and $\kappa > 1$, satisfies (8) for all large t . For in that case $I(r) = (\log_{l+1} r)^{-\eta}$ and

$$\begin{aligned} \frac{r\phi'(r)}{\phi(r)} - \frac{(1 + (\kappa - 1)/\lambda)r}{\phi(r)I(r)} \\ = 1 + \frac{1}{\log r} + \dots + \frac{1}{\log r \log_2 r \cdots \log_l r} + \frac{1 - (\kappa - 1)\eta/\lambda}{\log r \log_2 r \cdots \log_{l+1} r}. \end{aligned} \quad (12)$$

Thus, for this ϕ , the result of Theorem 2 holds for any $\kappa > 1$ and $\lambda > 0$. If $\beta > 1$ we may choose κ and λ satisfying $\kappa/\lambda = \beta$ with $\kappa > 1$ and $\lambda < 1$, and then (10) contradicts the conclusion of Theorem 1.

Remarks: 1. The case $\beta = 1$ is open.

2. Suppose that $0 < \beta < 1$ and that in Theorem 2 we choose $\kappa > 1$ arbitrarily and $\lambda = \kappa/\beta$. The difference between the radii in Theorem 1 and Theorem 2 is $\sqrt{I(a(r, u))^{1-\beta/\kappa}}$. To get some idea of the significance of this factor, consider ϕ of (11). The radius of the disc in Theorem 1 is

$$\eta^{-1}t \cdot \log t \cdot \log_2 t \cdots \log_l t \cdot (\log_{l+1} t)^{1+\eta}|_{t=a(r, u)}$$

and in Theorem 2 is

$$\eta^{-1}t \cdot \log t \cdot \log_2 t \cdots \log_l t \cdot (\log_{l+1} t)^{1+\beta\eta/\kappa}|_{t=a(r, u)}.$$

Numerous applications of the theories of Wiman-Valiron and Macintyre exist, in areas including complex dynamics ([2], [5], [9]), complex differential equations ([6], [10], [11], [16]) and the zero distribution of derivatives ([3], [12]). We note also that functions of the form (5) are connected to the electrostatic fields generated by positively charged wires which meet the complex plane at the points z_j [4].

2 Proof of Theorem 1

First, with u given by (5), we have $a(r, u) \rightarrow \infty$ as $r \rightarrow \infty$. For otherwise $B(r, u) = O(\log r)$ and therefore $\mu_u(D(0, r)) = O(1)$ as $r \rightarrow \infty$, which gives that $\sum_{j \in \mathbb{N}} c_j < \infty$. By (6) the c_j would be bounded below, and this would mean that there are only finitely many c_j , which is a contradiction.

We need two lemmas. The first is based on a well-known result about real functions and can be found in ([2], Lemma 6.10).

Lemma 3 *Let $v : \mathbb{C} \rightarrow [-\infty, \infty)$ be subharmonic, and let $\epsilon > 0$. Then there exists a set $F \subseteq [1, \infty)$ of finite logarithmic measure such that*

$$a(r, v) \leq B(r, v)^{1+\epsilon}$$

for $r \geq 1$, $r \notin F$.

The second lemma summarises, with a small change, the lemmas and discussion in ([1], Section 2). The change we make concerns two functions, σ_1 and σ_2 , that occur in the proof of Lemma 2.2 of [1]. Rather than (in the notation of [1]) $\sigma_1 = \sigma_2 = V^{K/2} \sqrt{\psi}$, we take $\sigma_1 = \sqrt{\psi}$ and $\sigma_2 = V^K \sqrt{\psi}$; the proof is unchanged except that the restriction on the constant L needs to be strengthened (from $L < 2$ to $L < 3/2$).

Lemma 4 *Let $t_0 > 0$, let $v : \mathbb{C} \rightarrow [-\infty, \infty)$ be a subharmonic function, and let $\phi : [t_0, \infty) \rightarrow (0, \infty)$ be a differentiable function satisfying (3) and (4) for certain constants $0 < K < 1 < L < 3/2$. Then there exists a set $F \subseteq [t_0, \infty)$ of finite logarithmic measure such that*

$$B(s, v) \leq B(r, v) + a(r, v) \log \frac{s}{r} + I(a(r, v))^K$$

for

$$\left| \log \frac{s}{r} \right| \leq \frac{1}{\sqrt{\phi(a(r, v))}},$$

uniformly as $r \rightarrow \infty$, $r \notin F$.

We apply Lemma 4 with $K = \beta$ and L satisfying $1 < L < 3/2$, and we apply Lemma 3 for $\epsilon = \beta$. Let F be the union of the exceptional sets of these lemmas. We put $\rho = 2r/\phi(a(r, u))^{\beta/2}$ whenever r is so large that $a(r, u) \neq 0$. Let C be a positive constant sufficiently large that for all large r and any $z_j \in D(z_r, C\rho)$ we have

$$c_j > (\log C)^{-1} I(a(|z_j|, u))^\beta, \quad (13)$$

which is possible by (6). We consider the function

$$v(z) = u(z) - u(z_r) - a(r, u) \log \frac{|z|}{r} = u(z) - B(r, u) - a(r, u) \log \frac{|z|}{r}. \quad (14)$$

For $z \in \overline{D}(z_r, C\rho)$ we have

$$\left| \frac{z - z_r}{z_r} \right| \leq \left| \frac{C\rho}{r} \right| = \frac{2C}{\phi(a(r, u))^{\beta/2}} = o(1)$$

as $r \rightarrow \infty$, by (4) and since $a(r, u) \rightarrow \infty$. Thus since $\beta < 1$,

$$\left| \log \frac{|z|}{r} \right| = \left| \log \left| 1 + \frac{z - z_r}{z_r} \right| \right| \leq 2 \left| \frac{z - z_r}{z_r} \right| \leq \frac{4C}{\phi(a(r, u))^{\beta/2}} \leq \frac{1}{\sqrt{\phi(a(r, f))}}$$

for large r , and the hypotheses of Lemma 4 are satisfied. Since

$$v(z) \leq B(|z|, u) - B(r, u) - a(r, u) \log \frac{|z|}{r}$$

by (14), we conclude from Lemma 4 applied to u that

$$v(z) \leq I(a(r, u))^\beta \quad (15)$$

as $r \rightarrow \infty$, $r \notin F$.

For large r , we have $0 \notin D(z_r, \rho)$ so that the difference of v and u as defined in (14) is harmonic in $D(z_r, \rho)$, and hence their Riesz measures in the disc coincide. Thus

$$\mu_v(D(z_r, \rho)) = \sum_{z_j \in D(z_r, \rho)} c_j \quad (16)$$

where the z_j are matched with the c_j in (5).

On the other hand, we have $\mu_v(D(z_r, \rho)) \log C \leq \int_0^{C\rho} \frac{\mu_v(D(z_r, t))}{t} dt$, and by (15) and the fact that $v(z_r) = 0$,

$$\begin{aligned} \int_0^{C\rho} \frac{\mu_v(D(z_r, t))}{t} dt &= \frac{1}{2\pi} \int_0^{2\pi} v(z_r + C\rho e^{i\zeta}) d\zeta - v(z_r) \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} I(a(r, u))^\beta d\zeta \\ &= I(a(r, u))^\beta. \end{aligned}$$

Combining this with (16) gives

$$\sum_{z_j \in D(z_r, C\rho)} c_j \log C \leq I(a(r, u))^\beta.$$

In view of (13), we deduce that there do not exist any $z_j \in D(z_r, C\rho)$, and thus in $D(z_r, r/\sqrt{\phi(a(r, u))})$.

The remainder of the proof, including the derivation of (2), reproduces the arguments in ([2], Theorem 2.2).

3 Proof of Theorem 2, preliminaries

We follow Bergweiler's intricate construction with slight changes, first introducing certain auxiliary functions in terms of which the example of Theorem 2 is defined. The main concern in this section is to show that certain key calculations that Bergweiler makes carry over in their modified form. Wherever possible, results from [1] have been simply quoted, with appropriate references.

Let ϕ be as in the statement of Theorem 2 and define $\Psi, \chi : [t_0, \infty) \rightarrow (0, \infty)$ by

$$\Psi(t) = \lambda\phi(t)I(t)^{1+(\kappa-1)/\lambda} \quad (17)$$

and

$$\chi(t) = \lambda\phi(t)I(t)^{1-1/\lambda} \quad (18)$$

where λ and κ are constants satisfying $\lambda > 2$ and $\lambda/2 \geq \kappa > 1$, and $I(t)$ is given by (7). Note that

$$\int_{t_0}^t \frac{ds}{\Psi(s)} = \frac{1}{\kappa-1} \left(I(t)^{-(\kappa-1)/\lambda} - I(t_0)^{-(\kappa-1)/\lambda} \right), \quad \int_t^\infty \frac{ds}{\chi(s)} = I(t)^{1/\lambda}. \quad (19)$$

From the first of these we have

$$I'(t) = -\frac{\lambda I(t)^{1+(\kappa-1)/\lambda}}{\Psi(t)}. \quad (20)$$

In what follows it is helpful to know that $I(t) \leq 1$ for $t \geq t_0$, which can always be achieved by taking a somewhat larger value of t_0 . Let us suppose that this has been done.

As in [1], we define $A_1 : [1, \infty) \rightarrow [t_0, \infty)$ by

$$\log t = \int_{t_0}^{A_1(t)} \frac{1}{\Psi(s)} ds. \quad (21)$$

From the first part of (19) we have

$$I(A_1(t)) = ((\kappa - 1) \log(C_1 t))^{-\lambda/(\kappa-1)}, \quad (22)$$

where $C_1 = \exp\{(\kappa - 1)^{-1} I(t_0)^{-(\kappa-1)/\lambda}\}$. We also introduce

$$A_2(t) := tA_1'(t), \quad A_3(t) := tA_2'(t), \quad A_0(t) := \int_1^t A_1(s) \frac{ds}{s}. \quad (23)$$

Differentiating (21), we have

$$A_2(t) = \Psi(A_1(t)); \quad (24)$$

also (see [1], (3.4))

$$A_3(t) \geq A_2(t) \geq A_1(t) \geq t, \quad t \geq 1, \quad (25)$$

and (see [1], (3.10))

$$A_2(t) = o\left(A_0(t)^{\frac{L}{2-L}}\right) \quad (26)$$

as $t \rightarrow \infty$, where L is the number of (8). Define $G : [1, \infty) \rightarrow [0, \infty)$ by

$$G(t) = \int_1^t \sqrt{\chi(A_1(s))} \frac{ds}{s}, \quad (27)$$

and let $H : [0, \infty) \rightarrow [1, \infty)$ be the inverse function. We have, using (17), (18), (24), (25) and the fact that $I \leq 1$,

$$\frac{H(t)}{H'(t)} = \sqrt{\chi(A_1(H(t)))} = \sqrt{\frac{\Psi(A_1(H(t)))}{I(A_1(H(t)))^{\kappa/\lambda}}} = \sqrt{\frac{A_2(H(t))}{I(A_1(H(t)))^{\kappa/\lambda}}} \geq 1. \quad (28)$$

Also, observing that from (20) and (24),

$$I'(A_1(t)) = -\frac{\lambda I(A_1(t))^{1+(\kappa-1)/\lambda}}{A_2(t)},$$

we obtain after some calculation, and using (23),

$$\begin{aligned} \frac{d}{dt} \left(\frac{H(t)}{H'(t)} \right) &= \frac{d}{dt} \left(\sqrt{\frac{A_2(H(t))}{I(A_1(H(t)))^{\kappa/\lambda}}} \right) \\ &= \frac{A_3(H(t))}{2A_2(H(t))} + \frac{\kappa}{2} I(A_1(H(t)))^{(\kappa-1)/\lambda}. \end{aligned} \quad (29)$$

Since $(H/H')' = 1 - HH''/(H')^2$, we have from (29),

$$\frac{H''(t)}{H'(t)} = \sqrt{\frac{I(A_1(H(t)))^{\kappa/\lambda}}{A_2(H(t))}} \left(1 - \frac{A_3(H(t))}{2A_2(H(t))} - \frac{\kappa}{2} I(A_1(H(t)))^{(\kappa-1)/\lambda} \right). \quad (30)$$

From (29), (25) and the fact that $I \leq 1$ we have

$$\frac{d}{dt} \left(\frac{H(t)}{H'(t)} \right) \leq (\kappa + 1) \frac{A_3(H(t))}{2A_2(H(t))}. \quad (31)$$

We need a related estimate, for $\frac{d}{dt} \left(I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left(\frac{r}{H(t)} \right) \right)$. Using (20), (24) and (31) we have, for r fixed and t such that $H(t) \leq r$,

$$\begin{aligned} & \left| \frac{d}{dt} \left(I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left(\frac{r}{H(t)} \right) \right) \right| \\ &= \left| -\kappa (I(A_1(H(t))))^{(2\kappa-1)/\lambda} \log \left(\frac{r}{H(t)} \right) \right. \\ & \quad \left. + I(A_1(H(t)))^{\kappa/\lambda} \left(\log \left(\frac{r}{H(t)} \right) \frac{d}{dt} \left(\frac{H(t)}{H'(t)} \right) - 1 \right) \right| \\ &\leq \left(\kappa + (\kappa + 1) \frac{A_3(H(t))}{2A_2(H(t))} \right) \log r + 1. \end{aligned}$$

Since, as Bergweiler has shown (see [1], (3.15) and (3.2)), $A_3/A_2 \leq Lc^{1/L}A_2^{1-1/L}$, where $c = \Psi(t_0)t_0^{-L}$ and L is the number of (8), the preceding estimate gives

$$\left| \frac{d}{dt} \left(I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left(\frac{r}{H(t)} \right) \right) \right| \leq \Lambda A_2(r)^{1-1/L} \log r + 1 \quad (32)$$

for $H(t) \leq r$, where $\Lambda = \kappa + L(\kappa + 1)c^{1/L}/2$.

Finally, from (30) we have (as in [1] between formulas (3.13) and (3.14))

$$\left| \frac{H''(t)}{H'(t)} \right| \leq \frac{1}{2}(\kappa + 3) \frac{A_3(H(t))}{A_2(H(t))^{3/2}} = o(1)$$

as $t \rightarrow \infty$, and we deduce from this (cf [1], (3.14)) that

$$H'(t+s) = (1 + o(1))H'(t) \quad (33)$$

as $t \rightarrow \infty$, uniformly for $0 \leq s \leq 1$.

4 Proof of Theorem 2

Consider

$$u(z) = \sum_{j=1}^{\infty} c_j \log \left| 1 + \left(\frac{z}{H(j)} \right)^{\left[\frac{H(j)}{H'(j)} \right]} \right|, \quad (34)$$

where $\left[\frac{H(j)}{H'(j)}\right]$ means the integer part of $\frac{H(j)}{H'(j)}$, and with the notation of the preceding section,

$$c_j = I(A_1(H(j)))^{\kappa/\lambda}. \quad (35)$$

Since the terms of the series (34) are subharmonic and the series converges locally uniformly (as will be evident from our subsequent calculations), u is subharmonic.

As in [1] we write

$$A_j = \log \left(1 + \left(\frac{r}{H(j)} \right)^{\left[\frac{H(j)}{H'(j)}\right]} \right),$$

for $r \geq 0$, and

$$S_1 = \sum_{j=1}^{[G(r)]} c_j A_j, \quad S_2 = \sum_{j=[G(r)]+1}^{[G(\rho r)]} c_j A_j, \quad S_3 = \sum_{j=[G(\rho r)]+1}^{\infty} c_j A_j,$$

where G is as in (27) and $\rho = 1 + A_1(r)/(2A_2(r))$. Then

$$B(r, u) \leq S_1 + S_2 + S_3, \quad (36)$$

and we estimate S_1 , S_2 and S_3 in turn.

For S_1 , since $H(j) \leq r$ for $j \leq G(r)$ we have, using (28), (32) and Bergweiler's Lemma 3.1 of [1],

$$\begin{aligned} S_1 &\leq \sum_{j=1}^{[G(r)]} I(A_1(H(j)))^{\kappa/\lambda} \left(\left[\frac{H(j)}{H'(j)} \right] \log \left(\frac{r}{H(j)} \right) + \log 2 \right) \\ &\leq \sum_{j=1}^{[G(r)]} I(A_1(H(j)))^{\kappa/\lambda} \frac{H(j)}{H'(j)} \log \left(\frac{r}{H(j)} \right) + G(r) \log 2 \\ &\leq \int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left(\frac{r}{H(t)} \right) dt \\ &\quad + (\Lambda A_2(r)^{1-1/L} \log r + 1) G(r) + G(r) \log 2 \\ &= \int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left(\frac{r}{H(t)} \right) dt \\ &\quad + O \left(G(r) A_2(r)^{1-1/L} \log r \right) \end{aligned} \quad (37)$$

as $r \rightarrow \infty$. With the change of variable $s = H(t)$ we obtain

$$\begin{aligned}
& \int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log\left(\frac{r}{H(t)}\right) dt \\
&= \int_1^r I(A_1(s))^{\kappa/\lambda} s G'(s)^2 \log\left(\frac{r}{s}\right) ds \\
&= \int_1^r I(A_1(s))^{\kappa/\lambda} \chi(A_1(s)) \log\left(\frac{r}{s}\right) \frac{ds}{s} \\
&= \int_1^r \Psi(A_1(s)) \log\left(\frac{r}{s}\right) \frac{ds}{s} \\
&= \int_1^r A_2(s) \log\left(\frac{r}{s}\right) \frac{ds}{s} \\
&= \int_1^r A_1'(s) \log\left(\frac{r}{s}\right) ds \\
&= A_0(r) - t_0 \log r,
\end{aligned} \tag{38}$$

in view of (23), the last step following after integrating by parts. Concerning the error term in (37) we have, from (22) and (24), and using $\chi = \Psi/I^{\kappa/\lambda}$,

$$G(r) \leq \sqrt{\chi(A_1(r))} \log r = O\left(\sqrt{A_2(r)} (\log r)^{\frac{3\kappa-2}{2\kappa-2}}\right) \tag{39}$$

as $r \rightarrow \infty$, and $A_0(r) = \int_1^r (A_1(s)/s) ds \geq r - 1$ from (25), so that

$$\log r \leq (1 + o(1)) \log A_0(r) \tag{40}$$

as $r \rightarrow \infty$. Thus, using (26),

$$\begin{aligned}
G(r) A_2(r)^{1-1/L} \log r &= O\left(A_2(r)^{(3L-2)/(2L)} (\log r)^{(3\kappa-2)/(2\kappa-2)}\right) \\
&= O\left(A_0(r)^{(3L-2)/(4-2L)} (\log r)^{(3\kappa-2)/(2\kappa-2)}\right) \\
&= o(A_0(r))
\end{aligned} \tag{41}$$

as $r \rightarrow \infty$. Combining (37), (38) and (41), and using the fact that $L < 6/5$, we conclude that

$$S_1 \leq (1 + o(1)) A_0(r) \quad (r \rightarrow \infty). \tag{42}$$

Turning to S_2 , we have, using (27) and (24), and since $H(j) > r$,

$$\begin{aligned}
S_2 &\leq I(A_1(r))^{\kappa/\lambda} G(\rho r) \log 2 \\
&\leq I(A_1(r))^{\kappa/\lambda} \sqrt{\chi(A_1(\rho r))} \log(\rho r) \log 2 \\
&= I(A_1(r))^{\kappa/(2\lambda)} \sqrt{\Psi(A_1(\rho r))} \log(\rho r) \log 2 \\
&\leq \sqrt{A_2(\rho r)} \log(\rho r) \log 2.
\end{aligned}$$

As Bergweiler shows ([1], p. 28), $\sqrt{A_2(\rho r)} \log(\rho r) = o(A_0(r))$ as $r \rightarrow \infty$, and thus

$$S_2 = o(A_0(r)) \quad (r \rightarrow \infty). \quad (43)$$

Finally, for S_3 , since $H(j) \geq \rho r$ for $j \geq G(\rho r)$,

$$\begin{aligned} S_3 &\leq \sum_{j=[G(\rho r)]+1}^{\infty} c_j \left(\frac{r}{H(j)} \right)^{\left[\frac{H(j)}{H'(j)} \right]} \\ &\leq \rho \sum_{j=[G(\rho r)]+1}^{\infty} I(A_1(H(j)))^{\kappa/\lambda} \exp \left(-\tau \frac{H(j)}{H'(j)} \right), \end{aligned}$$

where $\tau = \log \rho \geq 0$. Thus, since, from (31), H/H' is increasing,

$$S_3 \leq \rho \left(1 + \int_{G(\rho r)}^{\infty} I(A_1(H(t)))^{\kappa/\lambda} \exp \left(-\tau \frac{H(t)}{H'(t)} \right) dt \right).$$

Making the change of variable $s = H(t)$, and using (28) and the fact that $\chi(A_1) \geq \Psi(A_1) = A_2$ and $I \leq 1$, we have

$$\begin{aligned} S_3 &\leq \rho \left(1 + \int_{\rho r}^{\infty} I(A_1(s))^{\kappa/\lambda} \sqrt{\chi(A_1(s))} \exp \left(-\tau \sqrt{\chi(A_1(s))} \right) \frac{ds}{s} \right) \\ &= \rho \left(1 + \int_{\rho r}^{\infty} I(A_1(s))^{\kappa/(2\lambda)} \sqrt{A_2(s)} \exp \left(-\tau \sqrt{\chi(A_1(s))} \right) \frac{ds}{s} \right) \\ &\leq \rho \left(1 + \int_{\rho r}^{\infty} \sqrt{A_2(s)} \exp \left(-\tau \sqrt{A_2(s)} \right) \frac{ds}{s} \right). \end{aligned} \quad (44)$$

As in [1], p. 29, we deduce from (44) that $S_3 = o(A_0(r))$, and combining this, (42), (43) and (36), we obtain

$$B(r, u) \leq (1 + o(1))A_0(r) \quad (r \rightarrow \infty). \quad (45)$$

To establish the reverse inequality, we use Jensen's inequality and Lemma 3.1 of [1], and obtain

$$\begin{aligned} B(r, u) &\geq \sum_{j=1}^{[G(r)]} c_j \left[\frac{H(j)}{H'(j)} \right] \log \left(\frac{r}{H(j)} \right) \\ &\geq \int_{G(r)-[G(r)]}^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \left(\frac{H(t)}{H'(t)} - 1 \right) \log \left(\frac{r}{H(t)} \right) dt \\ &\quad + O \left(G(r) \sup_{0 < t < G(r)} |F'(t)| \right), \end{aligned} \quad (46)$$

where

$$F(t) = I(A_1(H(t)))^{\kappa/\lambda} \left(\frac{H(t)}{H'(t)} - 1 \right) \log \left(\frac{r}{H(t)} \right).$$

Since, for $H(t) \leq r$,

$$\begin{aligned} & \left| \frac{d}{dt} \left(I(A_1(H(t)))^{\kappa/\lambda} \log \left(\frac{r}{H(t)} \right) \right) \right| \\ &= \left(\kappa I(A_1(H(t)))^{(2\kappa-1)/\lambda} \log \left(\frac{r}{H(t)} \right) + I(A_1(H(t)))^{\kappa/\lambda} \right) \frac{H'(t)}{H(t)} \\ &\leq \kappa \log r + 1, \end{aligned}$$

from (20) and (28), we have, taking account of (32) and (26),

$$|F'(t)| \leq (\Lambda + \kappa) A_2(r)^{1-1/L} \log r + 2 = o \left(A_0(r)^{(L-1)/(2-L)} \log r \right)$$

as $r \rightarrow \infty$. We deduce from (41) that

$$G(r) \sup_{0 < t < G(r)} |F'(t)| = o(A_0(r)) \quad (47)$$

as $r \rightarrow \infty$. Also, from (40),

$$\int_0^1 I(A_1(H(t)))^{\kappa/\lambda} \left(\frac{H(t)}{H'(t)} - 1 \right) \log \left(\frac{r}{H(t)} \right) dt = O(\log r) = O(\log A_0(r)).$$

From this, (46) and (47),

$$B(r, u) \geq \int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \left(\frac{H(t)}{H'(t)} - 1 \right) \log \left(\frac{r}{H(t)} \right) dt + o(A_0(r))$$

as $r \rightarrow \infty$. We have

$$\int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \frac{H(t)}{H'(t)} \log \left(\frac{r}{H(t)} \right) dt = (1 + o(1)) A_0(r)$$

as $r \rightarrow \infty$, from (38) and (40), and with the change of variable $s = H(t)$,

$$\begin{aligned} & \int_0^{G(r)} I(A_1(H(t)))^{\kappa/\lambda} \log \left(\frac{r}{H(t)} \right) dt \\ &= \int_1^r I(A_1(s))^{\kappa/\lambda} \log \left(\frac{r}{s} \right) G'(s) ds \\ &\leq \int_1^r \log \left(\frac{r}{s} \right) G'(s) ds = \int_1^r \frac{G(s)}{s} ds \leq G(r) \log r = o(A_0(r)) \end{aligned}$$

as $r \rightarrow \infty$, from (39) and (40). It follows that $B(r, u) \geq (1 + o(1)) A_0(r)$, and combining this with (45) we conclude that

$$B(r, u) = (1 + o(1)) A_0(r)$$

and, as in ([1], p. 30), that

$$a(r, u) = (1 + o(1)) A_1(r) \quad (48)$$

as $r \rightarrow \infty$.

In the context of Theorem 2, $H(j) = |z_j|$ and so, from (35) and (48), $c_j = I((1 + o(1))a(|z_j|, u))^{\kappa/\lambda}$ as $j \rightarrow \infty$. From (8) we have $r_2/r_1 \leq \phi(r_2)/\phi(r_1) \leq (r_2/r_1)^L$ for $r_2 \geq r_1 \geq t_0$, and it follows from this that $\int_r^{(1+o(1))r} \phi(t)^{-1} dt = o(\int_r^{2r} \phi(t)^{-1} dt)$ as $r \rightarrow \infty$. Thus $I((1 + o(1))r) = (1 + o(1))I(r)$ as $r \rightarrow \infty$, and we have (9).

Bergweiler's argument ([1], section 3.3), which requires (33), shows that for all large z , a disc centred at z with radius $d(|z|) = 9|z|/\sqrt{\chi(a(r, u)/2)}$ contains one of the singularities of u , and thus no Wiman-Valiron disc can have radius greater than $d(|z|)$. Since $\chi = \lambda\phi I^{1-1/\lambda}$, and since, arguing as in the preceding paragraph, $\phi(r/2) \geq \text{const} \cdot \phi(r)$, and also $I(r/2) > I(r)$, we have $d(|z|) \leq \text{const} \cdot |z|/\sqrt{\phi(a(r, u))I(a(r, u))^{1-1/\lambda}}$. If we first obtain this result for a value slightly smaller than λ , and with κ adjusted so that the ratio κ/λ remains constant, the conclusion of Theorem 2 follows.

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