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The taxicab locus of Apollonius: promoting exploratory routes to the punchline using rich undergraduate tasks

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ABSTRACT

A key motivational tactic in undergraduate mathematics teaching is to launch topics with fundamental questions that originate from surprising or remarkable phenomena. Nonetheless, constructing a sequence of tasks that promotes students' own routes to resolving such questions is challenging. This note aims to address this challenge in two ways. First, to illustrate the motivational tactic, the taxicab manifestation of a locus attributed to Apollonius is introduced and a natural question arising from comparison with the analogous Euclidean locus is considered, namely, does the taxicab locus of Apollonius ever coincide with a taxicab circle? Second, a companion sequence of rich undergraduate tasks is elaborated using theoretical design principles, with the tasks culminating in this fundamental geometric question. This note therefore provides a design approach that can be replicated in undergraduate teaching contexts based around similarly motivating mathematical phenomena.

1. Introduction

Mathematicians are drawn to phenomena that they describe as 'surprising' and 'remarkable' [1]. A key motivational tactic in undergraduate teaching is to use such phenomena as starting points, often in combination with 'problems that the topic can resolve' [2, Ch. 7]. However, the process of designing a sequence of tasks that offers undergraduate students significant ownership over a route to resolving such problems is challenging. One of the principal aims of this note is to bring structure to this process by explicating a task sequence using theoretical design principles suited to producing rich undergraduate tasks that encourage exploratory student approaches. Here, a *rich task* is understood as one that is 'complex, non-algorithmic, and non-routine, allowing for multiple strategies and representations and no single pathway to a solution' [3, p.602]. The use of rich tasks is motivated by the particular set of cognitive demands they place on students; for example, they typically require students to conjecture, generalise and justify their solutions (see Section 3 for further discussion and [4] for wider background motivating the use of rich tasks).

In order to promote and illustrate the motivational tactic described above, a novel manifestation of a locus attributed to Apollonius [5, Ch. 6] is introduced and the task sequence presented here is designed to converge towards a fundamental question involving this geometric object. This question arises from a natural course of mathematical investigation whereby familiar loci are reinterpreted in terms of a new metric.

A popular choice of metric for deepening student awareness of abstract geometry is the taxicab metric (or ℓ_1-metric) [6], [7], [8], [9], [10], [11]. For points $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ in $\mathbb{R}^2$, the *taxicab distance* between P and Q is defined by

$$d(P, Q) = |x_P - x_Q| + |y_P - y_Q|$$

and the resulting function d is referred to as the *taxicab metric*. Under this metric, the manifestations of familiar curves such as conic sections are found to exibit characteristics that both complement and contrast with their Euclidean counterparts [12], [13], [14]. Beyond such purely geometric territory, Tisdell [15] has recently suggested pedagogical benefits to approaches that use the taxicab metric when establishing *a priori* bounds on potential solutions to differential equations.

The principal taxicab object of interest here is defined as follows. Given two distinct foci $Q, R \in \mathbb{R}^2$ and a positive real constant $k \neq 1$, the *locus of Apollonius* is defined as the locus of points $P \in \mathbb{R}^2$ such that

$$\frac{d(P, Q)}{d(P, R)} = k \tag{1}$$

where d can be any metric on $\mathbb{R}^2$. For example, when d is the Euclidean metric on $\mathbb{R}^2$, this locus is a Euclidean circle, referred to as the *circle of Apollonius* [5, Ch. 6]. This may be deduced geometrically [5, Th. 6.8.2] or algebraically by using completion of the square to arrive back at the familiar Euclidean equation of a circle. In what follows, d will always be the taxicab metric and the resulting locus defined by (1) will be referred to as the *taxicab locus of Apollonius*. The 'punchline' referred to in the title of this note is the fact that the taxicab locus of Apollonius never coincides with a taxicab circle.

This note is organised as follows. A brief introduction to the taxicab locus of Apollonius and a short proof of the 'punchline' is given in the next section. In Section 3, the relevant task design literature is reviewed and the task sequence is presented and explicated in light of that literature. Avenues for further development are considered in the conclusion. The overarching aim is to provide the reader with a design approach that can be replicated in undergraduate teaching contexts based around similarly motivating mathematical phenomena.

2. The taxicab locus of Apollonius

Let $Q = (x_Q, y_Q)$ and $R = (x_R, y_R)$ be distinct foci in $\mathbb{R}^2$ and let $k \neq 1$ be a positive real constant. Write $P = (x, y) \in \mathbb{R}^2$ and define $\lambda_x^Q(P) = \text{sgn}(x - x_Q)$, where sgn denotes the usual sign function. Define λ_x^R, λ_y^Q, and λ_y^R analogously. Then, (1) is equivalent to
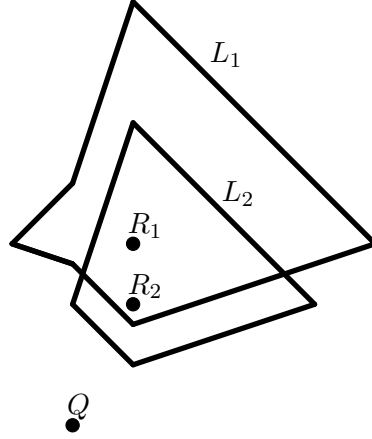
$$\lambda_x^Q(P)(x - x_Q) + \lambda_y^Q(P)(y - y_Q) = k\lambda_x^R(P)(x - x_R) + k\lambda_y^R(P)(y - y_R). \tag{2}$$

In order to organise an analytical construction of the locus, it is helpful to consider the 4-tuples defined by

$$\lambda(P) = (\lambda_x^Q(P), \lambda_x^R(P), \lambda_y^Q(P), \lambda_y^R(P)).$$

$(-1,-1,1,1)$	$(1,-1,1,1)$	$(1,1,1,1)$
$(-1,-1,1,-1)$	$(1,-1,1,-1)$	$(1,1,1,-1)$
$(-1,-1,1,-1)$	$(1,-1,-1,-1)$	$(1,1,-1,-1)$

Figure 1. Possible values of λ in relative position

Figure 2. Two taxicab loci of Apollonius, L_1 and L_2, with defining parameters $k = 2$, $Q = (0,0)$, and $R = R_1 = (1,3)$ and $R = R_2 = (1,2)$, respectively

Without loss of generality, it can be assumed that $x_Q \leqslant x_R$. In the case where QR has non-negative gradient, the possible values of $\lambda(P)$ describing the position of P relative to Q and R in the plane are given in Figure 1. Each 4-tuple in Figure 1 also corresponds to a potential line segment in the locus, with (2) giving the equation of the corresponding line. However, depending on the defining parameters Q, R and k, there are always some 4-tuples in Figure 1 such that no points P in the corresponding region satisfy (2). There are a number of ways to see this; the proof of the theorem given later in this section gives one example of appropriate reasoning.

Figure 2 shows two examples of taxicab loci of Apollonius. Since the taxicab circle of centre (a, b) and radius r comprises the four line segments consecutively joining $(a - r, b)$, $(a, b + r)$, $(a + r, b)$, $(a, b - r)$, $(a - r, b)$, locus L_2 in Figure 2 might entice one to think that an appropriate choice of parameters could yield a taxicab locus of Apollonius that is also a taxicab circle. The following shows that this cannot happen.

Theorem 2.1. *A taxicab locus of Apollonius cannot be a taxicab circle.*

Proof. First note that a taxicab circle comprises two line segments with gradient 1 and two with gradient -1.

Assume that QR has non-negative gradient. As previously described, the 4-tuples in Figure 1 may be used to identify potential line segments that form the locus of Apollonius. For any $\mu = (\mu_1, \mu_2, \mu_3, \mu_4) \in \{-1, 1\}^4$, set

$$m(\mu) = \frac{k\mu_4 - \mu_3}{\mu_1 - k\mu_2}.$$

Using (2), it can be seen that applying m to each of the 4-tuples in Figure 1 gives

μ	$m(\mu)$
$(-1,-1,-1,-1)$	-1
$(-1,-1,1,1)$	1
$(1,-1,1,-1)$	-1
$(1,1,-1,-1)$	1
$(1,1,1,1)$	-1

Table 1. 4-tuples μ for which $|m(\mu)| = 1$

the possible gradients of line segments in the locus. In particular, irrespective of the value of k, at most two potential line segments with gradient 1 may occur, as shown in Table 1.

However, using the definition of the locus,

$$k = \frac{|x - x_Q| + |y - y_Q|}{|x - x_R| + |y - y_R|}$$

Therefore, $k > 1$ if and only if there exists a point P on the locus for which

$$\lambda_x^Q(P)(x - x_Q) + \lambda_y^Q(P)(y - y_Q) > \lambda_x^R(P)(x - x_R) + \lambda_y^R(P)(y - y_R). \qquad (3)$$

Hence, if there is a locus point P such that $\lambda(P) = (-1, -1, 1, 1)$, then by eliminating x and y from (3), it follows that

$$k > 1 \Leftrightarrow x_Q - y_Q > x_R - y_R. \qquad (4)$$

On the other hand, if there is a locus point P for which $\lambda(P) = (1, 1, -1, -1)$, then (3) gives

$$k > 1 \Leftrightarrow x_Q - y_Q < x_R - y_R. \qquad (5)$$

The incompatibility of (4) and (5) implies that only one of the 4-tuples $(-1, -1, 1, 1)$ and $(1, 1, -1, -1)$ can arise as a value of $\lambda(P)$ for a given locus. It follows from Table 1 that there is at most one line segment in the locus with gradient 1. Thus, the locus cannot be a taxicab circle.

In the case that QR has negative gradient, one uses a completely analogous argument to show that there can be at most one line segment in the locus with gradient -1 instead. $\qquad \square$

3. The associated task sequence

In this section, relevant design considerations from the literature are reviewed and the sequence of tasks is presented (in italics) with supporting commentary elaborating how the tasks align with these considerations.

3.1. Design considerations from the literature

Mathematical task design has evolved to become a multi-faceted theory rooted in a much broader design research methodology [16]. A fundamental issue highlighted by Ruthven et. al [17] is the need to make a distinction between design as implementation and design as intention, and a case can be made for emphasis on both aspects of mathematical task design [16, Sec. 2.2.16]. Here, the emphasis is on design as intention, with the aim of constructing rich tasks that encourage exploratory student approaches. The work of Lithner et al. [18], dealing with task design to enhance creative mathematical reasoning (CMR), and the design principles of Lin et al. [19], dealing with tasks that encourage conjecturing and proving, are drawn upon together here as they address these design intentions in a number of complementary ways. Notably, both perspectives link to Brousseau's Theory of Didactical Situations [20] (TDS) which has been applied and adapted using design experiments in a university setting [21].

The tasks presented here have been designed for use in a seminar of at least two hours for a medium-sized (less than thirty, say) group of students, including periods of exposition and informal discussion. This type of seminar is somewhere between a 'dialogue course' in the sense of Hersant and Perrin-Glorian [22] (who describe the use of TDS outside of experimental situations) and the more traditional lecture-problem class format for mathematics teaching at undergraduate level.

In order to frame the tasks, exposition should include underlying theory (including definitions) and preparatory examples (in particular, taxicab circles) but should not provide a prescribed method or algorithm to complete the given tasks, as this relieves the student of the responsibility to attempt significant CMR [18]. Lithner et al. also argue that the absence of a prescribed method should be an implicit expectation between the teacher and the student. That is, in the terminology of TDS, it should form part of the *didactic contract*, otherwise the process of *devolution*, whereby the student takes on responsibility for the anticipated problem solving process, would be at stake. These considerations are thus assumed to be necessary contextual prerequisites in order for the tasks presented to meet the richness criteria of being non-algorithmic and non-routine.

While Lithner et al. do not primarily depart from the premise of constructing exploratory tasks, they do nonetheless describe the tasks they find to encourage CMR as 'exploratory'. These are further elaborated as tasks that require arguments to substantiate conclusions in order to 'guide and verify the construction of new insights' [18, p.228]. Indeed, exploratory tasks are recognised as a key initial phase leading to conjecturing and proof [23]. Lin et al. have developed a set of principles for designing tasks involving conjecturing and proof that encompass this initial exploratory phase, those that promote conjecturing and the transition to proof (*transiting*) being most applicable here. The principles assert that

(1) conjecturing is promoted by providing opportunities to engage in observation, to engage in construction (for example, of new rules, formulae and principles), to transform prior knowledge and for reflection;
(2) transiting is promoted by generating a need for students to engage in proof and by establishing norms in the classroom that 'allow discussions, under the facilitation of the instructor, about accepting/rejecting mathematical ideas, including conjectures, based on the logical structure of the mathematical system rather than by appeal to the authority of the instructor' [19, p.317].

This emphasis on establishing classroom norms within the transiting principle outlined

$(1, 1)$ $(3, 1)$
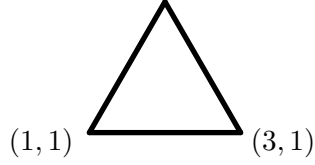
Figure 3. The Euclidean equilateral triangle for Task 1

above aligns with the work of Lithner et al. and TDS.

3.2. Taxicab equilateral triangles (Task 1)

The figure [Figure 3] shows a Euclidean equilateral triangle. Is this a taxicab equilateral triangle? More generally, given two vertices of a taxicab equilateral triangle, how do we locate a third vertex?

From the outset, this task requires multiple steps of reasoning as it is necessary to find the coordinates of the remaining vertex of the triangle $(2, 1 + \sqrt{2})$ in order to calculate the taxicab length of the slant sides. The given example also provides a basis to encourage further observations with the potential to construct a rule in an obvious special case, namely when the given vertices lie on a horizontal line.

The second part of the task is amenable to multiple solution strategies involving both algebraic and geometric representations. The phrasing of the question allows different types of construction as a solution; for example, a description of a general method or a solution formula. Based on the condition that the two unknown sides must have equal lengths, a model solution might begin with the system of equations

$$\left. \begin{array}{l} |x - x_Q| + |y - y_Q| = |x_Q - x_R| + |y_Q - y_R| \\ |x - x_R| + |y - y_R| = |x_Q - x_R| + |y_Q - y_R| \end{array} \right\}$$

where $Q = (x_Q, y_Q)$ and $R = (x_R, y_R)$ are given vertices and $P = (x, y)$ is the unknown vertex. This may be interpreted as an algebraic problem based on cases of the absolute value, or geometrically as an intersection problem between two taxicab circles. The latter provides an opportunity to transform prior knowledge; for example, the Euclidean compass-and-straightedge construction of an equilateral triangle could be interpreted in the taxicab plane with the aid of taxicab circles.

Finally, the second part can be explored more deeply if the ways in which taxicab circles can intersect is taken into account. For example, in contrast to the given example, if the gradient m of the line segment joining the given vertices satisfies $|m| = 1$, then infinitely many solutions are possible.

3.3. Taxicab ellipses (Task 2)

Draw and justify a diagram of a taxicab ellipse starting with two fixed foci lying on a vertical or horizontal line. What other types of taxicab ellipse are possible?

Prior to this task, it is expected that the abstract definition of an ellipse has been given and the example of a taxicab circle has been introduced. Along with the outcome of the previous task, these prerequisites are aimed at providing students with the experience to tackle this task without breaching the didactic contract.

Again, the problem can be approached using multiple solution strategies involving algebraic and geometric representations or a combination of both, the phrasing of the task being aimed at encouraging the use of a combination. The inclusion of the word 'justify' in the task statement is also intended to promote transiting (see [18]).

Motivated by Task 1, conjecture and construction might stem from considering whether or not of the foci lie on a vertical or horizontal line, distinguishing the cases of 'hexagonal' and 'octagonal' taxicab ellipses. The defining constant k in the ellipse equation

$$d(P,Q) + d(P,R) = k$$

introduces a layer of complexity not present in Task 1, and a surprising new phenomenon may be observed when $d(Q,R) = k$; if the foci Q and R are not aligned vertically nor horizontally in this case, then the resulting taxicab ellipse is in fact a rectangular region.

3.4. Taxicab loci of Apollonius (Task 3)

Apollonius was a Greek geometer who gave an alternative definition of a circle as 'the locus of points in a plane that have a specified ratio of distances to two fixed foci'. Draw and justify examples of how this locus appears in the taxicab plane. Is this taxicab locus of Apollonius ever a taxicab circle?

This final task addresses the design considerations in a similar way to the previous two tasks but allows greater scope for exploration and conjecture. For example, conjectures might involve the minimum and maximum number of line segments forming the locus or the number of line segments with gradient 1 or -1 (leading ultimately to the punchline of Theorem 2.1 and an answer to the final part of the task). Beyond this, it is clear from Figure 2 that the number of line segments forming the locus, s say, is no longer governed by whether or not the foci are vertically or horizontally aligned and the line of enquiry associated with predicting whether $s = 4$ or $s = 6$ from the defining parameters is much more involved than the analogous problem for an ellipse.

4. Conclusions and further work

In this note, my aim has been to illuminate the process of constructing a sequence of tasks that stem from a motivational mathematical phenomenon and which meet a particular set of design intentions rooted in theoretical principles of task design. The logical next step is to develop the initial design through a cycle of progressive refinement; Ruthven et. al [17] frame the need for the clarity and coherence of initial designs within this cycle, which they identify as a key feature of design implementation.

Although the design intention here emphasises exploration over formal proof, I had no particular motive to sideline the latter. That said, the task sequence presented originates from a first geometry course for undergraduates studying mathematics and education who, like learners of mathematics at many levels, experience difficulties with proof. Transiting has been found to mitigate this difficulty in a comparable situation [24], and it would be interesting to study the potential of a design framework directly targeting transiting in relation to the material presented here.

Finally, in terms of exploration, the use of technology plays a crucial role [25], and specifically tailored tasks that make use of dynamic geometry tools provide an

invaluable counterpoint to the tasks suggested here.

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