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EXPLORING CHILDREN'S CONCEPTIONS OF ZERO

Rona Catterall

This thesis is submitted in accordance with the requirements
of Sheffield Hallam University
for the degree of Doctor of Philosophy

July 2006

The candidate confirms that the work submitted is her own and that appropriate credit
has been given where reference has been made to the work of others.

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ABSTRACT

This abstract is submitted in accordance with the requirements for the degree of PhD

Rona Catterall

Exploring Children's Conceptions of Zero

> 'Zero remains, even in our times, an outcast among the natural numbers ...
> A symbol without much understanding of its meaning.' (Pogliani et al 1998, p742)

The overall aim of this study was to explore children's conceptions of zero. To determine whether children have more problems understanding and using zero than other single digit numbers and, if so, to investigate why these problems might arise. The focus areas of this exploration were:

(a) Zero as a number and its relationship to other numbers
(b) The zero number facts
(c) The empty set
(d) The language of zero.

Initial data was gained using questionnaire returns from 100 children, aged 10-11 years, in five UK primary schools. More detailed fieldwork was undertaken using task-interviews conducted with 136 children, aged 3 to 11, in one of these schools. The children's explanations for their answers, correct or not, and the analysis of their reasoning provided some unexpected results.

With regard to the children involved in this research this study concludes that a child's conception of zero consists of a series of *generally accepted notions* such as zero being a number, zero being worth nothing and zero being found in the number order, next to one. These generally *accepted notions* are subject to *diversity of thought* and an individual child's *diversity of thought* did result in high profile consequences. These were the ignoring of zero; the formation of *a personal zero rule(s)*; children's understanding of *nothing* as *nothingness* and the startling reaction of many young children (aged 3 to 5) to the empty set.

The research highlights and contributes new knowledge to an, as-yet, uncharted area of investigation that of children's conceptions of zero. As a consequence the findings are discussed in terms of their implications to primary mathematics education.

Pogliani, L., Randic, M. and Trinajstic, N. (1998) *International Journal of Mathematical Education in Science and Technology*, **29**, 729-744.

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ILLUSTRATIONS, LISTS, MODELS AND TABLES

Illustrations

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READER INFORMATION

The conventions used in this study are:

Child quotations:

- Child quotations are in italic type and are indicated by a square bullet point.
- [ ] indicate a researcher's explanation or question
- The brackets following a child quotation give the age of the child and whether the quotation was written or verbal.
- Written quotations - contain the spellings and grammar of the child.
- Verbal quotations – here the writer has used the words of the child and has tried to capture the child's nuances with the use of punctuation and underlining.

Numbers:

- When a written number symbol is referred to in the text it is written as '2, 4, 7'.
- When a spoken number word is referred to it is written as 'two, four, seven'.
- A quantity, a number of objects in a set, is referred to as 'an amount' and is written as **, ****, *******.

In the charts and tables:

- For clarity and to aid comparison the percentages in the tables have been taken to the nearest whole number.
- The numbers not in parenthesis are the first answers; those in parenthesis show the altered, final answers.

Where colour coding is used in the tables this is:

- red for the number of children
- black for the percentage within that cohort
- green for the number of incorrect answers.

CHAPTER 1

METHODOLOGY AND METHOD

Live as if you were to die tomorrow.
Learn as if you were to live forever.
Mahatma Gandhi

Outline of Chapter 1, Part 1

Part 1, of this chapter, deals with the influences the researcher brought to the study by reviewing the researcher's personal history in education and mathematics. Explanations are given for the selection of the research and why the subject of children's conceptions of zero was of particular interest. Background information on the research area leads on to the limiting of the research content to the areas of exploration covered by this study. Part 1 concludes with a discussion on the particular relevance of the research to primary mathematics education.

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Chapter One, Part One: Research History

Background of the researcher

> It is important to recognise that knowledge generation, within research, is understood as an active, context-based process influenced by the values, histories, and practices of the researcher and of the community in which the research is done. (Atwater 1996, p.3)

Acknowledgement must be given to the influences brought to every aspect of the research including the selection of data collection methods; the formulation of questions; the administration of tasks and interviews; the choice of categories for analysis and the interpretation of empirical data. It is important that the reader is informed of the researcher's background which could cause any predisposition, so the reader can appreciate the effect these may have had on this research.

The researcher spent twenty years teaching children in a primary and nursery school, first as a classroom teacher, later as Deputy Headteacher and Acting Headteacher. There followed a decade in the Mathematics Advisory Teacher Service working with children in mainstream primary schools, as well as with special needs children (with moderate and severe learning difficulties) and supporting the teachers of these children. For the next ten years she was engaged as a senior lecturer in the Mathematics Education department of a university. Here her role included working with students on various courses as well as involvement in Initial Teacher Training and In-Service Training in primary mathematics for practising teachers and educationalists.

Professional interests

Young children's understanding of number, together with the long-term effects caused by the way number is taught, have been of specific interest to the researcher during her years in Mathematics Education. The number zero in particular appeared to create its own mathematical problems for children, for students and for teachers. Over the years pieces of children's work connected with zero have been collected and incidents, involving both adults and children, recorded. In 1984, while attending a conference of the British Society for the Psychology of Learning Mathematics (BSPLM) at Manchester Polytechnic, Didsbury, the researcher heard Martin Hughes present a paper on the topic of young children dealing with single digit numbers. Particularly fascinating were his findings on children recording the empty set (this research was subsequently published in Hughes, 1986).

Between 1998 and 1999 the researcher wrote a series of articles about the teaching and learning of single digit numbers, with particular reference to young children. The title of one article was *Zero* (Catterall, 1999). Reading around the subject (Seife, 2000; Reid, 1992; Katz, 1998) revived the researcher's interest in zero but it also revealed that there were limits to her own understanding. Did this limited comprehension apply to others? In a number of informal discussions with students, teaching staff and adults their confusion with zero was noticeable. All expressed surprise that they could be so uncertain in their thinking; so unsure of answers to questions involving '*nothing*'.

Illustrative incidents

One teacher said that her 13 year old daughter, '*does struggle enormously with the concept of zero*'.
The teacher went on to add that she was the same when she was young. Then she quietly admitted, '*now, when I see a zero in a sum my heart sinks a bit. I'm still not confident that my answers are right*'.

A second teacher said, '*I do think I'm quite good at mathematics but I do have some problems explaining how you get the answer when you multiply by zero*'.

Those who were working with children provided examples where zero created cause for concern in various areas of children's mathematics. These examples mirrored the experience of the researcher. Many questions were raised. Why are children and teachers having problems? What are the problems they are having? Are there common misconceptions? Are they misconceptions?

The scope of the research

As would be expected, zero may appear wherever there are other numbers; however, compared with other single digits zero brings with it more complicated issues. These issues have many facets including the language and description of zero, zero and infinity, indices, positionality, zero as a place holder, the empty set, partitioning, zero as a number, the relationship of zero with other numbers, zero in calculations and zero in areas of measurement. In all these areas zero appears to cause more problems than any other number. An example of this is seen in computation. Guedj (1998) states that if you wish children to succeed in arithmetic tests then you omit the zeros. Suydam and Dessart (1978) found one of the seven most frequent whole number errors was located in errors with zero, for each operation. Other publications substantiate the general thinking that zero difficulties permeate whole number computation but it would appear that few research projects address children's understanding of zero unless it is in a place value context and in algorithms of two or more digits (Lappan, 1987). No study appears to have attempted to discover if the foundations for some of these errors lie in a child's conception of zero and if and how these conceptions develop.

The central theme of this research was to explore primary children's understanding of zero; how young children conceive zero with regard to other single digits numbers; how these conceptions affected the use of zero in the zero number facts (outside the areas of place value and two digit algorithms). Permeating this investigation was the contextual selection, use and understanding of the language used for zero.

The research explored the conceptions of individual children, aged 3 to 11, within the focus areas of,

1. Zero as a number and its relationship to other numbers
2. The zero number facts

$a + 0$	$0 + a$	$a - 0$	$0 - a$
$a \times 0$	$0 \times a$	$0 \div a$	$a \div 0$

3. The empty set
4. The language of zero

It was recognised that aspects of mathematics are inter-linked and that it might be difficult to separate, entirely, these focus areas. So, as the inquiry continued, in certain areas it became appropriate to address some of the other zero aspects.

The relevance of the research

It is difficult to see when research on some aspect of the teaching, learning and understanding of number has not been relevant but there is a particular relevance at this time. Over the last two centuries, there have been many changes to the content of the mathematics curriculum for young children in the English education system. Similarly, there have been many changes in pedagogical approaches to the implementation of this curriculum (Gordon and Lawton, 1978). Over recent decades, the numerical aspect has been subject to particular emphasis. At times, changes in mathematics for children in the education system were closely allied to change within the wider context of the primary school curriculum. At other times, for example with the introduction of the National Curriculum for Mathematics (1988), the model of the secondary school mathematics curriculum impacted on curriculum change even for the youngest children (Anning, 1997).

The researcher, throughout her career, has experienced the impact of the frequent changes to content and pedagogy that characterised the primary curriculum in the last quarter of the 20th century. The greatest changes in the mathematics curriculum, for children aged 5 to 11, were brought about by the implementation of two major national documents.

- The first was the National Curriculum for Mathematics (NC), various editions were implemented between 1988 and 2000, and each version made the number content more prominent. The National Curriculum for Mathematics stated, in general terms, what should be taught but not how; methods were left to schools and staff. The content of the NC was underpinned by national assessment, known as the Standard Assessment Tasks (SATs).

- The second was the National Numeracy Strategy (NNS, 1999). This was prescriptive, stating when, in what order and how aspects of number should be taught. Its focal point, as the title suggests, was that of number. While this document does not carry the weight of law, unlike the National Curriculum for Mathematics which is on statute, in a very short space of time the impact of the NNS was seen on the children's number experience in the classroom.

In these documents are seen two common elements, the appearance of which explains why research into children's conceptions of zero is of particular relevance at this time.

1. The encouraged usage, by the teacher and the child, of correct mathematical terminology in all areas of mathematics. The belief being that the use of correct terminology reduces confusion and increases understanding. As a result the word *'zero'* came into common classroom use.

2. The emphasis upon mental calculations, as oral and mental work was seen as a secure foundation for numeracy. This brought the children into contact with *'the zero number facts'* much earlier in their number work.

Complex political and economic forces are impacting on the development of curricula (DfEE, 1997). There is, still, rapid curriculum change. This researcher believes that any reformulation of mathematical goals and guidance on pedagogy should be informed by detailed knowledge and understanding of the younger children's conceptions of our number system in order to produce effective approaches to teaching and learning. Where goals are prescriptive, as with the National Curriculum for Mathematics, it is particularly important that both the goals for children's learning and the guidance about development and implementation of the curriculum are underpinned by relevant research. One can speculate that these national incentives have played a part in conceptual research in mathematics being replaced by social research. Whatever the reason, the literature search leads one to believe there has been very little cognitive research in recent years and, while it is difficult to illustrate omissions, there has been no significant research on the child's conceptions of zero.

A personal journey

Throughout the research the aim was to be concentrated, informed and systematic and to contribute new information to a significant area of mathematics education. Underlying the study was the personal journey of the researcher following Monly's (1978) five empirical steps (in Cohen & Manion 1994, p.13) experience, classification, quantification, discovery of relationships, approximation of truth.

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Chapter One, Part Two: The Theoretical Field

> Does a paradigm not only include a philosophical worldview but a linkage to a certain type of research method … if so then a paradigm does determine the method?
>
> (Cook and Reichardt 1979, p.1)

Outline of Chapter 1, Part 2

Part 2 begins with an overview of the literature which informs this study with pointers to where ideas are more fully discussed in later chapters. It moves on to reflect upon the researcher's personal perspective on the creation of new knowledge. A discussion takes place on the nature of research in mathematics education, including Action Research and Exploratory Research. The merits of qualitative and quantitative methods of data collection in the field of education research are considered and the rationale for decisions made within this study is explained.

Literature overview

> It is nigh on impossible to 'finish a literature review and then start research'. (Jones and Pope, 2004, p.67)

It is unusual to find a book solely on the subject of 'Zero' yet at the turn of the 21st century two major works were published. These were Kaplan's book, *The Nothing That Is: A Natural History of Zero* (1999) and Seife's book, *Zero: The Biography of a Dangerous Idea* (2000). These publications chart the history of zero and assess the importance of zero in the mathematical world as well as its considerable impact upon everyday life. Reading these helped to set the scene at the start of the research process.

Extensive reading was undertaken with regard to the history of zero. The reasons for zero's turbulent history and its long, complicated journey until it finally became recognised in the western world had a strong impact upon the researcher. The works of Datta and Singh (1993), Dilke (1993), Flegg (1989), Granger (2000), Ifrah (1994), Katz (1998), Menninger (1992), and Schimmel (1993) added detail and depth as they explained zero's development. For the purpose of this thesis there were four important historical aspects. How zero moved from a symbol which marked an empty space to a number (Chapter 2, part 2), how the '0' symbol evolved (Chapter 6, part 1), how the language of zero developed (Chapter 6, part2) and how the inclusion of zero caused computational problems (Chapter 4, part 1). Within the data collected in this study were glimpses of zero's past as are seen in the two following examples. The first is when a child used a blank space as a sign for emptiness which reflected the ways of the Mayan and Sumerian people. The second example was seen during Task-Interviews, when a child and an adult expressed their understanding that zero could be used as a stop or marker between numbers; this usage was seen in zero's early history. Whether such instances are coincidence or demonstrate the 'principle of parallelism', where individual development is claimed to mirror the historical development of the subject matter, are discussed in Chapter 7, part 2. The main literature source used in this discussion is Rogers (1997) article, *Ontogeny, Phylogeny and Evolutionary Epistemology.*

It is to be expected that a literature search on zero would be mathematics and science specific and in such material the main discussion points tended to centre round zero's role in physics and pure mathematics. Three articles written by Pogliani et al (1998) and Rotman (1985 and 1993) at first reading seemed inappropriate to this study; they contained issues such as the vanishing point and imaginary money. However, in the early stages of the research they challenged this researcher's understanding of the wider concept of zero and this, in turn, later helped her to clarify the children's conceptions of zero.

Mathematical language and definitions proved to be a constant source of perplexity. Is zero a number? What is a number? Is zero a natural number, a rational number, an integer, a whole number, a cardinal number and an ordinal number? It was most naive of the researcher to expect to find a definitive source from which to gain answers to these questions. It was possible to empathise with the frustration of others as seen in Wolfram (accessed 2004) who wrote, 'Due to lack of standard terminology…' and then proceeded to define his own terms. It had been expected that there would be an up-to-date source where mathematically agreed standard terminology could be found, where changes in definition were noted. Without this how does one know what is currently acceptable? Mathematical dictionaries, and other literary sources of definitions, to which one might refer, all have a problem with regard to the age of the publication and of 'being of the moment'. By accessing web sites, such as Hyper-dictionary (Accessed: 2005), Oxford English Dictionary (Accessed: 2005), The Mathematics Forum (Accessed: 2004), Wikipedia. Natural Number (Accessed: 2004), xreferplus. Dictionaries (Accessed: 2005), Zona. Types of Numbers (Accessed: 2004), it was hoped that the vocabulary explanations from these sources would contain current thinking. Chapter 6, part 5 considers the development of the language of zero together with the changes in dictionary definitions and grammatical usage over the past 50 years.

A literature search in the areas of mathematics education and zero did not reveal any significant research on children's conceptions of zero within the areas covered by this study. This meant the researcher had to search wider sources of theoretical literature to include research which had a zero connection but whose aims did not match those of this study and/or was undertaken with people from a different age band. An example of this was the work of Wheeler and Feghali (1983) whose research aim was to determine the concept of zero of a small group of preservice elementary school teachers by asking the question 'What is zero?' Though the information was from adults not children the data classifications appeared of interest and value to this research. However, closer examination of their research data raised important issues as to its value. These are discussed in Chapter 1, part 2.

It was expected that a key researcher, Piaget, would throw light on the children's use of and their reactions to zero. As a result of this expectation Piaget's work was consulted early in the research process and at various stages during the study his work was reviewed. Why did the work of Piaget not inform this thesis? There are many examples of writers who level criticism at Piaget. These include Sugarman (1987), who sees in Piaget's 'levels of knowledge' that Piaget believed number concerns only concrete operations, and Hyde (1970) who noted in Piaget's 'study of concept formation', the importance or lack of importance Piaget attached to spoken language. Both of these are reasons why Piaget's work did not directly inform this thesis.

Considering the first of these critiques, that Piaget saw children as concrete operational thinkers concerned with manipulating things. These 'things' may be concrete or they may be 'in the mind'. One then asks could this equate with the number zero? Zero is not 'concrete' and so manipulation, either as a physical or as a mental action, cannot apply (Chapter 2, part 2 and Chapter 6, part 4). Considering the second of these critiques, that Piaget placed little importance on spoken language. It was found throughout this study that the children elected to explain zero through the use of language. These are two reasons why Piaget's work did not inform this thesis.

Furthermore, Piaget believed that the origins of thought are to be found in action and he was not prepared to allow language as the source of thought. In order to refute this Martin Hughes's (1986) research concentrated on the importance of spoken and written language in mathematics. He focused his attention on young children and number, on the children's ability to record, to read and to interpret numbers and equations within a series of set exercises. One section of his research was particularly applicable to this study; this was where he worked with the younger children (within the age range 3 to 5) on a game involving an empty set (Chapter 5, part 1). Hughes published his research in the book *Children and Number: Difficulties in Learning Mathematics* (1986); in this book he includes details of an unpublished study by Anna Stallard. She, too, had the same focus as Hughes but asked some older children (age range 6 to 11) to explain '0' and '4 + 0 = 4'. When considering the representation of zero her findings made a useful comparison with the findings of this research; this debate is in Chapter 4, part 5.

Chapter 5 reports on the younger children's reactions to the empty set. These reactions ranged from unease to anxiety. There was a notable absence of relevant research and literature in mathematics education on the important topic of the empty set. In order to understand the children's strong responses a search was made in the area of emotional psychology. A psychologist was consulted; she suggested emptiness might stir the emotions in two areas, that of rejection psychology which could include solitude, isolation and loneliness and that of motivational psychology such as deprivation, scarcity and insufficiency. Writers such as McKee (1980), Miller (1987), Langmeier and Matejcek (1975) and Laming (2004) were consulted but it was decided that to follow such lines of enquiry would make this a different thesis. Hence this aspect was left to await further research.

The presence of zero in simple equations is seen to create different problems from those generated by equations that do not contain zero. Because of this most of the researchers whose work is based on simple number bonds rarely include those of the zero number bonds. One journal article by Oesterle (1959) entitled *What about those 'zero facts'?* does deal with these zero facts but its theme and discussion area are whether they should be taught or left until a child meets them in two digit algorithms. Oesterle stated that a child will have fewer problems if the inclusion of zero is left until two digit algorithms are understood. To substantiate this statement it was necessary to review research undertaken on the problems children encounter with two and three digit algorithms. Dickson, Brown, and Gibson (1993), in their book *Children Learning Mathematics: A Teacher's Guide to Recent Research* included the research findings of a number of educationalists who, over the years, have looked at children's computational errors. Pertinent to this study are the work of Baroody (1984), Brown and Burton (1978), and Burton (1981) who include interesting findings relating the problems children experience if there is a zero in these algorithms. The work of these researchers highlights, though it does not explain, the zero problems. However, when their work was allied to the findings in this research the outcome was particularly thought provoking, as is seen in Chapter 4, part 4.

One section of this study concentrated on children aged 3 to 5 so the reading of publications of those who have an interest and expertise in the field of young children was undertaken. These included Haylock and Cockburn (1989), Atkinson (1992), Montague-Smith (1997), Maclellan (in Thompson 1997), Gifford (2003), Aubrey (1994 and 2000) and Pound (1999). In these publications zero, if included at all, was given only a cursory mention. This in itself was noteworthy. With the exception of Hughes (1986) it was surprising that there is little reference to zero and young children. As is discussed in Chapter 2, part 2 and in Chapter 8, part 1, the explanation may be that young children are seen to deal in practical situations, with counting numbers, while zero is abstract and not a counting number. In their book, *The Child's Understanding of Number*, Gelman and Gallistel (1986) include a thought provoking section entitled 'What numerosities can the young child represent?' where the practical and theoretical schools of counting are considered. While zero was not specifically included in this work this researcher felt there was a strong association with the problems surrounding the move from the practical to the theoretical use of number and the abstract nature of zero.

If zero was to be found included in a mathematics education publication invariably it was to state that problems associated with zero were language related, though the nature of these problems was not expanded upon. The difficulties surrounding the language of zero was to be one of the main aspects of this research. Literature in this area was sparse indeed. Only one publication, a psychology book, was found on this subject. This was Brann's (2001) book, *The Ways of Naysaying No, Not, Nothing and Nonbeing.* This is an American publication so allowances had to be made with regard to the differences in zero language usage in America and Britain (Chapter 6, part 5). Even so the in-depth, comprehensive language debate in this book widened the researcher's understanding of words such as *nothing* and *none*. This in turn gave her insight into how a child's use of such words might determine a child's conception of zero.

Because this is exploratory research the researcher had to use many, wider literature sources in order to gain small pertinent pieces of relevant information. As the research progressed and as the researcher's expertise and knowledge increased then new connections and related areas opened up. The task was one of continuous reading and listening. The lack of major pieces of relevant research in turn acted as a strong indicator that this work was moving in an under-researched area of investigation.

Reflecting on the nature of knowledge creation

> Does a paradigm not only include a philosophical worldview but a linkage to a certain type of research method ... if so then a paradigm does determine the method?
>
> (Cook and Reichardt 1979, p.1)

Most research adds to what has gone before with the researcher referring to other people's contributions as if they were a storehouse of conversations contributing to an ongoing debate. It maybe that the nature of a piece of research is to explain how what is about to be said is significantly different from the work of others, or how it concurs, underpins or elaborates on previous findings. Possibly the researcher intends to correct some error or aims to challenge previous research by repeating experiments using a different design or a completely different sample range.

Whatever the topic or style of research many researchers first follow the procedure of undertaking a literature search and reading appropriate books and published papers. The eminent academic scientist, theoretical biologist and mathematician John Maynard Smith questioned this practice in a BBC 2, TV programme, Seven Wonders of the World (3rd May, 1995).

> There are troubles about reading the papers first. I mean, you go into a new field you'd think the right thing to do would be read about what everybody else has said about it. If you do that you finish up accepting their questions as well as their answers as being what matters in that subject. And maybe they're wrong. And maybe they are asking the wrong questions. If you go into it as ignorant, as I do, and read as little as possible and plunge in and try and solve things occasionally you make a complete idiot of yourself. But occasionally you ask a question that other people haven't asked and provide an answer that other people haven't provided and that's much more fun. [1]

This researcher finds these views refreshing and is in agreement with what he says though with reservations. She feels where there is appropriate literature then to ignore this is not only time-consuming but is likely to lead to a repeat of work already undertaken, akin to re-inventing the wheel. But what is essential is that first one explores ones own views, beliefs, understandings and concerns and, for future reference, that these are written down. Throughout the research

[1] Transcribed from a video of the BBC 2, TV programme 'Seven Wonders of the World', 3rd May, 1995.

process one should not loose sight of these initial thoughts so that when immersed in the work of others ones own questions are not forgotten. These personal opinions are necessary as bench marks against which to compare the views of others, so that one can debate the work of others in the light of one's own beliefs. It is to be expected that during the research process and literature review ones own notions will develop but without the initially inspired and possibly naïve thinking new knowledge is less likely to be created.

In this research as was seen in the literature review (Chapter 1, part 2), in the areas of zero covered by this study little previous research had been conducted. This is an instance where the researcher had no alternative but to 'go into it as ignorant'. It meant she was put in the position where she had to 'ask a question that other people haven't asked and provide an answer that other people haven't provided'. Not only is this likely to be as Maynard Smith suggested 'more fun' but this researcher strongly believes that this is the way new knowledge is created. To be in a position where one is not able to use other people's literature, where one does not have other opinions to agree or disagree with, can be worrying, frightening but it is also invigorating and liberating. The freedom of working in a 'new' area comes with a price, that of being one's own monitor, moderator, critic, assessor and evaluator.

Do certain styles of research attract certain types of researcher? Jones and Pope (2004, p.64), in their reflections upon what motivates a researcher, found that researchers rarely write about how they decided on which questions to address. However, Schoenfeld (1999) assures us that our background plays a large role in making this decision. For, he goes on to say that,

> while it may be a truism it is nonetheless true that much of what we do, individually and collectively, is shaped by our personal histories. (Schoenfeld 1999, p.4)

Bloomer, Hodkinson and Billett (2004) observed that throughout our lives we develop dispositions – to life, to education and, eventually to research, as part of our personal habitus. Etherington (2004) provides more examples of reflexivity in the research process and how the impact of history, experience, beliefs and culture impact upon the research process and outcomes.

The example given by Jones and Pope (2004, p.66) is that of Zoltan Dienes who, early in his life, wondered why most people had difficulties with mathematics. He recognised an issue and 'as the years rolled by' he kept on wondering about the possibilities; he then began to investigate and to write. Echoes of this are seen in Chapter 1, part 1 of this study, where this researcher described her background and recounted her fascination with the topic of zero and, like Dienes, she too mulled over the subject over many years.

Reflecting on the words of Maynard Smith quoted on the previous page, and being in sympathy with his views this researcher realised how her own personal habitus was reflected in her choice of research style. This lead her to acknowledge that she favoured research that allowed her to work in unknown territory, unknown to her and more especially unknown to others. This was probably the reason why the subject of zero had such an appeal for her. She selected a topic which not only fascinated her but which, because of the lack of literature, allowed her to work in a style she preferred, one that would allow her the freedom to explore, to be a pioneer.

In the 'Research Methodologies for Education' course at Sheffield Hallam University, 7th October 2001, Liz Barrett suggested that,

> The researcher is re-searching. This implies that you are seeking something lost, hidden; but it needs to be borne in mind that what you are researching may not be there to be found. (Barrett, *2001*)

On the surface this could be a pessimistic approach resulting in a negative outcome. This research must, by the nature of it being in an uncharted area, be productive for even if it produces a negative result this adds to the research conversation. In this research situation the researcher is not 'seeking something lost, hidden,' but is using the process of fieldwork, classification of data, analysis and to produce a platform for debate.

Selden (2002) believes that mathematics education researchers, just like mathematicians, put a great deal of thought into selecting their research questions. She says these questions must be,

> new, nontrivial, non-obvious and the potential answers should be interesting.
> Selden (2002, p.3)

Niss (1999) describes research in mathematics education as,

> … the scientific and scholarly field of research and development which aims at identifying, characterising, and understanding phenomena and processes…
> Niss (1999, p.3)

Is there conflict in that mathematics is seen to be a science while education belongs in the social world? Bassey (in Hitchcock and Hughes, 1995) asks of education, is it a social or a scientific world? Bassey likens the social and scientific approaches to those taken by engineers and explorers entering new territory. The engineers set out to map the terrain; maybe they are looking for geographical features that will tell them there is oil below the surface. They are the scientists. The explorers are in an uncharted wilderness, open to anything they might find and how it relates to the landscape. They are the social researchers. A useful, visual approach, though as with most analogies it lacks definition. These people will have set out with different aims, but, in un-chartered wilderness, explorers will also map the terrain, engineers will be open to and affected by things they might find.

Wiliam (1998) sees the complexity of education being such that it is difficult to see educational research ever reaching a state of scientific clarity. He goes on to suggest that much educational research is at an early stage in research terms where little is certain and there is much to be explored before any hypothesis can be considered.

> The complexity of education demands the use of very many different research techniques and models. (Hitchcock and Hughes 1995, p.5)

With regard to this study, once it had been recognised that educational research demands its own theoretical paradigms then the tension of having to choose between scientific hypothesis and the need to investigate was lifted. This was especially so when considering the use of qualitative or quantitative methods of research. By treating qualitative and quantitative methods as incompatible it was felt that this encouraged researchers to use only one or other model when it may be a combination of the two that is best suited to their research needs. This is expressed and expanded upon in Glaser and Strauss (1967) in whose opinion,

> There is no fundamental clash between purposes and capacities of qualitative and quantitative methods or data ... We believe that each form of data is useful for verification and generation of theory, whatever the primacy of emphasis.
> (Glaser and Strauss 1967, p.17-18)

This study used a mixed approach and provided a methodology match. The quantitative and 'scientific' model with the qualitative, more interpretative model are seen not as alternatives but as simply different aspects of the search for knowledge and understanding: complementary rather than opposing. This researcher's views are compatible with those of Aubrey (2000); she writes that we,

> ... need the quantitative perspective to keep a grip on complexity and the qualitative perspective to keep a grip on subtleties. (Aubrey 2000, p.100)

While using both qualitative and quantitative methods can be time consuming one notable advantage is that each method builds upon the other; this aids triangulation and helps to correct inevitable biases present in each method (Denzin, 1970; Garner et al, 1956; Webb et al, 1966 cited in Aubrey 2000).

Every aspect of practice in this study relied upon the development of the researcher's own thinking as to the best way to achieve trustworthy knowledge of or a worthwhile commentary on the chosen phenomena. When undertaking a Master of Science study this researcher experienced action research, both in the course philosophy and in the modes of research. This model closely matched the philosophy of the researcher, for all spheres of her career have been founded on the cycle of systematic planning, action, observation and reflection. This basic model would influence this study but serious consideration needed to be given as to whether this research project belonged in the domain of Action Research.

Cohen and Manion (1997) see the purpose of Action Research as,

> … small-scale intervention in the functioning of the real world and a close examination of the effects of such intervention… [It is] concerned with diagnosing a problem in a specific context and attempting to solve it in that context … the ultimate objective being to improve practice in some way or other. (Cohen and Manion 1997, p.86)

In education, action research appears to have the express purpose of improving practice and involving teachers as participants within a school or community of schools. This study did not primarily involve teachers and whilst ultimately the research could contribute to the improvement of practice it was not a stated focus. By accepting the above definition of Cohen and Manion (1997) it would appear that this research lies beyond familiar Action Research territory. Nevertheless, this researcher's model of working based on systematic planning; action, observation and reflection would be maintained.

An area where this research fits more comfortably is within the field of exploratory research and within the three categories defined by Robson (1993),

1. Exploratory	To find out what is happening To seek new insights To ask questions To assess phenomena in a new light Usually, but not necessarily, qualitative
2. Descriptive	To portray an accurate profile of persons, events or situations Requires extensive previous knowledge of the situation etc to be researched or described so that you know appropriate aspects on which to gather information May be qualitative and/or quantitative
3. Explanatory	Seeks an explanation of a situation or problem, usually in the form of causal relationships May be qualitative and/or quantitative

The title and central theme of this research is 'to explore children's conceptions of zero'. The elements of exploratory research highlighted by Robson (1993) are clearly seen within this study. Finding out what was happening, seeking new insights and asking questions were accomplished by means of Questionnaires, Task-Interviews and Activity-Interviews. Assessing the phenomena in a new light was achieved though comparing the analysed data from each of the data collection methods and viewing the results in the light of zero's place in the world of mathematics and in the field of education. All these were underpinned by the influence of related research and readings within the appropriate spheres of literature and the researcher's experience.

Ethical issues

Information needed for this research was considered beneficent and non-malfeasant. The researcher had permission from the Local Education Authority to work in the schools. The Questionnaires were administered in schools where the consent of the headteacher and the teaching staff had been given. In the Task-Interview school the consent of the school governors had also been given. In the Questionnaire and Task-Interviews participants were asked to supply their name and age but the giving of this information was at the discretion of the participant. The researcher felt that she had informed the participants as to the nature of the research and had asked if they wanted to be involved (Chapter 1, Part 5, Preparing for the Task-Interviews - Setting the Scene). All information was treated as classified information and, in order to allow access to data without betraying confidentiality, identities were deleted and replaced by a coding system. In writing up the work this confidentiality and anonymity have been maintained.

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Chapter One, Part Three: The Empirical Field

> Facts are not there waiting to be gathered therefore can you collect data or do researchers 'produce' data through the research process? (Barrett, 2001)

Outline of Chapter 1, Part 3

This section discusses the strengths and weaknesses of using triangulation. The decisions as to which methods of data collection would be used for each focus of the research are presented. The cross-sectional aspects of the main study are explained.

Triangulation

One of the major criticisms in using triangulation is that there will always exist the epistemological paradox of differing settings and different methods. In education this is likely to be stronger than in the controlled laboratory experiment because the educational settings change and children's learning is always evolving. Whether this learning is in a formal classroom situation or in an informal setting (with children playing, reading, watching TV) children will be 'learning' and in the learning changes will be taking place. As it is impossible to remove the element of children learning and of change taking place then this expressed criticism of using triangulation must be accepted.

However, a strong case can be made for using triangulation. Robson (1993) succinctly states and reviews what he sees as the three advantages in a multiple methods approach.

- Firstly, the use of multiple methods should result in a 'reduction of inappropriate certainty'.
- Secondly, different methods can be used to focus on 'different but complementary questions within a study.'
- Thirdly, the use of multiple methods can be used to 'enhance interpretability' at the stage when results are being analysed. (Robson 1993, p.290)

These were the three reasons why triangulation was used in this study. Triangulation allowed for the possibility of multiple perspectives in the yielding of both qualitative and quantitative data. It was accepted that all methods of data collection have their shortcomings and so, to counteract and rebalance these, it was prudent to use more than one approach. Cohen and Manion (1989) affirm the importance of triangulation and recommend the using of multiple methods in order to minimise the potentially distorting effects of a single methodology. It was the intention that this research would employ multiple methods and in an empirical, social or educational enquiry the three most commonly used ways of collecting information are observation, interviews and questionnaires. This research included all three of these methods.

The focus of the research

The research explored the conceptions of zero of children aged 3 to 11. The focus areas of this exploration were,

1. Zero as a number and its relationship to other numbers
2. The zero number facts
3. The empty set
4. The language of zero.

Data collection was through a questionnaire, through the administration of tasks (activities), which involved interviews and observation. Each focus of the research used the following modes of data collection,

1. Zero as a number and its relationship to other numbers where the mode of data collection was through a questionnaire and tasks (the latter involving interviews and observation).

2. The zero number facts where the mode of data collection was through a questionnaire and tasks (the latter involving interviews and observation).
3. The empty set where the mode of data collection was through activities (involving interviews and observation).
4. The language of zero where the mode of data collection was through the written language from a questionnaire and the spoken language from activities and tasks (involving interviews and observation[2]).

The study, data collection order and timescale

This was to be a cross-sectional study focusing on knowledge and understanding in order to assess the cognitive functioning of different age groups within the age range 3 to 11 years. Table 1 shows the three aspects of this cross-sectional study,

Research phases		
Phase	**Data collection method**	**Age range**
Phase 1	A questionnaire	10-11
Phase 2	Task-Interviews	5 - 11
Phase 3	Activity-Interviews	3 – 4

Table 1

The total number of children participating in this research was 234; the breakdown is shown in table 2.

Number of research samples	
Research methods	**Number of children**
Questionnaire	100
Activity-Interview	25
Activity-Interview plus Task-Interview	109
Total number of children	234

Table 2

The questionnaire was administered, over a one-week period at the end of July 2002, to all the Year 6 children present in research schools A, B, C, D and E. The Activity-Interviews and Task-Interviews, for each age range, were completed within a continuous period. The overall time-span taken depended on the availability of the children, the time required by each individual child and the number of children within the age range. The maximum number of days for the completion of the data collecting within an age range was a period of ten school days, the norm being a school week. The order and timescale is seen in table 3.

[2] Details of the style of interview used and the collection of information through observation are discussed in part 5 of this chapter.

Data collection timescale			
Date	**Data collection method**	**Age**	**Number of Children**
July 2002	Questionnaire	11	100
Spring – Summer 2003	Activity-Interviews	3	11
		4	14
Spring – Summer 2004	Task-Interview plus Activity-Interview	11	20
		10	11
		9	20
		8	16
		7	13
		6	12
		5	17

Table 3

Each of the data collection methods, the Questionnaire, Task-Interviews and observations, and Activity-Interviews and observation, are discussed in the next section of this chapter.

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Chapter One, Part Four: The Questionnaire

Outline of Chapter 1, Part 4

Part 4 concentrates on the cross-sectional study, phase one, which involves the use of a questionnaire. An explanation is given for the selection of this data collection method together with an acknowledgement of its strengths and weaknesses. The research sample selection and the piloting, the questionnaire content, the types of questions and conducting of the questionnaire are reviewed.

Questionnaire sample selection

In order to gain a feel for the field, to help problem pose, to form conjectures and to inform the design of other research methods, a survey was undertaken in the form of a questionnaire (a copy of which can be found in appendix 1).

Permission was kindly given by a Local Education Authority (L.E.A.) to approach the schools and teachers in their area. Whichever schools were selected it was recognised that the children would be affected by internal and external factors. Within the school there would be a variety of teaching styles and mathematical activities all influenced by a range of professional interpretations by teachers with their own mathematical ability and philosophical viewpoint. It is impossible in school based research to remove all the variables which affect children's learning (chapter 1, part 3). The selection of a 'main' research school was important. The researcher was looking for a research school where the mathematics was not 'outstandingly different' to that of others in the area. Rather than select one school at random it was decided to collect data from a number of primary schools. Five schools, an average of 20 children per school, seemed a reasonable number in order to provide a comparison of data. The five schools were selected by convenience sampling,

> Captive audiences such as pupils or student teachers often serve as respondents in surveys based upon convenience sampling. (Cohen and Manion 1994, p.86-89)

The selection rationale was that the schools were all known to the researcher, they were convenient to visit and they took children from a similar catchment area. The schools were situated within a square mile, in a small town on the Lancashire, Cheshire and Yorkshire borders. In this study these five schools are referred to as research schools A, B, C, D and E.

Questionnaire piloting

The initial questionnaire was piloted on a relevant population of primary children, in a school not involved in the research programme, with children aged eight (5 samples), nine (4 samples), ten (5 samples) and eleven (7 samples). The results showed that while the content was suitable for children within this age range the questionnaire format was more appropriate for use with the 10 to 11 year old children. The piloting allowed the design and content to be checked. The initial questionnaire was reviewed and some alterations were made in the order and style in which the questions were presented. This revised questionnaire was piloted in a different school with 7 children, aged 10-11.

Questionnaire sample size

Due to factors of expense, time and accessibility, it is rarely possible or practical to obtain data from a large, wide population. As with the majority of research there was a need to compromise between ideals and practicalities. Empirical research involves drawing a sample. Sampling decisions were made early in the overall planning, as were considerations with regard to the range and sample sizes. The question of how large should a sample be in order to conduct an adequate survey, needed to be considered. Decisions as to the correct sample size are influenced by the purpose of the study and by the nature of the population under scrutiny. However, according to Cohen and Manion (1994), a sample-size of thirty is held by many to be the minimum number of cases if a researcher plans to use some form of statistical analysis.

The questionnaire was given to all the Year 6 children (aged 10-11 years) in research schools A, B, C, D and E who were present in the week before the summer holidays (July, 2002); this was at the end of their primary school phase of education. In total there were 100 returns from children.[3]

[3] The members of staff, from the five primary schools, were also asked to complete the same questionnaire.

Questionnaire Returns					
Research School	School A	School B	School C	School D	School E
Age of children	10-11	10-11	10-11	10-11	10-11
Number of children	20	22	19	19	20

Table 4

Questionnaire content

In writing the questionnaire (see appendix 1) the aim was to ensure that each question generated useful data while the goals of the research were always borne in mind. The questionnaire enquired into four main content areas.

1) Research goal - What did the children understand by the word 'zero'?
Questionnaire component – To answer the question, 'Is zero a number?' and to provide an explanation for the answer.

2) Research goal - How did the children see zero in its relationship to other numbers?
Questionnaire component – To reorganise sets of numbers beginning with the smallest.

3) Research goal - What were the children's responses to the zero number facts?
Questionnaire component – To answer 'the zero number facts' questions and to provided an explanation for each answer.

4) Research goal - To explore the written language used for zero.
Questionnaire component – To answer the question, 'What word would you use instead of zero?' (Further data on the language of zero might come from the answers to the other questions.)

Types of questions used in the questionnaire

The respondents were children who were more at ease answering questions which asked for knowledge, such as 3+0=; such questions gained more responses than the open-ended questions or those which asked for opinions and reasons. This was seen in the first pilot questionnaire where one question used was that asked by Wheeler and Feghali (1983), 'What is zero?' In the piloting of the questionnaire for this study a high percentage of the children's responses were uninformative. Many of the children did not attempt to answer that question or they wrote, *I don't know*. At the time this was seen as a disappointing development; for the intention, in using this particular question, had been to compare the results with those of Wheeler and Feghali (1983)[4]. On reflection this feedback, from what appeared to be 'negative' responses, was in itself an interesting result. It underpinned the notion that zero is a source of conceptual difficulty. The subject had to be explored from a different direction. It was decided to ask a more user friendly binary choice question with a *'yes' or 'no'* answer. The question now asked was, *'Is zero a number?'*

[4] This is discussed in greater detail in chapter 2, part 2.

As with many closed questions this might have be considered a threat to the validity of the findings in that they were not representing what the participants wanted to say, as the participant was restricted to choosing from the alternatives given by the compiler of the questionnaire. However, as there were only two possible answers (*'yes'* or *'no'*) to the question, *'Is zero a number?'* then the argument of restricted alternatives being a threat to validity was not applicable. The possibility of adding a *' don't know'* choice was dismissed as this would be returning to the problems found with the initial pilot question. Dismissed for similar reasons was another consideration of a *'yes and no'* choice (though this answer was given by one child when both *'yes'* and *'no'* were circled). This closed question was immediately followed by an open-ended question: *'Why do you think zero is/ is not a number?'*

The 'quintamensional questionnaire design' was introduced by Gallow in 1974 (in Cohen and Lawton, 1994). A closed question is used to capture specific information and then an open question to explore the participant's justification. This was the style adopted for the second trialling and its success meant it was used throughout the final Questionnaire and also in the Task-Interviews and Activity-Interviews.

Administration of the Questionnaire

In each of the five schools the questionnaire was administered to children who were in their normal classroom situation. They sat at desks arranged in small groups. The researcher presented the questionnaire to the children. An explanation was given about surveys and that this survey was connected with some research about small numbers (intentionally the word 'zero' was not mentioned). It was also explained that children of their age, in other schools in the area, were being asked to help by completing the questionnaire. It was made clear that it was important they give only their own answers or the survey data would not be of any value. They might think some questions were very easy, others harder but this was because, later, the same questions would be used with people older and younger than they were. The children were asked to complete as much of the questionnaire as they could but not to worry if they could not answer all the questions. No time limit was given for the completion of the work. If a mistake was made the children were asked to put a line through the error and not to use an eraser (this was so that the researcher could see the original answer). The class teacher continued to oversee the work and the papers were collected the next day. All the questionnaires from research schools A, B, C, D and E were administered and collected from the school within the same week.

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Chapter One, Part Five: The Tasks-Interview

> *Understanding,* the effort to interpret or make sense, is seen as a definitive feature of human existence. As the British psychologist Frederic Bartlett (1932) put it, human beings are motivated by an *effort after meaning*; more portentously, Maurice Merleau-Ponty's (1962) view of the human condition has us *condemned to meaning.* (Hall, 2004)

Outline of Chapter 1, Part 5

The following section of the study explains the limitations of the Questionnaire and the decision to use a Task-Interview. Considerations are given for the selection of the interview and observation as data collection modes. Details of the Task-Interview are examined. These include the sample selection, the piloting, the role of the interviewer, the preparation for the conducting of the Task-Interview and the recording of the data.

Limitations of the Questionnaire

The interesting, valuable collated data from the questionnaire provided a feel for the field and helped problem pose, but its limitations as a method of data collection needed to be addressed (Oppenheim, 1992). This researcher noted several limitations which applied to the use of the Questionnaire in this study. By the nature of the questionnaire the language and vocabulary was that of the written word. In order to gain access to the spoken word of the *'zero language'* it was necessary to work directly with children. A further consideration was that the second phase would be to work with younger children for whom a questionnaire would not be suitable. The main limitation of the questionnaire was that it did not allow the researcher to add supplementary questions, to hold a discussion with the child as to the reasons for their responses. Explaining one's reasoning is a difficult exercise for an adult and even more so for a child. Access to a child's reasoning, thinking and understanding was a major issue. The method was to be through tasks and interviews, known as *'Task-Interviews'* and through observation. These will be considered separately.

The Task

An interview can be a stressful means of collecting information, especially for children who may be feeling insecure. Over the years this researcher has found success in using Mant's (1983) 'third corner approach'. When one person tries to elicit information from another person the situation can appear confrontational. This can be alleviated if the two people are encouraged to look not at each other but at the third corner of the triangle. This requires the selection and use of a third element upon which both the interviewer and the interviewee can concentrate. The third corner was to be 'a task', or a series of tasks. Putting the focus on tasks rather than the child opened the field of discussion in a non-threatening mode. The tasks also helped to capture and hold the child's attention.

The tasks were firmly based upon the questions in the questionnaire but the child experienced them in a practical way. This strong link with the questionnaire, the using of the same instructions, the same questions, in the same order, with the same number content meant that the answers could be compared with those given in the questionnaire. Each task is explained in detail in further chapters in this study. These tasks were structured so they could be repeated with other children. As with the questionnaire, the answer from each task was quantifiable data. These answers were recorded on a prepared recording sheet ready for coding and analysing.

In order to gain insight into the child's understanding alongside the tasks were asked a series of set questions. Dependent upon the child's response supplementary, semi-structured questions were used to explore further the reasoning and the thinking behind the child's answers. This was the interview section of the 'Task-Interview'.

The Interview

Kieran (1994) stresses the influence of the interviewer on the interviewee and, hence, on the outcomes of the interview. He notes how the close proximity of the interviewer must lead to a changed performance, albeit slightly, of the interviewee and, added to this, there are the dangers of misleading 'interpretation' in light of the researcher's own philosophical stance.

> ...it isn't the performance, it's the watcher that's carrying in his mind all of what the performance ought to be. And that's a very different thing. (Kieran 1994, p.7)

These difficulties will always apply to interviews. This researcher believed that having a 'task' as the focus of the interview, adding observation as another method of collecting data and not analysing the material until all the interviews were completed would help to keep the impact of the research more standardised.

It was intended that the outcome of the qualitative data from the Task-Interviews should provide a description and rationale that would be clearer and more open to understanding than the individual children would have been able to provide unassisted. At the same time it was important that the structure did not restrict the collection of data but provided the flexibility to explore further, to encourage the children to extend their ideas, explain their reasoning and follow fruitful tangents. To fulfil these objectives a semi-structured interview was selected with the tasks as the focus. Drever (1995) warns of the danger of simply seeing 'semi-structured' as a convenient compromise or even using the term to cover an interview where the intention was to stick to a structure but the questions were insufficiently planned. These Task-Interviews were planned, the tasks and the set series of questions gave structure but there was sufficient *'adjustability'* to allow for the *'ad hoc'* to happen. To the interview and the task was added a further method of data collection: observation.

Observation

Observation, a simply matter of watching others, would seem an obvious way of collecting data in a social or educational enquiry.

> Observation skills are generally not caught they are taught. Trained observers may be sensitised to pick up snippets of verbal or non-verbal interaction from an event which a lay person or untrained person, such as a parent, has not noticed. It's a bit like attending a 6- hour course on identifying fungi, learning how to tell one fungus from another and noting favourite habitats from pictures and then walking through a wood and spotting a fly agaric or a common inkcap. Someone else walks through the wood, enjoys the scenery, and when afterwards asked about the fungi says 'What fungi? I didn't see any.' They didn't see any because they were not trained to see any. Their eyes were open but not focussed on fungi; perhaps they were focussed on the path or on the pattern of trees or foliage. So the exclamation 'I didn't see that happen' is not unusual from untrained observers in classrooms. (SHU Book 3, 2001, p.6)

Whether 'caught' or 'taught' it is prudent for the reader to examine the observer's expertise. The researcher gained these skills as a result of many years of involvement in observing and reporting as part of the task of a teacher, advisory teacher, lecturer, tutor and audit moderator for the government's National Standard Assessment Tasks (SATs). These roles involved observation of a class, group and individual; observation of children, teachers and students; observation of situations that included teaching, learning, interviewing, selection and appraisal. From these varied experiences the researcher learned the techniques and skills required to assess such things as the questions to ask, the proportion of silence, off task as opposed to on task behaviour, body language, knowing when to ask subsidiary questions and how to probe more deeply.

While appreciating the value of observation data there are also weaknesses to consider for non-participatory and participatory observation must affect the data outcome, the latter even more so. The observer also needs to question the reliability of the data, in the sense that other observers, with similar experience, may record different things. Had observation been the only method of data collection then the reliability of the findings of this research would have been less robust; but observation was used to help to contribute to the process of triangulation.

The strength of using observation lay in the fact that it was a flexible means of collecting first hand information, a method suitable in a wide variety of situations. In this research it was an appropriate method to select as often children, particularly younger children, have difficulty expressing themselves. They resort to 'doing' rather than, or as an extension of, verbalising; hence a card would be pointed to rather than read, a gesture made rather than a feeling expressed verbally. Davis (1994) and Mason (2002) both refer to observation as the methodology of the *'Discipline of Noticing'. Discipline* suggests orderliness, self-control, form, knowledge, method, practice, training while *noticing* offers a feel of perception, discernment, regard, attention. This researcher intended to go beyond observing and embrace the 'Discipline of Noticing' where she would be in a position to note *'a critical incident'*. Within this research observation of a 'critical incident' included instances of a 'eureka' moment; of frustration; of misuse of a procedure; of puzzlement.

The observations were collected as unstructured data; the researcher was aware that such data were dependent upon subjective interpretation. A record was made of what was seen, using notes and jottings, in order to capture facts and perceptions, reminders and puzzling thoughts that needed to be checked further. Interpretative, additional notes were added at the time and expanded notes added immediately after the event, while images were clear.

The Task-Interview sample

Who were to be the sample children from whom data would be collected using the Task-Interviews and observation methods? It was impracticable to work with the same cohort of a hundred children who had completed the Questionnaire as they were now in many different secondary schools. Nor was it feasible to work in all five schools due to the schools having staffing and other logistical problems, and also it was impractical from the point of view of the researcher.

The responses from the questionnaires, from each child and from each of the five individual schools had been analysed. A comparison was made between the results from the five schools. No one school showed any significant, singular result, quite the opposite in that there were many common features[5]. The decision was made to concentrate on one of the original five schools. This school had a stable staff; the results from the questionnaires had revealed the same pattern of responses as the other schools, with no answers specific to that school. It appeared to be representative of the five schools. This was Research School D.

Task-Interview piloting

The content of the Task was that of the questionnaire and this had already been piloted with

[5] This data was used as part of the triangulation process and is included in later chapters of this study.

children in the age-range eight to eleven. The piloting of the Task-Interviews was with children aged 5 to 10, in two schools with three children, randomly selected from each of the age ranges five, six, seven, eight, nine, ten and eleven. These were schools not involved in the research programme.

The piloting confirmed the need for the researcher to quickly assess the level of attainment of a child and match this to the suitability of each task content. This was necessary as, because of the wide age range, not every child was asked to attempt or to complete all the tasks; for instance a six year old was not asked about ordering decimals while a nine year old may understand ordering negative numbers. The researcher's experience of making accurate judgements was brought to this initial assessment task. Further indications came from the method of asking probing questions. Asking a child to explain his/her answers alerted the researcher to the fact the child did not understand the question. An example of this being a 7 year old boy who gave answers to the number facts $3 \div 0 =$ and $0 \div 3 =$ and then explained that $\div$ was – without the dots and that he had *'done take-aways'*. The decision as to whether a task was suitable frequently came from was the children. They were happy to try the tasks presented to them, to give answers and they themselves felt comfortable in aborting a task. The original fear there may be muddied data did not arise. On the rare instances where a child provided an answer to a question they did not understand then, as will be seen, the data collection, reporting and analysis process was able to handle such resulting data appropriately.

Role of the Interviewer

The interviewer is the data collection instrument and decisions as to what is said (or not said), the reaction to responses made and even the seating arrangements can all influence the outcome. While there was a need to strike a balance between that of an active listener and a probing questioner it was important to remember that an interview is a social situation with expectations on both sides and where issues of power, trust and self-presentation are present. The Task-Interviews were being conducted with children the researcher had not met before. The nature of the interview and any ground rules that were necessary needed to be set out. At the same time, with each interview, it was crucial each child should feel at ease, especially as they were working with a stranger, away from the rest of their class. Hence the researcher placed great emphasis on making each child feel secure and at ease as is demonstrated in the following section.

Preparing for the Task-Interviews – Setting the Scene

A high priority was the establishing of a rapport with each child. A child's apprehension and nervousness were to be expected for not only was the researcher a stranger but a stranger who was asking them about mathematics, which some children openly said they did not understand.

The opening phases were important, discussion about their clothes, pets, their painting on the wall was encouraged. No child refused to come and to engage in the Task-Interview; on the contrary many children appeared to enjoy the experience.

Illustrative incidents

The teacher of the Y1 class explained that she was concerned about one child taking part in the research. This 5 year old girl had hardly spoken since coming to the school two months previously. The researcher did not want the child to be treated differently from the other children in the class yet she was aware that this was a delicate situation. So, with the consent and in the presence of the class teacher, the researcher asked the girl if she wanted to come into the next room and do the work the others had been doing. To everyone's surprise the girl held the researcher's hand, went into the next room and completed all the tasks, offering spoken, though concise, answers.

Some children were most enthusiastic and many asked if they could do the work again. One 7 year old boy gave his name as Matthew Hall (not his real name). Having a feeling that she had met a similar child she asked if he had a relative in another class. The boy said, *no*. The researcher then recalled another 7 year old boy called John Hall from the same class as Matthew. Possibly they were twins. The researcher asked Matthew if he was related to John Hall. There was no immediate answer. Then the boy admitted his name was John Matthew Hall. He had wanted so much to come and do the work again that he had given his middle name.

In the time it took to complete the task, between 20 and 40 minutes depending on the child's ability and age, it was possible for rapport to be established. When the researcher returned to the school, after a gap of six months, on crossing the playground she was greeted with smiles and inundated with requests *to do some more of that work with you.* An indication that many children found the experience positive and pleasant. When working with the children it was essential to put them at ease so that they talked freely; it was believed that this had been successfully achieved.

Margaret Donaldson (1987) in her book *Children's Minds* sees children as 'meaning makers' who will try to make meaning of the situation and of the questions they are asked. This researcher felt it important that the children were helped to make meaning of the situation they were in by involving them in the research process and by explaining what was expected of them. In simple terms the children were told about research and the collecting of information and that it was hoped the information would be useful to teachers to help children understand their work better. It was explained that they would be asked questions, some very easy and some not so easy. If they did not know the answer then it was fine to say so but that it would be more useful if they would '*have a try*'. They were assured that it was not a test, and need not be concerned that their teacher would know the answers they had given. Being aware that children assume that when asked for a reason for their answer this is 'a hint' that that the answer is incorrect and so the answer is changed. It was made clear to each child that he/she would not be told if an answer was correct or not. There was no evidence of a child changing his/her answer as a result of this question. The researcher told them she was really interested in what they were thinking when they answered a question and the only way for her to find this out was to ask lots of questions and that sometimes they might think the questions asked were silly. Whatever they said was important and so it would be written down.

Recording the Task-Interviews data

The children could see the recording taking place and could read it if they wished to do so. Indeed if a child gave an answer and then changed their mind they would be asked which answer they wanted and would be shown that their answer had been altered accordingly. If they said they were not sure then they would be asked if it was acceptable for both answers to be written, with a question mark added. Frequently the writer checked with the child as to what should be written. This made the child comfortable with and part of the recording process and it was beneficial to the recorder, for the child would pause and wait until the writing was completed before attempting the next task. It was sometimes necessary to make further notes. These were written as soon as possible after the completion of the Task-Interview, in the knowledge that time and space can produce different recollections.

All answers were accepted and an explanation was asked for correct answers as well as for incorrect answers. At no time was the child told whether their answer was or was not correct. This also meant the children could not confidently tell their friends *the right answers*. Being aware that children like to talk, and boast, about what they have done, at the end of the Task-Interview each child was asked if they could keep a secret. It was explained that if they told their classmates what they had done and the answers they had given then it would ruin all the research work. Each child was asked, before beginning the tasks, if any of their friends had told them what we would be doing. It appeared that they *kept the secret*, as there was no indication of collaboration within the Task-Interview data.

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Chapter One, Part Six: The Activity-Interview

How a person reasons is not open to direct inspection.
(Brown and Dowling 1998, p.61)

Outline of Chapter 1, Part 6

Part 6 concentrates on the Activity-Interviews. The Task-Interviews were used to explore the children's conceptions of zero, the Activity-Interviews were designed to explore the children's understanding of 'emptiness'. The Activity-Interviews were initially intended to be used with children aged 3 to 5 but this was reviewed and they were used with the full age range of children.

The empty set

To reiterate, the scope of this research was in four focus areas:

1) Zero as a number and its relationship to other numbers
2) The zero number facts
3) The empty set
4) The language of zero.

The third focus area was that of *the empty set*. It was intended that the work would be with children who had yet to start or who were at the beginning of compulsory education; nursery and reception children (aged 3 and 4). The objectives were to look specifically at emptiness and the empty set, the language of the empty set and the children's recording of the empty set. As young children meet emptiness prior to their ability to count and recognise numbers, a critical area of the research was to explore how children who had little, if any, experience of zero as a word or as a symbol would react to *nothing*, to *the empty set.*

There were specific issues with regard to the collecting of data from young children. These included the children's limited ability to explain themselves together with the depth of their understanding and their interpretation of each question. It was recognised that these needed to be carefully addressed. This was done in the knowledge that a minor change in the task could result in a significant change in a child's response. To address the '*empty set*' aspect involved developing new activities.

The Activity-Interviews

Activities were designed to explore children's conceptions of emptiness. To differentiate these from the Tasks and the Tasks -Interviews these were called the Activities and the Activity-

Interviews. The rationale, the conducting of the activities, the role of the interviewer, the recording followed that of the Task-Interviews (see part 5 of this chapter).

There were three Activities in the Activity-Interviews,

a) **The Bottle Activity**, an activity using sets with no numerical quantity. In this activity the children were presented with an empty set (details in chapter 5).
b) **The Ribbon Activity**, an activity which involved sets with a numerical quantity. In this activity the children produced an empty set as a result of sorting (details in chapter 5).
c) **The Number-Letter Recognition Activity,** an activity involving reading some cards on which were written number or alphabet letter symbols (details in chapter 6).

Activity-Interview piloting

The activities were piloted in a nursery (5 children ages 3-4) and two reception classes (4 children aged 4).

Activity-Interview sample

Initially it was intended that the Activity-Interviews would be used with the children aged 3 to 4 (nursery and Reception classes) and the Task-Interviews with the children from 5 to 11 (children in Y1 to Y6 classes). Evaluation of the data and initial interpretation from the Activity-Interviews showed some fascinating results and highlighted a need for further collection of data to substantiate and clarify the findings. This resulted in the decision to include the three Activity-Interviews with the Task-Interviews used with the older children as it was thought this might highlight cognitive function changes in the knowledge and understanding of '*nothing*'.

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Chapter One, Part Seven: Analysis of the Data

Analysis is the activity that turns data into knowledge.
(SHU Book 2, 2001, p.12)

Outline of Chapter 1, Part 7

Part 7 examines the question of data analysis. This includes data confidentiality, the important decision of when the data was analysed and the ways the qualitative and quantitative data were analysed

Data confidentiality

For the purposes of the research each child's data were recorded using a code; this code preserved anonymity. It was essential to have a system of accessing each child's raw data and this access was used frequently to check, recheck and crosscheck information from all the different modes of data collection. The code also gave information as to the age and sex of the child as well as the school and class teacher. It was thought that there might be a sex bias in the data, though this proved not to be so. The name of the school and teacher was used to track if children from any school or class showed any significant, singular result. This, again, did not occur. The coding was of benefit in tracing children who changed their mind and gave a second, even a third answer. All these answers, together with the rationale given by the child, were noted down. The way all these answers were treated, whether the changes were or were not considered significant, is addressed in the analysis of the various pieces of data in the appropriate chapters.

The data analysis schedule

According to Ashworth (1993) decisions about how the data will be analysed should not be left until the research has been completed but should be an ongoing process throughout the study. In this study it was felt that at some stages of the research the data required immediate analysis in order for decisions to be made; in other instances it was important that the data were not analysed until the data collection was completed.

Analysis of the Questionnaire was undertaken as soon as the papers were returned, for upon these results depended the next stage of the study. They confirmed the relevance and appropriateness of the selection of questions and informed the design of Task-Interviews.

Unlike the results from the questionnaire, which were 'cold' data, the Interviews were conducted by the researcher and were dependent upon the researcher for their impartiality. This researcher felt it was important to ensure that, as the interviewer, she was not coloured by any analysis and that preconceptions did not impact on her observations and on the interviews. Findings could have affected her perceptions, affected the running of the tasks and activities and distorted the results, threatening the validity of the outcome. The main core of the data collection was the Task-Interview and these data were not analysed until all the individual Task-Interviews had been completed. This approach is articulated by Giorgi (1985) and also by Wertz (1983). Summaries of their key points are:

(a) the need for the researcher to continually bear in mind that it is the informant's meanings that are being sought, and
(b) the withholding until late on in the analysis of the decision about whether there are themes in the data which are common from person to person.

The Questionnaire and the Task-Interviews had the same content but the work with the younger children, the Activity-Interviews, was seen as independent from the other fieldwork data. Preliminary collation and analysis of the data from Activity-Interviews took place when all the children in the 3 to 4 year old groups had completed the activities. It was during the classification and the preliminary analysis when the importance of the findings was noted. Further analysis was suspended, for the reasons given above, as the decision was made to include the Activity-Interviews with the Task-Interviews conducted with children aged 5 to 10.

Type of data being analysed

At the end of the fieldwork various types of data had been collected.

- The Questionnaire produced quantitative data in the form of answers to algorithms and qualitative data as answers to set questions.
- Task-Interview data produced quantitative data in the form of answers to algorithms, qualitative data as answers to set questions together with sets of notes.
- The Activity-Interview produced qualitative data as answers to set questions together with sets of notes.

All answers were recorded; however, there was an issue to be addressed. Where children had changed their minds a judgement had to be made as to which answers would be used. In the Questionnaire it was decided that the child's first recorded answer would be used, as this was more likely to be the child's first reaction to the question. As no explanations were available it was not possible to know the reason for the change of answer; there was also the possibility that any changes might have been the result of the child being influenced by his/her peers. When the Questionnaires were examined it was seen that few children altered their answers. Again, the issue had to be addressed in the Task-Interviews, as there were children who changed their mind as to the answer. The decision was made to ask the child for the answer he/she wished to be recorded and why they selected this answer. Again these changes were noted and where the changes and the accompanying rationale become significant they flagged in the relevant sections.

Analysis of quantitative data

The Questionnaire and the Task-Interviews provided quantitative data from:

- questions with a multi-choice selection, such as whether or not zero was a number
- questions with a limited number of answers, as with the ordering of a set of numbers
- algorithms, where there were a finite number of answers given.

These answers were organised in a structured, consistent way using a tabular form. The responses from each child and from different age ranges were analysed to see if there were any common features of similarities or differences. Analysis of the quantitative data was seen as analytical; it was the comparison of this data which aided the identification of themes.

Analysis of qualitative data

> Understanding is not based on transposing oneself into another person, on one person's immediate participation with another. To understand what a person says is to come to an understanding about the subject matter, not to get inside another person and relive his experiences ... (Gadamer 1989, p.383)

The Questionnaire, Task-Interview and Activity-Interview provided qualitative data from:

- semi-structured set questions with open-ended answers
- unstructured, open ended, probing questions
- observations.

Here a balance was needed between the complexities involved for descriptive purposes and the rigour required for explanatory purposes. This qualitative data were in the form of answers to set questions and sets of notes. The latter, in particular, needed to be organised and re-organised in order to move beyond description. Each task was taken separately and the exact words, from each child's response, were recorded. This textual data were seen as a source of factual information. Text from the notes made from each child was searched, key-words and phrases highlighted. A search within a cohort of children was made for common words or phrases and for common incidents such as how many times something was said; common perceptions and common gestures were observed and noted. These, then, became the basis for the classification headings. The headings formed would be used to sort the data from the next age range responses to that task.

Category headings were reviewed to see if there was any duplication and to see if categories could be combined. In the event of headings being merged it was necessary to re-check the response of each child affected by the merging. As each child was recorded by the use of a code it was possible to access the raw data. The reverting to and re-checking of the raw data was used repeatedly. The reasons for each classification are explained in detail within each task analysis.

Analysis of the qualitative data was seen as a two-stage process. First that of highlighting, organising and catagorising. The data could then, in the second stage, be treated as analytical and analysed in a similar way to the quantitative data described on the previous page.

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Chapter One, Part Eight: Interpretation, Evaluation, Generalisation

> In interpreting, we do not, so to speak, throw a 'signification' over some naked thing which is present-at-hand, we do not stick a value on it; but when something within-the-world is encountered as such, the thing in question already has an involvement which is disclosed in our understanding of the world, and this involvement is one which gets laid out by the interpretation. (Heidegger 1962, p.190)

Outline of Chapter 1, Part 8

Part 8 of chapter one considers the possible interpretation of the data analysis and how reliability and validity may affect the overall generalisations and possible outcomes.

Interpretation

From the result of the analysis stemmed interpretation. Discussion of interpretative details will be more relevant at the end of each area of enquiry and will be left to future chapters. There were, however, common elements in the evaluation of the data and interpretation of the results.

The result of the analysis whether from a qualitative or quantitative source was classified, synthesised information. This information was then searched for misconception, for pattern and for paradox. This search was undertaken through cohort analysis; by searching for correlation within an age range; and by development across age ranges. The analysed data was examined for ways in which particular meaning was communicated in the search for evidence of ways of thinking. It was hoped that the process of data generation, analysis, testing and interpreting might reveal general features of the situation and, possibly, lead to theory building; whether this is tentative, provisional or robust could not be predicted.

Evaluation

Whether the research questions can be answered depends upon the reliability and validity of the assessment methods used. In turn assessment techniques must themselves be evaluated. This involves, primarily, the determination of reliability and validity. Reliability often refers to consistency of scores from the testing while validity provides a check on how well the test fulfils its function, normally through independent external criteria. This research did not begin with a research question that required 'testing'; its aim was that of exploration. Consideration was given to reliability and validity though the selection of the data collection methods. Reliability and validity were bound up in decisions with regard to the design and administration of the Questionnaire, the Task-Interviews, the Activity-Interviews, and in the scheduling and analysis of the resulting data. The nature of this research included qualitative and quantitative data.

It was intended that the persuasiveness of findings would be strengthened by the knowledge that information was obtained from various sources in various ways and thus the variety of research techniques would enhance the validity and reliability of findings. It is for these reasons that triangulation was considered a priority, as was discussed in part 3 of this chapter.

Generalisation

This study does not aim to test a hypothesis but to undertake an exploratory investigation of an as-yet-uncharted area of student experience. There is no intention, within this study, to assume generalisations beyond the area of data collection contained in the research. Open generalisations would be dangerous but local generalisation might be possible. It is not appropriate at this point to anticipate any outcomes of the research for the research process has only just begun. This chapter has considered methodology and method; it has set the scene for the start of this research journey to *Explore Children's Conceptions of Zero.*

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CHAPTER 2

What is Zero?

> Why should zero, that O without a figure, as Shakespeare called it, play so crucial a role in shaping gigantic fabrics of expressions, which is mathematics? Why do most mathematicians give it pride of place in any list of the most important numbers?
>
> (Kaplan 1999, p.2)

Outline of chapter 2

Part 1, of this chapter, provides a brief account of the turbulent history of zero. Mathematics is a cumulative science with its past assimilated in its present and future. To understand the nature of zero it is necessary to look at its history. This section sets the scene for the discussion in Chapter 7, part 1 as to whether an historical review, in the ways zero evolved, developed and was used, along with epistemological analysis, may provide an insight into why the concept of zero can be difficult to understand. Part 2, of this chapter, explores the definitions of zero and discusses the disputed place of zero in various number sets. As well as discussing zero in the world of mathematics the educational aspect is reviewed. Highlighted are the cardinal and ordinal values of number with particular emphasis on zero, counting numbers and the empty set. Empirical data, from the children's responses to the questions 'Is zero a number?' and 'Why do you think zero is/is not a number?' are analysed.

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Chapter Two, Part One: The history of zero

The importance of mathematics' history

> A number system consists of various sets of symbols and rules for using them to express quantities as the basis for counting, comparing amounts, performing calculations, determining order, making measurements, representing value, setting limits, abstracting quantities, coding information and transmitting data.
>
> (Encyclopaedia Britannica CD, 1994)

After the long familiarity with the number system used today, which is a clear, efficient, easily learned method for handling numbers, it is often forgotten that its reliance on zero is crucial. Yet zero encountered resistance for several centuries. The history of zero is long, complicated and traumatic. While it is not feasible to include a comprehensive history of zero in this study the researcher feels it is important that the main aspects are highlighted; for, as J.W.L. Glaisher (1848-1928), in a Presidential Address to the British Association for the Advancement of Science in 1890, said,

> ... no subject loses more than mathematics by any attempt to dissociate it from its history. (Moritz 1993, p.45)

The great French mathematician Henri Poincare (1854–1912) reinforced this belief,

> If we wish to foresee the future of mathematics, our proper course is to study the history and present conditions of the science. (Moritz 1993, p.47)

A counting system

The basis of the numeration system appears to stem from the need to count real objects. The sophisticated procedure of counting required the appreciation of one-to-one correspondence, consistency in the order of counting and a need to invent words for numbers. This *named system* required a different name for each quantity and a consistency of use so that communication could be relied upon. Later oral communication expanded into a written system, beginning with the elementary representation of number through tallying and developing into the use of symbolic numerals.

At this early stage in the evolution of a number system would there be a number for zero? Counting arose from necessity. Counting was and is of reality, zero is of the abstract, intangible. Indeed, one may ask if a practical counting situation required a number zero. Providing the information that a farmer possesses four hens and three sheep is the result of counting and stating a number. To give information about the lack of cows the farmer's reliance on phrases and words, such as *none, no cows, not got any,* would suffice. The importance lies in conveying what one has (or had) rather than what one has not got in the sense that the list of what one has not got would be endless. But at the same time it seems inconceivable that early man did not know the concept of emptiness. Does this mean that zero was a concept to be discovered or to be invented?

Zero: invented or discovered?

> **discover** - disclose, expose to view, reveal, make known, exhibit, manifest
> **invent** – devise, originate, fabricate. (The Oxford English Dictionary, xreferplus, Dictionaries. accessed 2005)

For zero to be discovered would suggest that it had been in existence; it was a natural phenomenon. Whereas if zero was invented then people brought it into being; it was artificial. As numbers are considered to be an invention then, as with the other numbers, zero was invented. Numbers were devised as tools and zero, though a latecomer in the number system, was a needed tool.

Early evidence of zero

The following sections, on the history of zero, draw on the works of Datta and Singh (1993), Dilke (1993), Flegg (1989), Granger (2000), Ifrah (1994), Menninger (1992), Kaplan (1999) Katz (1998) and Schimmel (1993).

Zero's first format would most likely be found on the dust board, a writing surface upon which sand was spread. Calculations were designed to have the figures erased at each step, for numbers and symbols were for immediate use not for keeping records. This illustrates the first barrier of attempting to trace zero's early history, that of evidence. Also, one needs to be cautious of the first single items of evidence for while this evidence may provide a date but it is a survival date of the evidence, not evidence of the start of something. Accurately dating the available evidence is fraught with difficulties.

> When it comes to the pedestrian matter of dating such stories or tracing their antecedents, we must give it up. An attitude more poetic than ours towards when events occurred and towards the events themselves, makes hazy chronicles of these distant times. Even an early edition of the Surya Siddhanta – the first important Indian book on astronomy – claimed the work to be some 2,163,500 years older than it has been shown to be. (Kaplan 1999, p38)

When trying to trace the development of zero researchers have uncovered many discrepancies; this is particularly so when dealing with zero in India. Why such discrepancies? Early documented evidence is mainly of a poor quality. Added to this are the processes of translating and editing such works. These occurred mainly in the 19th century, particularly between 1850 and 1885 with much re-editing of these translations occurring in the 1900s to 1920. The same problems arise as the Hindu number system is traced into Islamic mathematics where there are no extant Arabic manuscripts of works such as those of Muhammad ibn-Musa al-Khwarizmi (c.780-850AD). Instead there are several different versions made in Europe in the twelfth century which describe the algorithms and the characters using our familiar place value system.

The difficulties of reliable evidence, of dating the evidence and of the problems caused by using material from translated sources means that historians disagree on developmental details[1]. However, most sources agree that three cultures independently invented zero, the Sumerians (Babylonians), the Maya and the Hindus. Unlike other numbers zero has a dual role. It is a place marker and a number. Only the Hindus took on the step of thinking of zero as a number.

The Sumerians

The area of Sumeria, between the Tigris and Euphrates rivers, is the site of the earliest known

[1] A chronology of inventions and progress in the history of mathematics is to be found in appendix 3.

civilization. Later this was known as Babylon and now forms part of modern southern Iraq, in the area between Baghdad and the Persian Gulf. In the Old Babylonian Period (1800-1600 BC) their number system was based on 60 (sexagesimal). Only two symbols were used

 for 1 and  for 10

All other numbers were seen as multiples of these. For sixty or more a positional notation arrangement was used.

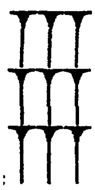
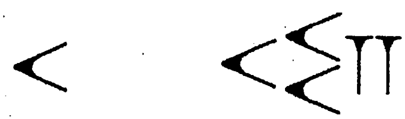
Examples of Old Babylonian Numbers	
a)  $9 \times 60^{\circ} = 9$	b)  $43 \times 60^{\circ} = 43$
c) 	$1 \times 60^1 + 45 \times 60^{\circ} = 105$
d) 	$10 \times 60^2 + 0 \times 60^1 + 32 \times 60^{\circ} = 36032$

Illustration 1

In example d) a large space represented an empty place in a horizontal arrangement. One would expect the use of a space to be confusing, especially if two adjacent spaces were required, but Cajori (1991) seems to think otherwise, as the larger sexagesimal base reduced the risk of ambiguity.

> Of the numbers from 1 to 216, 000 fewer than 1.7% are of this kind [that is require the use of an empty space] as compared with over 30% that require one zero and over 6.5% that require two or more zeros in the decimal system. (Cajori 1991, p.3)

This *empty space* method was used for over a thousand years until, as historian Lisa Jardin (Melvyn Bragg, BBC Radio 4, 2004) graphically explains, there occurred an incident in Sumeria, some 5,000 years ago, a quiet eureka moment, when a scribe used two diagonal wedges,  , to represent *nothing here.*

The Mayans

The Mayans lived in and around the Yucatan Peninsula of Central America. They placed great importance in the calendar. It was the priest's task to determine auspicious days through complex astrological calculations. Mainly they used a base twenty (vigesimal) number system.

• for 1 and — for five

A dot symbol represented one and a line represented five, thought to represent a pebble and a stick, although Ifrah (1994) suggests the dot represented the cocoa bean, which was the Mayan unit of currency. Numbers of twenty or more were represented in vertical columns beginning at the top.

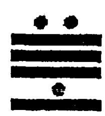
Examples of Mayan Numbers	
a)  $9 \times 20^0 = 9$	b)  $17 \times 20^0 + 6 \times 20^1 = 137$

Illustration 2A

At the next level one would expect to be represented by 20^2 but this is not so. The structure of the Mayan numbers is seen below,

The Mayan number structure		
Multiples of:	20^0	(1)
Multiples of:	20^1	(20)
Multiples of:	18×20^1	(360)
Multiples of:	18×20^2	(7200)
Multiples of:	18×20^3	(144000)

Illustration 2B

Initially this change, at level three, seems strange but the $18 \times 20^1 = 360$ was the number of days in the (normal) Mayan calendar (the other 5 days were not included in the months and were referred to as days of chaos). It was the priests' role to calculate the complex astrological computations to decide which days were auspicious, or not. The priests were the main users of the number system and by using 18 at the third level calculations involving the calendar became simpler. However, as was seen with the old Babylonian system, a problem arose when there was an empty space. This is demonstrated in Illustration 2C where it can be seen that the problem of using an empty space appears even more acute in a vertical arrangement.

Example of Mayan numbers showing the empty space

Which of the above is: $7 \times 20^0 + 14 \times 20^1 = 287$

$7 \times 20^0 + 0 \times 20^1 + 14 \times 18 \times 20^1 = 5047$

$7 \times 20^0 + 0 \times 20^1 + 14 \times 18 \times 20^2 = 100807$

Illustration 2C

Again, like the Babylonians, the solution came with the use of a symbol to represent the empty space. The symbol was that of a snail shell examples of which are seen in illustration 2D.

Various forms of the Mayan zero (Guedj 1996, p.111)

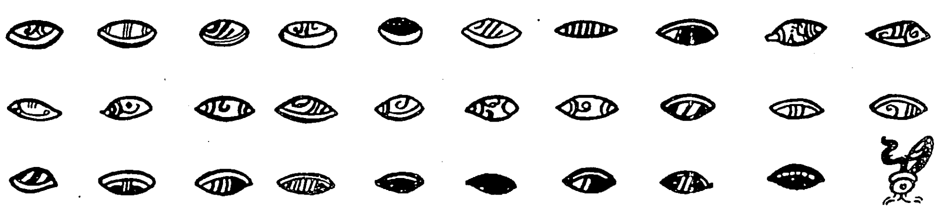
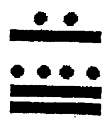


Illustration 2D

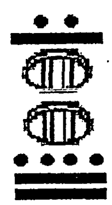
This made the examples in illustration 2C, with the inclusion of the shell symbol, appear as seen in illustration 2E.

Examples of Mayan numbers using the shell symbol



$7 \times 20^0 + 14 \times 20^1 = 287$

$7 \times 20^0 + 0 \times 20^1 + 14 \times 18 \times 20^1 = 5047$



$7 \times 20^0 + 0 \times 20^1 + 14 \times 18 \times 20^2 = 100807$

Illustration 2E

There is no accurate data when the use of a symbol to denote an empty place began. Joseph (1990) puts it at about 250AD, at the beginning of the Classic Period when the Mayans were adopted hieroglyphic writing, while McQuillin (1997) puts the date slightly later at 357AD. It is important to note that the Maya did not use the 'zero' as a number in its own right.

The Hindus

Around the year 600AD, the Hindus evidently dropped the symbols for numbers higher than 9 and began to use their symbols for 1 through 9 in our familiar place-value arrangement. Here again is found the use of the empty space system later filled with a symbol, often a dot. The earliest dated evidence does not come from India itself but is found in a fragment of the work of a Syrian priest Severus Sebokht, dated 662AD. Severus remarks that the Hindus have a valuable method of calculation 'done by means of nine signs' (in Katz 1998, p.231). However, Severus only speaks of nine signs, and there is no mention of a sign for zero. Flegg (1989) feels this may have been because Severus did not see that zero had the same status as 1 to 9. Our own ciphers were imported from the Arabs and those of the Arabs were imported from India. In India no ciphers were imported, there was no need for such importation '... as the old system of numerals was sufficiently cipherised' (Flegg 1989, p.110).

When considering these three cultures two important aspects emerge. The first is the early use of a symbol to denote an empty space. This symbol was not a number but a type of punctuation mark used to ensure that the numbers had the correct interpretation. The second is to wonder why the brilliant mathematical advances of the Greeks did not include the adoption of the empty place indicator and all the advantages this gave. The reason may well be that the Greek mathematical achievements were based on geometry. Numbers needed for notation, so that records could be kept, was the province of merchants. The part played by merchants and by trade will appear again as the story of zero moves to Europe.

The Sumerians, the Maya and the Hindus independently used zero as a place marker. However, as Kaplan (1999) explains only the Hindus took the next step and embraced zero as not just a marker, a gap or a space, but as a number; a number on a par with the other number symbols 1 to 9 and one that could be used in calculations.

According to Datta and Singh (1962) the earliest use of a symbol for zero (sunya) was seen in metrics by Pingala (before 200 AD) in his *Chandah-sutra*. Two symbols were used to distinguish between two kinds of operation, halving and absence of halving. Why were *two* and *zero* selected? *Two* as it links with the process of halving as division by two. *Zero* probably because of its being associated with *absence or subtraction.* Datta and Singh (1962) state that both the notions of absence and subtraction were common in Hindu mathematics from earlier times.

Pulisa was said to be conversant with the concept of zero as a numeral (*c.*400AD). Datta and Singh (1962) again provide the information that in the *Panca-siddhantika* (*c.*505AD) zero is mentioned in several places where it is conceived as a number of the same type as three, two or one. The writings of Jinabhadra Gani (529 –589AD), a contemporary of Varahamihera, appears to offer conclusive evidence of the use of zero as a distinct, numerical symbol,

> ... the zero of Jinabhadra Gani is not a mere concept of nothingness but a specific numerical symbol used in arithmetic calculation. .... All known Hindu treatises on arithmetic and algebra contain a section dealing with the fundamental operations of zero, including involution and evolution. ....[these] presuppose its existence as a numeral denoted by some specific symbol. (Datta and Singh 1962, pp.79-80)

The treatises referred to by Datta and Singh (1962) no doubt include those of the astronomer Varahamihira's Panchasiddhantika (c. 575AD), Bhaskhara's Aryabhatiya (629AD) and Brahmagupta's Brahmasphutasiddhanta (628AD). The latter defines zero as,

> ... the result of subtracting a number by itself (a – a = 0) and [he]described its number on the following terms: When zero (shunya) is added to a number or subtracted from a number, the number remains unchanged; and a number multiplied by zero becomes zero. (Ifrah 1994, p.439)

Evidence of zero used as a number in mathematical equations shows a different understanding. This development would, eventually, change the thinking of the world's mathematicians.

Of the three cultures who used zero as a marker for an empty position, why were the Hindus the only ones to take the important step of seeing zero as a number?

In India numbers began to be used without the use of objects; it was the number symbols which were manipulated, patterns were seen, games and puzzles developed. Zero emerges at this point from the hands of the traders into those of the mathematicians; Hindu mathematicians discovered what we understand in mathematics, that the numbers have a reality of their own. Why should this happen in India? Why should the leap into the abstract be convenient and comfortable to the Hindu mind?

It is known that there existed in Hindu philosophy the concept of emptiness, of a void. This is indeed still within the basis of Hindu belief, while the Buddhists hold the concept of Nirvana, which is attaining salvation by merging into the void of eternity.

> ... emptiness is not a new development or creation. It is not a product of philosophical analysis nor an invention of Buddha. Emptiness has been the actual nature of all phenomena from the very beginning. (Gyatso 2001, p.54)

It is clear that the Hindu religion could encompass the principle of zero and, as the priests were among the greater users of the number system, the religious concept of zero would be transferred to a number zero. Zero held an element of eastern mysticism.

Zero moves west

Despite differing cultures and beliefs the adoption of zero, and the mathematical changes which ensued, were taken up by Islamic mathematicians. One reason proposed by historian Lisa Jardine (Melvyn Bragg, BBC Radio 4, 2004) was that Mohammed was a merchant and zero was used by traders. Less than 100 years after Mohammed's death (632 AD) an Islamic empire had arisen with its cultural and political centre in Baghdad. While native customs persisted, the Arabic language was generally spoken and written in the Islamic domains. Here foreign texts, Greek, Hindu and Persian, were collected, translated and adapted. These texts then travelled west and, after translation into Latin, became accessible to Western Europe. This helps to explain why we refer to what is clearly the Hindu method of writing numerals as Arab numerals.

By the 10th century zero was in widespread use throughout the Arab Mediterranean, being actively transmitted and promulgated by Arab merchants. There the zero sign stayed, mainly within the confines of Arab culture, until the 13th century. The first person to bring zero to Europe is thought to have been Abraham be Meir ibn Ezra (1092 – 1167AD), a Rabbi from the then Muslim Spain. He travelled widely in Europe and wrote on diverse subjects including mathematics. In c. 1150 he wrote *The Book of Number* in which he introduced zero. This was the time of the Crusades and the Christian world tended to be resistant to new ideas from the 'Muslim Infidels'. Yet it was to be the Crusaders who brought back knowledge learned on their travels and thus spread the influx of Muslim culture into Europe. However, zero was not so readily accepted in the Christian world.

Christianity accepted readily the notion of infinity as this mirrored the belief that God and Christ were infinite. Not so with zero as this represented emptiness, a void. But if God was everywhere there could not be a void. If there could conceivably be a void then this would be where the devil's work took place. Emptiness was seen as a vacuum and represented annihilation. The use of the finger and thumb to form a circle was an evil sign. So the sign for zero, the circle or ring, represented eternity with nothing inside it. This was, indeed, a frightening concept.

In Christian Europe zero continued to be dismissed, by those whose function it was to handle numbers, as incomprehensible, as unnecessary, but also as evil. One may wonder how the church could hold such control. The reason being that the handling of number was in the hands of clerks who were Church-educated, immersed in Latin and under the pay and influence of the Catholic Church. This explains why the Church was so influential in matters of numbers and could enforce resistance to the 'infidel symbol'. It took many centuries for this resistance to be overcome. Apart from the Church there was another group in society who opposed the adoption of the Hindu numeral; these were the abacists.

The abacists or the algorists

In the 10th century Europe was still using the Roman numerals. Throughout its history there is no evidence that Roman numerals were used, or ever intended to be used, for calculation: instead, calculations were carried out, not as manipulations of written numerals, but as operations on the beads of a counting-board or abacus.

In the period between the 10th and 13th centuries the abacists, who wrote Roman numerals but calculated with the abacus, were in conflict with the algorists who both recorded and calculated with Hindu numerals. As Hindu numerals correspond just as naturally to the state of the abacus as Roman ones, where was the objection?

The fundamental obstacle of the abacists was the essential use by the algorists of the zero sign, a sign which affected the values of numerals wherever it occurred but had no value itself and which appeared as a number in calculations though it answered to no positive or real quantity. Pictures of numbers on paper, as the representation of the abacus, were acceptable as a record. However, the algorists had abandoned the abacus and calculated using only symbols. There were public contests as to who were the quickest and most accurate, the abacists or the algorists. Having more experience the abacists tended to win. The engraving (illustration 3) from the early 16th century shows clearly which Lady Arithmetic favours. On the left is Boethius, calculating by using Arabic numerals; on the right is Pythagoras, using an abacus.

With the expansion of trade, there was a growing need to write down and keep permanent records. During the 13th century, with the beginning of mercantile capitalism in Northern Italy, the handling of numbers was passing from Church-educated clerks immersed in Latin to merchants, artisan-scientists and architects. These were educated in the vernacular and to them arithmetic was a pre-requisite for trade. They appreciated the versatility of Arab/Hindu numerals. These writers, such as Fibonacci in his treatise *Liber Abaci* of 1202 [2], were becoming more influential. Still zero's passage was far from smooth, indeed zero was banned by the Florentine Government in 1299. But the renaissance was a practical as well as an intellectual period: the advisers of new rich merchants were seeing the benefits of Italy's double book-keeping system, which showed whether their clients were in debt or in profit. Here zero played a key role.

How did zero move from double book keeping to manipulating zero in algebra, as if zero was a number? Lisa Jardine (Melvyn Bragg, BBC Radio 4, 2004) says it was by serendipity that the

[2] In *Liber Abaci* Fibonacci described the nine Indian symbols together with the sign 0 but it was not widely used for a long time after that. It is significant that Fibonacci is not bold enough to treat 0 in the same way as the other numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 since he speaks of the "sign" zero while the other symbols he speaks of as numbers.

The engraving, dated 1508, shows Pythagoras, on the right, using an abacus in the form of a table. He is competing with Boethius, on the left, who is calculating using arithmetic signs and Arabic numerals. Above is *Margarita philosophica* who appears to look favourably on the speedy arithmetic calculations of Boethius.

Gregor Riesch, *Arithmetic Personified as a Woman* (detail) wood engraving, from the Margarita philosophica, Friedberg, 1503, BNF, Paris. (Guedj, 1996, p.52-53) intellectuals, at some point, recognised the power of this idea in the mathematics world. There is no doubt that the calculation demands of capitalism played a large part in breaking down any resistance still remaining to zero. By the beginning of the 17th century Arab/Hindu numerals had replaced Roman ones as the dominant mode of recording and manipulating numbers throughout Europe. So we find mathematicians in the far corners of Europe, such as John Napier in Scotland (1588), in his "equations to nothing", using zero in its widest sense.

Chapter 6, The Language of Zero, contains further insight into zero's chequered and in some parts mysterious history as the development of a symbol to denote zero and the numerous words used for 'zero' are discussed. What can be seen, within its turbulent history, is the evolving of some of the different facets of zero listed below,

1. Emptiness as a result of absence and as a result of subtraction.
2. The need to represent emptiness.
3. Emptiness to nothing.
4. The discomfort with zero and emptiness shown by certain cultures.
5. The zero word and the '0' symbol.
6. Zero and its place in calculations.
7. Zero as a number and its relationship with other numbers.
8. Zero and place value.

With the exception of the last item, which is not included in the research terms of reference, all these elements will be considered within this study. The second part of this chapter looks at the definitions of zero in today's world and whether the children in this research considered zero to be a number.

0 ~ 00 ~ 000 ~ 00000 ~ 000 ~ 00 ~ 0

Chapter Two, Part Two: Zero the number

Definitions of Zero

> To a young child an "understanding-definition" develops through the use in situations that have meaning to him. Hence his understanding of zero should be a development that includes the different uses and roles of zero. His definition should be one that may be enlarged and refined and of such a nature that it need not be denied or dis-allowed at a later stage. Frequently we may do an injustice to a child by trying to be too adult in our definitions. (Oesterle 1959, p.111)

Oesterle (1959) voices his concern that when teaching children we may be 'too adult in our definitions'. What might these definitions be? *Common usage* definitions were found in dictionaries, a summary is given in list 1 (full extracts can be found in appendix 2).

Definitions of Zero

- the symbol '0',
- the element of a number system, the integer denoted by the symbol '0'; nought,
- the cipher '0', indicating an absence of quantity or magnitude,
- the ordinal number between +1 and –1, starting point in scales from which positive & negative quantity is reckoned, the line or point on a scale of measurement from which the graduations commence,
- in thermometers, freezing point of water or other point selected to reckon from; *absolute* - in temperature, point at which the particles whose motion constitutes heat would be at rest, estimated at - 273.7 *C.*, the temperature, pressure, etc., that registers a reading of zero on a scale,
- no quantity or number, nothing; nil,
- a person or thing of no significance; nonentity,
- the lowest point in any standard of comparison; bottom of scale, nullity, nadir,
- Mathematics:
  a. the cardinal number of a set with no members,
  b. the identity element of addition that leaves any element unchanged under addition, in particular, a real number 0 such that $a + 0 = a$, $0+a=a$ for any real number a

(from Collins Interactive Dictionary,1995; The Concise Oxford Dictionary, 2004; Webster's Dictionary,1998)

List 1

One may ask what definition adults, particularly teachers and students who will have a strong influence on children's mathematics, would put on zero. Which of the above would they elect to emphasise? The only research information found in this domain was a study by Wheeler and Feghali (1983). This provided an interesting insight into what the *then* next generation of teachers' understanding was of zero. They interviewed Preservice (Initial Teacher Training) students, asked the question 'What is zero?' and identified eight categories from the responses.

1. Number
2. Empty set (including null set)
3. Nothing (including absence of things)
4. Symbol (including numeral, representation, digit)
5. Placeholder
6. Identity (for addition)
7. Other (different from above)
8. Unclassifiable (ambiguous response)

It was this researcher's intention to use these eight categories when classifying the raw data from the children's responses to the question 'What is zero?' The categories of Wheeler and Feghali were presented as headings. In order to identify with these eight headings it was necessary to obtain detailed clarification as to what was contained within each category. This was done by carefully exploring the examples of *Selected responses, typical of a category* provided by Wheeler and Feghali. These are shown in list 2.

Selected responses, typical of a category.

1. Number
   Zero is the number found between –1 and +1 on the number line.
   Zero is a number that indicates nothing.
   Zero is a whole number that is used to express that there are no elements in a particular set.
2. Empty set
   Zero is the set with no elements in it.
   Zero stands for the empty set or the null set.
3. Nothing
   Zero is nothing, no objects.
4. Symbol:
   Zero is the first symbol or numeral in the Hindu arabic numerative system.
   Zero is a numeral denoting an empty set or nothing, null.
   Zero is a digit (0) which has face value but no place or total value.
5. Placeholder:
   In our place value system, it is a "place holder".
6. Identity:
   When added on or subtracted from a number the number says the same.
7. Other:
   Zero is used to start the counting numbers off.
   Zero is the dividing point between positive numbers and negative numbers.
8. Unclassifiable:
   Zero can't be counted, but can be seen.
   In multiplication and division, zero causes the answer to be unfounded.
   Zero keeps us from getting confused.

(Wheeler and Feghali 1983. p. 151)

List 2

While these provided greater detail, on closer examination, the category positioning of the selected responses appeared contradictory and this, in turn, resulted in ambiguity. The following are incidents illustrating the cause for concern.

- The response – *'Zero is the number found between –1 and +1 on the number line.'* is placed in the 'Number' category. While *'Zero is the dividing point between positive numbers and negative numbers.'* is considered to belong to the 'Other' category.
- The response – *'Zero is the first symbol or numeral in the Hindu Arabic numerative system.'* is placed in the symbol category but one may ask whether the respondent was saying that zero was a symbol or that zero was a number.
- The difference between the 'Empty Set' (including null set) category and the 'Nothing' (including absence of things) category was blurred. According to the selected responses the difference hinged on the use of the word 'set'.

It is likely that most adults appreciate some, though maybe not all, of the aspects of zero found in the dictionary definitions (list 1) and being asked to define zero means selecting and emphasising one aspect. It is not surprising that some of the selected responses contained more than one element which could place them in more than one category as is seen in one of the selected responses, *Zero is a digit (0) which has face value but no place or total value.* A further consideration was that Wheeler's research was undertaken in the USA where the use of the word 'zero' differs from that in the UK. This is an issue which is addressed in 'Zero Language' found later in this study (chapter 6). Unfortunately it was not possible to gain more information about Margariete Montague Wheeler's categorisation decisions.[3]

Children's Responses to the question 'Is Zero a number?'

As stated above, the original intention, in this study, had been to use Wheeler's question 'What is zero?' and to adopt her eight categories when analysing the data. As explained in chapter 1, part 4, the responses to this question in the pilot study were uninformative and as a result the question was changed to, 'Is zero a number?'

Children meet the word 'number' long before entering school. Young children inform you that they *know their numbers* and will recite (or attempt to recite) the number names in order – one, two three, etc. Children hear the word *number* used in contexts such as telephone number, bus number, car number and house number. While these are considered to be in the 'non-numerical context and are used as identifying names or codes', (Maclellan in Thompson 1997, p35) nevertheless it is an association of a symbol, or set of symbols, with a word number. Often a child will read '3' as *'number three'* and when writing '3' will say 'I will write *number three*'. Hence there was no concern in this research as to whether the children would be able to understand a question involving the word *number*. Indeed there was no indication that any of the children who were asked 'Is zero a number?' did not understand the question. This closed question was then immediately followed by an open-ended question, '*Why do you think zero is/ is not a number?*' Data collation and analysis for these two questions will be discussed separately.

The responses to the question 'Is zero a Number?' came from the children in the Questionnaire and the Task-Interviews and these were placed in four categories,

1. <u>Yes</u> – a child stated zero was a number
2. <u>No</u> - a child stated zero was not a number
3. <u>Yes and No</u> – a child stated that zero was a number in some instances and was not a number in other instances
4. <u>Don't know</u> – a child stated he/she did not know whether or not zero was a number.

[3] Attempts were made to reach Dr Wheeler. It was with sadness that the researcher was informed, by her husband (who is the Associate Provost at Northern Illinois University) that Dr. Margariete Montague Wheeler had 'passed away' in 1988.

'Is zero a number?'						
	Questionnaire	Task-Interview				
Age of children	11	11	10	9	8	7
Total number of responses	85	20	12	20	16	13
Yes	69 81%	17 85%	9 75%	16 80%	10 63%	9 69%
No	15 18%	2 10%	1 8%	1 5%	2 12%	
Yes and No	1 1%	1 5%	2 17%	3 15%	4 25%	2 15%
Don't Know						2 15%

Table 5

As the information came from two different data collection methods a comparison of the results from the Questionnaire (from children aged 11) and Task-Interview (for the 11 year old group) was undertaken; the differences were felt to be sufficiently small to be acceptable for the purposes of triangulation.

As can be seen from table 5, the percentage of children who thought zero was a number was high, consistently high across all the ages. The range being from 63% to 85%, with an overall percentage of 76.

How might one interpret the children's responses to the question 'Is zero a number?' The natural reaction is to ask which one of these groups of children was correct. At this point it seems appropriate to examine the question asked of the children, whether or not zero is a number. In order to do this one needs to explore the question 'What is a number?'

What is a number? The world of mathematics

Considering that finding an answer to this question would not be too difficult showed the naivety of the researcher. The starting point was to look at dictionary definitions. Harper Collins Dictionary CD (1995) was typical:

> **number n.**
> a concept of quantity that is or can be derived from a single unit, the sum of a collection of units, or zero. Every number occupies a unique position in a sequence, enabling it to be used in counting. It can be assigned to one or more sets that can be arranged in a hierarchical classification: every number is a complex number; a complex number is either an imaginary number or a real number, and the latter can be a rational number or an irrational number; a rational number is either an integer or a fraction, while an irrational number can be a transcendental number or an algebraic number.

Other dictionaries, mathematical dictionaries and web-sites [4], showed that numbers come in

[4] These sources included Hyperactivedictionary (accessed March 2005), xreferplus dictionaries (accessed March 2005), The Oxford Advanced Dictionary (2000), Collins Dictionary (2000), Harper Collins Dictionary on CD (1995), Daintith and Nelson (1989).

various types and belong to various classes including:

Complex	Integer	Natural	Irrational	Positive
Imaginary	Negative	Whole	Cardinal	Transcendental
Ordinal	Real	Rational	Algebraic	Fraction

The numbers taught by most primary teachers and encountered by most primary children are rational numbers, which include natural (counting) numbers, positive and negative integers and rational numbers (fractions and decimals). The set of rational numbers encompasses the needs of this study.

How are natural numbers, real numbers, rational numbers and integers defined? Each of these will be reviewed separately. This researcher was surprised at the lack of consensus when seeking definitions. Indeed Wolfram (accessed 2004) showed his frustration by writing, 'Due to lack of standard terminology...' and proceeded to define his own terms.

Natural Numbers

> Zero remains, even in our time, an outcaste among the natural numbers.
>
> (Pogliani et al 1998, p.742)

An example of this lack of consistency in agreed definitions is seen with natural numbers which have their origins in the words used to count objects and thus begin with the number one. The logic being that, 'if there are zero objects you don't count them' (The Mathematics Forum, accessed 2004). Hence, *natural numbers,* because of this background, are often known as the *counting numbers*. While the majority of sources accessed agreed that *natural numbers* begin with 1, 2, 3, ... Wikipedia (accessed 2004) found that, 'The precise definition of the natural numbers has not been easy.' Here Wikipedia refers to the work of Giuseppe Peano who proposed five axioms for the natural numbers. These are known as 'the Peano axioms' or 'the Peano postulates' which state conditions that any successful definition of a natural number must satisfy: 'the first of these being that there is a natural number 0.' Wikipedia concluded that 'A natural number is a non-negative integer 0, 1, 2, 3, 4 ...' but this clear statement becomes confused when it continues, '... zero is sometimes excluded.'

Only a few children used the natural number/counting number debate in their rationale for deciding whether or not zero was a number. As an 11-year-old boy explained, zero is a number because,

- *When you count you begin with zero unless you miss it out.* (Aged 11, verbal)

Integers

Daintith and Nelson's (1989) mathematical dictionary defines integers as,

> Integers - the positive and negative whole numbers –1, -2, -3, etc +1, +2, +3 etc.

From this definition it could easily be assumed that zero is not included; an assumption strengthened by the knowledge that zero is neither positive nor negative.

Kaplan (1999) speaks of zero being the fulcrum, situated on the number line between the positive and negative integers. It was the opinion of the majority of sources, that zero was included with the integers... –3, -2, -1, 0, 1, 2, 3... This extended number line was the reason given by some of the children to explain why they thought zero was a number.

- *It's halfway between negative and positive numbers, got to be a number because it's there between negative and positive* (Aged 10, verbal)
- *Because the numbers below zero like –3 are called numbers and so it has to be a number* (Aged 11, verbal)

Rational Numbers

Rational numbers are numbers that can be written as a ratio of two integers; this can be expressed as a fraction or a decimal (the decimals must terminate or repeat). Is zero a rational number? An 11-year-old boy was very logical in his explanations as to why he thought zero was and was not a number.

- *Zero can be a number because it's in a decimal but not a number in a fraction.* [The child expanded upon this, using written illustrations, to explain ...] *You have 0.3 and 2.05. So that means zero is in a decimal. But you don't have 0/2 or 0/4.* [The researcher asked 'Why not?'] *Well, you just wouldn't write it.* [What does 0/4 mean?] *Zero over four means you don't have any.* [Any what?] *Any fourths, quarters, none so you don't write zero over anything. You don't use zeros in fractions.*[5] (Aged 11, verbal)

Returning to the question as to whether or not zero is rational number. A rational number is any number (zero being a number) in the form a/b where a and b are integers (zero being an integer). However, (there is the zero proviso) b should not equal 0. So one cannot have a/0 but one can have 0/b. Thus, while zero is considered a rational number it is unique in its behaviour; for while rational numbers inverse (c/d, d/c) zero does not.

The relationship between natural number, integers and rational numbers can be clearly displayed in diagrammatic form (Illustration 4).

---

[5] The use of [ ] indicates an explanation or a question asked by the researcher.

Rational Numbers ... –3, -2, -1, 0, 1, 2, 3, 4, 5,...
$\frac{1}{2}$, $3\frac{1}{4}$, 4.3, 0.37, - 0.13

Integers ...-3, -2, -1, 0, +1, +2, +3, ...

Natural Numbers
1, 2, 3, 4, 5, ...

Illustration 4 (Adapted from Haylock and Cockburn 1989, p 27)

As one moves from the centre of the diagram to the next set certain mathematical properties change.

As Haylock and Cockburn so aptly explain,

> In fact some things that are true become no longer true, some things that are false become true and some things that were not possible become possible.
>
> (Haylock and Cockburn 1989, p26)

Taking '1' as an example. '1' is a natural number, an integer and a rational number, as it can be represented in the class of fractions as 2/2, 3/3, 4/4. All the digits 1,2,3,4,5,6,7,8,9, fit into this sphere. However,

- *Zero is so tricky.* (Aged 8, verbal)

Illustrative incident

An interesting, recent example of this confusion was heard on a BBC Radio 4 programme 'The Learning Curve'.
On the programme the question was posed as to which letters of the alphabet are not used in number names. The answer included alphabet letters one of which was 'Z'.
At the beginning of the next week's programme, the presenter reported that during the week many listeners had queried the inclusion of 'z' in the answer, indicating that 'z' is in the number word 'zero'.
The response by the presenter was that, 'while zero is a fully fledged number we meant the counting numbers'.

(BBC Radio 4, The Learning Curve, 4.30 pm, Tuesday, 15th, 22nd March 2005)

The reasons for this confusion could be a result of the word *number* lacking specificity or another example of *zero* still causing problems.

While recognising the conflict in terminology for the purpose of clarity, in the present discussion, it was decided to take the definitions agreed by the majority of sources examined. These are,

- Zero is not in the set of natural numbers.
- Zero is in the set of integers.
- Zero is in the set of rational numbers.

According to these mathematical terms the answer to the question 'Is zero a number?' could be 'yes', 'no' or 'yes and no'. This in turn has the result that all the children who gave one of these answers (table 5) were correct in their response.

While mathematicians may speak in terms of natural numbers, integers, rational numbers and real numbers, the educationalists look at how to teach children about numbers. When children are asked why zero is/is not a number they are very unlikely to respond using these mathematical terms but are more likely to refer to an aspect of number they have encountered in the learning process. In order to place zero in the context of other single digits the next section considers the different aspects of the teaching and learning of early number[6].

Aspects of teaching and learning early number

> There are three children in class four who are five.
>
> (Haylock and Cockburn 1989, p.23)

The teaching of number can be a complicated and daunting task for to have an understanding of single digits a child needs the following skills.

- To say the number names in order, the correct order of the spoken names is essential as these are necessary for counting: one, two three, four, five, ...

- To use this number word order (together with a one to one correspondence, and with the knowledge that the last word spoken tells you how many objects have been counted) to find the number of objects in the set. This develops into the understanding that the objects can be counted; the final result remains the same.

- To match the spoken number words to the symbol 1, 2, 3, 4, 5, ... The number word order is matched to the number symbol order found on the number line. The child then needs to be able to match the individual symbol and word independent from the other numbers: e.g. 5 as five.

---

[6] *For clarity within this study reference to,*

- *a number symbol 2, 4, 7, is written as 2, 4, 7.*
- *the spoken number word is written as two, four, seven.*
- a quantity, a number of objects in a set, is referred to as 'an amount' and is written as **, ****, *******.

- To combine the above and understand the relationship of the amounts *, **, ***, to the number symbols 1, 2, 3, and to the words one, two, three. So that ***** can be communicated orally as five and recognised as the number symbol 5.

(Adapted from work in Fuson and Hall, 1983; Gelman and Gallistel, 1986; Maclellan, in Thompson, 1997)

The task of learning and applying such a complicated skill is broken down into small elements and these are then woven back until '*the threeness of three*' is understood. The amount, the number symbol and the spoken number name are a package 'three is three is three', that is 3 is three, 3 is ***, three is ***. All the children, aged 6 to 11 in this study, demonstrated (within the Task-Interviews) that they knew the numbers to ten. That they,

- Knew the counting order of words to ten
- Knew that 3 was the symbol for the spoken word 'three'
- Could count *** and give this set the name three and represent it with the symbol 3
- Could order sets of objects *, **, ***, ****.
- Could order the symbols - 1, 2, 3, 4, 5, 6, 7, 8, 9,
- Knew the positional order - first, second, third, fourth.

Early in the Task-Interview the children were asked '*Is three a number?*' This was a verbal question; '3' was not shown. Every child said it was a number, the reasons given were,

- The number word order

You say three when you say your numbers – one, two three, four, ... (Aged 7, verbal)

- A set of three

You can have three apples. (Aged 7, verbal)

- The number symbol '3' and the number symbol order

Three is on the number line with the other numbers. (Aged 7, verbal)

Understanding number involves a relationship or 'network of connections'.

> The concept of a number like three appears to involve a network of connections between the symbol 3, the word 'three', concrete situations of sets of things using the cardinal aspect and pictures of number involving the ordinal aspect, such as the number lines.
>
> (Haylock and Cockburn 1989, p23)

The reasons the children gave, as to why '3' is a number, depended upon which aspect they stressed. Whether they emphasised the word, the amount or the number symbol. Some children gave more than one reason. All the reasons are included in table 6.

Why do you think *three* is a number?				
Total number of children	Total number of responses	Number word order	Amount in set	Number symbol on the number line
81	104	9 9%	7 7%	88 84%

Table 6

A high percentage of children gave the symbol on the number line as the reason why *three* was a number. The children's reasoning provided a useful comparison with the data from the parallel question, *Why do you think zero is/is not a number?*

What makes you think zero is/is not a Number? Data analysis

Table 5 shows the percentage of the children who thought zero was a number was consistently high across all the ages. What reasons did these children give for their answer? Would they mirror those for *three* (table 6) in content and in the percentage of answers in each category?

From the critique of Wheeler and Feghali's research (1983), discussed in part 2 of this chapter, had grown an appreciation of the difficulties in analysing information and defining categories. In order to group the data a search was made of the children's answers for common words or phrases. These formed the list of categories. Later these were reviewed and combined to reduce the number. An illustration of this being that there were initially separate classifications for 'positive and negative', 'the number line' and 'the symbol order', these were merged to form one category headed 'the number line order'. Combining headings was undertaken only after returning to the original data to re-check the response of each child, to monitor that there was a consistent match with the data and the final category heading. The original list was kept available for reference (see appendix 4). The final seven category headings were – Number line order, Place value, Algorithms, Context, No-value, Non-numerical and Meaning unclear [7]. Most children gave only one reason but if more than one reason was given they were all included [8]. A complete table is in appendix 4.

To aid clarity of analysis the following sections are reviewed separately under the three aspects:

(i) zero is a number
(ii) zero is not a number
(iii) zero is and is not a number

(i) Zero is a number – rationale

Why zero is a number						
	Questionnaire	Task-Interview				
Age of children	11	11	10	9	8	7
Total number of reasons	92	26	12	20	17	11
'YES' Reasoning	**75** **82%**	**22** **84%**	**9** **75%**	**16** **80%**	**10** **59%**	**9** **82%**
Number line order	42 46% *	13 50%	8 67%	13 65%	10 59%	8 73%
Place value	8 9%	4 15%		2 10%		
Algorithms	13 14%	2 8%		1 5%		
Context	4 4%					
Non-numerical	3 4%	2 7%	1 8%			
Meaning unclear	5 5%	1 4%				1 9%

* The percentage has been calculated across all the reasons.

Table 7A

[7] Explanations of each category together with illustrations of children's responses are to be found in Appendix 4.
[8] Hence there were slightly more reasons than there were responses to the question, 'Is zero a number?' (Table 5).

Table 7 shows the reasons given by the children as to why zero is a number. It can be seen that across all the age ranges, the main reason for saying that *zero is a number* was the number line order. Most of these explanations were based on the number line and the extended number line. Two typical examples being,

- *Zero is a number because zero is in the number line.* (Aged 8, verbal)
- *Because zero is in the number line between negatives and possitivse and it is the first number of the possitives.* (Aged 11, written)

Only four children referred to the order of the number words,

- *You say it with the other numbers. Because when you count down from say ten you will eventually reach zero.* (Aged 7, verbal)
- *Because you say zero, one, two, three to show it's the start. You see it in the number line.* (Aged 10, verbal)

Other reasons (given by the 11 year old and 9 year old children) were that of place value and algorithms. In place value the rationale was the use of zero with other numbers,

- *Zero goes with other numbers to make bigger number.* (Aged 9, verbal)
- *If zero wasn't a number there wouldent be 10, 20, 30 ect.* (Aged 11, written)
- *If you look in 100 and other numbers like 120 then it's in that number, so it must be a number.* (Aged 11, verbal)

While a small number of children used algorithms as justification,

- *Because you see it in sums* (Aged 9, verbal)
- *It is used in sums like all other numbers.* (Aged 10, verbal)

It would appear that the majority of the children saw zero as a number because the symbol '0' is seen on the number line with other known number symbols. Similarly, when zero is in association with other known numbers such as in algorithms, on door numbers, making larger numbers such as 50, then it is a number. The word *zero* is said in conjunction with the other number words – zero, one, two, three, ... so again this reinforces the acceptance that zero, by association, is a number. Previously it was established (table 6) that all the children said *three* was a number with two of the reasons given, the number word and the number line order, being mirrored in the *zero* answers. Interestingly the third reason was not common to both numbers, that *three* is an amount and that *zero* is used to form larger numbers.

(ii) Zero is not a number - rationale

Table 8 shows the reasons given by the children as to why zero is not a number.
There appears to be no dominant rationale in the table charting the children's reasoning for why zero is not a number.

Why zero is not a number						
	Questionnaire	Task-Interview				
Age of children	11	11	10	9	8	7
Total number of reasons	92	26	12	20	17	11
'NO' Reasoning	16 **17%**	**3** **12%**	**1** **8%**	**1** **5%**	**3** **18%**	
Counting order					1 6%	
Algorithms	8 9%		1 8%			
Place value				1 5%		
No Value	2 2%	3 12%			2 12%	
Non-numerical	4 4%					
Meaning unclear	2 2%					

* The percentage has been calculated across all the reasons. Table 7B

While zero is not included in the set of natural or counting numbers only one child used the counting order as the justification for saying zero was not a number. The category labelled *Algorithms* appears in both the 'yes' and the 'no' tables. The children in the 'no' algorithm category referred to zero as being '*nothing*', having '*no value*'.

- *If you ad it* [zero] *to 3 you ad nothing and so on because you can't ad it up can you.* (Aged 11, written)

The *'no-value'* category contained a rationale that overtly referred to zero's lack of value,

- *It is not* [a number] *because there's nothing there.* (Aged 10, verbal)

When each of the categories for zero not being a number were reviewed words and phrases such as *'blank', 'doesn't do anything', 'none', 'not have an effect', 'can't be counted', 'nothing there'*, were used. This theme of 'nothing', that zero cannot be counted, that it is not a quantity was found 'hidden' within the categories and was the dominant reason given as to why zero is not a number.

Because of this *covert factor* the data from the children who said (or wrote in the case of the Questionnaire) that zero is a number was reviewed. It was seen that in concentrating on categorising the elements of the children's 'yes' responses the covert comments, which referred to zero's lack of value, had been masked. An example is seen from a boy, in the questionnaire, who wrote,

- *Zero is a number because it's there halfway between the negative and positive numbers but zero is basically nothing.* (Aged 11, written)

This statement had been classified as a 'yes' answer with the rationale being that of the number order. The reference that *'zero is basically nothing'* had not been recorded in the tables. All the raw data was reviewed and it was revealed that 23% of the 'yes' group of children had included, within their explanation, a rider that zero was '*nothing*'.

Section III, Zero is/is not a number – rationale

Table 9 shows the reasons given by individual children as to why zero is/is not a number.

Why zero is/is not a number						
	Questionnaire	Task-Interview				
Age of children	11	11	10	9	8	7
Total number of reasons	92	26	12	20	17	11
'Yes and No' Reasoning	1 1%	1 4%	2 17%	3 15%	4 24%	2 18%
Yes No	NL A/NV	NL NV	NL NV	NL CO	PV NV	CT NV
Yes No			NL NV	NL NV	NL CO	NL NV
Yes No				NL NV	NL NV	
Yes No					PV/NL CO/NV	

* The percentage has been calculated across all the reasons. Table 7C

Key *NL – number line order* *A – algorithm* *PV – place value* *CO - counting order*
NN – non-numerical *CT – contextual* *NV – no value* *U – unclear meaning*

From each of the age ranges came similar reasoning:

- *A kind of number but not a number, it is shown in the number table but not a number of anything. Yes, because it is in the number line and no because there is nothing there.* (Aged 11, verbal)
- *Yes and no. A bit of both. It's nothing but it's a 0 ...* [the child draws a circle]... *in front of 1, 2, 3.* (Aged 10, verbal)
- *Is zero a number? No, yes, not sure. It's on the number line but it's none. I guess it's both.* (Aged 9, verbal)
- *It's worth nothing. It's just a blank. It's just a zero like an "oh". It's not a number because numbers are 1, 2, 3, 4. That's just a zero* [points to '0'] *but it comes before 1 on the number line. There it's number.* (Aged 8, verbal)
- *It doesn't matter what you call it it's worth nothing, so it can't be a number, it can't be anything but it is a number on the number line unless it's in the alphabet as an 'oh'.* (Aged 7, verbal)

The 'yes' reasons, from this third group of children, reflected the main reason in table 7 *Zero is a number – rationale'*, that of zero being in the sequence of number symbols on the number line. The 'no' reasons echoed the main reason of those children in table 8, *Zero is not a number – rationale'*, who felt that zero expressed '*nothing'*.

In summary:

- the main reason why zero was considered to be a number was the number line
- the main reason why zero was considered not to be a number was that zero was *nothing*
- 23% of the children who said that zero is a number included in their explanation a reference to zero being 'nothing'

The tension appeared to lie with zero being *nothing* and with zero being in the number order rather than the fact that zero was not considered a natural or counting number. Because of this it seems an appropriate place to discusses further the subject of zero and counting and whether zero can be regarded as a cardinal and ordinal number.

Cardinal and ordinal numbers

Collins Interactive Dictionary (1995) defines a cardinal number as,

- a number denoting quantity but not order in a group
- a measure of the size of a set that does not take account of the order of its members Compare natural number. Compare ordinal number.

In the Mathematics Dictionary of Daintith and Nelson (1989) the definition is succinct,

- A cardinal number indicates the number of elements in a set.

Montague-Smith (1997) in her book *Mathematics in Nursery Education* explains the cardinal principle. That the counting numbers are used in a one-to-one correspondence - one, two, three - the final number in the count represents how many are in the set. This is the cardinal number of the set. The number symbol 3 and a picture of three frogs show the link between the numeral and its cardinal value. While the counting numbers do not contain zero one can have a set with no elements. It is the empty set that gives zero its cardinal value.

There seems to be no disagreement (Haylock and Cockburn 1989, Gelman and Gallistel, 1986, Hughes 1986) that the cardinal number 'describes the numerosity of a defined set of objects' (Maclellan in Thompson 1997, p 35).

It is with the ordinal context that different points of view emerge. Collins Interactive Dictionary (1995) defines an ordinal number as,

- a number denoting relative position in a sequence, such as first, second, third
- a measure of not only the size of a set but also the order of its elements. Compare cardinal number.

It is the first aspect of ordinality, *denoting the position in a sequence,* which seems to be emphasised in education, possibly because it has its own set of words to be learned - *first, second, third* ... and it links with the natural, counting numbers. Fuson and Hall (1983) maintain that children experience number words in the ordinal context much less than they do in the cardinal context. From this researcher's experience she too has found this to be so. Ann Montague-Smith declares, there is no position which comes before 'first'; consequently zero does not appear in this aspect of ordinality (Montague-Smith, 1997).

It is the second aspect of ordinality given above, *a measure of not only the size of a set but also the order of its elements,* which seems to be overlooked, though not by Rotman (1985). For Rotman associates zero with the counting process and, as a consequence, he puts forward a persuasive argument for zero being both a cardinal and an ordinal number.

> ... what from a semiotic point of view might be so difficult and alien about the number zero? ... there is in all discussions a *sine qua non*: the activity of counting. It seems to be impossible to imagine any picture, characterization, or description of numbers that does not rely on a prior conceptual familiarity (whether this be explicitly formulated or assumed as background knowledge) with the process of counting. (Rotman 1985, p.24)

Rotman describes counting as requiring the repetition of an identical act as is seen in marks such as /, //, ///, ////, where /, //, ///, become the numerals 1, 2, 3, and where zero also functions as a number in the way it indicates the absence of a tally mark[9].

> Such an absence can be construed in two different ways depending on whether the numbers produced by counting are seen cardinally or ordinally. (Rotman 1985, p.25)

He goes on to explain that if we interpret counting cardinally, then the tallying groups correspond to a set of counted objects; 'a process that makes zero the cardinal number – nought', as it signals 'the absence of any such corresponding mark' (Rotman 1985, p.25).

> If counting is interpreted ordinally ... 1, 11, 111, etc, appear as records which mark out by iconic repetition the sequence of stages occupied by a counting subject. Zero then represents the starting-point of the process; indicating the virtual presence of the subject at the place where the subject begins the whole activity of traversing what will become a sequence of counted positions. It is presumably this trace of subjectivity – pointed to but absent – that Hermann Weyl was referring to when, in his constructivist account of the mathematical subject, he characterised the origin of co-ordinates, represented by 0 on the line and by (0,0) in the plane and so on, as the "necessary residue of ego extinction".
> (Rotman 1985, p.25)

This researcher agrees with Rotman, who views zero as a cardinal and an ordinal number,

> Thus, zero points to the absence of certain signs either by connoting the origin of quantity, the empty plurality, or by connoting the origin of ordering, the position which excludes the possibility of predecessors. (Rotman 1985, p.25)

In this study the data indicates that the main reasons given by children who thought that zero was a number (table 7) was the number line order, the ordinal aspect of number. While the main reason given by the children for zero not being a number was that it was 'nothing', the cardinal aspect of number. This corresponds with the views of Haylock and Cockburn (1989),

[9] This absence was noted in the early number systems in part 1 of this chapter.

> This fixation that zero is nothing is, of course, part of the over-emphasis on the cardinal aspect of number ... Once we consider the ordinal aspect, zero is then not just as good a number as any other but becomes a very significant and important number. It is the point before one on the number line, and sometimes the starting point; it is the ground floor of the department store; it is midnight on a digital watch – and a temperature of zero degrees is certainly not an absence of temperature!
>
> (Haylock and Cockburn 1989, p.25)

It is clear that the number line and the empty set play a large part in forming children's conceptions of zero. The children's attitudes to '*nothing*', in the form of the empty set, are discussed later in this study (chapter five). The children's conceptions of zero in relationship to other numbers on the number line are the subject of the next chapter.

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CHAPTER 3

ZERO AND ITS RELATIONSHIP TO OTHER NUMBERS

> The paradoxes posed by an innocent number (zero), rattling even this century's brightest minds and threatening to unravel the whole framework of scientific thought. ... It (zero) provides a glimpse of the ineffable and the infinite. That is why it has been feared and hated - and outlawed. (Seife 2000, p.2)

Outline of chapter 3

Part 1 of this chapter provides a critical analysis of the visual aids found in schools which are used to present the number order to children, this is undertaken with particular emphasis on the problems of including zero. The recommendations of the National Numeracy Strategy for the use of such displays are also reviewed. Part 2 explains the fieldwork, the selection and presentation of the questions, the collection and classification of the data and its analysis. Part 3 reflects on the main points, which arose from the fieldwork, and each of these are discussed in detail.

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Chapter Three, Part One: zero and the number order

The research aim: zero and its relationship to other numbers

The aims of the research were to explore the conceptions of individual children, aged 3 to 11, within the areas of,

1. The empty set
2. Zero as a number and its relationship to other numbers
3. The zero number facts
4. The language of zero

Chapter Two focused upon 'zero as a number'. This chapter is concerned with the second part of this aim, 'its relationship to other numbers'. To pursue this 'relationship with other numbers' a series of questions was devised for use in the Questionnaire (appendix 1) and in the corresponding Task-Interviews. The final questions were based round an *ordering numbers* exercise. This decision, to use *ordering numbers*, was made quite early in the research project, so that the questions could be included in the Questionnaire. It was not appreciated, at that time, that the collation and analysis of the question '*Why do you think zero is/is not a number?*' would reveal the main reason as *the number line order* (table 7). This provided an added facet to the findings of this chapter.

The introduction of The National Numeracy Strategy (DfEE, NNS, 1999) is said to have led to significant changes in primary mathematics teaching in England and is seen as a successful initiative (Earl *et al*, 2003). However, Earl and his co-writers add that the NNS has 'not yet produced the needed depth of change in teaching and learning' (p.140) and that,

> ... evidence is mixed about which teaching has actually changed beyond the adoption of structure and format of the ... daily mathematics lesson. (Earl et al, 2003, p.133)

As will be demonstrated in chapter 4 and chapter 6 two major common elements in the NC and the NNS had a significant effect upon the teaching of 'zero', the use of correct terminology and emphasis upon mental calculations. While the National Curriculum for Mathematics (1995) stated the mathematics content, the National Numeracy Strategy (DfEE, NNS, 1999) specified how aspects of number should be taught, including the use of mathematical equipment and resources. This is why a third element to have significant effect upon the teaching of zero is found in the NNS and not in the NC. This is the NNS recommendation on the display and use of a number line in each classroom, the details of which are given below.

What resources do we need?

Beside a board, each classroom should have a large, long **number line** for teaching purposes, perhaps below the board, and at a level at which you and the children can touch it. A 'washing line' of numbers strung across the room, and which can be added to or altered, is useful. Provide table-top number lines, marked and unmarked, for individual use. For Reception and Year 1, number tracks with the spaces numbered to 20, rather than number lines with the points numbered, are helpful, including those made from carpet tiles. You could also have a floor 'snake' for children to move along in corridors, the hall and the playground. For Year 2, lines need to extend to 100; by Year 4 they should include negative numbers; Years 5 and 6 need marked and unmarked lines on which decimals and fractions can be placed.

(DfEE, 1999, NNS Y123 p.29-30)

The use of a display showing the order of the number symbols has been used for many years. The change being that now '0' is added to this order, however, this small change is problematic. This would seem to be a suitable point to review the various perceptions and aspects of what constitute a *number line*.

The number order - a review of the display resources

In the role of an advisory teacher for primary mathematics one of the tasks of this researcher, together with her colleagues, was to review and evaluate the many commercially published mathematics schemes. This task was ongoing and covered a period of about eight years, after the introduction of the first NC document but prior to the NNS. One noteworthy feature was how the display of the number order, in both the pupils' workbooks and in the resources suggested in the teacher's handbook, differed according to the age of the child. It was common for the staff in schools and the writers of commercially produced mathematics schemes to refer to the *number line*, whether this was a line of cards, the number symbols written in a line, a number bar or a number line. The name, *number line,* seems to be a generic term used to encompass any visual display which shows the number symbols in order but with no consideration as to the limitations of each display. It is these limitations, with reference to zero, which form the basis of the following discussion.

The number order, children of 5 and under

Children of nursery and reception class ages (3 to 5 year olds) would often be introduced to the order of the number symbols by the use of a series of commercially produced *number cards*

Number card

4
* * * *
four

Illustration 5

These would show the number symbol, illustrations of sets of similar objects and the number word (Illustration 5). These *number cards* would be displayed, in order, in the classroom. This 'discrete' number line would begin at '1' and end at '10'. It was not the norm to have a card for zero. This lack of a zero card was frequently the subject of discussion (amongst the researcher, her colleagues and teachers of young children) the consensus of opinion being that there were two main reasons for this omission,

1. The difficulty of illustrating an empty set.
2. The contextual use of 'the zero words'. There was a variety of words including 'nothing', 'none', 'nought' which were in common use for the empty set and for the '0' number symbol. (The language of zero is the subject of chapter 6.) The word used depended on the context of the '0' symbol. Zero was, at that time, not a commonly used word.

The number order, children of 5 to 7

Teachers of infant children (children aged 5 to 7, Key Stage 1) tended to use the form illustrated below (Illustration 6A). This will be referred to as a *number bar.*

1	2	3	4	5	6	7	8	9	10

Illustration 6A

Number bars made use of 'the spaces' or 'intervals'. These *number bars* combined the use of the number counting, number quantity as well as the number symbol for often these *number bars* were designed to match, in size, blocks and rods from resources such as Stern and Unifix. The number symbol 3 was on space three and it would also match block three. It took ten spaces or blocks to reach 10. If zero had been used then this would have meant that the symbol 3 was on space or block 4; it also would have meant that '0' was on block 1. Hence the *number bar* began at '1'. There was no '0'. As the *number bar* was designed with equal spacing then an element of length was involved. Indeed this 'numbering of the spaces' extended to the 'infant ruler', which was identical in form to the *number bar* and was often used as such. These rulers were notoriously difficult to use for child and adult. This is demonstrated in Illustration 6B where it is unclear, unless the spaces are counted, whether the red line is 4 or 5 units long.

1	2	3	4	5	6	7	8	9	10

Illustration 6B

The same principle, of numbering spaces, applied in using grids as in the use of addresses [1], seen in 'A to Z' maps, or more in keeping with the children's ages and interests in the treasure island maps.

The number order, children of 7 to 11

In published mathematics schemes for the junior children (aged 7 to 11, Key stage 2) the *number bar* changed to the *number line* (Illustration 7).

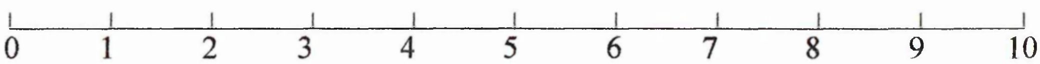

Illustration 7

[1] This researcher and her colleagues were involved in many a debate with teachers who, in their assessment of children, recorded that the children could use co-ordinates when they were using addresses

Here the marks and not the spaces were numbered, the '0' being included as the starting point. Again this development from numbering spaces to numbering the lines was seen in the move from addresses to the use of coordinates and the marking of the origin. The *number line* equates with the measurement scale seen on a ruler or tape measure, on a capacity jug and on an analogue clock; though on most measurement scales the zero is assumed, it is not shown. The continuous *number line* also corresponds with co-ordinates and the axes on graphs.

The familiar mode of reference, used by teacher and child, to each of these three modes was *the number line*. With the NNS recommend that zero be used at an early age did the NNS also recommend the use of the number line, and its inclusion of zero, with these younger children? The next section provides an answer to this question by providing a review of the NNS number order display recommendations.

The number order display - recommendations in the NNS

The NNS, states that children should appreciate the relationship between single digits and advocates, that each classroom should display a *number track or number line*. Of particular interest is how the NNS include the position of zero in this number order relationship. The NNS uses three formats, here labelled illustrations 8A, 8B and 8C.

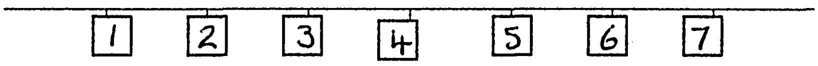

Illustration 8A

1	2	3	4	5	6	7	8	9	10

Illustration 8B

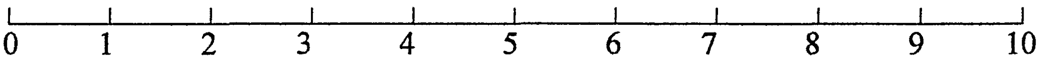

Illustration 8C

The NNS calls,

- Diagram 8A - a number track and a number line
- Diagram 8B - a number track
- Diagram 8C - a number line.

The NNS terminology used for these formats is confusing and inconsistent (a detailed critique is to be found in appendix 5). In essence Diagram 8A is the *number card* format, Diagram 8B is the *number bar*, Diagram 8C is the *number line* described on the previous page. It would appear that the three forms and the age of the children who use them show little change from those in use prior to the NNS. What is noteworthy is that, while the NNS states that children should appreciate that zero is in the space before 1 on the number track, zero is not included in any of their number track illustrations. Possibly this is connected with the problems mooted earlier with the use of '0' on *a number bar.*

Whether, one uses a *number bar* or a *number line*, whether the numbers are in the spaces or on the marks may be seen as placing too much emphasis on minutiae and could be construed as being pedantic or flippant. This may be so until one considers the inclusion of '0' then the cardinal, ordinal problem appears again. As Seife (2000) writes,

> We don't have to worry about mixing up the value of the number – its cardinality – with the order in which it arrives – its ordinality – since they are essentially the same thing. For years this was the state of affairs and everybody was happy. But as zero came into the fold the neat relationship between a number's ordinality and its cardinality was ruined. The numbers went 0, 1, 2, 3: zero came first, one was second in line, and two was in third place. No longer were cardinality and ordinality interchangable.
> (Seife 2000, p.59-60)

This is the root of the calendar problem and of the measurement problem, explains Arsham (Accessed 2002), and he uses a ruler to illustrate that the first inch is the interval between 0 and 1. Considering whether the millennium starts in 2000 or 2001 depends on whether you look at a number as a points in time or a time interval. Measurement is as an interval; number as a point.

> Thus the millennium ended with the passing of the two-thousandth year, not with its inception.... Zero is a cardinal number and indicates value; it does not name an interval... Continuous data come in the forms of Interval or Ratio measurements. The *zero point* in an Interval scale is arbitrary. (Arsham, Accessed 2002)

The confusion is between the notion of 'time window length' and a 'point in time'. A confusion compounded when a Christmas television programme was entitled 'Bethlehem Year Zero' (see appendix 13).

Illustrative incident

A well known error, when children are learning to use a ruler or tape measure, is that they align the '1' to the start of the object being measured. They begin measuring at '1' as they would begin counting.

This is still a problem by the age of 11 as was noted in an APU (1980) practical test when 85% of all 11 year olds measured the length of a 13 cm line to within 5mm. 5% aligned the end of the line to the '1' on the ruler thus obtaining an answer 1cm longer than it should have been. (In Dickson, Brown and Gibson (1993, p.98).

This researcher, in a conversation with a Further Education lecturer in engineering, was recounting this common mistake and the resulting answers being '1 unit' too big. There was a minute of silence then the Further Education lecturer said, in astonishment, 'that probably explains a problem I encounter most years with a small number of my students. I could never understand why they were a mm out in their drawings and measurement.'

All the children in this research project worked in a classroom with the number order displayed on the wall. The displays were in the form of *number cards*, *number bars*, *number lines* or just *the number symbols written in order*. All the displays seen in the research schools included zero. Many children, when arranging the number cards, made reference to the '*number line*'. It is important to note that *the number line* was a general term used by the children in which they may have been referring to any one of the various number order formats previously discussed. It was important for the researcher to be aware of these various formats and the children's access to them. There was no evidence that the use of zero in these different formats had any effect on the children's arranging of the numbers.

0 ~ 00 ~ 000 ~ 00000 ~ 000 ~ 00 ~ 0

Chapter Three, Part Two: Ordering numbers

Questionnaire and Task-Interview content and data collection methods

The Questionnaire and Task-Interview number symbol ordering questions were to provide an insight into the children's understanding of zero in relationship to other numbers and into how the children viewed zero within the number order.

The sets of numbers used were,

- single digits
- simple fractions plus zero
- decimals plus zero
- negative numbers plus zero

The actual numbers and the order they were presented in both the Questionnaire and Task-Interview is in list 3.

Ordering numbers questions
a) ½, ¼, 2, 1, 0
b) .3, .4, 0, .5, .1
c) -9. 0 - 7, -1, -4
d) ¾, ¼, 0, ½
e) 0.4, 5, 1.2, 8, 0
f) 3, 0, 5, 4, 7
g) 8, 5, 7, 1, 0, 4, 3, 2, 9, 6

List 3

As the task-interview material needed to be accessible to a wide age and ability range the integers were kept small. The fractions and decimals selected were those which young children meet early in their mathematical work. In the Questionnaire the researcher had no knowledge as to whether the child understood all the types of numbers in the questions, such as decimals, fractions, negative numbers. It could only be assumed that the children did not attempt to answer a question they did not understand.

The same series of numbers, in the same order as in the Questionnaire, were presented to the children in the Task-Interview situation. For the Task-Interview each of the numbers was written on a piece of card and the child was asked to read the symbol on each card and then arrange the set of cards beginning with the smallest.

Number ordering, card set A

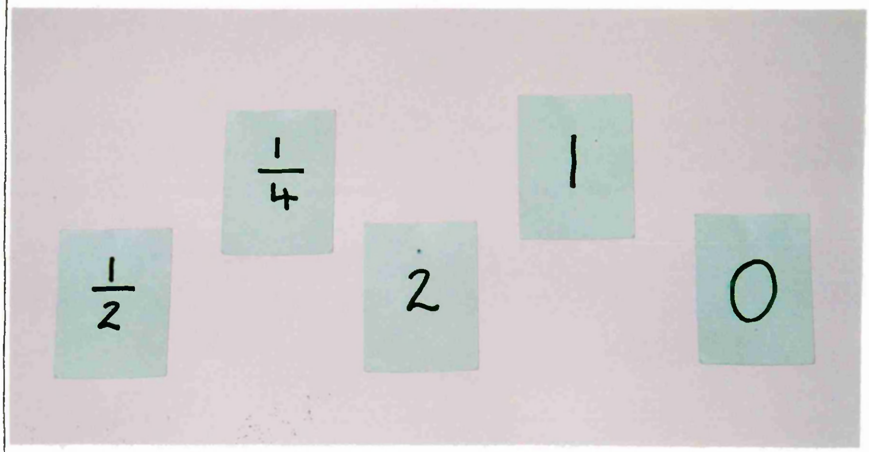

Illustration 9

If the child could not read the symbols and did not appear to be familiar with fractions, decimals or negative numbers then the researcher did not continue with the relevant question. The Task-Interview allowed participants to expand on their answer, to provide an explanation as to how the answer was reached. It also allowed the researcher to ask questions in order to gain clarification. Whatever the answer, correct or not, the child was asked for an explanation.

When a child had completed a sequence he/she was asked to read their number order from the smallest to the largest amount in order to note,

- whether a child had ordered from left to right, or vice versa, a child who ordered the sequence as 5, 3, 2 may have intended the order to read, from the smallest to the largest, 2, 3, 5

- how a child read any '0s' in the sequence, for example 0, 0.4 being read as *naught, zero point 4*

- if the emphasis was placed on certain numbers or parts of numbers such as reading 0.4 as ***zero point four*** with stress on the words ***zero*** and ***four***

Collating and analysing the data

When classifying the answers there was an acute awareness that the results contained the child's depth of understanding of fractions, decimals and negative numbers as well as that of zero. This is demonstrated using some of the responses given to question (d) ordering ¾, ¼, 0, ½.

Some responses to number ordering cards (set A)	
response 31...	**0, ¼, ½, ¾**
response 32 ...	0, ½, ¼, ¾
response 33 ...	0, ¼, ¾, ½
response 34 ...	0, ¾, ¼, ½
response 35 ...	0, ¾, ½, ¼

List 4

The mis-ordering of the fractions had to be taken into consideration when isolating the child's positioning of zero from the other factors. Responses 31 to 35 (list 4) place zero at the beginning of the order sequence. Regardless of the fraction order, the answers in list 4 were classified as '*zero then fractions*'. It was this element which was of significance.

In the Questionnaire most answers were first answers, few changes were made but it did appear that those few changes were possibly a result of a less mathematically secure child being influenced by another child or by seeing another child's answer. The decision was made to use only the first answers from the Questionnaire. In the Task-Interviews children would give an answer and then by providing an explanation for their answer would change their mind. In the charts for the Task-Interviews the numbers not in parenthesis are the first answers; if changes were made then the final answers are those shown in parenthesis. This structure is used throughout the data recording and is illustrated in the following example showing the results of the Task-Interview with two groups of children.

Example of number ordering presentation format		
Question A ½, ¼, 2, 1, 0	Task-interview	
Age of children	Age 9	Age 8
zero/ fractions/other digits	35% (25%)	40%

Table 8

- 35% of the 9-year-old children gave zero*/fraction/other digits* as a first response. After explaining their reasoning this was reduced to 25%. The final percentage is shown in brackets.

- 40% of the 8-year-old children gave *zero/fraction/other digits* as a first response. After explaining their reasoning this percentage had not changed.

Both sets of numbers are of value. First answer: the children's first response gave their immediate reaction to the ordering of the sets of numbers, this is the answer they are likely to have given in an assessment situation such as the SATs tests (and the Questionnaire). Using the first answers means that a comparison could be made with the data from the Questionnaires and the data from the Task-Interviews. Second answer: the second response (which applied only in the Task-Interview situation) was a result of the researcher asking each child to provide an explanation for his/her answer. It required a child to reflect. In the process of the explanations some children changed their mind and re-ordered the cards. There was no apparent pattern in the answer changes, either with individual children or with specific explanations. Some changes were from correct to incorrect answers, some from incorrect to correct. As all these changes were a result of children explaining their reasoning it was important to record these changes

In many of the sections it is interesting to see the strong connection, thrown up by the researcher's triangulation strategies, between the percentage of answers from the Questionnaire and the first answer given in the Task-Interview. Sometimes this is across the age ranges but it is especially noticeable when comparing the results from the Questionnaire (completed by 11 year old children) and the Task-Interview data from the same age range. This can be seen in the first fraction and the first decimal ordering (table 9A and table 11A). In the second fraction and second decimal question (table 10A and table 12A) the comparison is not as close. The researcher felt that this was due to the Task-Interview children explaining their reasons and often noting their errors. This had an effect on their approach, later in the interview, to the second related question. In this second question some children made reference to the first question and gave the same rationale. While the connection between the Questionnaire results and those of the Task-Interview in the second question is not as strong, it is still noteworthy. This strong link in the quantitative data is reassuring as this suggests that the Questionnaire and Task-Interview are both exploring the same phenomena.

The following sections review the results of the children's responses to the ordering numbers questions. It is not possible to know the rationale for the Questionnaire answers and no claim is being made that there is any correlation between what may have been their reasoning and the reasons given by the children in the Task-Interview group.

Presentation of the classified data and its analysis

The data and analysis will be presented under the following four headings:

A) fractions and zero
B) decimals and zero
C) negative numbers and zero
D) single digits

A) Fractions and Zero

The fractions included in the ordering numbers questions were those young children meet early in their mathematical work, namely halves and quarters. There were two questions containing fractions: fractions and whole numbers plus zero ½, ¼, 2, 1, 0 **(A1)** and simple fractions plus zero ¾, ¼, 0, ½ **(A2).**

A1) First fraction ordering - the responses to ordering: ½, ¼, 2, 1, 0

Fraction card set A

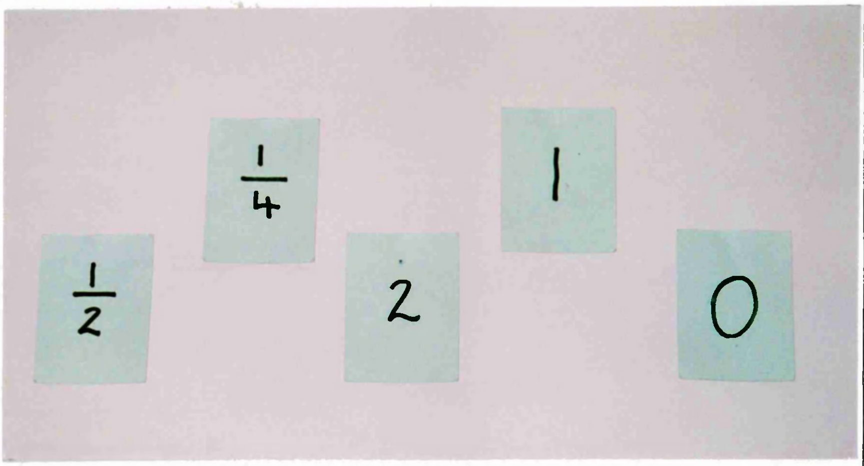

Illustration 10

Question A ½, ¼, 2, 1, 0	Questionnaire	Task-Interview				
Age of children	Age 11	Age 11	Age 10	Age 9	Age 8	Age 7
Number of children	100	20	12	20	16	13
Total number of responses	100	20	12	19	15	9
zero/ fractions/other digits						
response 1 ... 0, ¼, ½, 1, 2	**41**	**9 (12)**	**4 (5)**	**6 (5)**	**3**	**1**
response 2 ... 0, ½, ¼, 1, 2	2			1 (0)	3	3
fractions /zero/other digits						
response 3 ... ¼, ½, **0, 1, 2**	47	8 (5)	6	4 (5)		
response 4 ... ½, ¼, **0, 1, 2**	1			1 (3)	0 (1)	1
zero/other digits/fractions						
response 5 ... **0, 1, 2,** ½, ¼	3	1	1	3 (2)	5 (4)	4
response 6 ... **0, 1, 2,** ¼, ½	1			1		
Other						
response 7 ... ¼, ½, 1, 2, 0	1					
response 8 ... 0, ½, 2, ¼, 1	1					
response 9 ... 0, ¼, 1, ½, 2	1	1				
response 10 ... 0, 1, ¼, ½, 2		1	1 (0)	3		
response 11 ... 0, 1, ½, 2, ¼					3	
response 12 ... ½, **0, 1, 2,** ¼					1	
response 13 ... 0, ½, 1,2, ¼				1		
?	1					
....	1					

Table 9A[2]

[2] The numbers not in parenthesis are the first answers; those in parenthesis show the altered, final answers.
? – the child wrote numbers other than given in the number order.
..... the child did not use all the numbers given in the question

Three frequent response categories were identified and are shown as percentages[3] in table 9BCD.

- zero/ fractions/other digits
- zero/other digits/fractions
- fractions /zero/other digits

The three frequent response categories							
Question A ½, ¼, 2, 1, 0	Questionnaire	Task-interview					
Age of children	Age 11	Age 11	Age 10	Age 9	Age 8	Age 7	Age 6
Number of children	100	20	12	20	16	13	13
Total number of responses	100	20	12	19	15	9	2
zero/ fractions/other digits	43%	45% (60%)	33% (42%)	35% (25%)	40%	44%	
fractions /zero/other digits	48%	40% (25%)	50%	25% (40%)	7%	11%	
zero/other digits/fractions	4%	5%	8%	20% (15%)	33% (27%)	44%	

Table 9BCD

There follows an analysis of these three response categories and of the children's explanations. It needs to be stressed that it is not possible to know the rationale for the.Questionnaire answers. All the reasons given are from the children in the Task-Interview group.

Analysis of the first response category: *zero/ fractions/other digits.*						
Question A ½, ¼, 2, 1, 0	Questionnaire	Task-interview				
Age of children	Age 11	Age 11	Age 10	Age 9	Age 8	Age 7
zero/ fractions/other digits	43%	45% (60%)	33% (42%)	35% (25%)	40%	44%

Table 9B

Here the children's reasons for putting *zero/fractions/other digits* involved comparing the size of zero with the other numbers.

- ❑ 0, ¼, ½, 1, 2, ~ *Zero is lower than 1, zero is a whole one lower than one.* (Age 9, verbal)
- ❑ 0, ¼, ½, 1, 2, ~ *¼ and ½ are more than zero.* (Age 8, verbal)
- ❑ 0, ¼, ½, 1, 2, ~ *Zero is less than a quarter.* (Age 8, verbal)
- ❑ 0, ¼, ½, 1, 2, ~*'0' is nothing, ¼ is something, zero isn't anything.* (Age 7, verbal)
- ❑ 0, ¼, ½, 1, 2, ~ *¼ is more than nothing.* (Age 9, verbal)

One rationale that zero is ***a whole number,*** was used by children to explain why '0' went before the fractions: 0, ¼, ½, 1, 2, was

- ❑ *Zero is a whole number. Zero has no parts to it.* (Age 10, verbal)

As we shall see below, the idea of ***a whole number*** was also used to explain other orderings.

[3] Whole number percentages are used.

Analysis of the second response category: ***fractions /zero/other digits***						
Question A ½, ¼, 2, 1, 0	Questionnaire	Task-interview				
Age of children	Age 11	Age 11	Age 10	Age 9	Age 8	Age 7
fractions /zero/other digits	48%	40% (25%)	50%	25% (40%)	7%	11%

Table 9C

Observing the children ordering the cards the initial reaction of many children was to place ***fractions /zero/other digits***. The explanations were not that children were putting zero after the fractions but that they saw **0 as being before 1.** Putting zero next to one was often the initial reaction, though some children, on reflection, changed their order to zero then fractions.

- ¼, ½, 0, 1, 2, ~ *You have to have 0 in front of 1.* (Age 10, verbal)
- ¼, ½, 0, 1, 2, ~ *It goes 0 then 1, then 2.* (Age 9, verbal)
- ½, ¼, 0, 1, 2, ~ *Because 0 is more than ¼, zero comes before one.* (Age 9, verbal)
- ¼, ½, 0, 1, 2, ~ *Zero is after ½ and zero is the next smallest, then you get 1, like with 0,1,2,3.* (Age 10, verbal)
- ¼, ½, 0, 1, 2, ~ *These ¼ ½ are parts of a whole, '0' is a whole number.* (Age 10, verbal)

Analysis of the third response category: ***zero/other digits/fractions***						
Question A ½, ¼, 2, 1, 0	Questionnaire	Task-interview				
Age of children	Age 11	Age 11	Age 10	Age 9	Age 8	Age 7
zero/other digits/fractions	4%	5%	8%	20% (15%)	33% (27%)	44%

Table 9D

The children, whose ordering placed zero before the other digits with the fractions at the end, also used the ***zero before one*** rationale.

- 0, 1, 2, ½, ¼, ~ *0 always goes first, before 1 and then 2.* (Age 7, verbal)
- 0, ¼, 1, ½, 2, ~ *Zero is before 1, but <u>one</u> quarter comes in between.* (Age 9, verbal)
- 0, 1, ¼, ½, 2, ~ *Zero is before 1, then you get the fractions.* [The child then read ¼ as one and a quarter, ½ as one and a half, then 2.] (Age 10, verbal)

Whether, as in table 9C, the children put ***fractions /zero/other digits*** or as in table 9D they put ***zero/other digits/fractions*** a common explanation given was that '0' was to be put before '1'. For clarification of this important point table 9E [4] combines the data where the children placed '0' next to '1'. A comparison shows that a high percentage of children, across the age ranges 7 to 11, placed '0' before '1'.

'0' next to '1' responses						
Question A ½, ¼, 2, 1, 0	Questionnaire	Task-interview				
Age of children	Age 11	Age 11	Age 10	Age 9	Age 8	Age 7
Number of children	100	20	12	20	16	13
Total number of responses	100	20	12	19	15	9
Put <u>012</u> in order	52%	45% (30%)	58%	45% (55%)	40%	56%

Table 9E

[4] Taken from table 9A, responses 3, 4, 5, 6 and 12.

Observing children ordering the cards: ½, ¼, 2, 1, 0, revealed three approaches,

1. Children who put 0 ¼, ½, 1, 2, (*zero/ fractions/ other digits*) saying that '0' was smaller than the fractions and the other digits or that the fractions and other digits were larger than zero.

2. Children who placed the fractions together as a set ¼, ½, and the single digits together as a set 0, 1, 2 and then either put fractions then digits or digits then fractions, the common remark being that '0' goes next to '1'.

3. Children who ordered the fractions ¼, ½, then the single digits 1, 2, resulting in the order ¼, ½, 1, 2 but retained the '0' card and made a remark such as ¼, ½, 1, 2 ~ *is right but I don't know where to put the zero.* A few added that they had ordered numbers and fractions before but had not been asked to *order numbers and fractions with a zero as well.* Most frequently the final decision was to put the '0' next to '1'.

The second set of *ordering numbers* which included fractions was ¾, ¼, 0, ½. Only fractions and zero were used, no other single digits, and when the data were analysed it was not expected to find that the children were influenced by the 0,1, 2 order.

A2) Second fraction ordering - the responses to ordering: ¾, ¼, 0, ½

Fraction card set D

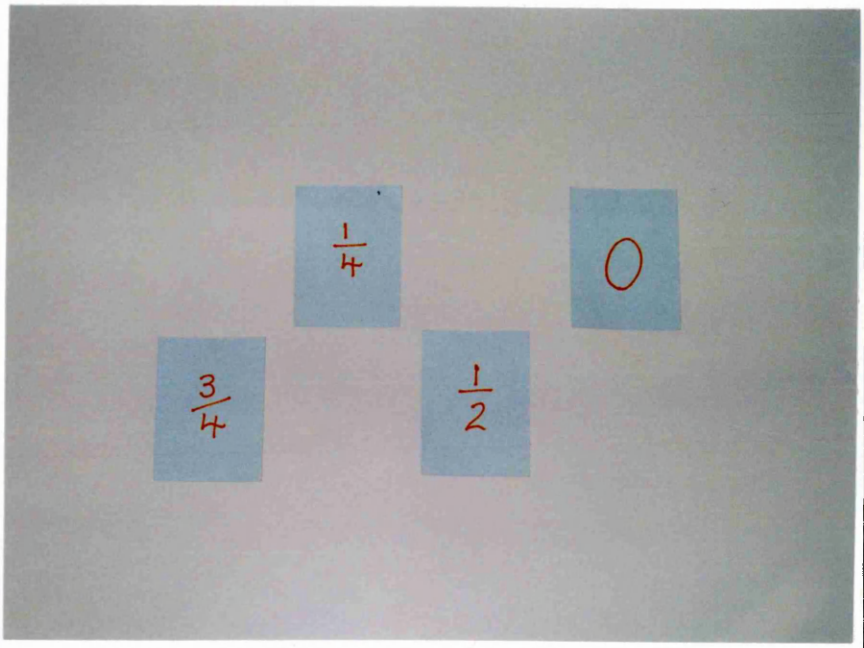

Illustration 11

Question D ¾, ¼, 0, ½		Questionnaire	Task-Interview[5]		
Age of children		Age 11	Age 11	Age 10	Age 9
Number of children		100	20	12	20
Total number of responses		99	20	12	16
zero then fractions					
response 31……	**0 ¼ ½ ¾**	**35**		**13**	**9 (10)**
response 32…..	0 ½ ¼ ¾	5	1		
response 33…..	0 ¼ ¾ ½	3			2 (3)
response 34……	0 ¾ ¼ ½	7	2		
response 35…..	0 ¾ ½ ¼	2			
fractions then zero					
response 36…..	¼ ½ ¾ 0	31	1	4 (6)	3 (2)
response 37…..	½ ¼ ¾ 0	2	1		
response 38…..	¼ ¾ ½ 0	6	1		2 (1)
response 39…..	¾ ¼ ½ 0	5	1	2	
Other					
response 40……	¼ ½ 0 ¾	2			
?		1			
~					

Table 10A

The two frequent response categories were identified as,

- zero then fractions
- fractions then zero

The two frequent response categories				
Question D ¾, ¼, 0, ½	Questionnaire	Task-interview		
Age of children	Age 11	Age 11	Age 10	Age 9
Number of children	100	20	12	20
Total number of responses	100	20	12	16
zero then fractions	53%	80%	50% (33%)	69% (81%)
fractions then zero	44%	20%	50% (67%)	31% (19%)

Table 10BC

Analysis of the first response category: *zero then fractions*				
Question D ¾, ¼, 0, ½	Questionnaire	Task-interview		
Age of children	Age 11	Age 11	Age 10	Age 9
zero then fractions	53%	80%	50% (33%)	69% (81%)

Table 10B

Looking at the children's explanations for the first of these classifications, ***zero then fractions,*** a high proportion of the reasons given by the children involved comparing zero to fractions,

- ❑ *0 ¼ ½ ¾ ~ 0 is lower than all these numbers.* (Age 10, verbal)
- ❑ 0 ¼ ½ ¾ ~ *Zero is nothing,* [Draws the fractions as the fractions of a circle] *With zero you've got none of the circle there.* (Age 9, verbal)
- ❑ 0 ¼ ½ ¾ ~ *Because these ¼ ½ ¾ are more than zero.* (Age 11, verbal)
- ❑ 0 ¼ ½ ¾ ~ *Zero has no parts to it.* (Age 11, verbal)

[5] With the 7 and 8 year old children only one response to ordering, ¾, ¼, 0, ½, was recorded, this was because the children could not read the card showing the fraction ¾. Thus, for this question, data is only available from the age ranges 9 to 11.

The other reasons for putting zero and then the fractions were,

- *Zero always goes first.* (Age 9, verbal)
- *Because 0 is not a number. I'm putting it at the front but it could go at the end.* (Age 10, verbal)
- *Not sure where the zero goes.* (Age 9, verbal)

The children's explanations for the second frequent response category, ***fractions then zero,*** were most revealing.

Analysis of the second response category: ***fractions then zero***				
Question D ¾, ¼, 0, ½	Questionnaire	Task-interview		
Age of children	Age 11	Age 11	Age 10	Age 9
fractions then zero	44%	20%	50% (67%)	31% (19%)

Table 10C

While question **A1** included fractions and single digits (½, ¼, 2, 1, 0) question **A2** contained only simple fractions plus zero (¾, ¼, 0, ½). The reason for this was to remove the temptation for the child to put '0' with other digits. Nevertheless the most common reason given for putting the '0' after the fractions was linked to the **number order**.

- *zero is more than the fractions, it starts the whole numbers, zero, one, two, three ...* (Age 9, verbal)
- *zero is a whole number – like one and two and three.* (Age 11, verbal)
- *¾ is the smallest, then we go bigger up to zero, then we say one, two and we count up.* (Age 10, verbal)
- *zero always goes first. When you have loads of numbers then 0,* [the child pointed to the symbol on the card] *you always write this first. It's like zero, one, two, three, four, five, six, seven, eight, nine, ten. You always say zero first.* [The child put firm emphasis on the word 'first'.] (Age 10, verbal)

The other explanations for placing ***fractions then zero*** made reference to zero being 'nothing'.

- *Nought is just nothing* [puts the zero card after the fractions] – *it could go anywhere.* [Why?] *Because 0 is not a number. I'm putting it at the front but it could go at the end.* [Why?] *Because it is nothing, so it can go anywhere. It doesn't mean anything so it doesn't matter where it goes.* (Age 11, verbal)
- *Not sure where the zero goes.* [Why not?] *I could put it there.* [The child moves the card to the end of the sequence. Could you put it anywhere else?] *Yes, there.* [The child points to the space between two fractions.] O*r there.* [The child points to the space between two other fractions. Why could it go in all these different places?] *It's nothing so it can go anywhere – it's nothing so it makes no difference where you put it.* (Age 10, verbal)
- *Nought is just nothing* [The child puts the zero after the fractions] – *it could go anywhere.* [Why?] *Because it is nothing.* [Here the word 'nothing' was said with great emphasis.] (Age 9, verbal)

These children saw '0', as being nothing, of no importance. Thus it did not matter where it was placed in the sequence or even if it was omitted.

Summary of fraction and zero findings

Generally the reasoning of children who were correct, in that they placed the fractions after '0', involved comparing the size of zero and the fractions. With the children who were incorrect and place '0' in other positions two main themes dominated the children's reasoning; that of keeping the **single digit number order** with '0' next to '1' and the effect caused by **zero being nothing.**

0 ~ 00 ~- 00000 ~- 00 ~ 0

B) Decimals and Zero

Two ordering questions contained decimals: decimals plus zero, .3, .4, 0, .5, .1 **(B1)** and decimals and single digits plus zero, 0.4, 5, 1.2, 8, 0 **(B2)**.

Before looking at the data from these questions the children's references to the '0' or lack of '0' in front of the decimals will be reported first. The two ordering questions containing decimals each had a different focus. Apart from the positioning of the single digit '0' card there was also the inclusion of '0' as a place marker. The first decimal question **(B1)** had simple decimals with no 0-place marker (.3, .4, 0, .5, .1) and the written form '.3' was used, not '0.3'. Here the intention was that would be no other '0' in the decimals being ordered apart from the single digit '0' card. In the second decimal question **(B2)** '.4' was written as '0.4'. The aim in using the two versions of the decimals, with and without the '0', was to see if '0' as a single digit and '0' as a place marker would cause changes in the children's answers and explanations.

As the Task-Interviews progressed the significance in the writing of the decimals as '.1', '.3', rather than '0.1', '0.3' became apparent. When the '0' was seen as part of the decimal then the decimal order had a pattern, a pattern which helped in the placing of the single digit '0' card, a pattern that could be seen to follow the number order, such as:

0, 0.1, 0.2, 0.3... and 0, 1.0, 1.1, 1.2...

So the original decimals (.3, .4, .5, .1) were physically re-written by some children to include '0' (0.3, 0.4, 0.5, 0.1) but the single digit '0' was not altered. However, a number of the children had a different approach. Their solution was to remove the single '0' card and make '.3' into '0.3' by placing the single '0' card in front of each of the decimals in the sequence.

- *You've got no numbers there – in front of the point – you need to have this* [lifts the '0' card] *in front of the point. You don't have zero on its own you have nought point one (0.1).* (Aged 11, verbal)
- *Zero is before 0.1, and 0.2, and 0.3 that's where I'd put the zero.* (Aged 10, verbal)
- *Zero is not actually a whole number so it goes 0.1 0.3 0.4 0.5 , it is the only place it can go as you need a point to separate the decimals and you put zero in front of the point.* (Aged 11, verbal)
- *What do these dots mean?* [Answers himself] *Oh they're decimals. You could put '0' in any space as you say zero point one, say zero point two, say zero point three, say zero point four.* [Could you put it behind the number?] *You can't have .50 with a zero at the end, it is the wrong way round.* [Could you have the zero card on its own?] *Not with decimals.* (Aged 9, verbal)

It was not expected to encounter this problem in the second ordering decimals question (0.4, 5, 1.2, 8, 0) as the decimal was written as '0.4' rather than '.4'. While there were some references made to the '0' in front of the decimal this certainly did not occur as frequently as when the zero was missing as in '.4' in the previous decimal ordering. Though again there were children who would keep the single digit '0' in their hand because there was already a '0' on the decimal '0.4'. One child, ordered 0.4, 1.2, 5, 8 but withheld the '0' card saying,

- *There's already a zero there. Zero goes before 0.4 (zero point four) because it (0.4) has a zero in front.* (Aged 11, verbal)

B1) First decimal ordering - the responses to ordering: .3, .4, 0, .5, .1,

Decimal cards set B

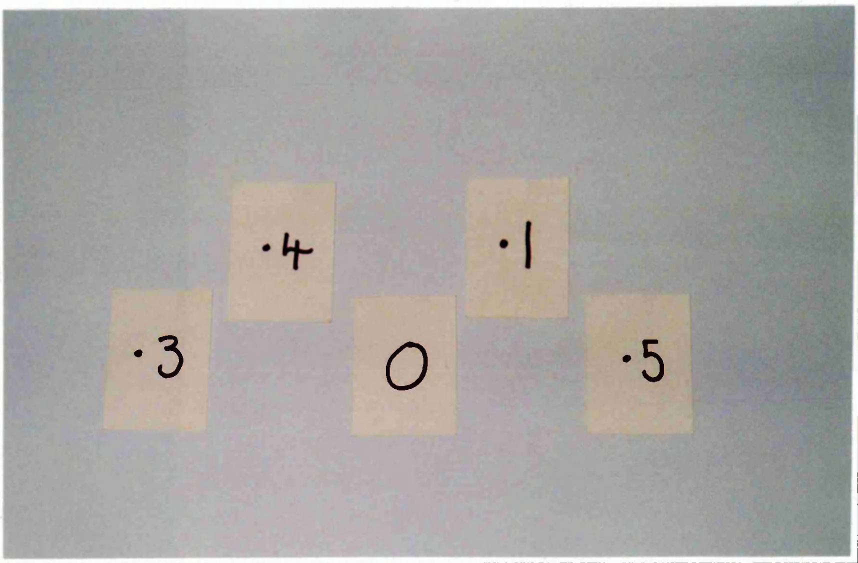

Illustration 12

Question B .3, .4, 0, .5, .1	Questionnaire	Task-interview		
Age of children	Aged 11	Aged 11	Ages 10	Ages 9
Number of children	100	20	12	20
Total number of responses	93	20	10	16
zero then decimals				
response 11..... 0 .1 .3 .4 .5	**45**	**9 (11)**	**6 (8)**	**6 (12)**
response 12..... 0 .5 .4 .3 .1	1			
decimals then zero				
response 13..... 1 .3 .4 .5 0	13	8 (6)	3 (1)	8 (2)
response 14...... .1 .4 .3 .5 0	1			
response 15..... 5 .4 .3 .1 0	26	2	1	(2)
Other				
response 16... –5 –4 –3 –1 0 saw decimal as negative	1			
response 17, zero anywhere between numbers		1		
response 18..... 0 1 3 4 5 – no decimals	5			
response 19...... 1 3 4 5 0 – no decimals	1			
?				
....				

Table 11A

The decimal ordering numbers questions were asked of the 9, 10 and 11 year old children. From the data in table 11A two frequent response categories were identified,

- zero then decimals
- decimals then zero

Both of these categories are considered together as the final answer appeared arbitrary. About 50% of children aged 9 and 30% of children aged 10 began by arranging the decimal as .1 .3 .4 .5. They said that they were sure that was right but didn't know whether to put the zero at the beginning or the end. Their final decision was tentative.

Analysis of the two response categories				
Question B .3 .4 0 .5 .1	Questionnaire	Task-interview		
Age of children	Age 11	Age 11	Age 10	Age 9
Number of children	100	20	12	20
Total number of responses	93	20	10	16
Zero then decimals	50%	45% (55%)	60% (80%)	38% (75%)
Decimals then zero	50% (13%)	50% (40%)	44%	

Table 11BC

Overall about half of the explanations compared the size of zero with the other numbers, though this did not always provide the correct order.

- 0 .1 .3 .4 .5 ~~ *0 is nothing* [The child points to '0' card] *but .1, .3 there is something there.* (Age 9, verbal)
- 0 .1 .3 .4 .5 ~~ *.1, .2 is under zero.* (Age 11, verbal)
- 0 .1 .3 .4 .5 ~~ *As .1 is more than zero.* (Age 10, verbal)
- 0 .1 .3 .4 .5 ~~ *Because zero is the smallest.* (Age 11, verbal)
- .1 .3 .4 .5 0 ~~ *Because these* [The child points to the decimals] *are less than zero.* (Age 9, verbal)

- .5 .4 .3 .1 0 ~~ *0 is smaller than 1, .1 is larger than .5 so zero comes after .1, because .1 is larger than zero.* (Age 10, verbal)

While explanations of the decimal and zero sequence sometimes threw light on a child's understanding of zero it also linked closely with the child's understanding of decimals and his/her view of the number line in decimal form. It proved difficult to differentiate between the two. What was of interest was the reference to zero being **a whole number**, particularly as this was used as the main reason for thinking the decimals were smaller than zero:

- .1 .3 .4 .5 0 ~~ *Because zero is a whole number* (Age 9, verbal)
- .1 .3 .4 .5 0 ~~ *It's points not like the whole, so it has to make the numbers up to get to zero which is not a point but a whole one.* (Age 11, verbal)
- .1 .3 .4 .5 0 ~~ *When you go into points then you go below zero because 0 is a whole number.* (Age 10, verbal)
- .1 .3 .4 .5 0 ~~ *The point should come before zero - if halving one then you get .5 - then you get full numbers.* [What are full numbers?] *Whole ones with no parts, like you have parts of whole ones with fractions.* (Age 11, verbal)
- 0 .1 .3 .4 .5 ~~ *Zero comes first then you have zero point 1 (0.1) then you go up to 0 on its own. The number before the point is a whole one.* [A whole one, what do you mean?] *A whole number like one and two.* (Age 9, verbal)

B2) Second decimal ordering - the responses to ordering: 0.4, 5, 1.2, 8, 0,

Decimal card set E

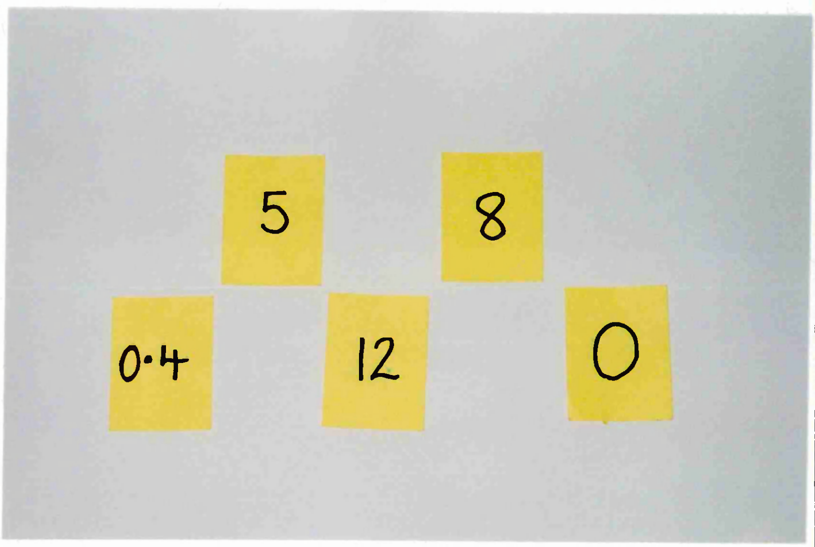

Illustration 13

All the responses to ordering: 0.4, 5, 1.2, 8, 0, are seen in table 12A.

Question E 0.4, 5, 1.2, 8, 0	Questionnaire	Task-interview		
Age of children	Aged 11	Aged 11	Aged 10	Aged 9
Number of children asked	100	20	12	20
Total number of responses	97	20	10	16
zero first in the order				
response 41...... 0 0.4 1.2 5 8	**51**	**12 (11)**	**6**	**12**
response 42..... 0 0.4 5 8 1.2	1			
response 43...... 0 5 8 0.4 1.2	5	1		
response 44...... 0 1.2 0.4 5 8	2			
zero at end of order				
response 46..... 0.4 5 1.2 8 0	1	1		
response 47..... 0.4 5 8 1.2 0	1			
zero infront of the other digits				
response 50...... 0.4 1.2 0 5 8	13		1	2
zero between the decimals				
response 52..... 0.4 0 1.2 5 8	15	6 (7)	3	2
?	1			
~	7			

Table 12A

From the data in table 12A three frequent response categories were identified,

- zero first in the order
- zero between the decimals
- zero in front of the other digits

For ease of comparison these are shown in detail in tabular form in Table 12BCD

The three frequent response categories				
Question E 0.4, 5, 1.2, 8, 0	Questionnaire	Task-interview		
Age of children	Aged 11	Aged 11	Aged 10	Aged 9
Zero first in the order	61%	60% (55%)	60%	75%
Zero between the decimals (response 52)	16%	30% (35%)	30%	13%
Zero in front of other digits 0, 5, 8,	19%	5%	10%	13%

Table 12BCD

Each of these frequent response categories will be analysed in turn.

Analysis of the first frequent response category: ***zero first in the order***				
Question E 0.4, 5, 1.2, 8, 0	Questionnaire	Task-interview		
Age of children	Aged 11	Aged 11	Aged 10	Aged 9
Zero first in the order	61%	60% (55%)	60%	75%

Table 12B

Again, as with the first decimal question (B1) reference to the size of zero or to the other numbers is the main explanation given by the children for their number order. In the majority of cases this resulted in placing zero at the beginning (Table 12B):

- ❑ 0 0.4 1.2 5 8 ~ *Zero is lower than the others, it is smaller.* (Age 9, verbal)
- ❑ 0 0.4 1.2 5 8 ~ *Zero is the lowest, I think.* (Age 9, verbal)
- ❑ 0 0.4 1.2 5 8 ~ *0.4 is more than zero.* (Age 10, verbal)

The greatest surprise was the number of responses that placed zero amongst decimals seen in table 12C, response 52.... 0.4, 0, 1.2, 5, 8,

Analysis of the second frequent response category: *zero between the decimals*				
Question E 0.4, 5, 1.2, 8, 0	Questionnaire	Task-interview		
Age of children	Aged 11	Aged 11	Aged 10	Aged 9
Zero between the decimals (response 52)	16%	30% (35%)	30%	13%

Table 12C

At first this was viewed as the children placing '0' after 0.4, as had occurred in the first decimal ordering. Understanding came as the children read their order for, while the children did read '0, 1.2' as 'zero, one point two' the emphasis was placed so that the numbers became '**zero, one,** point, **two'**. On closer examination it can be seen that the zero had been placed next to the '1' and the '2 ' (giving an order of **0, 1.2**). This, despite the presence of a decimal point, gives the appearance of the number order 0, 1, 2, thus possibly following the normal counting system.

Analysis of the third response category: *zero in front of the other digits*				
Question E 0.4, 5, 1.2, 8, 0	Questionnaire	Task-interview		
Age of children	Aged 11	Aged 11	Aged 10	Aged 9
Zero in front of other digits 0, 5, 8,	19%	5%	10%	13%

Table 12D

The number order is seen again with children who put '0' with 5 and 8 (table 12D), the number order being used as the rationale.

- 0.4, 1.2, 0, 5, 8 ~ *Zero has to go with 5 and 8, like on the number line but there are other numbers missing.* (Age 9, verbal)

Combining the results from table 12C and table 12D illustrates that was a substantial percentage of children who arranged the numbers with reference to the *number order* (whether this was 0, 1.2 or 0, 5, 8). This high percentage applied across these three age-ranges (Table 12E).

Preserving the number line order				
	Questionnaire	Task-interview		
Age of children	Aged 11	Aged 11	Aged 10	Aged 9
Preserving the number line order	34%	35% (35%)	40%	25%

Table 12E

A thought-provoking issue, raised by a number of children, was whether or not zero was **a whole number**. Some children, when arranging the fraction ordering, had also voiced this concern. Returning to check the raw data it was seen that (with the exception of two children) it was not the same children who had voiced their thoughts on this issue in the fraction section. The following explanations are typical of these children's responses.

- [The child puts zero then 0.4, then changes to 0.4 then zero. You don't seem too happy with the answer?] *I don't know if the zero is a whole number or not.* [Which zero? The child points to the '0' card.] (Age 11,verbal)
- [The child places the 0.4 card followed by the zero card, then changes the order to zero then 0.4. Are you unsure?] *I think that's right, it depends if zero is a whole one.* [A whole one? Can zero be one? The child laughs] *I mean a whole number.* (Age 11,verbal)
- [The child orders zero then 0.4, then changes to 0.4 then zero. The child points to the 0.4 card and explains that] *nought point four is smaller than zero.* [Why did you change?] *Well, I thought that zero is a whole number so a decimal must be smaller.* (Age 10,verbal)

Summary of decimal and zero ordering

Generally the children who correctly ordered the two decimal sets of numbers gave an explanation which involved comparing the size of zero and the decimals (this also occurred with the zero and fraction ordering). As was anticipated the findings, from the ordering of decimal and zero, were complicated. While acknowledging that the children's knowledge and understanding of decimals did affect their judgement, interesting aspects involving zero did occur.

There was the inclusion and exclusion of '0' in the writing of the decimals and the dual role of '0' to be considered. The inclusion of zero as in '0.3' aided ordering for a number of children who saw a pattern of 0, 0.2, 0.2, 0.3, ... However, the single digit '0' confused other children who felt that the zero had already been used in the decimals, as in'**0**.3'. When distinguishing between zero '0' as a single digit and zero in the decimal '0.3' no child made any reference to zero and place value, zero as a place marker or zero as being *nothing.*

A theme which dominated the children's reasoning in the fraction ordering was also dominant in the decimal ordering that of zero and its position in the **number order**. A second area of particular interest, which appeared briefly with the fraction ordering but became more pronounced with the decimal ordering, was the children's need to know whether or not zero is a **whole number**.

0 ~ 00 ~~ 00000 ~~ 00 ~ 0

B) Negative Numbers and Zero

The reason for including negative numbers was to collect data on the child's conceptions of zero within the extended number line. These questions were restricted to the children in the age ranges 9, 10 and 11. In line with the NNS recommendations, these children's classrooms had a number line which contained - and + integers, written with the negative integers on the left and the positive integers on the right.

... -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9...

The responses to ordering -9, 0, -7, -1, -4 are found in table 13A

The majority of the children saw '0' as the symbol between the positive and negative numbers and placed '0' at the end of the sequence, the other numbers were ordered as to the child's understanding of negative numbers or to their recall of the extended number line order. Recall was the first strategy. The Task-Interviews were conducted outside of the children's classroom and frequently children would say that they were trying to remember the order.

Negative numbers, cards set C

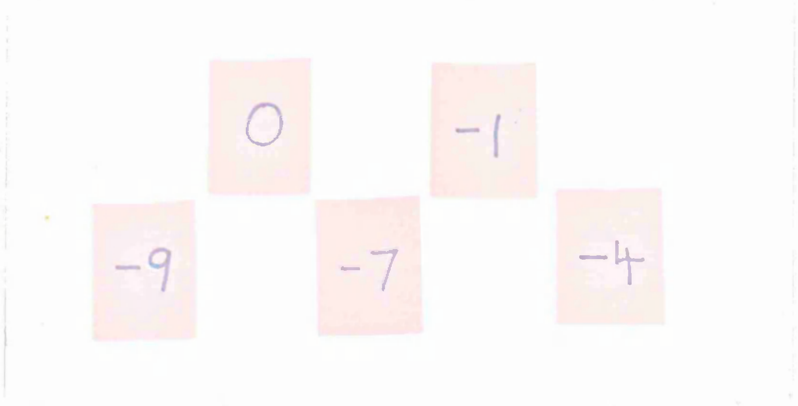

Illustration 14

Question C -9, 0, -7, -1, -4	Questionnaire	Task-interview		
Age of children	Age 11	Age 11	Age 10	Age 9
Number of children	100	20	12	20
Total number of responses	99	20	12	17
negative numbers then zero				
response 21...... -9 -7 -4 -1 0	**80**	**12**	**8 (9)**	**14 (16)**
response 22..... -1 -4 -7 –9 0	6	4 (3)	4 (3)	2 (1)
zero then negative numbers				
response 24...... 0 -1 -4 -7 -9	8	2 (3)		1 (0)
response 25..... 0 -1 -4 -9 -7	1			
response 26..... 0 -9 -7 -4 -1				
zero between negative numbers				
response 27..... -1 -4 0 -9 -7	1			
response 29...... 1 4 7 9 0 with no negative signs	1			
response 30...... zero anywhere between numbers		1		
?	1	1		
----	1			

Table 13A

A review of the responses showed that the majority of the children's explanations included <u>either</u> a reference to the size of '0' or the negative numbers,

- *9 is the highest, -9 is the lowest as you take off the most so zero goes before –1.* (Age 11,verbal)
- *Because 0 is the highest.* (Age 10,verbal)
- *-1 is one less than zero.* (Age 9,verbal)

<u>or</u> a reference to the position of the number on an extended number line,

- *Numbers come back from zero* [points to the left] *instead of forward.* (Age 11,verbal)
- *You get 1 then zero then the minuses.* (Age 11,verbal)
- *When you go into minus numbers then after 1 is zero then you get minus numbers starting with –9.* (Age 10,verbal)

There were just two exceptions from a child aged ten and a child aged eleven who felt that zero could go between any of the negative numbers.[6]

Illustrative incident

As the rationale for both the 10 and 11 year old was the same the incident involving the older child is cited.
[The 11 year old child explained] *Zero can go anywhere between these numbers.* [The child pointed to the cards showing -9, -1, -7, -4. Why can it go anywhere?]
These are minus numbers. Zero will go anywhere so you get zero minus one, zero minus four, zero minus seven, zero minus nine. [Is it possible to explain this a bit more or could you show me what you mean?]
[The child selected pencil and paper from the resources on the table. He explained] *Zero minus one is* [he wrote 0-1], *zero minus four,* [0-4], *zero minus seven,* [0-7], *zero minus nine,* [0-9]. *You can have zero minus any number. You see them in sums.*
It would appear that these two children did not see the – 2 as a negative number but the subtraction sign. This in turn had an impact upon the way that they viewed the role of zero.

While these comments and examples may give some insight into the problems associated with zero, overall the ordering of zero and the negative numbers appeared to add little information to the children's conceptions of zero. The main reason for this being that it was difficult, with any degree of accuracy, to extrapolate the understanding of negativity and the understanding of zero from the data and from the children's explanations.

What was informative was that the majority of children saw '0' as the symbol between the positive and negative numbers.

> Zero is the point of equilibrium … without prior concept of zero negative numbers could not exist. (Guedj 1998, p.80)

[6] These were not the children (referenced on p.79) who said that, as zero was worth nothing, it could go anywhere.

No doubt the format of the extended number line in the classroom influenced the children, all of whom insisted that the negative numbers went on the left of the '0'. No child would accept the order 0, –1, –2, –3, –4 because the right side was reserved for the '*ordinary*', positive numbers. The left, right orientation was seen in the ordering of single digits, the subject of the next section.

0 ~ 00 ~- 00000 ~- 00 ~ 0

D) Single Digit Numbers and Zero

Two number ordering questions which contained only single digit numbers, (**D1** and **D2**) were asked of children aged 5 to 11. **D1** contained some single digit numbers (3, 0, 5, 4, 7). The reason for this selection was to see the children's reaction to the gaps, the missing numbers. **D2** contained all the single digit numbers (0 to 9). The results can be seen in table 14A and table 14B. These have been placed together to aid comparison.

D1 The responses to ordering: 3, 0, 5, 4, 7

Question F 3, 0, 5, 4, 7	Questionnaire	Task-interview						
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6	Aged 5
Number of children	100	20	12	20	16	13	13	18
Total number of responses	96	20	12	20	16	13	13	18
response 0 3 4 5 7	93	20	12	20	16	13	12	16
Missing number confusion							1	2
?	1							
----	3							

Table 14A

D2 The responses to ordering: 8, 5, 7, 1, 0, 4, 3, 2, 9, 6

Question G 8, 5, 7, 1, 0, 4, 3, 2, 9, 6	Questionnaire	Task-interview						
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6	Aged 5
Number of children	100	20	12	20	16	13	13	18
Total number of responses	96	20	12	20	16	13	13	18
response 0 1 2 3 4 5 6 7 8 9	92	20	12	20	16	13	13	18
?	1							
----	3							

Table 14B

Two children aged 5 and one child aged 6, could not order 0, 3, 4, 5, 7 because of the '*missing numbers*'. Otherwise, and not unexpectedly, the ordering of the single digits produced 100% of correct answers[7]. When the children were asked for their reasons for placing zero 0, 3, 4, 5, 7 and 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 the explanations were all connected with the number order,

- *You say zero, one, two, three ...* (Age 6, verbal)
- *It's that order on the number line.* (Age 8, verbal)

[7] This percentage excludes the spoiled answers from the Questionnaire where children missed out numbers in their sequence.

In the Task-Interview sessions, when a child had completed the ordering sequence, each child was asked 'Could you put the zero card in any other place?' With the numbers 0, 3, 4, 5, 7 the answer was always 'no'. However, with all the single digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, some children said that '0' could go after 9 (see table 15A). They explained they had seen this order in places such as on computer keyboards and on telephone keypads.

- *I've seen 1,2,3,4,5,6,7,8,9,0 in lots of places – it's on the computer keyboard.* (Age 10, verbal)
- *Because when we were in Mrs ~ class we had a number line on the wall that had 0 at that end, after the 9.* (Age 9, verbal)

Accepting zero in places other than *'in front of 1'*							
Age of children	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6	Aged 5
Number of children	20	12	20	16	13	13	18
NO	15	9	15	11	9	8	18
YES							
Computer, phones, etc	3	3	4	3	1		
Zero can go anywhere	2		1	1	1	4	
Don't know					2	1	

Table 15A

In table 15A it can be seen that a small number of children felt that '0' could go anywhere. In all but one incident the reason given was centred round zero being worth *nothing.*

Illustrative incident

After ordering the single digits as *0123456789* the children were asked if '0' zero could be put in any other place.
An eleven year old boy replied that *You could leave it out.* [He removed the '0' card.]
Or it could go anywhere.
[He moved the '0' card to different places between the other digits. Why can it go in all these different places?] *Because zero is worth nothing it doesn't matter where it goes or whether you put it in or not.*
[Can you do this with other numbers?] *No, because they're something and zero is nothing.*
(Aged 11, verbal)

Only one child felt that zero could be used anywhere when used as a stop, as a marker between numbers.

- *If you put 9 zero, or 7 zero it could mean a stop. It* [zero] *could be a break so it could go anywhere.* (Aged 6, verbal)

What was unexpected was the outcome of an informal discussion with a member of staff, a teacher in Research School D, who had not taught this child as she had a class of older children. It appeared that this teacher had the same understanding and used the same rationale.

- *For example you can put ¼ ½ 0 1 2 if you are using zero as a marker between the fractions and whole numbers.*(Teacher, verbal)

0 ~ 00 ~ 000 ~ 00000 ~ 000 ~ 00 ~ 0

Chapter Three, Part Three: Discussion points

Zero's position in the ordering of numbers (whether they were fractions and zero, decimals and zero, negative numbers and zero or single digits) created three main areas of tension for the children. Each of these three aspects will be considered in turn.

1) In the area of preserving the number order
2) In the question as to whether zero was a whole number
3) In the understanding of zero to be *nothing*

1. Preserving the number order

The teaching and learning of early number is a complicated, multifaceted task. Part of this is the learning of the number order. This includes the ability to recite the number words, (zero), one, two, three, four five, six, seven, eight, nine, ten ... and to know the order of the number symbols, (0), 1, 2, 3, 4, 5, 6, 7, 8, 9 ...

While the difference between understanding and pure memorisation is widely accepted there are areas where there is a need for deliberate memorisation. How else would one learn how to write one's name or the date of one's birthday or the number order? It is common, in schools, for young children to spend a short time each day reciting the number words in order. Frequently this is done in conjunction with the teacher or child pointing to the number symbols. The intent is to put the word and symbol connection and order into memory. [8]

Most mathematical educators are reluctant to discuss memory when considering mathematical learning and understanding, Gagne (1970) being one of the few exceptions. This researcher's views are in sympathy with those of Morris (1981) that reliance upon memory can have deleterious effects, as reliance upon memory adds significantly to 'mathematics anxiety', especially when memorisation replaces understanding which can lead to confusion. However, Freedmont (1971, cited in Byers and Erlwanger, 1985) goes to one extreme when he describes rote learning as one of the time-honoured enemies of effective mathematics learning. At the other extreme is Krutetskii (1976, cited in Byers and Erlwanger, 1985) who considers mathematical memory to be one of the abilities which distinguish the 'mathematically capable' from the 'mathematically incapable' students.

[8] The reliance on recall is discussed in chapter 4, part 3.

This researcher's experience in education is in keeping with Byers and Erwanger (1985) as they see the discrepancies between theory and practice. They see teachers in mathematics, far from ignoring memory, as being very cognisant with the problems it presents. Classroom teaching and learning of mathematics departs significantly from what theorists have proposed. The main differences may be summed up in two words: repetition and practice.

> The importance of memory for doing mathematics, from the lowest computations to the more sophisticated proofs is almost self-evident. The most crucial question is not whether memory plays a role in understanding mathematics but what it is that is remembered and how it is remembered by those who understand it – as well as those who do not. (Byers and Erwanger 1985, p.261)

When considering children learning the number order and the number symbol sequence then memory is of paramount importance. The activities presented to the child to aid this learning are numerous, ranging from chanting nursery rhymes, playing number games, using computer software, to placing visual material in the classroom. These provide the repetition and practice needed to place the number order and number symbol order into memory. Most children learn the 1 to 10 range of number words by rote with exposure to the sequence of number names and the experience of moderate amounts of sequence production activities (Fuson and Hall in Byers and Erwanger, 1985; Maclellan in Thompson, 1997). As a result many children can recite the number sequence to 100 by the time they are about six years of age (Maclellan, 1997). The learning of the number and order of the number names is dependent upon aural and verbal memory while the learning the number sequence of the number symbols, the number line order, relies upon visual memory. The marrying of the number names and number symbols does require considerable effort 'because nine words and symbols have to be associated, and none of them is predictable from any of the others' (Wigley 1997, p116). It is interesting to note that as Wigley speaks of nine symbols one can only assume that he was not including zero.

As far back as 1883 Galton referred to the visual number line. A hundred years later, Ernest (1983) found that, as a result of a questionnaire given to teacher training college staff, 65% had an internalised number line. All but 5% of these were straight-line number forms, which he deduced were,

> ... possibly stimulated by the greater use of graded rulers and physical number lines since Galton's time. (Ernest in Thompson 1990, p116)

Almost two decades later most schools are following the NNS recommendation that each classroom display the number symbols in order, including the zero number symbol (chapter 3, part 1). Also many computer software programmes now display the number symbol order in their mathematics programmes[9]. The different types of number displays used in the school (chapter 3, part 1) did not appear to affect a child's answers in the ordering of the number cards. Possibly this was due to the fact that all the classroom displays included '0' and thus the children's visual memory included '0' in the number symbol order.

While the task-interviews were conducted in a room other than the classroom each classroom did have the number order on the wall, all began with 0 and there was a left to right ordering:

0 1 2 3 4 5 6 7 8 9...

Reading and writing, in European languages, moves in a left-right direction. It is natural that when working with children who are learning to read and write that the left-right movement is emphasised. Thus it would appear to be the norm, verging on universal convention, that the number line is written so the eye movement follows the left to right rule.

The children in this section of study were very confident in knowing the oral number order zero, one, two, three, to nine and the number symbol order 0, 1, 2, 3, 4, to 9 to such an extent that they did not like the number order to be disturbed. When the researcher ordered the digits as 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, the children would not accept this change of direction.

- *You must have it going this way.* [The child rearranged the cards to the left-right order.] (Age 7, verbal)
- *Zero is the smallest and you go from the smallest to the largest, that way* [The child pointed from left to right.] (Age 8, verbal)

The order had to have a left-right direction and zero '0' went next to one '1'. An interesting observation in the Task-Interview, was that many of the 5 year old children and some of 6 year old children, when asked to order the cards 5,7,0,4,3, put: 0 - - 3 4 5 - 7 explaining that they were leaving spaces for the 1, 2 and 6.

As was seen in the ordering of *fractions and zero* and *decimals and zero* the numbers were often placed in such a way as to preserve the single digit number order. In this situation there appeared to be a conflict with the child's security in the memory of the number order and their logical reasoning based on the size of the numbers being ordered. In these situations children would amend their reasoning and the number order would take precedence.

[9] This was observed by the researcher in her involvement with the evaluating and designing of computer material for children in both Key Stage One and Key Stage Two.

- [The child orders 0, ¼, ½ then stops. The researcher asks if there is a problem.] *That means quarter and half are bigger than nothing.* [The child looks at the two cards, 1 and 2, in her hand. She then changes the order to ¼, ½, 0 followed by 1, 2.] *You have to put zero with one and two.* (Age 8, verbal)

Once the association and predictability of the number and symbol order are in place then it would appear difficult to alter.

2. The phrase "whole number" as applied to zero

Children enjoy classifying objects, names, etc and numbers. Many of the children were keen to say whether zero was or was not in the set of even numbers, though this took them no nearer to finding answers to where zero was when ordering the number sets given, they just wanted to make the statement. Similarly they wanted to know if zero was, or was not a whole number but this question had a purpose for the information help their reasoning as to where to place zero in relationship to the other numbers in the sets.

Children, throughout the ordering tasks, raised the question as to whether or not zero was a *whole number*. The problem appeared more acute when ordering zero with fractions. The whole number, which was seen as 'the large number in the front of a fraction', helped to order mixed fractions so that the 1 in 1½ placed this between 1 and 2. But ½ and ¼ were not written as 0½, which would indicate that they went between 0 and 1.

To try to gain further insight into children's understanding of *whole numbers* the Y6 task-interview children (aged 11) were put into four random groupings of four/five children for an informal discussion. The debate centred round how one recognised whole numbers, where they were to be found and what their attributes were.

The following are statements which represent the commonly agreed opinions of these 11 year old children.

What are whole numbers?

- *Full ones, not part of something like fractions.*
- *Like the numbers on a number line ... but negative numbers are not whole numbers ... because the minus sign meant that they are less than zero.*

The researcher asked about decimals and gave 3.42 as an example. The children referred to this in terms of money.

- *With £3.42 you have 3 pounds and 42 pence and these are whole things. It's the same with metres* [remarked another child] *it would mean metres and centimetres.*

The discussion moved on to the topic of fractions and whole numbers as the children compared '0' with other numbers known, by them, to be whole numbers.

- *Whole numbers are full numbers not bits like fractions.*
- *Whole numbers appear in front of fractions.*
- *You write it big in front of the fraction like one and a half, or three and a half.* [This child wrote 1½ or 3½.] *But you don't write zero and a half.* [The child wrote 0½, the other children laughed.]

The researcher asked why you didn't write zero and a half? After a pause the responses were:

- *You just don't. It's silly.*
- *You can cut whole numbers into pieces but you can't do that with zero. You can't cut nothing.*
- *You can't have half of nothing*

Here we have examples of children joining pieces of knowledge and applying them to a new situation. Within each of the groups the children's reasoning was in the practical, concrete rather than in the abstract. Their explanations were logical within the limits of their knowledge, within the conventions they had learned, including the way to say, to do, to write and to order numbers. At the end of this discussion the consensus of opinion, of these children, was that zero is not a whole number.

Is zero a whole number? The answer to this question would appear to be hazy. While the phrase 'whole number' is in common usage the precise language of the mathematician and the everyday language of mathematics could be seen to be at variance. Indeed, within the world of mathematics there is conflict in the use of this terminology; whether one should be using the term 'whole number' and if so what it means. Wolfram (Accessed 2004) illustrates the lack of conformity:

> Whole numbers is one of the numbers 1, 2, 3, ... (Sloane's AOOOO27). '0' is sometimes included in the list of 'whole' numbers (Bourbaki 1968; Halmos 1974) but there seems to be no general agreement. Some authors also interpret 'whole number' to mean a number having 'fractional part of zero,' making the whole numbers equivalent to the integers.
>
> (Wolfram, Accessed 2004, p.1)

Most sources seemed to agree that zero is accepted as a whole number though their reasoning varied.

- Whole numbers – this group has all the Natural Numbers in it plus the number 0. (Zona. Types of Numbers, Accessed 2004, p.3)
- ... the whole numbers arise from the need to be able to subtract any two numbers ... (University of Chicago, Newton BBS, (Accessed 2004, p.1)
- Whole numbers (the counting numbers and 0) (Annenberg. Learning Math, Number Sets, Accessed 2004, p.1)

Illustrative incident

The following excerpts are from two teachers responding to the topic of '0 as a number' on The Mathematics Forum web site (Accessed 2004). They give a flavour of how the understanding and use of the term *whole numbers* adds confusion and may have an affect on classroom practice.

1. To be honest, I never learned a nice way to think of the whole numbers, but I think the best way is to think of them as 'quantity numbers'. Someone could ask 'how many pork chops do you have?' and then you could answer with any 'counting number,' or if you don't have any you tell them 'zero'. The set of numbers you can use to answer this kind of question is the set of whole numbers.

2. I tell my kids that Zero has many meanings: Behind whole numbers it [zero] increases the whole number quantity by one place value for each zero'

The first teacher appears to include zero under the umbrella of whole numbers while the second teacher appears to exclude zero.

(The Mathematics Forum, Dr Mathematics p.2, Accessed 2004)

There is also the added dimension of the everyday connotations of completeness the phrase 'whole number' implies and one that would, surely, preclude negative numbers as being part of the *whole number* set. One may ask with a child's logic, indeed with adult logic, how –3 can be considered 'whole' when it is not there. This abstractness can be extended further to apply to zero. How can a non-entity be 'whole'?

The generally accepted reasoning seems to be that integers are *whole numbers* and as zero is an integer then it is a *whole number.* As Haylock and Cockburn succinctly state,

> The set of integers comprises all whole numbers, positive, negative or zero:
> (Haylock and Cockburn, 1989, p.32)

Maor illustrates using the example of an extended number line,

> If we add to the set of natural numbers the notion of direction, we get the set of integers, or whole numbers. These are made up of the natural numbers, their negatives, and the number zero. (Maor 1987, p. 41)

Again, as in the discussion in chapter 2, part 2 on '*What is a number?*' this researcher asks whether the underlying problem stems from a lack of standard terminology. Or, whether the causation might be in the opposite direction. Are the problems themselves the reasons for the lack of standard terminology?

3. Understanding zero to be *nothing*

During the initial *warm-up* conversation with a young child she very excitedly told the researcher about the new baby boy in the family. A little later when preparing to start the Task-Interview the researcher asked the little girl how old she was, she answered, *Five but our new baby is zero because he's just been born.*

All the children in this study were very familiar with the word *zero.* In the Number Ordering Task-Interviews when a child had completed a sequence he/she was asked to read their number order. As table 15B shows there was almost unanimous use of *zero.*[10]

Reading of '0' in the number order sequences							
Age of children	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6	Aged 5
Number of responses	20	11	20	16	12	10	11
Zero		11 100%	20 100%	16 100%	12 100%	9 90%	11 100%
Mixed						1 10%	

Table 15B

It was during their explanations for their number orders that the word *nothing* was used. It is common for zero to be described as *nothing* (see chapter 6 for details on the 'Language of Zero'). However, some children took this understanding of *nothing* to a different level.

Illustrative incident

In a school in Yorkshire a boy of six was arranging a set of single digit number cards in order. He put 1 2 3 4 5 6 7 8 9 but kept the '0' in his hand. The teacher asked where this card should go. The child held the '0' card between 4 and 5, but kept hold of the card. The teacher asked if he wanted the card to stay there. The child moved the card so that it was between 7 and 8 but again kept hold of the card. When the teacher asked if he was happy to leave the card there the child again moved the card, this time between 2 and 3. The teacher asked the boy where he wanted to put the card.
The child replied, *It can go between 4 and 5, or 7 and 8, or 2 and 3 or any of the numbers because it stands for 'nowt' and there is 'nowt' between 4 and 5, or 2 and 3, or 7 and 8. There's 'nowt' between any of these numbers, it can go anywhere.*
('Nowt' is the Yorkshire vernacular for nothing.)

This incident illustrates the reasoning that as there is nothing between the numbers 1,2,3,4,5,6,7, 8,9, then zero, which represents *nothing*, can be placed between any of the numbers. A strong statement, that one can put the '0' symbol in these places, as '0' is the symbol that represents *nothing*. In her experience this researcher had found the incident described, while not common,

[10] Not every children read all the number sequences, for example the younger children did not recognise decimal or negative numbers.

did occur with younger children. Even so, it was surprising to find children aged eleven still holding the same opinion (table 15A). Was this connected with the image of the number order discussed earlier in this chapter? Was it that these children's conceptual image was that of the number bar and discrete data rather than the number line and continuous data?

This chapter has concentrated on the children's ordering of number symbols. However, in the explanations of their number orders children sometimes described the value of zero. Words and phrases such as *nothing, not worth anything,* or indeed *nowt* were used. The train of thought of a few children was reminiscent of the riddle of assumptions – a bird has wings, a bird can fly, a penguin has wings so a penguin can fly. This transferred to zero as - zero is nothing, nothing is worthless and of no value, if it is worthless it is of no significance, if it is of no significance it has no effect, if it has no effect then it can be ignored. The concept that zero is of *no significance* was seen in the ordering of single digit numbers and with ordering fractions and zero. To quote Rotman, zero serves as,

> ... the site of an ambiguity between an empty character . . . and a character for emptiness: a symbol that signifies nothing. (Rotman, 1993, p26)

The next chapter moves from dealing with zero as a symbol on a number line to focus on the zero number facts where children are dealing with amounts. Here the conception of zero as *nothing* may become a higher profile feature.

0 ~ 00 ~ 000 ~ 00000 ~ 000 ~ 00 ~ 0

CHAPTER 4

THE ZERO NUMBER FACTS

> … for zero to be a power of equal status with other numbers we must first understand how to add, subtract, multiply, divide with it for a start. (Kaplan 1999, pp.70-71)

Outline of Chapter 4

Part 1 begins with a brief history of zero in equations leading to a discussion of the relevance of the zero number facts in today's classroom. The design and delivery of the zero number fact Task-Interview and the collation and analysis of the data are considered in detail. Part 2 presents the quantitative and qualitative data, for addition, subtraction, multiplication and division. These are reviewed separately followed by a summary of the findings for that specific operation. Part 3 reflects on the four main areas of tension, which arose from the zero number facts analysis. Part 4 looks at the wider picture of the zero problems in algorithms.

0 ~ 00 ~ 000 ~ 00 ~ 0

Chapter One, Part One: The number facts and zero

Zero in equations - a brief history

Chapter Two presented a condensed history of the emergence of the concept of zero and its acceptance as a number. Compared with the other single digits, zero came into our number system quite recently so it is to be expected that zero in equations has a relatively short history. Mathematicians in seventh-century India reasoned that since zero is a number it followed that number operations could be performed with it. However, it was apparent that different problems arose when one tried to consider zero's interaction within the operations of arithmetic - addition, subtraction, multiplication and division. Three Indian mathematicians Brahmagupta, Mahavira and Bhaskara tried to address these problems. In the seventh century AD, Brahmagupta (b 598 AD) attempted to give the rules for arithmetic involving zero and negative numbers. He gave the following rules for addition,

> The sum of zero and a negative number is negative, the sum of a positive number and zero is positive, the sum of zero and zero is zero. (A History of Zero, MCS. accessed 2002)

With regard to subtraction he explained that if you subtract a number from itself you obtain zero. He did appreciate that subtraction gets more complicated when negative numbers and zero are involved.

> A negative number subtracted from zero is positive, a positive number subtracted from zero is negative, zero subtracted from a negative number is negative, zero subtracted from a positive number is positive, zero subtracted from zero is zero.
> (A History of Zero, MCS. accessed 2002)

Brahmagupta moved on to multiplication and division. He stated that any number when multiplied by zero is zero, but he struggled when it came to division. He asserted that,

> $0 \div 0 = 1$. He did not see the logical complications produced by this assertion. However, he did seem aware that the division of a nonzero number by zero was a touchy matter, because he did not offer any comment or possible values for $a \div 0$ when it does not equal zero. (Smith, 1996)

While today we know that his choice led to inconsistencies with the established rules of arithmetic, it must be acknowledged that, from the first known person to try to extend arithmetic to include zero, it was a brilliant attempt.

In 830AD, around 200 years after Brahmagupta, Mahavira wrote *Ganita Sara Samgraha*, designed as an updating of Brahmagupta's work but, in the main, it reaffirmed Brahmagupta's findings. However, it is seen that, 500 years after Brahmagupta, Bhaskara (1114-1185AD), the leading Indian mathematician of the twelfth century was still struggling to explain division by zero. His text *Vija-Ganita* (c 1150) was the first to suggest that $a \div 0$ is infinite (assuming a does not equal 0). At first sight it is tempting to believe that Bhaskara was correct but if this were true then 0 times infinity must be equal to every number n, thus making all numbers equal. It may be that the Indian mathematicians could not bring themselves to the point of admitting that one could not divide by zero.

The brilliant work of the Indian mathematicians travelled westward to the Islamic and Arabic mathematicians. However, it was not until the 12th century, when Ibn Ezra wrote three treatises on numbers, that the Indian symbols and ideas of decimal fractions came to the attention of some of the learned people in Europe. As has been noted, in chapter 2, part 1, the Italian mathematician Fibonacci was the central figure in bringing these new number ideas to the attention of Europeans. He was an important link between the Hindu-Arabic number system and the European mathematics, though it is significant that he hesitates to address '0' as he does '1, 2, 3, 4, 5, 6, 7, 8, 9' for he speaks of the 'sign' zero and not of the number zero. From his treatment of zero he had not embraced the sophistication of the Indians Brahmagupta, Mahavira and Bhaskara nor of the Arabic and Islamic mathematicians such as Ibn Ezra and al-Samawal.

One might have thought that the progress of the number systems in general, and zero in particular, would have been steady from this time onward. This was far from the case. Cardan solved cubic and quadratic equations without using zero. He would have found his work in the 1500s so much easier if he had had a zero but it was not part of his mathematics. By the 1600s zero began to come into widespread use but still only after encountering a lot of resistance.

The problems encountered by the great eastern mathematicians, in defining the outcomes when zero is used in number operations, are mirrored in the difficulties found in classroom arithmetic (Suydam and Dessart, 1978; Lappan, 1987; Guedj1996). This researcher's experience points to some children, Initial Teacher Education students and primary school staff having problems, while others showed insecurity, when working with zero in simple arithmetic situations. She recalls the following illustrative incident which occurred early in her teaching career.

Illustrative incident

A new boy joined the class of 10 year olds. Alan was capable of complicated calculations but made occasional errors. On initial analysis it appeared that the errors involved zero. In order to investigate the problem further Alan was given work to take home. This work consisted of simple algorithms and number facts all with a zero element. The next day Alan's mother arrived and demanded to know why Alan had been given such simple sums to do and that he found them so easy that he'd finished them in ten minutes. The mother was asked if the work was correct? She had assumed it was but had not checked. It was suggested that they marked the sums together. Over 70% were incorrect. Alan responded by saying that he sometimes got mixed up when there were nothings in the sums. The mother responded by demanding that her son had more such work to do at home.

The number facts - relevance in today's classroom

Knowing the number facts is particularly relevant in today's classroom, since the two major documents, the National Curriculum for Mathematics and the National Numeracy Strategy brought 'mental arithmetic' back in fashion for children aged 5 to 11. With the arrival of the first National Curriculum (1988) there were considered to be three approaches to dealing with a calculation: first mentally, then resorting to using pencil and paper and finally using a calculator. The reason why this emphasis on mental calculations found in the NC did not materialise in practice was, according to Thompson,

> ... because of subject knowledge demands of the curriculum, mental arithmetic did not receive the emphasis that it deserved. (Thompson 1999, p.146)

This stress on mental methods was seen again in 1996 with the National Numeracy Project and continued with the National Numeracy Strategy (1999). The latter not only brought mental calculations into the primary classrooms but also kept them high profile, the result being that 'mental methods' have become one of the most important components in the mathematics lesson.

Given that you cannot calculate unless you have something to calculate with, the phrase 'mental calculation' was seen to encapsulate the two important aspects of mental work, namely, recall and strategic methods (Thompson 1999, p.147). Mental recall is the 'knowing', the recalling a fact 'learned by heart', or being able to work out very quickly specific number bonds or tables facts. 'Known facts' then become the basis for mental strategies as these known facts are used to work out unknown facts.

It is the 'known facts' which are of interest in this study. Some number facts need to be *remembered* so that there is instant recall without hesitation. The answers given to these number facts and their use in mental strategies rely on the accuracy of mental recall. What constitutes the 'number facts'?

> These will include addition and subtraction facts as well as multiplication and division facts. ... simple number operations, two single digits – addition number facts (also known as addition bond) such as 3 + 5, 7 + 2; the inverse operation of subtraction number facts (bonds) 5 –3, 7 – 2; multiplication facts 5 × 3, 2 × 4 (up to 10 × 10); the inverse operation of division and the division facts 8 ÷ 4, 3 ÷ 3. (Harris and Spooner 2000, p.4)

Included in these number facts are those that contain the number zero; these are the focus of this chapter.

Zero number facts: fieldwork

The aims of the research were to explore the conceptions of children, aged 3 to 11, within,

1. The empty set
2. Zero as a number and its relationship to other numbers
3. The zero number facts
4. The language of zero

This chapter is concerned with the third element, focusing on children's mental recall of the eight number operations (a +0, 0 + a, a – 0, 0 - a, a × 0, 0 × a, a ÷ 0, 0 ÷ a). These will be referred to as '*the zero number facts*'.

Each class teacher was asked if the children, as a whole, would be capable of answering simple number facts using numbers up to 5. The zero facts were not mentioned specifically as the staff were not informed of the details of this research so as not to alter their teaching. All the teachers of the children in the age range 6 to 11 were happy for their children to be working with addition and subtraction of single digit numbers with multiplication and division being restricted to the more mathematically able or to the older children. Decisions then had to be made as to the content, format and order presentation of the zero number facts.

- ***Selection of the single digit for 'a'***

The selecting of the number for 'a' needed careful consideration, it would be kept constant and so used in all the *zero number facts*. As was discussed earlier in this study (the section on 'Aspects of teaching and learning early number', chapter 2, part 2,) there are many skills required in understanding single digits and children acquire these skills with the numbers 1 to 5 before 6 to 9. In this research the zero facts were being used across a wide age and ability range and equations with smaller digits would be more inclined to receive an answer. The researcher wanted the children to feel comfortable with using the number selected, for that number symbol to be familiar. According to Geldman and Gallistel (1978, p.51), when asked to reason about a large set the young child does poorly. In contrast, they say, a set of three objects does lead to numerical reasoning in the 3 year old as the child at this age is able to obtain an accurate numerical presentation of small sets. The decision was made to use 3 for 'a'.

- ***Selection of the zero number fact format***

The selection between the horizontal or vertical format had to be made. The horizontal format, particularly when using single digit number bonds, is widely used with younger children, possibly as the left to right style 3+4= links with the reading of text. As Hughes et al suggest, the horizontal format of the equation precedes the vertical,

> When automaticity is achieved in the horizontal, pupils might advance to the vertical.
> (Hughes et al 2000, p.11)

It was felt that the horizontal format would be acceptable to the wide variety of age ranges and ability ranges. No child in the research appeared to have any difficulty in using this format.

- ***Selection of the zero number facts presentation order***

The number facts were presented in the following order,

1) a +0	2) 0 + a	3) a - 0	4) 0 - a
5) a × 0	6) 0 × a	7) a ÷ 0	8) 0 ÷ a

The order was selected as it was seen as the usual order a child would learn the four rules of number, that is moving from addition, to subtraction, then multiplication and on to division.

> All complex procedures in mathematics can be analysed down to the four basic rules of number. First basic addition bonds - the so called number bonds - to 10, subtraction bonds are then learned ... (Hughes et al 2000, p.11)

This order meant it would be less likely for a child to be overawed, by being presented with a number fact equation containing one of the four rules that they had not met in class. Also, if a child was having problems, it was easier for the researcher to decide not to proceed with subsequent questions.

Zero number facts: data collection methods

The zero number facts were included in the Questionnaire. The children were asked to, 'Write in your answers to the following sums and explain how you got your answer' (appendix 1). The same zero number facts, in the same format and order were offered to the children in the Task-Interview situation with each of the zero facts being written on a piece of card (illustration 15).

Task-Interview: The set of zero number fact cards

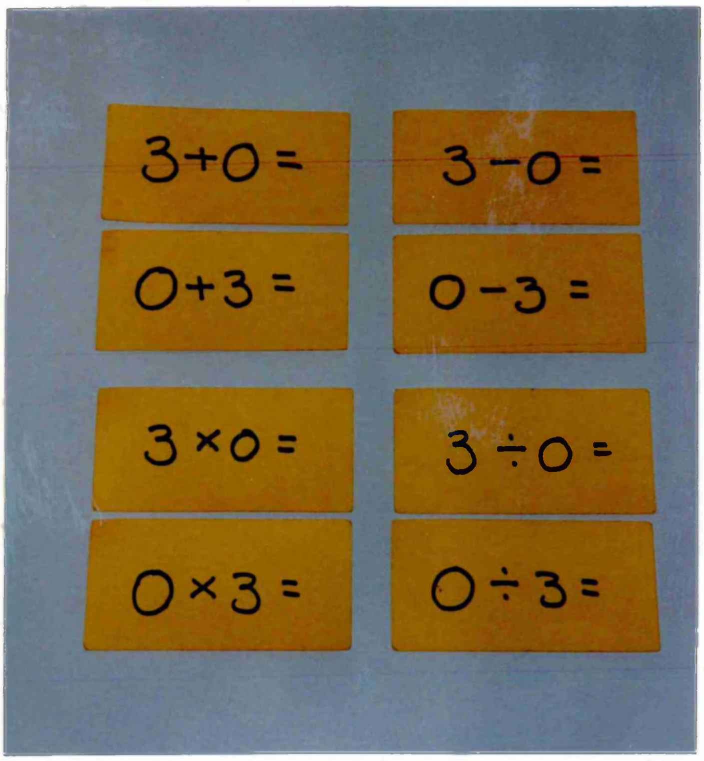

Illustration 15

There was a set presentation procedure, used in the Task-Interviews, which was followed for each of the zero number facts.

Task-Interview - zero facts presentation procedure

1. The child was shown a card and asked to ***'Please read what it says on the card'.*** There were a number of reasons for asking the child to do this:
   - To see that the child was not misreading the operation, if a child misread + for – then he/she would be asked to check what he/she had read. Roberts (1968) and Cox (1975), in their work on children's mistakes in algorithm and number bonds, found that 20% of errors were due to children applying the wrong operation. Also, this gave the researcher the opportunity to close the section of questions when the child's knowledge of the four rules appeared to have been reached.
   - To check that the child was answering the zero number fact on the card, for example not reading 0 +3 for 3 + 0.
   - To note the language the child used to read '0' in the zero number fact (this information will be used later in this study in chapter 6, The Language of Zero).
2. The child was asked ***'What is the answer?'***
3. The child was asked ***'What makes you say that the answer is...?'*** Whatever the answer, correct or not, the child was asked for an explanation.
4. There was a collection of resources on the table: a number line from 0 to 10, a box of cubes, a box of counters, pencil and paper. A scenario was presented to the child that, *My friend's little boy/girl doesn't know how to do that...* [the researcher would point to the specific zero fact number card]. ***How would you show him/her how to do it?*** The children were encouraged to use any of the resources. Here the intent was to see how the child would represent *zero* and for the researcher to gain further insight into the child's understanding of how he/she had arrived at each zero number facts answer.

The researcher was aware that when children are asked to give a reason for their answer they assume this to be 'a hint' that the answer was incorrect thus the answer is then changed. Surprisingly this did not occur, possibly because of the time taken with each child in the pre-task *setting the scene* (chapter 1, part 5). Throughout the task the children became accustomed to being asked for explanations, so much so that many explanations were offered before being requested. The children quickly got used to reading a zero number fact from the card, giving their answer, explaining how they knew the answer and illustrating it. While being systematic there was also informality as the researcher asked questions in order to gain clarification.

Zero number facts: collation and analysis of the data

The information collected was in the form of quantitative and qualitative data.

1. Quantitative data

The quantitative data were the answers to the zero number facts. Here the classifications were determined by the children's answers. In order to assist the comparison of data between the Questionnaire (collected from children aged 11 within five schools) and the Task-Interviews (conducted with children aged 6 to 11 in Research School A) the quantitative data are presented in tabular form. In keeping with the methodology decisions (chapter 1, part 7) the children's first answers from the Questionnaire were used; in the Task-Interviews the child was asked for the answer that he/she wished to be recorded, though changes in answers were noted and reported on as appropriate.

2. Qualitative data

In catagorising the qualitative data, the children's explanations of their answers to the zero number facts, proved a lengthy process and quite an ordeal. Categorisation began with the raw data from the Questionnaire and the children's explanations for their answers to the first addition zero number fact (3 + 0). A search was made of the raw data for common words or phrases. These were used as the basis for the classification headings. As each of the other seven zero number facts was examined further common words and phrases were added. The result was unmanageable. The categories were reassessed for duplication but this had little effect (examples of these initial classifications are to be found in appendix 6). A third attempt, by combining and refining headings, was undertaken. Though cumbersome this was an improvement and, initially, these headings, from the Questionnaire, were used to undertake the first categorisation of the Task-Interview.

However, by the nature of the Task-Interview data collection method, which encouraged compound answers, the Task-Interview explanations were more complex with some responses fitting into two categories. The ability to compare data across age ranges was losing its clarity. A long reflective time was needed. Further thought resulted in an approach being made from a different direction. Instead of using narrow categories, wider categories were used; if necessary these could contain subcategories. Categories with no zero association could be set aside. The outcome was the required rigor within a simpler system. Using these wider categories the data from different age ranges and from different number operations could more readily be compared and analysed.

The categories used for the children's explanations for their *zero number facts* answers are itemised in list 5. This is followed by an explanation of each category.

Categories for the zero number facts - explanations
a) gave no explanation *b) reiterated the question* *c) made a statement of fact* *d) used the number order* *e) related to a paired zero number fact* *f) used a derived fact* *g) contained a zero connection*

List 5

a) *Gave no explanation (Questionnaire only)*

The child provided the answer to the zero number fact but gave no reason for their answer.

***b) Reiterated the question** (Questionnaire only)*

A considerable number of children in the Questionnaire wrote, as their explanation, the zero number fact in number symbols or in words. For example for $3 + 0 = 3$ as,

- *three and zero is three* (Aged 11, written)
- *three and zero make three* (Aged 11, written)
- *3 + 0 = 3* (Aged 11, written)
- *three and zero together equal 3* (Aged 11, written)

In the Task-Interview a small number of children also repeated the zero number fact but placed the emphasis in such a way - *three and zero __is three__* - that they were making a statement of fact. This could also have applied to the Questionnaire explanations but as there was no means of knowing this to be true, or not, the information was included under a separate heading of 'reiterates the question'.

***c) A statement of fact** (Task-Interview only)*

These were children who were sure of their answers but who offered no explanation except to say *it is* or *I know*.

- *It just is - three and zero equals three.* (Aged 6, verbal)
- *I just know. I learned it.* (Aged 7, verbal)
- *I was told. I know. If you have zero and three then the answer is zero.* (Aged 8, verbal)
- *Every body knows that when you put three with zero you get three.* (Aged 10, verbal)

Included in this category were the children who repeated the question with an emphasis or inflection in their voice,

- *Three and zero, three.* [How do you know?] *Three and zero __is three.__* (Aged 7, verbal)
- *Three and zero __make three.__* (Aged 8, verbal)

In the above three categories - *gave no explanation, reiterated the question, made a statement of fact* - an answer was given but there was no rationale provided to underpin the answer. It was decided that they contained no further information as far as this study is concerned.

d) Number order

The children used a number order/number line explanation that involved counting forward or backward from zero, using zero as the starting point.

e) Relates to the paired zero number fact

This category was used when a child made reference to a previous answer which was the paired number fact, such as 3×0 and 0×3.

- [3 + 0 compared with 0 + 3] *It's the other way round.* (Aged 11, written)
- *Three add zero gives the same answer as zero add three.* (Aged 8, verbal)

Possibly some children were using the commutative concept while others saw this as matching the symbols 3+0 and 0+3 as was clearly demonstrated by a small number of children who rotated the second card and placed it under the first card to illustrate the connection (illustration 16).

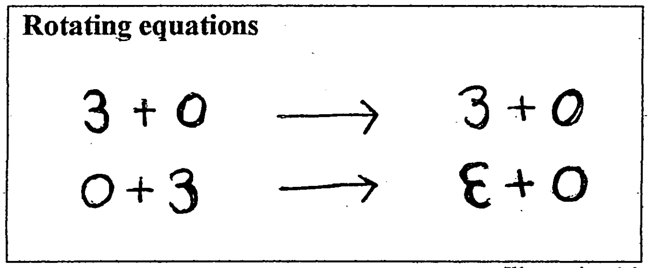

Illustration 16

Children who appreciated the connection between 3+0 and 0+3 were successful if the base answer was correct; if not, then the base error was compounded hence 3 + 0 = 0 became 0 + 3 = 0. Frequently children assumed this paired zero number fact connection in all four operations.

f) A derived fact

The child used a known fact, other than the commutative law, to explain the answer to the zero fact.

- 3 × 0 = 0, *Well, three times one equals three and it can't be the same so must be zero.* (Aged 9, verbal)

So many of the children's explanations were thought provoking, they made one yearn to have time to pursue further each section. But the aim of the study, to explore the children's conceptions of zero, had to take priority. It was considered that the explanations in the three categories, labelled d) *the number order,* e) *relates to the paired zero number fact,* f) *derived fact,* are methods a child was likely to use whether it was a basic number fact or a zero number fact. Hence these will not be discussed in any detail in the analysis with the exception of a very small minority that did have a zero content and these will be highlighted at the appropriate point. While the above categories, a) to f), may not have a zero connection they have been included in all the data tables so that the reader may set in context those explanations which did have a zero connection and those which did not.

g) A zero connection

In this category were placed explanations which contained a direct connection to zero, where zero affected the child's approach to and treatment of the number fact. This category is displayed in the tables, with the other main categories,

(a) *gave no explanation*
(b) *reiterated the question*
(c) *made a statement of fact*
(d) *used the number order*
(e) *related to a paired zero number fact*
(f) *used a derived fact*
(g) *contained a zero connection*

This final section, *a zero connection*, is sub-divided into three headings,

i) *Confused*
ii) The effect of nothing
iii) The zero symbol.

There follows an explanation of these sub-headings.

i) Confused. In a number of instances a child said he/she was confused because of the zero, in other instances it was difficult to understand the explanation except to appreciate that the confusion was a direct result of the equation containing a zero.

- 0 + 3 = 0, *I would say 3 or zero. If you get three on your fingers and if one then that makes four. But not one it's zero, then it's three. Could be both. Answer is three or both. There's a zero there. I say it's zero.* (Aged 7, verbal)

The remaining information, in the *zero connection,* at first appeared to be a single classification; that of zero being *nothing*. However, as the analysis progressed two aspects emerged - the element of the *'0' number symbol* and the element of *nothing*. These were valuable category distinctions which, after reviewing the addition, subtraction, multiplication and division sections, led to some interesting conclusions. Hence two sub-categories were formed, ii) The effect of nothing and iii) The zero symbol.

ii) The effect of nothing was where a child explained that, as zero was worth nothing, it need not be considered. It was the value of zero, or the lack of value, which was emphasised and included words and phrases such as *nothing, none, worth nothing,*

- 0 + 3 = 3, *You can't add it on because it's not worth anything.* (Aged 11, verbal)

iii) The zero symbol was where the child's explanation was connected with the manipulation of the '0' symbol in the equation. The child's explanation, while based on zero being nothing, took this one stage further to the point where the child had developed a practice that, by usage, had become a personal procedure or rule. The following illustrates a child using *a personal zero rule* of taking the first number in the equation and putting it as the answer.

- 0 + 3 = 0, *I would say the answer is about zero. You take the zero from the front and put it at the end, in the answer.* [Does this always work?] *Always with zero adds.* (Aged 7, verbal)

Comparing the Questionnaire and the Task-Interview zero number facts data

When attempting to compare the qualitative *zero number facts* data from the Questionnaire with that from the Task-Interviews a number of issues needed to be addressed. In the Questionnaire there were children, overall between 25% and 30%, across the eight *zero number facts* who gave an answer but gave no explanation. As the Questionnaire was a written document explanations were likely to be from those children who had a reasonable command of written English; from those who felt relatively confident in mathematics and in their own ability to explain their actions. In contrast, in the Task-Interview, which was a verbal exercise, every child provided an explanation. This was an important distinction that needed to be recognised. As a consequence, while the two sets of data from the Questionnaire and the Task-Interviews are shown in each table, direct comparisons can be made between the quantitative data but no direct comparisons are made between the qualitative data.

Part 2 of this chapter presents the collated data and analysis from the children's answers to the zero number facts for addition, subtraction, multiplication and division.

0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0

Chapter Four, Part Two: Data classification and analysis

Addition

1) Addition, answers to the equations (3 + 0, 0 + 3)

Tables 16 and 17, Addition A1 and Addition A2, contain the answers to 3 + 0 =, and 0 + 3 =. Each table shows the number of responses (in red) and the percentage (in black).

3 + 0 Answers	Questionnaire	Task-Interview					
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6
Number of responses	93	20	11	20	16	12	10
3 + 0 = 3	**88** **95%**	**19** **95%**	**11** **100%**	**17** **85%**	**13** **81%**	**12** **100%**	**10** **100%**
3 + 0 = 0	5 5%	1 5%		3 15%	3 19%		

Table 16, Addition A1

It can be seen that a high percentage of children gave the correct response, 3 + 0 = 3. The only other response given was 3 + 0 = 0; surprisingly this was not from the youngest children.

0 + 3 Answers	Questionnaire	Task-Interview					
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6
Number of responses	89	20	11	20	16	12	10
0 + 3 = 3	77	**19**	**9**	**14**	**13**	**11**	**9**
	87%	**95%**	**82%**	**70%**	**81%**	**92%**	**90%**
0 + 3 = 0	12	1	2	6	3	1	1
	13%	5%	18%	30%	19%	8%	10%

Table 17, Addition A2

While in the second zero addition fact 0 + 3 there was still a high percentage of correct answers it was notable that the number of children giving the answer 0 + 3 = 0 had increased. Was this because the zero is the addend and not the augend as in 3 + 0? Do the children have more problems adding to zero than adding zero? The following two tables 18 and 19, Addition A3 and Addition A4, show the explanations given by the children for their answers.

2) Addition, explanations for the answers to the equations (3 + 0, 0 + 3)

The children's explanations have been placed into the categories explained earlier in this chapter. Each category shows the number of children (in red), the percentage within that cohort (in black) and, in the corner, the number of incorrect answers (in green). Though the reasoning behind the incorrect answers is of importance merely following the explanations for the incorrect answers would omit vital information. It must be emphasised that this research aimed to study children's conceptions of zero and to do this entailed tracing the explanations for both the correct and the incorrect answers. The incorrect answers are shown only for the Task-Interview. In the Questionnaire many children offered no explanation so including the number of incorrect answers in the explanation tables seemed inappropriate and possibly misleading.

3 + 0 Explanations	Questionnaire	Task-Interview					
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6
Number of responses	93	20	11	20	16	12	10
No explanation offered	10 11%						
Reiterated the question	38 41%						
Statement of fact	10 11%	2 10%	11 100%	11 55% 1	12 75% 2	9 75%	7 70%
Derived fact	2 2%	1 5%					
A zero connection	33 35%	17 85% 1		9 45% 2	4 25% 1	3 25%	3 30%

Table 18, Addition A3

0 + 3 Explanations	Questionnaire	Task-Interview					
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6
Number of responses	89	20	11	20	16	12	10
No explanation offered	10 11%						
Reiterates the question	26 29%						
Derived fact	2 2%						
Statement of fact	5 6%	1 5%	7 64%	5 25% 2	3 19%	3 25%	1 10%
Relates to a paired zero number fact	25 28%	11 55%	1 9%	6 30%	8 50% 2	5 42%	5 50%
A zero connection	21 24%	8 40% 1	3 27% 2	9 45% 4	5 31% 1	4 33% 1	4 40% 1

Table 19, Addition A4

In the *derived fact* category the explanation had no particular reference to zero, as can be seen from the following example,

- 3 + 0 = 3 *Three add one equals four so don't add the one and you get three.* (Aged 11, verbal)

In the 0 + 3 explanations fewer children made a *statement of fact* as this was replaced by the category *relates to a paired zero number fact,* here the children were referring back to 3 + 0.

- *Same as 3 + 0 but the other way round.* (Aged 7, verbal)
- *Three add zero gives the same answer as zero add three.* (Aged 8, verbal)
- *0+3 is the same as 3+0.* (Aged 11, written)

Most of the incorrect answers for the addition *zero number facts* were to be found under the main heading *zero connection.* The information in this category was then divided into the three sub-headings and is to be found in table 20 and 21, Addition A5 and Addition A6. Annotations provide further details with regard to the content of the information under these sub-headings.

3 + 0 Zero connection	Questionnaire	Task-Interview					
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6
A zero connection	33 35%	17 85% 1	0	9 45% 2	4 25% 1	3 25%	3 30%
Effect of nothing	28	9		4			1
A zero symbol	4	8		3	2	1	2
Confused	1			2	2	2	

Table 20, Addition A5

0 + 3 Zero connection	Questionnaire	Task-Interview					
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6
A zero connection	21 24%	8 40% 1	3 27% 2	9 45% 4	5 31% 1	4 33% 1	4 40% 1
Effect of nothing	15	7		4	1	1	
A zero symbol	5	1	1	3			
Confused	1		2	2	4	3	4

Table 21, Addition A6

Confused: Comparing the two tables it can be seen that more children were confused with 0 + 3 than with 3 + 0, that is to say when '0' is at the beginning of the number fact.

- 3 + 0 = 3, *Three, in fact it is zero. No, three.*
  0 + 3 = 0, *This is changed round. I would say 3 or zero. I don't know which to choose. Could be both. Answer is three, or zero, or both. No, zero.* (Aged 7, verbal)
- 0 + 3 = 3, *I think the answer is 0 because you can't add 3 to nothing, maybe it's 3 because you don't need the zero. I'm not sure. The answers 3 or zero, I think. I say three.* (Aged 8, verbal)

***The effect of nothing*:** The researcher expected it to be the younger children who made references to zero being *nothing* but this was not so,

- 3 + 0 = 0, *You add nothing, nothing equals nothing.* (Aged 9, verbal)
- 0 + 3 = 3, *You can't add it on because it's not there.* (Aged 11, verbal)

The zero symbol: Where an explanation involved the manipulating of the '0' in the equation.

- 3 + 0 = 3, [The child covers up the +0 with her hand.] *This is no number at all. The answer is three.*
  0 + 3 = 3, *I take away the first number.* [The child puts her hand over the '0'.] *It's the same as the other sum* [3 + 0]. *It's three.* (Aged 6, verbal)

This child was using *a personal zero rule*; cover the + 0 and she was left with the answer.

- 3 + 0 = 3, *You take the three from the front and put the three at the end.*
  0 + 3 = 0, *You take the zero from the front and put it at the end.* (Aged 9, verbal)

This child was using *a personal zero rule*; take the first number and put it as the answer.

3) Addition - summary of the findings

In the addition zero number facts, 3 + 0 and 0 + 3, a high percentage of the children gave the correct answer. Most were confident in their response. No child hesitated to give an answer to 3 + 0 but there were hesitations for 0 + 3; this uncertainty applied to a few children in each age range. Why were there more mistakes with 0 + 3 than with 3 + 0? The children seemed surprised at being given the 0 + 3 equation; possibly children do not encounter 0 + 3 as often as 3 + 0. It will be interesting to see if the trend continues with the other zero number fact operations.

- 3 + 0 = 3, *Easy, I know that is three.* 0 + 3 = *I have none.* [Stops] *That's a hard one. I'm not sure.* (Aged 11, verbal)
- An eight year old had no hesitation in gave the answer to 3 + 0 = 3 but when shown 0 + 3 remarked, *I'll have to think about that one.* (Aged 8, verbal)
- Similarly a six year old, on being shown 3 + 0 immediately said *three.* [The researcher remarked, that was a quick answer.] *It's easy, because zero is at the end.* [On being shown 0 + 3 there was long silence.] *I'm not sure of this. The zero is not at the end. I think it is 3.* (Aged 6, verbal)

- In the Questionnaire one explanation for giving the answer 3 + 0 = 3 and 0 + 3 = 0, was that *You add nothing to three but you can't add three to nothing so it has to be nothing.* (Aged 11, written)

A trend was beginning to show in the *zero connection section.* The *effect of zero* in both 3 + 0 and 0 + 3 meant that zero was referred to as *nothing*, then the remaining number, three, became the answer. In the *zero symbol* procedures were used - *put your hand over the '0', take the largest number, it's the first number or the last number* - in order to comply with what appeared to be a child's *personal zero rule* used when there was a zero in an equation. The two procedures the *effect of zero* and the *zero symbol* resulted in the answer '3'. As '3' is the answer to 3+0 and 0+3 success was achieved. As the answer to 3 – 0 is also '3' then noting whether this reasoning trend continued would be interesting.

0 – 0 – 0 – 0 – 0 – 0 – 0 – 0 – 0 – 0 – 0 – 0

Subtraction

Those teaching the children in the age range 6 to 11 were happy for the children to be working with addition and subtraction of single digit numbers using number up to 5. They felt that the children would be capable of answering subtraction where the minuend was larger than the subtrahend. While 5 – 2 would be within the children's understanding few would have met 2 – 5. So it was appreciated that asking the children the answer to 0 – 3 could present difficulties. While the results from 3 + 0 and 0 + 3 could be presented together 3 – 0 and 0 – 3 need to be considered separately [1].

1) Subtraction (3 – 0), equation answers and explanations

Table 22, Subtraction S1 contains the classified answers to 3 – 0.

3 – 0 Answers	Questionnaire	Task-Interview					
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6
Number of responses	88	20	11	20	16	12	10
3 – 0 = 3	86 98%	16 **80%**	9 **82%**	17 **85%**	14 **87%**	9 **75%**	7 **70%**
3 – 0 = 0	2 2%	3 15%	2 18%	2 10%	2 13%	2 17%	3 30%
3 – 0 = -3		1 5%					
3 – 0 = 2						1 8%	
Can't be done, there is no answer				1 5%			

Table 22, Subtraction S1

There were two incongruous answers; each had a zero connection,

- 3 – 0 = 2, *You can't take away zero, take away one.* (Aged 7, verbal)
- 3 – 0 = *Can't be done, there is no answer.* (Aged 9, verbal) [Only one child insisted that you cannot *take away nothing* and so there could be no answer.]

[1] This has the effect of making the numbering of the subtraction tables appear non-sequential.

3 - 0 Explanations	Questionnaire	Task-Interview					
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6
Number of responses	88	20	11	20	16	12	10
No explanation offered	16 18%						
Reiterated the question	23 26%						
Statement of fact	2 3%	5 25% 1	4 36%	4 20% 1	3 19% 1	3 25% 1	1 10% 1
Number order		1 5% 1					
A zero connection	47 53%	15 70% 2	7 64% 2	16 80% 2	13 81% 1	9 75% 2	9 90% 2

Table 23, Subtraction S3

A high percentage of answers to 3 – 0 were correct. Table 23, subtraction S3 shows that, apart from an 11 year old child who used the *number order* and reached the answer –3, all the explanations given contained a zero connection.

The result of sub-dividing the *zero connection* into the three categories are displayed in table 24, subtraction S5.

3 - 0 Zero connection	Questionnaire	Task-Interview					
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6
A zero connection	47 53%	15 70% 2	7 64% 2	16 80% 2	13 81% 1	9 75% 2	9 90% 2
Effect of nothing	31	12	6	14	11	5	5
A zero symbol	14	2		1	2	2	3
Confused	2	1	1	1		2	1

Table 24, Subtraction S5

Confused: these were children who seemed to be misinterpreting the equation as is seen in the following example.

- 3 - 0 = 3, *Zero is the first number and you can't take away three so the answer is zero.* [The researcher felt the child was attempting to do 0 – 3. The child was asked to re read the question. The child insisted that she was doing *three take away zero.*] *The answer is zero. The answer could be three. I think it is three.* (Aged 7, verbal)

Thinking these confused children were having problems with subtraction the researcher wrote down 4 – 2 and 5 – 1. They had no difficulty in giving the correct answers. It would appear that zero was causing the confusion. The answers given were either zero or three.

***The effect of nothing*:** it was anticipated that children, when asked why their answer was 3 – 0 = 3 would include, in their explanation, a phrase such as 'you're taking nothing away'. This occurred and so most of the explanations came under *the effect of nothing* category (table 24, subtraction S5).

- 3 – 0 = 3, *Because you don't take anything away* (Aged 8, verbal)
- 3 – 0 = 3, *You can't take nothing because nothing doesn't exist* (Aged 11, written)
- 3 – 0 = 3, *Because nort can't be taken away from three* (Aged 11, written)

The zero symbol: explanations followed a similar pattern to those in addition, the removal of the '0' symbol to leave the three.

- 3 – 0 = 3, *Because not use 0 number 3 is on its own.* (Aged 11, written)
- 3 + 0 = 3, *The answer is 3, when you add nothing the answer is the first number.*
  0 + 3 = 3, *The first number goes away because it is nothing.*
  3 - 0 = 3, *You stay with the 3.* (Aged 7 verbal)

Why were most of the incorrect answers in the zero connection categories? They came from children who used the word ***can't***. While they said that you can't do the equation never the less an answer was given. Some children said that you *can't take away zero*, because zero is nothing, and this leaves the three. Other children's thoughts began with the same statement but they reached a different conclusion, *you can't take away zero* so then you take away the three giving the result 3 – 0 = 0. The use of the word ***can't*** is discussed in greater detail later in part three of this chapter.

2) Subtraction (0 - 3), equation answers and explanations

So far, with the zero facts of **3 + 0, 0 + 3** and **3 – 0** the children were confident in their answers and in their explanations. However, **0 – 3** was, to most children, unfamiliar. Surprisingly the expected responses that 'it can't be done … there is no answer', were used by only three children. The 0 – 3 equation resulted in the widest range of all the zero facts answers (table 25, subtraction S2).

0 – 3 Answers	Questionnaire	Task-Interview					
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6
Number of responses	86	20	11	20	16	12	9
0 - 3 = -3	**36 42%**	**8 40%**	**3 27%**	**5 25%**	**1 6%**	**2 17%**	
Two possible answers (0 or -3) (3 or –3)	6 7%	4 20%	5 45%	3 15%			
0 - 3 = 0	16 19%	2 10%	2 18%	5 25%	7 44%	3 25%	6 67%
0 - 3 = 3	21 24%	6 30%		4 20%	8 50%	6 50%	3 33%
0 - 3 = 0.3	7 8%		1 9%	1 5%			
Can't be done, there is no answer				2 10%		1 8%	

Table 25, Subtraction S2

Looking first at the children who gave the answer 0.3. These children appeared to have confused thoughts on decimals and negative numbers.

- 0 – 3 = 0.3, *0 – 1 = 0.1 and 0 – 2 = 0.2 and 0 – 3 = 0.3* (Aged 11, written)
- 0 – 3 = 0.3, *To take three off zero you go lower than zero. It's 0.1, then 0.2 then 0.3* (Aged 10, verbal)

The 0 – 3 = -3 response was given mainly by the older children who would have been introduced to negative numbers but within this older age range were children who gave two answers. *Two* *possible pairs of answers (3 and –3 or 0 and –3),* this was an intriguing group. These were not children who were giving an *either or answer*, nor were they children who were unsure or confused. These were children saying <u>there are two answers</u> the selection being dependent upon circumstances. These are some of the children's explanations for their two answers to 0 – 3.

- *–3 that's being technical, or zero.* (Aged 9, verbal)
- *That's zero. Can I go into the minuses? Then it is –3.* [Which is the answer?] *Depends whether you are going minus sums or not.* (Aged 10, verbal)
- *Zero take three equals three* [0 – 3 = 3] *like three take zero is three* [3 – 0 = 3]. *Three or if I took off the three it would be negative three. Yes, three or negative three.* (Aged 11, verbal)
- *0 – 3 = 3 or if you go into minus it is –3* (Aged 11, written)
- *Three, no zero. It's zero because if you've zero take three you can't so the answer is zero. Of course the real answer is minus three.* [I don't understand that.] *The answer is zero but if you are doing about minus numbers then you have to write –3. If you're not doing minus numbers then it is zero.* (Aged 10, verbal)
- *There is an answer if you go into the minus section, minus three, otherwise you can't do it so the answer is nothing, zero.* (Aged 11, verbal)

The alternative answers to –3, given by these 9, 10 and 11 year olds, were either '3' or '0'. The responses of younger children to 0 – 3 tended also to be '3' or '0'.

The explanations for the answers to 0 – 3 appeared, at first, as diverse as the answers, but they did fall within the categories already used with 3 + 0, 0 + 3, 3 – 0 (with the exception of *derived fact).*

0 – 3 Explanations	Questionnaire	Task-Interview					
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6
Number of responses	84	20	11	20	16	12	9
No explanation offered	18 21%						
Reiterated the question	15 18%						
Statement of fact	4 5%	7 35% 4	1 9%	4 20% 3	3 19% 2	3 25% 1	1 11% 1
Relates to a paired zero number fact	7 8%	1 5% 1		2 10% 2	6 38% 6	4 33% 4	2 22% 2
Number order	16 19%	5 25%	3 27% 1	4 20% 1	1 6% 1		
Can't be done, there is no answer				2 10% 2		1 8% 1	
A zero connection	24 29%	7 35% 7	7 64% 7	8 40% 7	6 38% 6	4 33% 4	6 67% 6

Table 26, Subtraction S4

Understandably children assumed that, as with addition, 3 – 0 = 3 would give the same answer as 0 –3. This, *paired zero number fact,* was one of the most common reasons for giving the answer '3' and it was one of the main areas that produced the most incorrect answers.

The Number order category contained those children who explained that they were counting down from zero and this took them into negative numbers. Using this method produced no incorrect answers (with the exception of those children, mentioned on the previous page, who confused decimals and negative numbers). Only three children said there was no answer,

- 0 – 3 = *you can't do that because zero is the smallest number and you can't take away three. There is no answer.* (Aged 7, verbal)

Though many children said 'you can't do that' most continued to provide an answer. The answer being either '3' or '0'. How did the child reach either of these answers? They did so mainly through the category containing the greatest number of explanations and the most incorrect answers, this was *a zero connection.*

0 – 3 Zero connection	Questionnaire	Task-Interview					
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6
A zero connection	24 28%	7 35% 7	7 64% 7	8 40% 7	6 38% 6	4 33% 4	6 67% 6
Effect of nothing	9	5	6	3	2		
A zero symbol	10	2	1	2	3	2	3
Confused	5			3	1	2	3

Table 27, Subtraction S6

Confused: it was understandable that 0 – 3 would confuse the younger children but fewer than the researcher expected appeared to be confused. All were happy to provide an answer and attempt an explanation. Indeed, in the Task-Interview, there were as many children confused with 0 + 3 as there were by 0 – 3.

- 0 – 3 = 0, *You can't take three from zero so it's zero. Zero take three is three. Is it three from zero? I think it's zero.* (Aged 9, verbal)
- 0 – 3 = 3, *Zero is the first number and you can't take away three so the answer is zero. Answer could be three. I think it is three.* (Aged 7, verbal)

The effect of nothing: using this explanation the children were confident of their answers. It had been expected that more children would have used this reasoning.

- 0 – 3 = 0, B*ecause there is no number less than 0 you can't take anything away from nothing.* (Aged 11, written)
- 0 – 3 = 0, *You cant take three from nothing it is a bigger number.* (Aged 11, written)

A zero symbol: again these were children who were manipulating the symbols in the equations. The illustrative incident below shows this being done. What is of particular note is how the manipulation of the number symbols surfaced. It was not initially apparent as the child confidently gave the answers to the addition zero number facts but when he met a situation where he attempted to explain something he found more challenging, then his methods became clearer.

Illustrative incident

Paul, on being shown the **3 + 0** card said, *I think it's three.* With the **0 + 3** card he hesitated, *I'm not sure. I think it's zero because it's different to three add zero. But it looks like it should be three.* He checked by using his fingers and decided the answer was three.
3 - 0 Paul covered up the –0 with his hand. *The answer is three.* [Why did you put your hand over some numbers?] *To make it* [the zero] *go away.*
0 – 3 Paul put his hand over the 0–. *You don't want the zero so you hide it.*
He then returned to the addition. He picked up the 3 + 0 card and covered up the +0, *The answer is three.*
He returned to the 0 + 3 and covered up the 0+. *Answer is three. You always cover up the zero.*
His final answers were: 3 + 0 = 3   0 + 3 = 3   3 - 0 = 3   0 - 3 = 3
(Aged 7, verbal)

3) Subtraction - summary of the findings

For the equation 3 – 0:

In the Task-Interview it was quite a surprise to find, across all the age ranges, at least 10% of the children who thought that 3 – 0 = 0. Some children felt the question to be rather strange which suggests that they do not often meet this zero number fact, though the older children would meet zero in algorithms such as:

$$\begin{array}{r} 835\ - \\ \underline{602} \end{array}$$

For the equation 0 - 3:

Here it was expected there would be a number of empty answers, and that the phrase 'can't do it' would play a dominant role. Yet there were only three children in total who thought that 0 – 3 could not be done and hence there was no answer. Maybe this was because children do not like to say there is no answer, it is akin to admitting failure. What did happen was that children would say 'you can't do it' but then they provided an answer,

> ... the student, unlike a typical computer program is not apt to just quit. Instead he or she will often be inventive, invoking problem-solving skills in an attempt to repair the impasse and continuing to execute the procedure, albeit in a potentially erroneous way.
> (Brown and Van Lehn, 1982, in Dickson, Brown and Gibson 1984, p. 261)

Some used *the zero is nothing* reason and this resulted in the answer being '0' or '3'. Others had a *personal zero rule* and the answer would be '0' or '3'. Choosing between '0' or '3' seemed to be arbitrary. This was particularly noticeable with the children who gave two answers one being '–3' the other '0' or '3'. An interesting phenomenon, noticeable in the addition zero number facts but which was strengthened in the subtraction, was the children expected that the answer to a zero number fact would be '0' or 'the other number'. This will be discussed in part 3 of this chapter.

In both the subtraction zero number facts (3 - 0 and 0 – 3) most of the explanations for correct and incorrect answers revolved round *the effect of nothing* with a few children using *the zero symbol* and a *personal zero rule*.

0 × 0 × 0 × 0 × 0 × 0 × 0 × 0 × 0 × 0 × 0 × 0

Multiplication

1) Multiplication, answers to the equations (3 × 0, 0 × 3)

The system of reporting the research information for multiplication will be the same as that used in addition and so the answers for 3 × 0 and 0 × 3, will be analysed together (tables 28 and 29) [2].

3 × 0 Answers	Questionnaire	Task-Interview				
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7
Number of responses	86	20	11	20	14	10
3 × 0 = 0	61 **71%**	18 **90%**	8 **73%**	12 **60%**	9 **64%**	4 **40%**
3 × 0 = 3	25 29%	2 10%	3 27%	7 35%	3 21%	6 60%
3 × 0 = 3 or 0				1 5%	2 14%	

Table 28, Multiplication M1

0 × 3 Answers	Questionnaire	Task-Interview				
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7
Number of responses	84	20	11	20	14	10
0 × 3 = 0	63 **75%**	15 **75%**	11 **100%**	14 **70%**	10 **71%**	4 **40%**
0 × 3 = 3	21 25%	5 25%		4 20%	3 21%	6 60%
0 × 3 = 0 or 3				1 5%	1 7%	
0 × 3 = -6				1 5%		

Table 29, Multiplication M2

The one anomaly within the multiplication answers was that of 0 × 3 = -6. This nine year old child gave the correct answer to 3 × 0 but he was confused with 0 × 3. His explanation was that you start at 3 and jump back in threes toward zero and reach -6.

Most children knew the answer to 3 × 0 and 0 × 3 would be '0' or '3' but a few children admitted to guessing while a small number of children could not decide on either one of the two answers.

- 3 × 0 = *I'm going to guess it is 3 or 0. It must be one number or the other it always is when there's a zero.* (Aged 8, verbal)

The number of children giving the answer '3' or '0' appeared to be relatively consistent across each age range, but it cannot be assumed that the same answers to these two equations came

[2] As few of the 6 year old children had any understanding of multiplication they were not included in the data.

from the same children. While one child would give the answers $3 \times 0 = 3$, $0 \times 3 = 0$, another would answer $3 \times 0 = 0$, $0 \times 3 = 3$. Tables 30 and 31 contain the explanations from which to try to examine this diversity of thoughts.

2) Multiplication, explanations for the answers to the equations (3×0, 0×3)

3 × 0 Explanations	Questionnaire	Task-Interview				
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7
Number of responses	86	20	11	20	14	10
No explanation offered	19 22%					
Reiterated the question	21 24%					
Statement of fact	16 19%	10 50% 1	5 45%	9 45% 3	5 36%	6 60% 3
Derived fact	3 3%	2 10%	3 27% 1	3 15%	2 14%	1 10%
A zero connection	27 31%	8 40% 1	3 27% 2	8 40% 5	7 50% 5	3 30% 3

Table 30, Multiplication M3

0 × 3 = Explanations	Questionnaire	Task-Interview				
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7
Number of responses	84	20	11	20	14	10
No explanation offered	22 26%					
Reiterated the question	14 17%					
Statement of fact	3 4%		4 36%	2 10% 1	3 21%	2 20% 1
Derived fact	2 2%	1 5%	1 9%			
Number order			2 18%	1 5% 1		
Relates to a paired zero number fact	22 26%	14 70% 3		8 40% 2	5 36% 1	6 60% 3
A zero connection	21 25%	5 25% 2	4 36%	9 45% 2	6 43% 3	2 20% 2

Table 31, Multiplication M4

The explanation *relates to a paired zero number fact* played a major role in the 0×3 answers as the children referred to their 3×0 answer. Understandably the children relied heavily upon their recall of the multiplication tables.

- $3 \times 0 = 0$, *I don't know how I know but I do.* (Aged 8, verbal)
- $3 \times 0 = 0$, $0 \times 3 = 3$, *I've done my tables.* (Aged 9, verbal)
- $3 \times 0 = 0$, $0 \times 3 = 3$, *You can't times zero by three or three by zero. As $3 \times 1 = 3$ and it can't be the same so must be zero.* Yes, *they're both in the 3 times table, it's the other way round. So that means $0 \times 3 = 3$* (Aged 9, verbal)

When using the multiplication tables as an explanation at least a quarter of the children added a codicil, *I think that it is 0*. Some used a derived fact as explanation and to re-assure themselves that the answer was zero. The derived fact took two forms, using the answer to 3×1 and using continuous (repeated) addition.

- $3 \times 0 = 0$, *If it was 3×1 it would be 1, so it must be zero.* (Aged 8, verbal)
- $3 \times 0 = 0$, *$3 \times 1 = 3$ take away 3 leaves zero.* (Aged 11, written)
- $3 \times 0 = 0$, *That's three zeros added, zero and zero and zero.* (Aged 10, verbal)

The children aged 7, 8, and 9 were those who were more unsure of their multiplication tables and who relied more on the elements in *the zero connection* (tables 32 and 33)

3 × 0 Zero connection	Questionnaire	Task-Interview				
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7
A zero connection	27 31%	8 40% 1	3 27% 2	8 40% 5	7 50% 5	3 30% 3
Effect of nothing	19	8	3	3	2	1
A zero symbol	8			4	3	2
Confused				1	2	

Table 32, Multiplication M5

0 × 3 Zero connection	Questionnaire	Task-Interview				
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7
A zero connection	21 25%	5 25% 2	4 36%	9 45% 2	6 43% 3	2 20% 2
Effect of nothing	13	5	4	3		1
A zero symbol	8			5	3	1
Confused				1	2	

Table 33, Multiplication M6

Confused: all the children were asked to read each equation before giving an answer and an explanation. While all children read the zero multiplication number facts correctly a small number appeared to be confused as to which number was the multiplicand and which the multiplier. The same three children were in both *confused* multiplication categories; an example of one child's reasoning follows,

- $3 \times 0 = 0$, *Because you are not 'timesing' anything- zero, zero, zero - not a number, it's nothing as you times it three times*
  $0 \times 3 = 0$, *Because none times no answer is 3. No, I'm changing the answer as I'm getting muddled up. Zero, zero, zero, is nothing, final answer is zero.* (Aged 8, verbal)

The effects of nothing: for children who relied on explaining their answer by using *the effects of nothing* the reasoning changed dependent upon the position of the zero. In 3×0, reasoned the children, you have three to begin with (some children used counters or fingers for illustration) then you 'times' by nothing but you still have the three, hence $3 \times 0 = 3$. With 0×3 you begin with nothing and multiplying by three means 'three lots' of nothing, so $0 \times 3 = 0$. This resulted in more incorrect answers for 3×0 than for 0×3.

- $3 \times 0 = 3$, *If you've got nothing times 3, then you've got three to start with. It must equal three.* $0 \times 3 = 0$ *if you've got 3 times nothing, that's nothing, 'Cause you've got nothing to start with and that's three lots of nothing. I think.* (Aged 10, verbal)

The zero symbol category contained a few children who explained how they manipulated the symbols, this may be an indication of *a personal zero rule* used only for multiplication or a more consistent rule which was appearing in more than one operation.

- $3 \text{ x } 0 = 3$, *When you times by zero the answer is what you started off with at first* (Aged 11, written)
- $0 \text{ x } 3 = 3$, *Because you can* $\times$ *a bigger number by 0. You get the bigger number.* (Aged 11, written)
- $3 \times 0 = 3$, *You've got three, it's at the front.*
  $0 \times 3 = 0$, *You've still got zero at the front.* (Aged 9, verbal)

Again, as with the subtraction, it was seen that when the child reached an operation where he/she was unsure then a *personal zero rule* was more likely to become apparent. This is seen in the illustrative incident below, which occurred during the Task-Interviews. For addition and subtraction the child had given the answer as a statement of fact, with no explanation. That he had, possibly, been using his *personal zero rule* to gain the addition answers came to light with the multiplication equations.

Illustrative incident

$3 + 0 = 3 \qquad 0 + 3 = 3 \qquad 3 - 0 = 0 \qquad 0 - 3 = 0$

For these four zero facts the child said, *I know,* no explanations were given.
When the 3×0 multiplication card was shown the boy said, *you take away the zero and leave the three.*
[The researcher remarked that she thought this was a multiplication not a take-away.]
No, you take it off. [The child covered up the zero with his hand.]
0×3, T*hat's the same* [The '0' symbol is covered over leaving the '3' for the answer.]
You can't do it with any other number. You can only do it with zero and add or times.
[The researcher said, 'show me what you mean.' He found the addition cards and covered up the zero with his hand.] *Add and times are similar. So those answers are the same, three. Take aways and divides are similar, if you take the dots off* $3 \div 0$ *you get* $3 - 0$. *So those answers are the same, zero.*
The final answers were:

$3 + 0 = 3$	$0 + 3 = 3$	$3 - 0 = 0$	$0 - 3 = 0$
$3 \times 0 = 3$	$0 \times 3 = 3$	$3 \div 0 = 0$	$0 \div 3 = 0$

(Aged 9, verbal)

3) Multiplication - summary of the findings

As 3 + 0 had more correct answers then 0 + 3 it was a possibility that this would occur with multiplication and that 3×0 would have more correct answers than 0×3. The reverse happened. Was this connected with the way the children had been taught their tables?[3]

$3 \times 0 =$	$0 \times 3 =$
$3 \times 1 =$	$1 \times 3 =$
$3 \times 2 = \ldots$	$2 \times 3 = \ldots$

Was it because they had met 0×3 in algorithms? Such as

$$\begin{array}{r} 304 \\ \underline{\times 3} \end{array}$$

The answer seemed to be found in the category *a zero connection* and *the effect of nothing.* In both 3×0 and 0×3 many children referred to zero as *nothing* in their explanations. With 3×0 the explanation was based on the premise that you have a set of three and no more sets of three so you have three: answer $3 \times 0 = 3$. Similarly, with 0×3 you have a set of none and then two more sets of none so you have nothing: answer $0 \times 3 = 0$. This resulted in more incorrect answers to 3×0.

Recall, in the form of the multiplication tables, played a major part in the children's reasoning. This was reflected in the high percentage found in the *statement of fact* and *derived fact* categories. Again the *effect of nothing* showed a particularly high profile as the children tried to use concrete examples to explain their answers. Often this was in the form of telling the equation as a story with objects such as sweets. A few children manipulated the '0' in the equation according to their *personal zero rule.*

$0 \div 0 \div 0 \div 0 \div 0 \div 0 \div 0 \div 0 \div 0 \div 0 \div 0 \div 0$

Division

- $3 \div 0 = 0$, *Three friends and nothing to give them.*
  $0 \div 3 = 0$, *Three cubes and share them into zero people no-one gets anything as no one there. It's a bit confusing as still got three cubes. Some people will know what to do.* (Aged 9, verbal)

1) Division, answers to the equations ($3 \div 0$, $0 \div 3$)

Most adults would be very wary at giving an answer to $3 \div 0$ and $0 \div 3$ and would find trying to explain their answer fraught with problems (see appendix 10).

[3] This is discussed later in this chapter; see also chapter 8 and appendix 15.

The intent, in presenting the division zero facts to the children, was not to become involved in right and wrong answers but to obtain their explanations. The children were puzzled but not perturbed at being presented with the two zero division facts and, within their mathematical understanding, most provided answers underpinned with logical reasons.

From tables 34 and 35 it can be seen that there is one anomaly, an 11 year old girl, who did not give an answer of '3' or '0' but gave the answer $0 \div 3 = 0.25$:

- .25 + .25 + .25 = .75 ~ *I'm trying to get to zero. I'm trying to divide zero into three.* (Aged 11, verbal)

All the other children followed the pattern of the other zero facts and expected '3' or '0' would be present in the answer. Most children found a reason for giving the answer '3' or '0'. Some children could not find the rationale that gave them a definite answer and would not make a choice between the two answers, so a category for '3 or 0' was necessary.

- $3 \div 0 = 0$ or 3, *Three. No it isn't because 3 ÷ 1 is three so you'd have the two sums with the same answer. If I keep adding zero then I'd never get to 3? This means that I'll never get the answer. I think it's 0 or 3.* (Aged 11, verbal).

3 ÷ 0 Answers	Questionnaire	Task-Interview			
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8
Number of responses	82	20	11	17	11
3 ÷ 0 = 0	42 51%	8 40%	5 45%	7 41%	2 18%
3 ÷ 0 = 3	38 46%	11 55%	3 27%	4 24%	5 45%
3 ÷ 0 = 3 or 0		1 5%	2 18%	4 24%	4 36%
There is no answer, it can't be done	2 2%		1 9%	2 12%	

Table 34, Division D1

0 ÷ 3 Answers	Questionnaire	Task-Interview			
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8
Number of responses	80	20	11	17	11
0 ÷ 3 = 0	54 68%	12 60%	6 55%	11 65%	2 18%
0 ÷ 3 = 3	21 26%	5 25%	2 18%	1 6%	5 45%
0 ÷ 3 = 3 or 0			1 9%	3 18%	4 36%
0 ÷ 3 = 0.25		1 **5%**			
There is no answer, it can't be done	5 6%	2 10%	2 18%	2 12%	

Table 35, Division D2

Understandably, few children gave their answer confidently, as *a statement of fact* (tables 36 and 37), most were trying to 'work out' the answer.

2) Division, explanations for the answers to the equations (3 ÷ 0, 0 ÷ 3)

3 ÷ 0 Explanations	Questionnaire	Task-Interview			
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8
Number of responses	82	20	11	17	11
No explanation offered	24 29%				
Reiterated the question	10 12%				
Statement of fact	2 2%	1 5%	3 27%	4 24%	
Derived fact	5 6%	4 20%	1 9%	1 6%	
There is no answer, it can't be done	2 2%		1 9%	2 12%	
A zero connection	39 48%	15 75%	6 55%	10 59%	11 100%

Table 36, Division D3

0 ÷ 3 Explanations	Questionnaire	Task-Interview			
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8
Number of responses	80	20	11	17	11
No explanation offered	24 30%				
Reiterated the question	8 10%				
Statement of fact	3 4%		2 18%	4 24%	
Derived fact	4 5%	1 5%			
Relates to a paired zero number fact	14 18%	8 40%		1 6%	
There is no answer, it can't be done	5 6%	2 10%	2 18%	2 12%	
A zero connection	22 28%	9 45%	7 64%	10 59%	11 100%

Table 37, Division D4

3 ÷ 0 Zero connection	Questionnaire	Task-Interview			
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8
Effect of nothing	30	15	5	5	3
A zero symbol	5			1	4
Confused	4		1	4	4

Table 38, Division D5

0 ÷ 3 Zero connection	Questionnaire	Task-Interview			
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8
Effect of nothing	19	9	5	6	3
A zero symbol	1			1	4
Confused	2		2		

Table 39, Division D6

Division was the one operation were *there is no answer, it can't be done* was used by a small, but significant number of children.

- 3 ÷ 0 ~ *Can't do it, the answer is zero, no there is no answer leave it blank.* 0 ÷ 3 *That's the same. No answer, can't do it.* (Aged 10, verbal)

Slightly more children felt there could not be an answer to 0 ÷ 3. The researcher thought that the reverse might have occurred as children would have met 0 ÷ 3 within algorithms such as 309 ÷ 3. Children's division explanations were very reminiscent of those in multiplication. With 3 ÷ 0 the children would begin with 'you have three' and after sharing three amongst no one the three would still be there. Answer $3 \div 0 = 3$. With $0 \div 3$ the children would explain 'you have none' and after sharing none amongst three then nobody would get any. Answer $0 \div 3 = 0$.

- $3 \div 0 = 3$, *I know the answer is 3. If one person wanted to share but there is no one there then he gets them all, three.*
  $0 \div 3 = 0$, *I have none and want to share them between three but don't have anything so the answer is nothing, zero.* (Aged 8, verbal)
- $3 \div 0 = 3$, *Got 3 marbles and try to divide it by nothing, it wont work so you've got three.*
  $0 \div 3 = 0$, *It wont work, nothing. Because it begins with a zero you can't divide nothing by 3. You've got nothing to start with.* (Aged 9, verbal)

A few children's explanations appeared to refer to their *personal zero rule*,

- $3 \div 0 = 3$, *Because if you divide by zero it* [the 3] *doesn't change* (Aged 8, verbal)
- $3 \div 0 = 3$, $0 \div 3 = 3$, *They all equal three because of the zero. There's no other answer.* (Aged 8, verbal)

3) Division - summary of the findings

A high percentage of the children put the explanation of the division equation in the context of sharing objects. $3 \div 0$ was more likely to be given the answer 3 as the children initially envisaged three objects while $0 \div 3$ was likely to be given a zero answer as there were no objects to share. As with the other zero number facts a number of children used the word 'can't' though still providing an answer (this will be discussed in detail in part 3 of this chapter). A few children said that there was no answer to either of the division zero facts. A small number of children gained their answer through the use of a *personal zero rule*, which involved manipulating the *zero symbol* in the two division equations.

~ 0 + 0 - 0 × 0 ÷ 0 ~ 000 ~ 0 ÷ 0 × 0 - 0 + 0 ~

Chapter Four, Part Three: Discussion points

The *zero* in the zero number facts created four main areas of tension for the children. Each of these four aspects will be considered in turn.

1) Recall
2) An *either/or* answer
3) Use of the word *can't*
4) The *zero connection*

1) Recall

Many children used the phrase 'I know' when recalling the answers to the zero number facts. Knowing the name of an object for identification purposes (such as number symbol) is a lower level skill than knowing the order of a set of these objects (as in the number order). These are normally 'in memory' put there through experience, practice and repetition, as discussed in chapter 3, part 3. Does one know the number facts by putting the information 'in memory' using these strategies?

- *I know three add zero and zero add three 'cause somebody told me but nobody told me three take away zero.* (Aged 6, verbal)

Some children appeared to rely heavily upon memory and having the experience, practice and repetition of the *zero number facts* is going to be significant to putting them in memory ready for recall. However, in the researchers experience (see chapter 1, part 1, 'The background of the researcher') when one looks at the teaching of the number facts the *zero number facts* are not always included. Similarly, it seems to be quite common when researching in the field of number bonds for the zero bonds to be omitted. One reason may be that researching the learning of the number facts is often undertaken with young children and includes the use of concrete material; hence work with zero would make it differ from other numbers (this is discussed further in chapter 8). In the book *Numeracy and Beyond* (Hughes, et al, 2000) there is a chapter which focuses on *Teaching for Application in Japan.*

> The lesson ... was concerned entirely with number bonds that added to six (3 + 3, 2 + 4 and so on). (Hughes, et al, 2000, p.97).

The lesson included these bonds which could be illustrated by the use of concrete materials (apples, sweets in tins), by patterns of blocks and by algorithms (flash cards showing the sums). The zero number facts, 0 + 6 and 6 + 0, were not included and neither the teacher nor the researchers commented about this omission. Two further illustrations of this omission are found in the first SATs mental arithmetic test and in the Staffordshire Arithmetic Test (see appendix 7).

The SATs section of mental arithmetic test, concerned with recalling the answers to the number bonds, gives instructions that the child should not work out the answer by using fingers or nodding the head (counting imaginary objects). A child is deemed to 'know' the answer if the correct response is given within a set time of 5 seconds for the KS1 children (7 year olds) and 3 seconds for KS2 (11 year old children).

The recall of number facts may be achieved through the recalling of a fact 'learned by heart', or through being able to work out very quickly specific number bonds or tables facts. For example 8 + 9 may be quickly derived from the pairs 8 + 8 and one more, or 9 + 9 and one less. Though a derived fact usually has a basis in a 'learned by heart' fact.

The Task-Interview category headed *statement of fact* contained the responses of children who 'knew' the answer by what appeared to be automatic recall. Using the SATs timing guidelines in the majority of cases the answers to the zero facts were given very quickly for addition, subtraction and multiplication (though division did cause some children to pause before answering). However, it was not possible to know whether each child 'knew' the answer or was able to work it out very quickly, there is a fine a borderline between the two.

A correct response to a number fact relies on the accuracy of the information that is in memory, whether the method is through mental recall or by the use of a mental strategies (which in turn is likely to be based on a recalled fact). One could ask if a 'rule' could also be in memory and be a strategy for providing a recall answer. The children who had devised their own *personal zero rule* could have applied their rule as others applied a derived fact. This application would produce a quick answer, but not necessarily a correct answer. Hence $3 + 0 = 3$ could be correct information in memory or result of using the personal zero rule (ignore the zero). In the same way $3 \times 0 = 3$ could be incorrect information in memory or result of using the personal zero rule (ignore the zero).

In this study it was in the multiplication section where most children made a reference to knowing (or not knowing) their multiplication tables. In the Questionnaire the children were asked to write down the first five lines of the five times table (the full results are seen in appendix 15). Only 28% used 5×0 or 0×5. The multiplication facts are an interesting case in that they are often the focus of the debate as to the value, or not, of rote learning. While the rote learning debate is not central to this thesis it would be valuable to know how often zero is included in the multiplication tables in schools. One could suggest that a child might use other methods to reach the answers to the multiplication facts, such as counting in 5s. In response to the questionnaire 25% of the children counted in 5s as a means of writing their 5 times table, all these children began at 5. No child began counting at zero. If zero is not included in the formal multiplication table format or in the counting in 5s then one may ask what other means are used to provide the practice and repetition required ensure the recall of the zero multiplication facts. This discussion is continued in chapter 8, part 1 in the section 'The teaching of zero'.

2) An *either/or* answer

If the eight equations in the zero number facts had the numbers replaced with 4 and 2 then the equations and answers would have been,

1) $4 + 2 = 6$	2) $2 + 4 = 6$	3) $4 - 2 = 2$	4) $2 - 4 = -2$
5) $4 \times 2 = 8$	6) $2 \times 4 = 8$	7) $4 \div 2 = 2$	8) $2 \div 4 = \frac{1}{2}$

That is to say there would be 5 different answers. With very few exceptions the children in this study realised that in a zero number fact, the answer was one or other of the numbers (in this study either '3' or '0'). Children said *the answer is either '3' or '0'* and many guessed which one of these answers to give. They had a 50/50 chance of being correct and this became a 'lazy' strategy used by a number of children. In written equations if a child puts the wrong answer and the work is corrected then the right answer is known. As the researcher did not tell the children whether their answer was, or was not, correct this may explain the reason for children giving an either/or answer, *it's either '0' or '3'* rather than selecting, albeit with a guess, one answer. It may seem a strange thing to say, but the knowledge that the answer is either/or did not simplify matters but seemed to cause confusion.

3) Use of the word *'Can't'*

There was a noticeable use of the word *can't* in the children's answers or explanations. Yet, while the child said *you can't*, an answer would be given. Only a very small number, within the division equations, said *can't* and followed this with *there is no answer.* Most teachers will recognise the phrase *you can't* from subtraction algorithms such as,

$$\begin{array}{l} 234 \\ \underline{125} \end{array} \quad \textit{four take away five you can't}$$

Children aged 10 and 11 are likely to be familiar with the more complicated algorithms such as the subtraction above or division, for example 804 ÷ 4. This may be the reason why the use of the word *can't* was particularly prevalent amongst the older children. The - *you can't* - phrase is tantamount to a 'chant', but the – *you can't* -'chant' is followed by something more ('borrow a ten', 'carry the number on') as the process continues. It is as if *you can't* sets up a model that cues in the next piece of the process to finding the answer. A case of *you can't so* what can be done about it? In the zero number bonds there are children who are saying *can't* whilst then giving an answer.

- 0 x 3 = 0, *You can't times nothing by a higher number.* (Aged 11, written)

The phase *you can't* was used for all four rules. In many instances it would appear that the phrase was used because the child was not thinking in the abstract but in the concrete, referring to objects. In the world of the concrete you literally cannot add 'no cubes' or take away 'no cubes'; you cannot physically add or remove '*nothing*'.

- 3 + 0 = 3, *You can't do it, you just don't add anymore.* (Aged 8, verbal)
- 3 + 0 = 3, *Three plus zero equals three. You can't add it on because it's not there.* (Aged 10, verbal)

It was expected that when presented with 0 – 3 the children would say, *you can't.* Many did but they also gave an answer. In the same vein as the Indian mathematicians, over 800 years ago, who could not bring themselves to the point of admitting that one could not divide by zero (part 1 of this chapter), so the children did not wish to admit that they had no answer to give. One can only speculate on the reasons why this occurred in this study. Reasons such as the fact that the children did not expect to be given an equation unless there was an answer; that it was not the norm for the children to be presented with equations where there was no answer. The use of the phrase *you can't* suggests inability; possibly the children felt that giving no response was a reflection on their ability or lack of it. Whatever the reason, while including the *you can't* phrase or chant, the children became quite determined to provide an answer.

- $3-0=3$, *Three take away zero, you can't take away zero from three.* [Stops. Gives no answer. The child is asked if there is an answer.] *Yes.* [Hesitates.] *Three. You can't take zero way as nothing to take away. I think the answer is three.* (Aged 9, verbal)
- $3-0=3$, *Really you can't take away zero.* [Why not?] *Because it's nothing, because no number to take away. So the answer is three.* (Aged 10, verbal)
- $3+0=3$, *You can't add 0 because you don't have anything to add.* (Aged 11, written)
- $3-0=3$, *You can't take nothing because nothing doesn't exist* (Age 9, verbal)

Can't do it was more likely to be followed by a zero answer. Hence a number of correct answers for multiplication came from a '*can't do it so the answer is zero*' reason. This is also seen in the high number of zero answers to $0 \div 3$.

- $3 \times 0 = 0$, *You can't do it so the answer is zero.*(Aged 9, verbal)
- $0 \div 3 = 0$, *If you can't do it then the answer is zero.* (Aged11, verbal)

Some children said *you can't do it* and this seemed to trigger a *zero rule*. Indeed, quite a number of children adopted a rule (this will be discussed more fully in the following section). Often such *zero rules* were not apparent until the child began to explain a number operation they were not too confident about, as is seen in the following example.

Illustrative incident

A boy, aged nine, used the word '*can't*' frequently. He gave the answers quickly but struggled to provide explanations.

$3+0=3$	*If you've got three sweets and you can't give them anymore.*
$0+3=0$	*You start with zero sweets and you can't get any.*
$3-0=3$	*Three sweets take zero off but you can't take zero off because it isn't a proper number.*
$0-3=0$	*Zero sweets you can't take three away.*
$3\times0=3$	*You can't times anything by nothing.*
$0\times3=0$	*Zero sweets you can't times it by three, you've nothing there.*
$3\div0=3$	*You can't divide three by zero. There's nothing there.*
$0\div3=0$	*Zero.* [How did you know?] *Because the zero is at the front.* [How does that tell you the answer?] *It's the quick way of doing it.* [Can you show me?]

He picked up the zero fact cards, read each one aloud and then he pointed to the first number of each equation as he gave the answer. (aged 9, verbal)

4) The zero connection

Within the *zero connection* category there were three aspects: i) confused, ii) the effect of nothing and iii) the zero rule.

i) Confused

In the *confused* category, there was evidence that in many cases zero was the cause of the confusion. Unfortunately it was not possible, from the data collected, to gain further insight into these children's responses and explanations.

ii) The effect of nothing

The effect of nothing was an aspect also present in the ordering of numbers, discussed in chapter 3, part 3. Again in the zero number facts it is seen how the children extended their understanding of zero from being worth nothing; to zero being of no significance; to zero having no effect; to zero being ignored. Examples of this extension of zero being nothing have been given in all four operations. A further illustration given below is of a child who was explaining 3 + 0 = 0. She wrote 0+0+0 on a piece of paper and continued,

- *That's not a sum because it doesn't make anything but zero. It is a sum because it's got 'adds' but it just makes zero, zero is nothing not something like ten. You see you just forget the zeros they don't matter.* (Aged 8, verbal)

iii) The zero symbol

The zero symbol categories contained the most surprising findings of this section of the research. Here was evidence of children acquiring a personal rule. It was the presence of the '0' in an equation that allowed the rule to be used; it became a *personal zero rule.*

Children pick up spoken patterns very quickly, even to the point of making false generalisations, in the same way as they use 'I goed' or 'I buyed' in learning to use spoken language (Dickson, Brown, and Gibson, 1984, p 201). These patterns are found in mathematics such as in the rhythmic counting in 5s and in 10s (though starting with zero does spoil the flow). There are also visual patterns for example those found on a 1 to 100 square (again starting with zero - in a 0 to 99 square – there is a different number pattern).

Most children search for patterns, for rules, and expect these rules to work in all situations. There are laws and structural properties which the children may have met but they are used in the wrong context, as has been seen in the category relates to a *paired number fact* when the commutative law is expected to work in all four operations. Children also combine laws or they produce their own *personal zero rule.* Added to this, as historian Lisa Jardine warns us,

> Zero is not like other numbers. People say, '*I hate zero because it is so tricky*'. You are not allowed to do certain things with zero that are legal with other numbers.
> (Melvyn Bragg, BBC Radio 4, 2004)

Analysing the addition and subtraction answers could easily mean personal zero rules go unobserved. When Sam, a 6 year old, gives the answers 3 + 0 = 3, 0 + 3 = 3, 3 - 0 = 3, there would appear to be no cause for concern. It is possible that a zero rule is being used in these three equations and, as a successful outcome is achieved, there is no questioning of the child's methods. This is the rationale for Sam's answers,

- *If you get a sum with a number and zero then it's the number* [that is the answer] (Aged 6, verbal)

Such an explanation was not confined to the younger children as is seen in the explanation from a child who was three years older.

- 3 + 0 = 3, 0 + 3 = 3, *Zero is absolutely nothing. The biggest number is the answer, forget about the zero. You don't need the '+0' or the '0+'.* (Aged 9, verbal)

As '3' is the answer to the first three equations both statements result in correct answers, their rule has worked. However, this rule may then become fixed and used in other situations. Some six and seven year old children could happily give answers to multiplication and division using their *personal zero rule* even though they did not fully understand how to multiply or divide. What astonished the researcher and was a cause of great concern was that a consistent *personal zero rule* was used by a small number of 11 year old children across all the zero number facts.

What were these rules? Where '0' and '3' were given as wrong answers these were classified and put in tabular form. (This table of *Incorrect Answers and Patterns* from the 11 year old children can be found in appendix 8.) While these results are of interest in their own right the value comes from following through the pattern of answers for individual children. Many children were found to be using a *personal zero rule* in some sections of the zero number facts. In the Questionnaire and in the Task-Interview there were a total of twenty-four children who explained and used the same rule throughout all the eight equations. Returning to the raw data the responses of these children were re-analysed. Previously analysis had been under each one of the four rules of number, now the explanations were viewed across the four rules. Two personal rules were used consistently: *largest number* and *first number* (see table 40).

Zero Rules						
	Questionnaire	Task-Interview				
Age of children	Aged 11	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7
Largest number	10	2		1	2	4
First number	2		1	2		
Total	**12**	**2**	**1**	**3**	**2**	**4**
Number of children who answered all 8 equations	77	20	11	17	12	7
Percentage *	**16%**	**10%**	**9%**	**18%**	**17%**	**57%**

* The percentage of children who gave either the largest number or the first number as answers to all 8 equations.

Table 40, Zero Rules

Below are examples from children whose data appears in table 40 and which illustrate each of the two rules being used across all four operations.

The answer is the largest number in the equation.

$3 + 0 = 3$	$3 - 0 = 3$	$3 \times 0 = 3$	$3 \div 0 = 3$
$0 + 3 = 3$	$0 - 3 = 3$	$0 \times 3 = 3$	$0 \div 3 = 3$

- *Zero is nothing so you don't need it really. You can forget it. I cover it up.* [Child puts a finger over the '0' for each of the equations.] (Aged 11, verbal)
- *Everything is three.* [Is it? Why?] *Because you wont add or take away or times or anything. A zero sum is easy; the answer is always the other number.* (Aged 7, verbal)

The answer is the first number in the equation.

$3 + 0 = 3$	$3 - 0 = 3$	$3 \times 0 = 3$	$3 \div 0 = 3$
$0 + 3 = 0$	$0 - 3 = 0$	$0 \times 3 = 0$	$0 \div 3 = 0$

- *Beginning with three you've got something to add to, take from, times by. Beginning with zero you can't add something to zero, take something from zero, times something to zero. You've got a zero at the beginning, the answer is zero.* (Aged 9, verbal)

One ten year old boy, whose answers are seen to be in the category of *the largest*, had developed or acquired an imaginative explanation as seen in the following illustrative incident.

Illustrative incident [4]

The researcher observed a Y6 10 year old boy illustrating his answers to 3 + 0 and 0 + 3, by placing three (green) counters and one (white) counter for the zero. This one counter represented zero but was not counted and the answer of 3 was given. This method of using one counter to represent zero continued with 3 – 0 and 0 – 3. However, as the counters had all been replaced in the box, the child very carefully searched to find a

[4] The recounting of this illustrative incident comes from notes and from a tape recording. When the boy began explaining about the ghost the researcher asked the boy's permission to tape record the task-interview.

white counter to represent zero. The child was asked if another colour could be used. *No, the counter for zero has to be white.* [Why?] *Because it stands for a ghost.* [I don't understand.] *Well, zero is like a ghost it zaps through the other numbers.* [I still don't understand.] *This is three* [he holds up three fingers on his left hand] *and the ghost zaps through the three* [the fingers on the right hand slid between the three fingers]. *A ghost can do that. Go through things and not touch them.* [How does this work with the 3 – 0 ?] *The ghost zaps through the three and you can't tell it's been. The answer is three.* [Does this happen with 3 + 0?] *Yes. Same thing.* [But what if the zero is at the beginning? The 0 +3 card was shown.] The boy used the fingers on the right hand to illustrate three and the left hand to 'zap'. *It's just the other way round.* The child continued with the same explanation for the multiplication and division zero facts. When he had finished all the zero number facts it was pointed out that all his answers had been given as three. He was asked if he was happy with those answers. *Yes, zero sums are always easy, when the ghost zaps the other number it doesn't do any damage, the other number is left OK. The answer to all those sums has got to be three.*

(Aged 10, verbal)

Looking at the answers of individual children it was difficult to know if a child was using *a personal zero rule* unless it was in the child's explanation and/or the rule continued into multiplication and division. Often a *personal zero rule* did not emerge until a child was explaining the rationale in an area where he/she was unsure of the answer. It is possible that there were other children, particularly in categories such as *statement of fact* and *no explanation offered,* where a rule was being used for some, if not all of the operations.

> When you say 'I just know' about something ... it means you've stopped thinking about it. You've reached the limits of your understanding. (Herron, 2004, p.56)

In this study children's conceptions have included notions that zero is *nothing*, that zero can be ignored, that zero can create a specific rule of use. How have these ideas developed? What might be the thinking of children, aged 3 to 5, who may not yet have acquired such conceptions? The next chapter contains the work from these young children.

This chapter has concentrated upon the zero number facts within the 0 to 9 number bonds. The final section looks at zero in algorithms to see if there is any linkage between the findings reported in this chapter and the 'zero problems' in algorithms.

~ 0 + 0 - 0 × 0 ÷ 0 ~ 000 ~ 0 ÷ 0 × 0 - 0 + 0 ~

Chapter Four, Part Four

The zero number facts - the wider picture

As was seen in the 'Memory' section of this chapter, while there is a wealth of literature and research related to number bonds few include the zero number facts. The researcher had to use wider sources, which were restricted to small pertinent pieces of information. This limitation was unavoidable and understandable as this was an exploratory investigation of an, as-yet, uncharted area of investigation. One of these restricted sources was in the field of algorithms and calculations where a 'zero problem' was in evidence.

Guedj states that if you wish children to succeed in arithmetic tests then you omit the zeros. He goes on to write,

> Zero has a bad reputation among schoolchildren. Students find it the most difficult number to understand. Up to the age of six and a half, 25% of children write $0 + 0 + 0 = 3$; up to the age of eight and a half, 50% write $0 \times 4 = 4$. (Guedj 1998, p109)

Unfortunately Guedj does not qualify these statements nor does he provide any sources for the data. Suydam and Dessart (1978) found one of the seven most frequent whole number errors was found to be errors with zero for each operation. However, this work was in the context of algorithms not single digit zero facts. Other publications substantiate the general thinking that difficulties about zero permeate whole number computation but few research projects address children's understanding of zero unless it is in a place value context and in algorithms of two or more digits (Lappan, 1987).

The work on algorithms does provided some information on the zero number facts, an algorithm being defined as being 'a mechanical procedure for solving a problem in a finite number of steps' (Daintith and Nelson, 1989, p.14). This researcher's experience has shown that children negotiate many of these different 'steps' in an algorithm as if they were single number facts. If a child is asked to complete,

$$\begin{array}{r} 2384 \\ -\ \underline{1103} \end{array}$$

It is very likely that the calculation will be broken down into 4 - 3, 8 - 0, 3 - 1, 2 - 1. Hence the knowledge and use of number bonds plays an essential role in mental and in pencil and paper calculating.

With the inclusion of zero in a calculation, what are the frequency and types of errors? Englehardt (1977) investigated the computational errors of 198 children, aged 9. From his 'Table of Frequency of Errors by Type' the following list has been extracted.

Englehardt's *Percentage of Error Type*

Basic Fact	33%
Grouping	22%
Inappropriate Inversion	21%
Incorrect Operation	4%
Defective Algorithm	18%
Incomplete Algorithm	7%
Identity	1%
Zero	6%

(Adapted from Dickson, Brown and Gibson, 1984, P.257)

List 6

However, there seems to be evidence of supposition in classifying some of the errors. For example Englehardt cites

$$\begin{array}{r} .2 \\ \underline{\times 3} \\ 5 \end{array} \qquad \text{as an } \textit{incorrect operation.}$$

But it could have been a mis- remembered *basic fact.* There was confusion in the *identity error* and *zero error* classifications. He describes an *identity error* as being a mistake, which indicates some confusion with the operation identities 0 and 1, while a *zero error* is where there is an indication of some difficulties with the concept of zero, such as 1 - 0 = 0. One may ask how it is possible to classify 1 - 0 = 0 using only the basis of the written response. Englehardt acknowledges these shortcomings and recommends an interview approach to help overcome this problem. He also admits the limitation of his study in terms of not considering the correct responses; some of which he feels were derived from 'erroneous approaches'.

In Ward's research, published in 1979 (in Dickson, Brown and Gibson, 1984, p260), he surveyed the errors of 10 year olds from 40 schools, asking teachers for examples of common mistakes which occurred in mathematics lessons. He found that by far the commonest mistakes mentioned were to do with subtraction, mainly to do with taking the smaller from the larger regardless of position, problems with zero (whether on top or bottom), and regrouping (borrowing). Again, it was not possible to spot any recognisable procedure because the children's explanations were missing.

Brown and Burton (Brown and Burton, 1978; Burton, 1981, in Dickson, Brown and Gibson, 1984, p.260) attempted to rectify this by investigating the errors arising in the subtraction algorithm using a total of 2500 American children.

They found that, in most cases, the errors were systematic. They reasoned that the child was following some well-defined procedure, which could be programmed on a computer, and which was, in general, identical to a 'correct' procedure except for one or more faulty steps in the program. They analysed them by using the analogy of 'bugs' in a computer program. Examples of seven of the commonest 'bugs' are shown in list 7, as summarised in 1982 by Resnick (in Dickson, Brown and Gibson, 1984).

Six of the seven most common 'bugs' listed involve zero. The 'bugs' are formed from the outcomes to the algorithms and do not include explanations of the children's thinking.

The seven commonest 'bugs' found by Brown and Burton (1978):
(Adapted from Dickson, Brown and Gibson, 1984, pp.261-2)

1) Smaller-From-Larger. The student subtracts the smaller digit in a column from the larger digit regardless of which one is on top.

326	542
-117	-389
211	247

2) Borrow-From-Zero. When borrowing from a column whose top digit is 0, the student writes 9 but does not continue borrowing from the column to the left of the 0.

602	802
- 437	- 396
265	506

3) Borrow-Across-Zero. When the student needs to borrow from a column whose top digit is 0, he skips that column and borrows from the next one. (This bug requires a special "rule" for subtracting from 0: either 0 - N = N or 0 - N = 0.)

602	804
-327	- 456
225	308

4) Stop-Borrow-At-Zero. The student fails to decrement 0, although he adds 10 correctly to the top digit of the active column. (This bug must be combined with either 0 - N = N or 0 - N = 0.)

703	604
- 678	-387
175	307

5) Don't-Decrement-Zero. When borrowing from a column in which the top digit is '0' the student rewrites the 0 as 10 but does not change the 10 to 9 when incrementing the active column.

902	205
- 368	- 9
344	1106

6) Zero-Instead-Of-Borrow. The student writes 0 as the answer in any column in which the bottom digit is larger than the top.

326	542
- 117	-389
210	200

7) Borrow-From-Bottom-Instead-Of-Zero. If the top digit in the column being borrowed from is 0, the student borrows from the bottom digit instead. (This bug must be combined with either 0 - N = N or 0 – N = 0.)

702	508
- 368	- 489
454	109

List 7

Looking at example 6 in list 7,

$$\begin{array}{r} 326 \\ -\underline{117} \\ 210 \end{array} \qquad \begin{array}{r} 542 \\ -\underline{389} \\ 200 \end{array}$$

from the work of contained within this study this researcher could envisage a scenario where a child is thinking, *'I can't do it and so I put zero'.*

This could also have been the response of a pupil in a particular case alluded to by Linda Dickson in her work with some low attaining 13 to 14 year olds (in Dickson, Brown and Gibson p.259) who attempted to work the division 216 ÷ 9 from right to left,

$$\begin{array}{r} 9\,\underline{)\,2\;1\;6\,} \\ 0\;0\;0 \end{array}$$

'9s into 6 is 0
'9s into 1 is 0
'9s into 2 is 0'.

Apart from stating that the pupil realised this was not a sensible answer but she could not remember how to do it, Linda Dickson did not pursue the zero answers as her research was centred on place value. While there are no zeros in the original algorithm one may speculate as to whether each of the '0s' in the answer was a result of the child whose rule was *'I can't do it so I put zero'.*

> The uncomfortable gap between numbers (which stood for things) and zero, which didn't, ... narrows the focus from what they were to how they behaved. Such behaviourism took place in equations ... the number which made it balance was as likely to be zero as anything else. (Kaplan, 1999, p.75)

It is felt that the findings in the area of the zero number facts, recounted in this chapter, add to the debate and thinking in children's approach to algorithms.

~ 0 + 0 - 0 × 0 ÷ 0 ~ 000 ~ 0 ÷ 0 × 0 - 0 + 0 ~

CHAPTER 5

THE EMPTY SET

A cup is made of bottom and sides. But its use lies in its emptiness. (Lao Tzu)

(A quotation seen at The Eden Project, Cornwall, 12th March, 2005, which referred to Green Gold of the East, a bamboo whose stems are used for making cups.)

Outline of Chapter 5

Part 1 of this chapter explains the decision to work with the younger children and the reasons for extending the study sample. Part 2 is concerned with the Bottle Activity-Interview. Details such as its design, presentation to the children, collation and analysis of the data, and summary of the findings are discussed. Similarly part 3 concentrates on the Ribbon Activity-Interview. Again its design, presentation to the children, collation and analysis of the data, and summary of the findings are explained. Part 4 compares and contrasts the information from both Activity-Interviews.

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Chapter Five, Part One: Setting the scene

The scope of this research was in four focus areas:

1) The empty set
2) Zero as a number and its relationship to other numbers
3) The zero number facts
4) The language of zero.

This chapter concentrates on the first of these areas, that of the empty set. Without exception, at some point in the Task-Interviews, each child (aged 7 to 11) referred to zero as *nothing*. The aim of this section of the research was to explore how children would react to *nothing* in the form of the empty set. Because of the older children's association with 'nothing and zero' the intent was to use a research sample of children who were unlikely to have this strong association; hence the work was intended to be undertaken with young children, nursery and reception children aged 3 and 4 years.

Collecting data from young children

There were issues about collecting data from young children; these included the children's limited ability to explain themselves, their interpretation of each question and the analysis of their understanding. It was recognised that these needed to be carefully addressed. Added to this was the knowledge that a minor change in any research task could result in significant changes in a child's response. Activities were developed and presented in the interview mode; to differentiate them from the tasks (with their link to the Questionnaire content) they are referred to as the *Activity-Interviews*. As with the *Task-Interview*, the *Activity-Interview* was semi-structured thus giving the researcher the flexibility to explore further, to encourage the children to extend their ideas, to explain their reasoning and for the researcher to follow fruitful lines of enquiry. There was a set pattern of presentation, the situation was informal and, when necessary, the 'curious questioning' model was adopted. Wood (1991, cited in Gifford, 2003) suggested this model. He found that using direct, closed questions with young children are less effective than with older children and that higher level of responses from the younger children come from speculative remarks such as 'I wonder why…?'. While there were times when a child's reaction would change the style of the questioning, or order of the Activity-Interview, the nature of the content did not change.

These Activities-Interviews were to cover four aspects,

- An empty set in a non-numerical situation
- An empty set presented to the child
- An empty set in a numerical situation
- An empty set formed as result of a task competed by the child.

The attention span of a young child is short and for this reason only two activities were used, *The Bottle Activity* and *The Ribbon Activity*. In order to capture and keep the young child's attention and interest the activities needed to be visual and to involve the child. A puppet was used as some psychological researchers found that using a puppets produce more responses from young children (McGarrigle and Donaldson 1974, cited in Gifford, 2003) their reasoning being that it reduces the reluctance to risk wrong answers and to lose face with an adult. The puppet acted as a 'go between' as the researcher's questions and queries were asked via the puppet. However, it was soon evident that the children were interested in each activity in its own right. They were more than happy to be involved in the work to the point where children would come and ask if they could have another go with the ribbons and bottles. There was no mention of the puppet. The puppet was used only with the nursery children. The following sections deal, separately, with the designing, the delivery of and children's responses to the Bottle and the Ribbon Activity-Interviews.

Extension of the sample range

The evaluation of the data and initial interpretation of the results was undertaken after the Activity-Interview work had been completed with the 3 and 4 year olds. The outcomes were unexpected and fascinating. They highlighted a need for further collection of data to substantiate and clarify the findings. This resulted in an extension of the research sample to include children aged 5 to 10 years in the hope that this might reveal cognitive function changes in knowledge and understanding of '*nothing*'.

While these Activity-Interviews were designed with young children in mind they needed no alteration to make them acceptable for use with the older children. Indeed as the Bottle and Ribbon Activities were presented to the older children before the Task-Interviews they became useful 'ice-breakers' and discussion points.

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Chapter Five, Part Two: Empty in a non-numerical context

The Bottle Activity-Interview

Young children's early mathematics begins in a qualitative rather than a precise numeric world and includes words such as *some, a lot, a bit, none.* The researcher wanted to use an activity that did not include number but could have an empty set. The intent was to give a visual image of *nothing,* if that is not a contradiction in terms. The activity was based in the realm of measures. While it would not be possible to present nil length or nil weight one can have 'visual' emptiness in capacity. The Bottle Activity would include an empty set presented to the child in a non-numerical situation.

Designing the Bottle Activity

It was important to keep as many elements as possible constant for, with young children in particular, aspects such as a colour change can affect an outcome. Clear plastic bottles, with tightly fitting screw caps, were used. The bottles were identical. To make the liquid very visible it was coloured with green food dye. It was necessary for the researcher to introduce comparison bottles to initiate the language of emptiness in order to note whether a child could differentiate, visually, linguistically and in writing between the bottles containing varying amounts of liquid. For example, a child might use the language of emptiness if there was a small amount of liquid in a bottle. The comparison allowed for more specific descriptions such as the one used by a 10 year old who described the empty bottle as, *fully empty*. The only variable was the amount of liquid in the four bottles; these were - full, approximately half-full, a small amount of liquid, empty (illustration 17).

Bottle Activity Resources

Illustration 17

Administering the Bottle Activity

In the Bottle Activity the child was asked to:

(a) ***Describe the bottles***

The child was shown all four bottles, arranged haphazardly. The researcher asked the child *to choose a bottle, hold the bottle and tell me about it.* This was repeated until all four bottles had been described.

(b) Provide a reason for the emptiness

Throughout the Activity-Interviews, in order not to put 'words in the child's mouth', the researcher used the same word or phrase used by the child. So, if a child said the bottle *had none in it* then the researcher used that same phrase: the researcher would pick up the empty bottle and ask, *Why do you think this bottle has 'none in it'?*

(c) Label the bottles

The researcher fastened a sticky label to the side of a bottle and gave the child a pencil. (The older children used a non-permanent marker and wrote on the plastic bottle.) The child was asked *to put something on the sticky label so we know how much is inside the bottle.*[1] This was repeated for all the four bottles.

(d) Order the four bottles

The child was asked to put the bottles in order and then *to touch the first bottle, the next bottle, the next bottle, the last bottle*; this was to establish the starting point of the sequence.

With all the age ranges the Bottle Activity was the first activity presented to the children as it did not contain any element of number. If the word 'zero' and/or the '0' symbol were used it was at a child's instigation and not as a result of him/her having been affected by the other research tasks and activities.

The Bottle Activity, collation and analysis of the data

(a) Describing the bottles

The noticeable, initial reaction to the set of bottles, particularly by the 3 and 4 year olds, was surprising. The majority of these young children exhibited one of two trends.

Trend one - first there were the children who appeared perturbed by the empty bottle. Their immediate reaction on seeing the set of bottles was to home in on the empty bottle and ask,

- *Empty, where's it gone?* (Aged 3, verbal)
- *Why is that one empty?* (Aged 3, verbal)
- *Where is it?* (Aged 3, verbal)
- *There isn't anything in that one. Why is there none in?* (Aged 4, verbal)
- *None in there, where's it gone?* (Aged 4, verbal)
- *That one is really funny.* [He points to the empty one, the researcher asks, Why?] *It has nothing in it. It's odd.* [The child laughs.] (Aged 5, verbal)

Some children were so anxious that they *wanted to put liquid in the empty bottle.* Two 3 year old children and one 4 year old were so concerned that they took the empty bottle to the sink to put water in it.

- *Empty. Have to put some in this* (Aged 4, verbal)

[1] Neither the word *writing* nor *drawing* was used in order not to suggest a mode of presentation; as Hughes writes this 'left open to the children as many options as possible' (Hughes, 1986, p.55).

Trend two - the second reaction was to ignore the empty bottle. The 3 and 4 year old children described the other three bottles but the researcher had to point to the empty bottle and ask the children *What about this bottle?* Some of the responses verged on the dismissive,

- *There isn't anything in that one.* (Aged 3, verbal)
- *But that one's got none in it. You don't need it.* (Aged 4, verbal)
- [Child laughs] *Odd one because it's got none in.* (Aged 4, verbal)

What words were used to describe the empty bottle? These are to be found in table 41.

Oral language used to describe the empty bottle								
Age of children	10	9	8	7	6	5	4	3
Number of children	12	20	16	13	13	17	14	11
Empty	9 75%	12 60%	6 38%	7 54%	6 46%	4 23%	1 7%	2 18%
None	2 17%	1 5%	3 19%	4 31%	5 38%	10 59%	6 43%	6 55%
Nothing	1 8%	3 15%	3 19%	2 15%		1 6%	1 7%	1 9%
Zero		3 15%	3 19%				3 21%	
Not full		1 5%			1 8%	1 6%		
Not got any in					1 8%			
All gone								1 9%
No response			1 6%				1 7%	1 9%
Other						1 6%	2 15%	

Table 41 - Bottle 1

The 3, 4, and 5 year old children found describing the bottles containing liquid quite challenging though the results were apt and inventive (with the focus of attention being on the bottle or on the liquid). The following, describing the bottles from full to empty, were used by 3 year olds.

- *Right to the top; right up to there; right up to there; none.* (Aged 3, verbal)
- *Green, green, green, none.* (Aged 3, verbal)
- [No answer. No answer. No answer.] *All gone.* (Aged 3, verbal)
- *Full; no; very no; empty.* (Aged 3, verbal)

What is particularly noteworthy is that, while a number of the 5, 4 and 3 year olds failed to describe the bottles containing liquid, only three children across the whole age range (3 to 10) did not provide a description of the empty bottle.

Three children used incongruous answers, the most perturbing being,

- *Daddy Bear* (full bottle), *Mummy Bear* (half-full bottle), *Baby bear* (small amount of liquid in the bottle), [Child stopped. The researcher indicated the empty bottle] Child said *The baby is dead. Mum shot it.* (Aged 5, verbal)

Overall the words used most frequently to describe the empty bottle were *empty, nothing, none.* The 3, 4 and 5 year olds tended to say *none,* the 7, 8, 9, 10 year olds tended to use the word *empty.* When the word *zero* was used it was because the child gave the other bottles a numerical value.

- *Ten, two, one, zero* (Aged 5, verbal)
- *Ten litres, five litres, two litres, zero litres* (Age 8, verbal)
- *Thirty millilitres, fifteen millilitres, five millilitres, zero* (Aged 9, verbal)

(b) Reason for the emptiness

Reasons given as to why the bottle was empty								
Age of children	10	9	8	7	6	5	4	3
Number of children	12	20	16	13	13	17	14	11
Liquid removed	5 42%	10 50%	7 44%	6 46%	6 46%	7 41%	7 50%	7 64%
Liquid not been put in	5 42%	8 40%	7 44%	3 23%	4 31%	3 18%		
Both	2 16%	1 5%	2 12%		1 8%	3 18%		
No reason given		1 5%		4 31%	2 15%	4 23%	7 50%	4 36%

Table 42, Bottle 2

Table 42 shows there were two explanations offered for the bottle being empty,

i) The liquid had been removed – while this was used by some children in each age range it was the only reason given by the 3 and 4 year old children.

- *The lid was off and someone knocked it over.* (Aged 10, verbal)
- *It's evaporated.* (Aged 9, verbal)
- *Somebody poured it out.* (Aged 8, verbal)
- *The water's all gone.* (Aged 4, verbal)
- *All gone.* (Aged 3, verbal)
- *None left.* (Aged 3, verbal)
- *Empty, all gone.* (Aged 3, verbal)
- *None, tipped all of it out, got none left.* (Aged 4, verbal)

Two children, both aged 5, felt that not only had the liquid been removed but that it had been put in the full bottle.

- *All been put in there* [points to the full bottle] (Aged 5, verbal)

ii) The liquid had not been put in – this reason was not used by any of the 3 and 4 year old children.

- *You forgot to put some in.* (Aged 8, verbal)
- *Nobody put any water in it.* (Aged 9, verbal)

(c) Labelling the empty bottle

After orally describing the bottles the children were asked *to put something on the label so we know how much is in the bottle.* Most of the children labelled the bottles using the same word/phrase they had used, orally, to describe them. Those who said *empty* tended to label the bottle with the word *empty* (allowing for spellings various).

(Verbal) *Full, half, nearly empty, empty*
(Written) Full, half, nearly emty, emty. (Aged 6)

Labelling the bottles								
Age of children	10	9	8	7	6	5	4	3
Number of children	12	20	16	13	13	17	14	11
Empty	10 83%	13 65%	6 38%	6 46%	6 46%	3 18%	1 7%	
Nothing	1 8%	1 5%	1 6%	2 15%				
None	1 8%		2 13%	2 15%	4 31%	3 18%	1 7%	
'0'		5 25%	5 31%	2 15%	1 8%	5 29%	8 57%	
Not full		1 5%			1 8%	2 12%		
Non litres			1 6%	1 8%				
No water					1 8%			
Less						1 6%		
Arbitrary marks[2]							2 14%	10 91%
10 (child said *this means none*)							1 7%	
Deliberately left empty								1 9%
No recording			1 6%			3 18%	1 7%	

Table 43, Bottle 3

Table 43 contains the categorised data showing the ways the children labelled the empty bottle. It can be seen that '0' appeared more frequently than had the oral word *zero* in table 41. With the 5 to 7 year old children it was mainly the verbal *none* which was replaced by '0' (referred to as *zero* by all the children); interestingly most of these children qualified the meaning and use of '0'.

- *I've put a zero, that's none, got none.* (Aged 4, verbal)
- *Got none. I've put zero.* (Aged 4, verbal)
- *Zero. Means none.* (Aged 7, verbal)

[2] Arbitrary marks - they contained squiggles, marks, part pictures or part of an alphabet letter.

Illustrative incident

The full bottle.
That's one whole one so I put one. [The child writes '1'.]
The half full bottle.
That's half so I put half. [The child writes '½'.]
The bottle containing a small amount.
That's about a quarter so I put quarter. [The child writes '¼'.]
The empty bottle.
That one's empty. [Pauses] *I put zero.* [The child writes '0'.] *Because I don't know what the fraction is for this zero.*

(Aged 8, verbal and written)

With the children aged 8 to 10 the tendency was that children who recorded a numerical value for the bottles containing liquid would then use '0' for the empty bottle. For example, with 1, ½, ¼, 0 and with 10 litres, 5 litres, 2 litres, 0 litres. However, a small number of the younger children had used a number to orally describe the amount of liquid in the bottles (such as *ten, five, two, empty*) and then the verbal *empty* became a written '0'. Whenever a numerical value was placed on the bottles containing liquid the empty bottle was given the '0' symbol.

Two other recordings, of 3 year old children, were noteworthy, as they were pertinent to the empty set. One child used arbitrary marks; these were, to the researcher, indecipherable squiggles on each of the four labels but the child scribbled out the marks on the paper for the empty bottle saying *None.*

The other 3 year old was the only child who did not record anything for the empty bottle leaving The label blank saying

- *Don't put anything, none in it so don't put anything.* (Aged 3, verbal)

(d) Ordering the bottles

The initial reason for asking the children to order the bottles was to see where the child placed the empty bottle in relationship to the other bottles. Only three children did not use the ordering sequence ***full to empty*** or ***empty to full*** (they did not provide a reason for their order). While most children ordered the bottles in a left – right direction, to ensure the researcher understood where the sequence started each child was asked to touch the first bottle, the next, the next and the last bottle.

Ordering the bottles								
Age of children	10	9	8	7	6	5	4	3
Number of children	12	20	16	13	13	17	14	11
Full to empty	5 42%	8 40%	7 44%	6 46%	7 54%	11 65%	11 79%	8 73%
Empty to full	7 58%	12 60%	9 56%	7 54%	6 46%	4 23%	3 21%	2 18%
Other						2 12%		1 9%

Table 44, Bottle 4

Unfortunately the children were not specifically asked why they had ordered *full to empty* or *empty to full*. However, as they worked some children gave a reason, directly to the researcher or by speaking aloud. In the age range 6 to 10 both the *full to empty* and *empty to full* order were used (see table 44). Children remarked that the order was *biggest to smallest, small to large,* or that *empty goes first, empty goes last.* A minority of the children saw the ordering of the four bottles as relating to the number order. This was interesting in that it suggested a move into the world of mathematics, linking the ordering of the bottles to the number line rather than to a real life situation as was seen with the younger children.

- *It's like counting down, the liquid is taken out.* (Age 8, verbal)
- *You always put empty first, on the left, like the number line.* (Age 8, verbal)
- *Smallest to largest - like counting.* (Age 8, verbal)

The younger the child the more likely the bottles were ordered full to empty. Was this because the child saw the order as demonstrating that the water had been removed?

- *All there, used some, used some more, used it all.* (Aged 5, verbal)
- *New Bottle* [full bottle], *old bottle, empty, the oldest* [the empty bottle]. (Aged 5, verbal)

Bottle Activity - summary of the findings

With the 3 and 4 year olds two extreme reactions, to the empty bottle, were identified; one was anxiety, the other was indifference. These reactions were not mutually exclusive; some children exhibited both, some one. The strong show of anxiety came as a shock to the researcher. These young children were very concerned and demanded to know why the bottle was empty, asking where the contents had gone. The ignoring of the empty bottle was more pronounced with the younger children, aged 3 and 4. It also occurred with a very small minority (one, occasionally two children) in the age range 5 to 8. This ignoring of the empty bottle was observed at different points in the Bottle Activity; in the describing of the bottles; in the labelling of the bottles and in the ordering of the bottles. At these times the researcher would have to remind the child to include all the bottles.

When the reasons given as to why the bottle was empty were collated there were two categories: because the liquid had been taken out (*removal*) or because no liquid had been put in (*omission*). With the older children both removal and omission were present in the reasoning. With the younger children there was a strong tendency for the emptiness to be seen as a result of removal. On reflection the researcher appreciated that this could have been due to the nature of the activity. Many children would be accustomed to seeing a bottle full of a liquid, possibly of a liquid they could drink, thus associating the empty bottle as being originally full and the emptiness being the result of the liquid been removed.

Illustrative incident

A five years old child, when shown the set of bottles, arranged them from full to empty. He then started to weave a story. Beginning with the full bottle he moved along the line of bottles, picking up each bottle in turn.

- *This is a new bottle, it's full.*
- *You used some.*
- *You used some more.*
- *This is the old bottle. All empty. You used it all. Buy a new one.*

(Aged 5, verbal)

Most children described the bottles in qualitative terms: *full, half full, a bit, empty.* The empty bottle was described using words and phrases such as *none, nothing there, not full.* Only a few children used a numerical description: *one, half, quarter, zero,* or *ten, five, two, zero.* When labelling the bottles the strong tendency was for a child to continue using the same mode (words or numbers) as he/she had used to describe the bottles' contents. The symbol '0' (read as zero) usually appeared when a child used a numerical approach to the labelling of the bottles. A number of the younger children who had described the empty bottle using the word *none* wrote '0' but each of these children qualified the use of the '0' symbol by explaining that '0' meant *none.* All the children who wrote '0' read this as *zero* (see the Language of Zero, chapter 6).

0 ~ 00 ~ 000 ~ 00000 ~ 000 ~ 00 ~ 0

Chapter Five, Part Three - Empty in a numerical context

The research of Martin Hughes

As mentioned in chapter 1, while attending a conference of the British Society for the Psychology of Learning Mathematics (BSPLM) at Manchester Polytechnic, Didsbury in 1984, this researcher heard Martin Hughes present a paper on the topic of young children dealing with single digit numbers. While Hughes' work (subsequently published in Hughes, 1986) focused on addition and subtraction and written algorithms, one section concentrated on written symbolism as representations of quantity. Particularly fascinating, and applicable to this study, were his findings on children recording the empty set.

One of his activities was the 'Tins Game'; this he conducted with twenty-five children aged 3 to 5 years. The Tins Game entailed using four identical tobacco tins with three, two, one and no bricks inside each tin. The child was shown each tin in turn and was asked: 'to put something on the paper (attached to the lid of the tin) to show how many bricks are in the tin'. The lid was put on the tin. This was repeated with all four tins. A week later the child was asked to identify the number of bricks in the tin using the information they had written on the label. (A full description of the Tins Game is given in appendix 9). It was not intended, in this study, to replicate this activity but to use it as a basis for formulating a similar activity.

Recognising the empty set

> One problem facing any system of representation is how to represent the *absence* of quantity, or 'none'. (Hughes, 1986, p.62)

When stating that 'there are none', that 'there is nothing', the phrase is incomplete. Rarely is it intended to convey that there are none anywhere, unless one is referring, say, to dinosaurs. There is normally a contextual situation, albeit unspoken, in mind: 'there are none - left in the building', 'there is nothing - on the plate'. When having three counters in one's hand and asking 'how many have I got?' it is likely the child will appreciate that you are asking about the counters. But to hold out one's hand and ask the same question can pose problems. 'How many have I got?' could refer to fingers or hands; it suggests an amount with a counting number as the answer. The question needs to be more specific and to include the unit name and define the boundary, 'How many *counters* have I got *in my hand*?' It is necessary to enclose the set, to show clearly the boundary within which the quantity, or non-quantity, lies. This is particularly relevant when dealing with the empty set as was demonstrated in Hughes' original activity involving bricks being placed on the table.

> We found that many children responded to the problem of presenting 'nothing' with puzzled looks and clearly found it hard to understand what we were asking them to do.
> (Hughes, 1986, p.63)

Hughes altered the activity and placed the bricks in tins and also changed the activity to a 'game' situation. He writes that between the two activities there was a notable difference which

> ... concerned the children's representation of zero. In our earlier study, the request to show that there was nothing on the table had led to several puzzled looks. In the Tins Game, representing the fact that there was nothing in a tin was treated just as straightforwardly as the other representations. (Hughes, 1986, p.65)

Hughes associates this change to his adoption of the 'game' situation but this researcher feels that there was a stronger reason. When the 'emptiness' was enclosed, the children could more easily understand that it was the empty set to which the interviewer was referring. Hence in formulating Activity-Interviews, which involved the empty set, it was important that the empty set had a clear boundary. With the non-numerical activity a bottle was used; with the numerical activity, which involved objects, opaque plastic containers were used.

Designing the Ribbon Activity.

In designing the Ribbon Activity, for this study, the researcher was mindful of the Tins Game used by Hughes. The numbers of objects used was kept small (Hughes, 1986, p.27-28 and chapter 4, part 1 of this study). Hughes uses 1, 2, 3, and 0 for his sets of objects and he speaks of the possibility of identification by default, that a child

> ...could identify the empty tin by default: that is, having identified the others as containing definite quantities of bricks, they would know that the remaining tin contained nothing. (Hughes, 1986, p.66)

In this study, in order that the children would not 'expect' the 1, 2, 3, counting numbers, and thus attempt to 'predict' one of the sets, the size of the sets used were 2, 3 and 0.

What objects would be used? Hughes had used plastic bricks. It was felt that using bricks or counters added the dimensions of noise and weight. Thus, as the tins were moved, a child could recognise the empty tin by its lack of weight and lack of sound. In the Horizon television programme 'Twice Five Plus the Wings of a Bird' (1985) in which Hughes demonstrated the Tins Game, it could be seen that he, rather than the child, handled the tins. In this research the children, not the researcher, needed to handle the containers. Another consideration was the interpretation of a visual representation of the contents of the tin and their reproduction on paper. How would a crudely drawn o or □ be translated? Was this to be interpreted as *one brick*, as the *circular form* of the container, as a representation of *an empty set* or indeed of the conventional *'0'* symbol? In an attempt to exclude some of these possibilities it was decided to use ribbons as objects. They were light, would make no noise if the containers were shaken and, in a child's recording, were less likely to be mistaken for a conventional number symbol.

The Bottle Activity fulfilled the first two aspects of the Activity-Interview (an empty set in a non-numerical situation and an empty set presented to the child). The Ribbon Activity fulfilled the other two aspects: an empty set in a numerical situation and an empty set formed as a result of a task completed by the child. Hughes had presented the tins to the children with the counters inside, as a *fait accompli.* In the Ribbon Activity sets would not be given to the child but would be created by the child as a result of sorting. The researcher felt that this change was most significant; it proved to be so.

Administering the Ribbon Activity

The Ribbon Activity looked at the sorting and the production of an empty set; the language used to describe the empty set and the representational recording of the empty set. The intention was to follow the structure below, with the understanding that, as with the Bottle Activity[3], the reactions of the younger children may result in a slight change of order.

(a) Creating the empty set

The child was shown three opaque plastic containers (coloured red, green and yellow) and was asked to identify the colours. A handful of ribbons (3 green, 2 red) was placed on the table and the child was asked to put the ribbons in the corresponding boxes *so that the colours of the ribbons match the colours of the boxes*. The child was asked: *How many ribbons there are in each box?* Most children counted the ribbons as they sorted them, only a small number needed to tip out the green and the red ribbons they had sorted.

The Ribbon Activity Resources

Illustration 18

(b) Describing and labelling the empty set

The child was asked to put the lid on each box and *use the special pen to put something on the top of the box so you will know how many ribbons are inside the box.*[4]

[3] As with the Bottle Activity a puppet was used only with the nursery children.

[4] Neither the word *writing* nor *drawing* was used in order not to suggest a mode of presentation; as Hughes writes this 'left open to the children as many options as possible' (Hughes, 1986, p.55).

(c) Recognising the empty set when it is not visible

When all three boxes were closed and labelled the researcher would point to a box. *Don't open the box. Tell me, how many ribbons are in that box?* [This was repeated for the other boxes.]

(d) Reasons for the emptiness

Finally the child was asked: *Why are there no ribbons in the yellow box?*

The Ribbon Activity, collation and analysis of the data

(a) Creating the empty set

The researcher felt that a significant aspect of the Ribbon Activity was that it was the child who created an empty set as a result of sorting. Having no yellow ribbons to put in the yellow container posed a dilemma, a dilemma a number of children tried to overcome. These two facets a) *The dilemma of the empty set* and b) *Overcoming the dilemma of the empty set* are the subject of the following sections.

(b) The dilemma of the empty set

There was an unexpected, strong, initial reaction, particularly from the younger children, about the lack of yellow ribbons. Table 45 shows the percentage of children who expressed this concern.

Voiced strong concern about the lack of yellow ribbons								
Age of children	10	9	8	7	6	5	4	3
Number of children	12	20	16	13	13	17	14	11
Concern about the lack of yellow ribbons		3 15%	3 19%	3 23%	4 31%	9 53%	6 43%	5 45%

Table 45, Ribbon 1

As soon as they saw the small pile of ribbons the majority of the children appreciated that there were no yellow ones. The younger children voiced their immediate concern as to the lack of yellow ribbons, this continued as they counted the green and the red ribbons. Some children began to search for the yellow ribbons in the researcher's case, and handbag. Some became so anxious that, at this point, the researcher had to tell them that she did not have any yellow ribbons.

- [The child's first reaction on seeing the pile of ribbons.] *Where's the yellow?* [While counting the green ribbons he asks,] *What about the yellow?* [He counts the red ribbons then asks,] *What about the yellow?* [He looks under the table.] *But where are the yellow ribbons? Where are they?* [He gets very agitated and so the researcher says she has no yellow ribbons.] (Aged 5, verbal)

- [The child looks carefully at the pile of ribbons. The researcher asks,] *Are you not happy with something?* [The child replies,] *Don't know. I can't find the yellow ones.* [She counts the three green and two red into the boxes. Stops.] *What about the yellow box? I can't find them. There must be some yellow.* [Becomes very concerned that there are no yellow ribbons in the yellow box. The researcher says that she has no yellow ribbons.] (Aged 4, verbal)
- [The child counts the green ribbons.] *Three green.* [Counts the red ribbons.] *Two red ribbons. Don't know this. Can't find them. Must be yellow ones.* [The child becomes most concerned and looks in the green and the red box.] *I want the yellow ones.* [The researcher says that she has no yellow ribbons.] (Aged 3, verbal)

The younger children found it difficult to accept that there were no yellow ribbons until they were told there were none. When the older children asked about the yellow ribbons they appeared to be checking whether there should be any yellow ribbons (possibly wondering if 'Miss' had forgotten to bring them) but then they realised, and accepted without being told, that there were no yellow ribbons.

- [On being given the pile of ribbons the child asks -] *Is there a yellow one?* [As she sorts the ribbons she says to herself -] *No yellow ones. Red and green got ribbons but not the yellow ones.* (Aged 6, verbal)
- *I'm looking for the yellow ones.* [Sorts the green and red ribbons. Then begins to smile, starts to laugh.] *I see. There's none in that box.* (Aged 7, verbal)
- [When given the pile of ribbons the child begins by asking -] *Are there no yellow ribbons?* [She looks towards the researcher's case. Then continues with the activity] (Aged 8, verbal)

(c) Overcoming the dilemma of the empty set

- *I don't like having no yellow ribbons. I don't like it. I don't like to write that there are no ribbons. Could I put something else in?* (Aged 9, verbal)
- *Yellow hasn't got any ribbons.* [Looks in researcher's briefcase and resources case. Demands that the researcher-] *Put some in.* (Aged 5, verbal)

As has been illustrated, quite a number of children were unhappy at the lack of yellow ribbons. Why would they want to have some yellow ribbons? For two possible reasons: because of the presence of a yellow container and because of the dislike of having an empty set. The concern at having an empty set was strong, so much so that some tried to overcome the dilemma. To overcome the emptiness of the yellow container two methods were attempted (table 46). These were *putting in other ribbons* of the wrong colour, a method used by children aged 5 to 9, and *imagining the ribbons* did exist, a method used by the children aged 3 and 4.

Overcoming the empty set 'dilemma'								
Age of children	10	9	8	7	6	5	4	3
Number of children	12	20	16	13	13	17	14	11
Put in other ribbon		1 5%	3 19%	1 8%	3 23%	3 18%		
Imagine the ribbons							5 36%	4 36%

Table 46, Ribbons 2

The following five examples are from children, in each of the age ranges 5 to 9, who tried to overcome the empty set dilemma by putting one of the other ribbons in the yellow box,

- *How come there are none for yellow?* [The child puts a green ribbon in the yellow box.] (Aged 9, verbal)
- [The child looks at the wide red and green ribbons.] *Can I call the thick ribbons yellow?* (Aged 8, verbal)
- [The child inspects carefully the red and green ribbons.] *Are there yellow spots on these?* (Aged 7, verbal)
- [The child sorts the ribbons then moves a red ribbon into the yellow box. The researcher reminds the child that the ribbons should match the colour of the box. The child moves the red ribbon back to the red box and says] *But the yellow box will be empty!* (Aged 6, verbal)
- [The child puts a green ribbon in the yellow box.] *I thought it should have one. I don't like it having none.* [The researcher reminds the child that the ribbons should match the colour of the box. The child smiles and moves the green ribbon into the green box.] (Aged 5, verbal)

The 3 and 4 year old children tried two different strategies, both of which involved imagining some yellow ribbons. The first was the chanting of part of the number order, such as 2, 3, 4 or 5, 6, as if something was being counted. The second strategy was to say that there was a ribbon (ribbons) in the box when it was empty.

- *Three, two, five, six. Six. Six yellow.* [The researcher asks the child to show her the yellow ribbons. The child looks bewildered, then says,] *But there are none.* (Aged 4, verbal)
- *Three, two, I've got two.* [The researcher asks the child to show her the two yellow ribbons. The child looks perplexed.] *They must be in your bag.* (Aged 4, verbal)
- *One yellow, two yellow, three yellow.* [The researcher asks the child to show her the yellow ribbons. The child looks mystified.] *There're in your hand.* [Researcher shows her empty hands.] *In your bag?* (Aged 3, verbal)
- *Got two.* [The researcher asks the child to show her the two yellow ribbons. The child looks very puzzled and looks inside the yellow box.] *None.* [The child still appears very puzzled.] *No yellow!* (Aged 3, verbal)

Describing and labelling the empty container

Oral description

When sorting had been completed the children were asked, *How many ribbons there are in each box?*

As can be seen in table 47, when describing the empty box over half of the children in each age range (3 to 10 years) used the word *none.* Surprisingly quite a number of children used the word *zero*, sometimes this was *zero* as a single word or as *zero ribbons.* With some of the 3, 4 and 5 year olds the response, to the question how many ribbons were in the yellow box, was a question or statement involving their removal,

- *Where have they gone? Can't find them.*(Aged 4, verbal)

Oral language used to describe the empty box								
Age of children	10	9	8	7	6	5	4	3
Number of children	12	20	16	13	13	17	14	11
None	8 67%	11 55%	10 63%	8 62%	10 77%	9 53%	7 50%	6 55%
Zero	2 17%	8 40%	5 31%	1 8%	1 8%	1 6%	3 21%	
Nothing	1 8%			3 23%	1 8%	2 12%	1 7%	1 9%
No yellow	1 8%	1 5%	1 6%	1 8%	1 8%	3 18%		
Refer to removal						2 12%	3 21%	3 27%
No answer given								1 9%

Table 47, Ribbons 3

(ii) Written Labels

The data for writing the label for the empty set have been placed in two tables due to the difference in the written responses from the 3 year olds compared with those from the other children. Table 48, Ribbons 4, shows that a high percentage of the 4 to 10 year old children used the number symbol '0'. These children followed a pattern when labelling all the boxes; they either wrote all number symbols – 2, 3, 0 or all number words – two, three, zero.

Labelling the empty box							
Age of children	10	9	8	7	6	5	4
Number of children	12	20	16	13	13	17	14
'0' symbol	11 92%	16 80%	11 69%	10 77%	10 77%	14 82%	10 71%
Word **'zero'**	1 8%	4 20%	1 6%		2 15%	1 6%	
Word **'none'**			4 25%	2 15%	1 8%		
Intentionally leaves empty							1 7%
Not know what to put				1 8%		2 12%	3 21%

Table 48, Ribbons 4

A small number of children in the 6 to 8 aged range used the number symbols 2, 3 and 'none' for the empty set. It was also noticeable that children felt the need to qualify the writing of the '0' symbol or the word 'zero'. All the children who wrote the word 'zero' said, *that means nothing* or *that means there are none there.* While the 4 and 5 year olds, on writing '0' said, *zero that means none*.

It is understandable that the 3 year old children found recording difficult and they recorded the 3 green ribbons and 2 red ribbons with marks, pictures, squiggles and arbitrary words (the latter being written by the researcher at the child's request). The researcher did think it was possible that the squiggles were intended to be drawings of the ribbons. However, each child's set of squiggles for the green and red containers were all very similar, the researcher had no means of distinguishing one 'number' or 'amount' from another. Whether the child could distinguish the meaning of the marks or squiggles it was not possible to say. It appeared to the researcher that most of these children, when asked to say how many ribbons were in the closed boxes, ignored their written label.

What was of interest to this study was the way some of the 3 year olds represented the empty set on the yellow label (table 49). While the boxes containing ribbons were labelled using pictures, marks and squiggles, with the empty set four of the children put a '0' while one left the paper blank because *that's an empty space.* One child asked the researcher to write *Batman, Robin, Alfred,* though there was no explanation given. The arbitrary marks and words were very similar whether used in a capacity (bottle activity) or numerical (ribbon activity) situation. However, with such a small sample of children using such arbitrary marks or words it was not possible to say whether there was a link between the one used for the empty bottle and the one used for the empty box.

3 year olds, labelling the empty yellow box	
Age of children	3
Number of children	11
'0'	4
'empty space'	1
Arbitrary marks	4
Arbitrary words	1
Not know what to put	1

Table 49, Ribbons 5

Recognising the empty set when it is not visible

When all three boxes were closed and labelled the researcher would point to a box. *Don't open the box. Tell me how many ribbons are in that box?* (This was repeated for the other boxes.) Across all the age ranges it was not possible to tell whether the label was being read or whether the child remembered how many ribbons were in a particular colour of box. When asked about the red and the green box all the 5 to 10 year old children gave the correct number answer. In the case of the 3 and 4 year olds these numbers were not always correct, some recited part of the number order, one showed the correct number of fingers, one child gave no response. The following illustrations were typical:

- The green box – *one, two, three*
  The red box - *seven*
  The yellow box - *five.* [Opens the yellow box, says] *None.* (Aged 3, verbal)

- The green box - *one, two, three, four*
  The red box - *one, two, three*
  The yellow box - *one, two, three.* [Opens the yellow box, says] *None.* (Aged 4, verbal)

- The green box - *five*
  The red box - *five, four*
  The yellow box – *one.* [Opens the yellow box, says] *Zero.* (Aged 4, verbal)

The researcher found this to be a surprising occurrence, that many of the 3 to 5 year olds needed to open the yellow box containing the empty set. No child in any age range opened the red or green boxes in order to check the contents. Only the yellow box was opened. Even when a child was confident of the number of ribbons in the other boxes he/she would open the yellow box.

- [Opens the box. Shakes it.] *None in. No yellow ribbons.* (Aged 5, verbal)
- [Looks in the yellow box.] *Empty. Got nothing in it.* [Picks up the box and shows the researcher the inside] (Aged 5, verbal)
- [Child reads the label '0' he has written on the container.] *It says zero.* [He opens the box to show the researcher inside.] *It says zero and that means nothing.* (Aged 4, verbal)
- *There's none in that box.* [She opens the box and looks inside.] (Aged 4, verbal)
- *None.* [Opens the box. Looks inside.] *None.* (Aged 3, verbal)

Reasons for the emptiness

The collated data in table 50 are the results of the child being asked, 'Why are there no ribbons in the yellow box?' The reason given by the majority of the older children, as to why there were no were no ribbons in the yellow box, was connected with the fact that they had no yellow ribbons to put in the yellow box.

- *No yellow ribbons in the pile.* (Aged 10, verbal)
- *I had no yellow ribbons to put in.* (Aged 9, verbal)
- *You only gave me red and green ribbons.* (Aged 8, verbal)

Reasons given as to why the yellow box was empty								
Age of children	10	9	8	7	6	5	4	3
Number of children	12	20	16	13	13	17	14	11
Initially there had been yellow ribbons	3 25%	4 20%	6 38%	6 46%	5 38%	9 53%	9 64%	8 73%
There were no yellow ribbons	9 75%	16 80%	10 62%	5 38%	4 31%	3 18%		
No reason given				2 15%	4 31%	5 29%	5 36%	3 27%

Table50, Ribbons 6

The explanation from most of the children aged 3, 4, and 5 involved an assumption that there had been yellow ribbons and they felt that the yellow ribbons had been removed. Some children searched for the yellow ribbons and asked where they were.

- [After looking under the table and in the researcher's cases the child says] *Can't find yellow.* (Aged 4, verbal)
- *No yellow because someone has emptied them.* (Aged 5, verbal)
- *Has someone took the yellow?* (Aged 5, verbal)
- [The child looks into the yellow box and says] *Empty. Got nothing in it.* [Pauses] *Because someone poured them out into special boxes.* (Aged 5, verbal)

Ignoring the empty set

It was noted that at various stages in the activity children had to be reminded to include the yellow container [the empty set]. While this was more prevalent with the younger children there were incidents recorded of children in every age range who ignored the empty container. This occurred at various stages during the activity such as when the red and green boxes had been labelled and the child ignored his/her labelling of the yellow container. At such a point a child would stop and the researcher would have to ask the child to include the yellow box. In the following incidents the children had counted the red and the green ribbons in the boxes and put on the lids. Then stopped. The researcher asks 'What about the yellow box?'

- *Very tricky. No in.* (Aged 3, verbal)
- *We haven't got any yellow ribbons?* [The researcher suggests that the lid is put on the yellow box. The child continues the activity.] (Aged 5, verbal)
- *There's nothing in the yellow box!* [Pauses] *So I put the lid on and I write zero.* [Child puts '0'.] (Aged 6, verbal)
- [The child looks in the yellow pot, does nothing. The researcher asks again, 'What about the yellow box?' The child laughs and says] *There's none in it.* [The researcher asks, 'What do you want to do?' The child replies] *Put the lid on and write zero. That means none, zero.* (Aged 7, verbal)

Illustrative incident

[The child counts the red ribbons, puts the lid on the red box. While counting the green ribbons he asks,] *What about the yellow one?*
[He picks up the green ribbons, counts them, puts them in the green box. Puts the lid on the green box and says,] *Done it.*
[He stops. Completely ignoring the yellow box. The researcher asks if he has finished.] *No.*
[He puts box 2 and 3 in a line.]
Researcher: *What about the yellow box?*
Child: *Leave it. There's nothing in it.*
Researcher: *What would happen if you put the lid on the yellow box?*
Child: *There's nothing in it.* [Reluctantly he puts the lid on the yellow box.]
Researcher: *What would you write on the box?*
Child: *I think it should be zero* [He writes '0'] *Zero means none in anything.*
(Aged 5, verbal)

The Ribbon Activity - summary of the analysis

The most notable occurrence was the concern shown, by a core of children, at the lack of yellow ribbons and the resulting empty yellow box. This was more prevalent amongst the younger children where the anxiety was so pronounced that some went in search of the missing ribbons and, in order to prevent further unease, the researcher had to inform them that there were no yellow ribbons. The aversion to the empty set meant that some children attempted to overcome the dilemma by putting other ribbons, from the red and the green set, in the yellow box or by imagining that there were ribbons in the box when it was empty.

When the lids had been put on the containers the children were asked how many ribbons there were in each box. No child opened the red or the green containers in which they had placed ribbons but a number of the 3, 4 and 5 year old children opened the empty yellow container, possibly to check if it was still empty.

There were two reasons given as to why there were no yellow ribbons in the yellow box. Either because, initially, there had been ribbons but that, for various reasons, they had been *removed.* Or because someone had forgotten to put them in, or provide them - *omission.* The younger children tended to see ***removal*** while the use of ***omission*** increased with the older age ranges.

Overall, when the children were asked to describe the empty set the word *none* was used while in the labelling of the empty set the use of the '0' number symbol was the most widely selected method.

Incidents were recorded of children in every age range who had to be reminded to include the yellow container at various stages in the Ribbon Activity-Interview though the ignoring of the empty container occurred more with the younger children.

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Chapter Three, Part Four: Discussion points

This section compares the findings of the Bottle Activity where an empty set was presented to the child in a non-numerical situation and the Ribbon Activity where the child, in a numerical situation, produced an empty set. From these two empty set activities three main discussion points arose. Each of these three aspects will be considered in turn.

1) Reaction to the empty set
    a) Ignoring the empty set
    b) Overcoming the empty set
    c) Checking for emptiness
2) Reasons for the emptiness
3) The language of emptiness

1. Reaction to the empty set

- [Throughout the sorting of the ribbons the child asks] *Where's the yellow? What about the yellow? What about the yellow? But where are the yellow ones?* [He begins to search for the yellow ribbons and to become distressed so that the researcher tells him she has no yellow ribbons. The child then ignores the yellow box.] (Aged 5, verbal)

Sources of concern, particularly to the 3, 4 and 5 year olds, were that there was no water in one of the bottle and that there were no yellow ribbons to put in the yellow box. In both cases the children's reaction ranged from puzzlement to great anxiety. The reaction from some was so extreme that it startled the researcher. It would have been expected that these children would have met an empty bottle situation, indeed the researcher had seen them, in their class, drinking with a straw from a bottle of milk or orange juice. However, there was a difference. Drinking the liquid produced the emptiness while with the Bottle Activity the child had been presented with the empty bottle. Nevertheless the children had had experience of an empty set in a capacity context.

In the Ribbon Activity the child produced the empty set. As the child was given the three boxes it is possible that he/she expected to have material to put in all the boxes. No child questioned whether there were red or green ribbons other than the ones they had been given. No child asked for blue ribbons or ribbons of any other colour. It is very likely that the presence of the yellow container indicated that yellow ribbons might be available. None of the older children remarked about the empty bottle but a number remarked about the lack of yellow ribbons. These remarks appeared to be more of a check that this omission was intentional. This would also underpin the expectation that a yellow box meant there should be yellow ribbons and this in turn may account for children asking and searching for yellow ribbons.

However, this does not explain the children's anxiety and their need to fill the empty set. Attempts to fill the bottle only occurred with the younger children but even the older children attempted to 'place' ribbons in the empty yellow box. The 3 and 4 year olds 'imagined' the yellow ribbons while the older children used ribbons from the other boxes. The researcher considered, as a possibility, that the children were intending material to be 'fairly shared ', that no one (or no bottle or box) should be without. She found no indication of this in any other activity or task.

As well as the children who made clear their unhappiness at having an empty set there were also children whose reaction was to attempt to overcome the empty set dilemma. Some children exhibited one of these traits while some exhibited both.

Dealing with the empty set dilemma took the form of a) *ignoring the empty set* or b) *overcoming the emptiness*. There was also the c) *checking for emptiness.*

a) Ignoring the empty set

- [Researcher had to remind a the child to include the empty bottle in the ordering] *But it's empty!* (Aged 6, verbal)

One way to deal with the emptiness was to ignore it. At various stages, in both the Bottle and the Ribbon Activity, reminders had to be given to include the empty set. This happened with a number of children in the age ranges 3 to 9 but occurred more frequently with the younger age ranges. It was an unexpected happening; the researcher found it paradoxical that a child could disregard one out of four bottles and one out of three containers. Was there another reason why a container was being ignored? As all the bottles and boxes were identical except for the contents then it can, surely, be concluded that it was ignored because it had nothing in it. Once the child had established that the container was *empty,* it held *nothing, none, zero,* then it would appear that the children saw the empty set as being of no significance; hence it was ignored.

b) Overcoming the empty set dilemma

- [Looks at the wide red and green ribbons. Smiles mischievously and says] *Can I put one of the other ribbons in the yellow box?* [The researcher asks, why?] *I don't like it to be empty.* (Aged 8, verbal)

Another way to deal with the emptiness was to put something there. It was only the 3 and 4 year old children who suggested overcoming the emptiness of the sealed bottle. However, the empty yellow box was easier, especially as the children were in control of the sorting which had produced the emptiness. There was the possibility that they felt responsible for its emptiness. The aversion to the empty set meant that some children attempted to overcome the dilemma by putting other ribbons, from the red and the green set, in the yellow box or by imagining that there were ribbons in the box when it was empty. In order to put ribbons in the yellow box children were prepared to flaunt the rules of the sorting or, in the case of the younger children, to count imaginary yellow ribbons.

c) Checking for emptiness.

There was a final surprising reaction to the empty set; checking that the set was still empty. The checking of the empty set applied only to the Ribbon Activity where, when the lids were placed on the containers, the contents were not visible whereas the bottles were transparent and the contents visible. The children were asked how many ribbons were in each closed box. It had been expected that providing the answer to the closed boxes containing two and three ribbons might cause some difficulties. The basis of this expectation came from the research of Inhelder and Piaget (1967). They reported that babies think that if a toy falls out of sight it no longer exists, while by the age of two most children appreciate that a ball which has gone under the chair still exists and they will search for it. The next stage, and one which is applicable to this study, is expressed succinctly by Anne Montague-Smith,

> Between three and four years children may not believe that objects which have been moved to a different position are still the same objects and take up the same amount of space.
>
> (Montague-Smith, 1997, p.79)

The anticipated difficulties did not appear. No child opened the red and green boxes to check and recount the contents. What did occur was the opening of the empty yellow container by children aged 3, 4 and 5. Why did they feel the need to open the empty box and not the others? Was it that the children were unsure that the container was originally empty? Were they expecting, like a conjuring trick, yellow ribbons to appear in the empty box? Did they find conserving emptiness more difficult than conserving an amount? Indeed one may ask whether one can conserve the empty set. It would appear that the Laws of Conservation refer to *some physical quantity:*

> Conservation laws: Laws requiring that, in an isolated or undisturbed system, the total amount of some physical quantity does not change in the course of time; the quantity is said to be *conserved.* (Daintith and Nelson, 1989, p.70)

Possibly the children who opened the yellow container were just reassuring themselves and checking on their original answer and/or they were wanting the researcher to corroborate the emptiness of the yellow box. The researcher can only suggest reasons why a child might need to check the empty set, but she has no means of knowing whether any are correct or if there were other factors involved.

The Ribbon Activity has its basis in the Tins Game from Martin Hughes's research (see appendix 9). Hughes does not mention any such strong reactions to the empty set. He presented the sets of tins together with the bricks inside, including empty tin, to the children while in this Ribbon Activity it was the children who produced the sets, including the empty set. Was this the reason why there was a strong reaction with the Ribbon Activity and not with the Tins Game? Would this explain the anomaly? There again, in the Bottle Activity, when the children were presented with the empty bottle the reaction from the young children was still one of concern. The researcher can only report her findings along with her strong impression that the children did not like to have an empty set but she has yet to find an acceptable reason for this.

2. Reasons for the emptiness

It had been expected that the majority of the children would have seen the empty sets as *omission*. There was no water in the bottle because the researcher had not put any in. There were no yellow ribbons to be sorted because the researcher did not include any yellow ribbons. However, none of the 3 and 4 year old children used the reason of *omission*. As Aubrey (1994) reminds us, the home is the child's first learning environment, everyday living provides the context and experiences for early mathematical knowledge and language. Most of a child's early experience of emptiness is in a *full to empty, empty to full* situation. The *full to empty* context evolves from having some to having none such as with a bottle of washing up liquid, drinking from a can or bottle, eating a packet of sweets or crisps. And conversely the *empty to full* context evolves from having none to having some such as the empty bowl or bath being filled with water; a cup before juice is poured in, an empty plate before food put on it. Certainly in the Bottle Activity it was understandable that at least 40% of the 5 to 10 year olds and all the younger children saw the situation as one of removal

In the Ribbon Activity no child saw, or was given, any yellow ribbons. Many of the older children had no problem in indicating omission by saying *there were no yellow ribbons; I had no yellow ribbons; there weren't any yellow ribbon to put in the yellow box*. In the Ribbon Activity all the reasons given by the 3 and 4 year olds for the empty yellow box were of removal and a notable percentage of 5 to 10 year old children (a mean of 37% with a range of 20% to 53%) gave an explanation involving removal. The researcher found the responses of these older children puzzling and intriguing for she had not presented yellow ribbons. No child suggested that any of the red or the green ribbons had been removed. Why did so many children give a reason involving the *removal* of ribbons they had not seen? Are situations, where omission occurs, less prevalent in these children's lives?

3. The language of emptiness

Spoken language, including that of mathematics, is situational. These two activities were set in different contexts and the researcher's questions mirrored these contexts. The Bottle Activity was in capacity and dealt with *how much* was in each bottle. The Ribbon Activity was set in a numerical situation and considered h*ow many* were in each box. The language of the empty set can subtly change according to the context of what is in the <u>other</u> containers in the set. Most of the children showed a great awareness of this and were sophisticated in their ability to treat the empty container in conjunction with the others in the set.

The children of 3, 4, and 5 tended to lean towards the use of the word *none,* a term applied to both situations. The trend with the older children was use *empty* for the lack of water in the bottle and *none* for the lack of ribbons in the box.

There was a high degree of consistency when transferring the spoken description to the labelling of the sets. Over 65% of the children in the age ranges 4 to 10 wrote '0' in the ribbon activity. This was to be expected as this was a numerical activity. However, children of 4 and 5, 8 and 9 used the '0' symbol on the bottle label, these were mainly children who had described the empty bottle using the word *none* or *nothing* and who then wrote the '0' symbol. Most children followed a pattern when labelling all the boxes; they either wrote all number symbols – 2, 3, 0 or all number words – two, three, zero.

One occurrence, which mirrored the findings of the *ordering* and the *zero facts* Task-Interviews, was the children's need to qualify the use of the word *zero* and the symbol *'0'.* This was in the form of an explanation in which word *none* (or less commonly the word *nothing*) was used.

The Language of Zero is a complicated subject. It is the focus of the next chapter.

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CHAPTER 6

THE LANGUAGE OF ZERO

> Names belong to things, but zero belongs to nothing. It counts the totality of what isn't there. By this reasoning it must be everywhere with regard to this and that. Then what does zero name? (Kaplan 1999, p37)

Outline of Chapter 6

The language of zero is a complex subject both to research and report upon. In order to present a cohesive account of the collection, data categorisation and analysis, chapter 6 has been divided into five parts,

- *Part 1, The zero symbol*
  To set the scene a brief history of the '0' symbol is presented. There is explanation of the fieldwork together with the outcomes on the language children used for the '0' symbol in various contexts.
- *Part 2, Alternative words for 'zero'*
  This section begins with a brief history of the word 'zero' together with the derivation of other alternative words for 'zero'. The children were asked what word they might use instead of zero. The results are presented and they are compared with the actual words the children used in the various research tasks, activities, interviews and in the questionnaire.
- *Part 3, The language explaining zero's value*
  *Three can mean the symbol '3' or the amount ***, zero often refers to the '0' symbol. The issue becomes more complicated with zero. How did the children describe zero's worth?*
- *Part 4, Other modes of communicating zero*
  In the zero number facts the children were asked to illustrate the equations. Part 4 considers the different signs and gestures used to denote 'nothing there'.
- *Part 5, Summary and discussion points*

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Chapter Six, Part One: The Zero Symbol

A brief history of the zero symbol '0'

> ... reading and understanding symbols ... in mathematics is a short and precise code for words. (Dickinson, Brown and Gibson, 1993, p.351)

Compared with other single digits, one of the conceptual difficulties with zero was using a sign in order to say that nothing is there. If there is nothing there then why not leave the space empty? As was seen in chapter 2, part 1, originally an empty space was used but this caused some difficult in knowing that a space, or indeed more than one space, had been left deliberately *empty* in order to communicate a message. The Sumerians used two diagonal wedges to represent *nothing here* while the Mayans used a snail shell. The next paragraphs, on the history of the zero symbol, draw heavily on the works of Datta and Singh (1993), Dilke (1993), Flegg (1989), Menninger (1992) and Katz (1998).

As a sign for zero the Hindus were known to have used a cross; however, the use of a dot was more common. So much so that, in Sanskrit, the zero was sometimes called *sunya-bindu* 'the empty-dot' after its physical meaning, that the position (originally on the counting board or abacus) was empty. Our modern custom of indicating a missing word or line of verse by a row of dots goes back to this practice[1]. Probably the first piece of evidence, of using a dot, is found in the writings of Subandhu, a poet who lived towards the end of the 6th century AD. From his poem *Vasavadatta* comes this beautiful, descriptive passage,

> ... and at the time of the rising of the moon with its blackness of night, bowing low, as it were, with folded hands under the guise of closing blue lotuses, immediately the stars shone forth, ... like zero dots (*sunya-bindu*), because of the metempsychosis, scattered in the sky as if on the ink-blue skin rug of the Creator who reckoneth the sum total with a bit of the moon for chalk. (Datta and Singh 1962, p.81)

When trying to trace the development of the zero symbol researchers have uncovered many discrepancies. Early documented evidence is mainly of a poor quality. Added to this are the processes of translating and editing of such works. These occurred mainly in the 19th century, particularly between 1850 and 1885 with much re-editing of these translations occurring in the early 1900s. These same problems arise as the Hindu number system is traced into Islamic mathematics where there are no extant Arabic manuscripts. Instead there are several different versions made in Europe in the twelfth century which describe the algorithms and the characters using our familiar place value system including, as the Latin version tells us, a circle to designate zero. Whether a circle was in the original document or is a result of translation is not known.

The dot must have been widely used for it to reach as far as China, for in the 8th century Chinese text *Khai-Yuan Chan Ching* there appears a description of an Indian numeral system using a dot for zero. On the other hand a dated inscription from the island of Banka in Indonesia, whose date converts to 686 AD, uses a small circle for zero. These are but two examples of the use of a dot and a circle coming from the eastern side of the subcontinent, Indo-China and south-east Asia. The dot later fell out of use leaving the circle we use today.

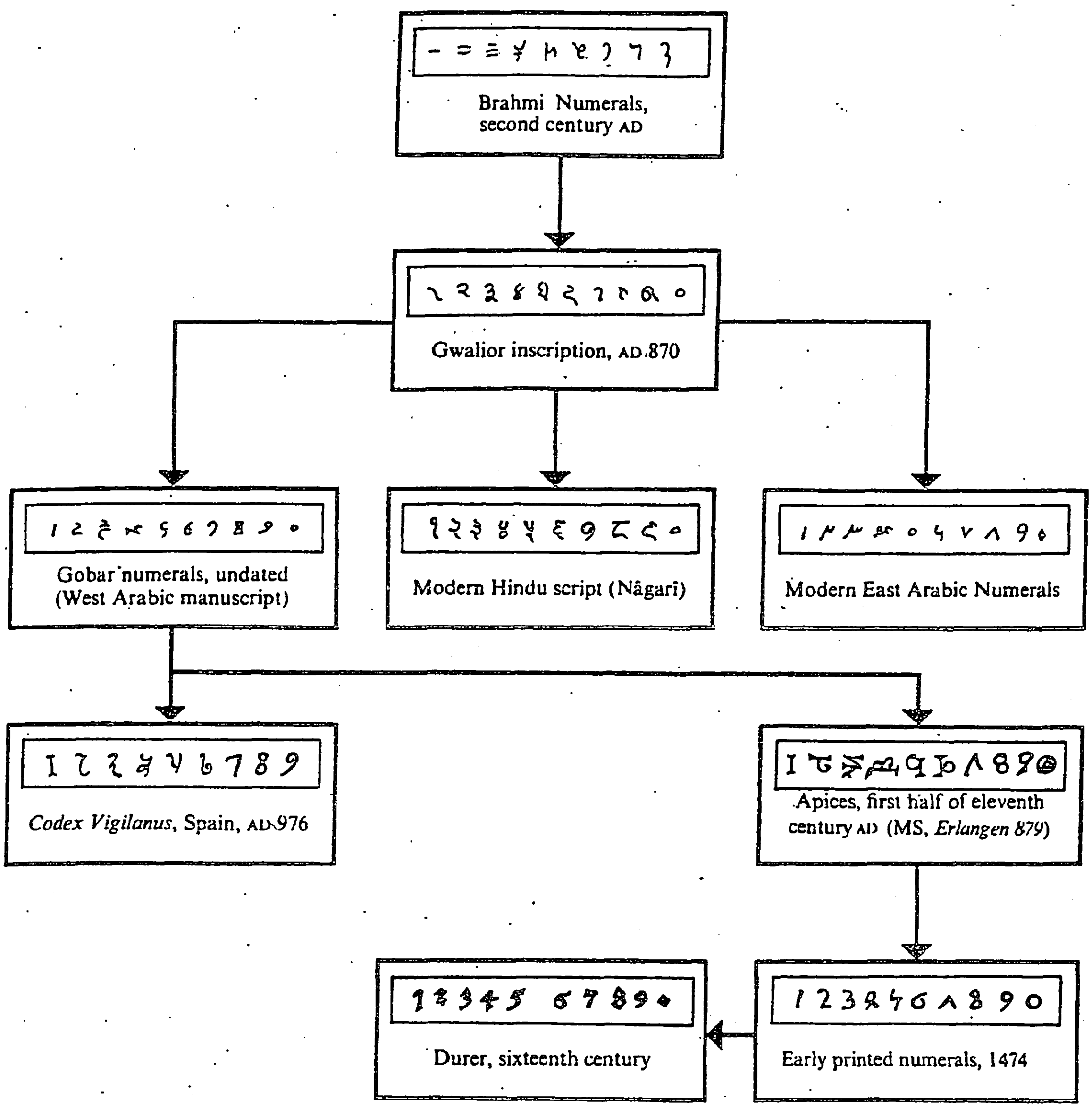
There are two early epigraphic and original records of the use of the circle. One is on the Ragholi plates (8th century), where zero can be seen in the form of a small circle. The second is an

[1] The Hindus had customarily used dots as a pledge to carry out unfulfilled tasks.

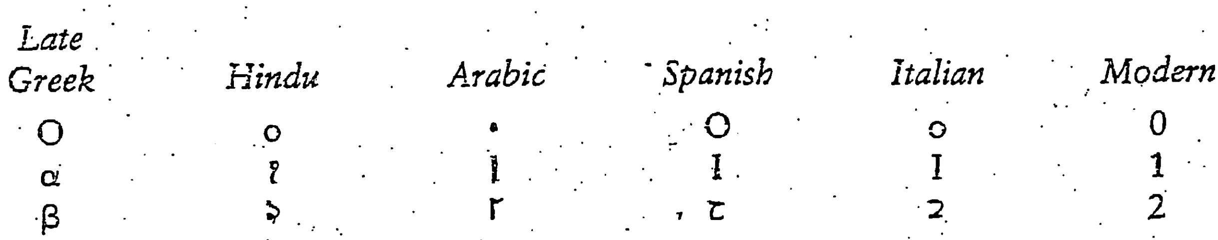
Illustration 19

Single number scripts

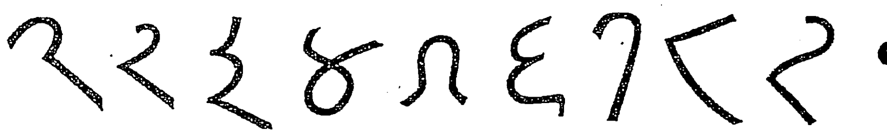
a) Family tree of Hindu-Arabic numerals (Flegg 1989, p.117)



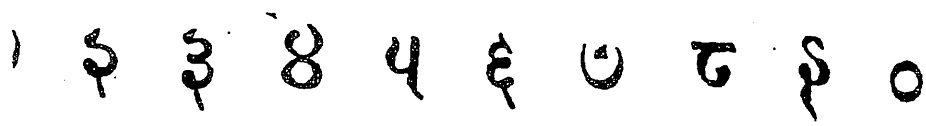
b) Comparative tables for 0 to 2 (Dilke 1993, p.98)



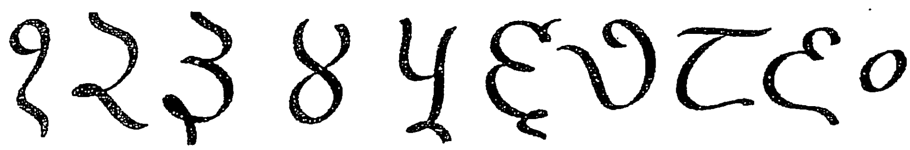
c) The evolution of Indian numerals: (Guedj 1996, p.50 – 51)



Ancient Nagari digits from 1 to 0

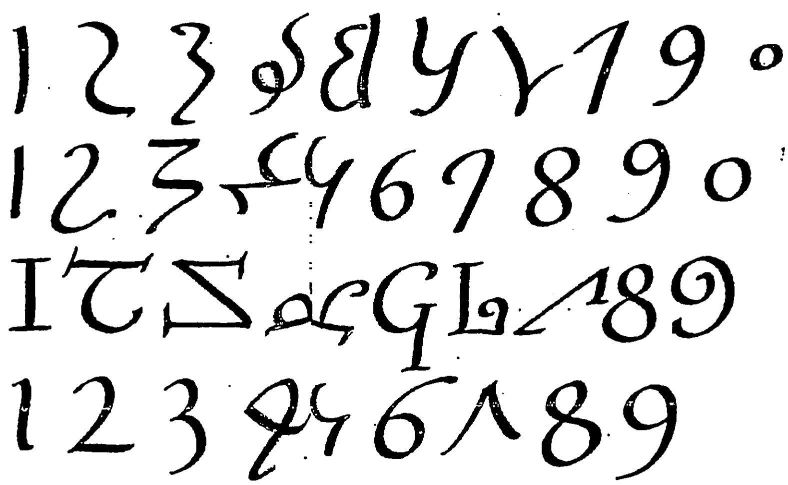


Sanskrit digits from 1 to 0



Ancient Hindu digits

d) The evolution of Arabic numerals in four stages: (Guedj 1996, p.50 – 51)



inscription on a stone tablet containing a date which translates to 876AD. The inscription concerns the town of Gwalior, 400 km south of Delhi, where they planted a garden that would produce enough flowers to allow 50 garlands per day to be given to the local temple. Both of the numbers 270 and 50 are denoted almost as they appear today although the 0 is smaller and slightly raised. A selection of the various number scripts can be seen in Illustration 19.

Why use a circle? Possibly the dot developed into the shape of our zero sign but there are other possibilities. One, which is favoured by this writer, is that of *The Hollow Trace Theory*. When pebbles are placed on a sand dusting board and then a pebble is removed a loosely circular depression remains. This is *the hollow trace* and this too may well be the origin of the '0' symbol. Menninger (1992) has other thoughts,

> The small circle may have been suggested by the Brahmi numeral for 10 or perhaps by the abbreviation of the Greek word *oudén,* 'nothing'. (Menninger 1992, p.403)

Dilke (1993) also looks to the Greeks for the basis of the '0' sign,

> Our sign seems to have arisen as a Greek astronomical zero coordinate, for example in the *Almagest* of Ptolemy, though there was no zero in other types of Graeco-Roman mathematics. (Dilke, 1993, p.14)

At this present time the details of the historical use of the '0' symbol for zero are incomplete. Fortunately there are still texts to be edited and translated though political difficulties continue to block access to many important collections. While these Arabic manuscripts continue to remain unread who knows what changes may take place in the history of number and of zero? An example, referred to in Katz (1998), which whets one's curiosity, is that of the earliest dated inscription using the place value system including the zero, dated 683 AD. This was found in Cambodia. Other areas of the world may also contain surprises.

Today '0' is the recognised symbol for zero. This study has included data on the children's use of '0' as a mark for the empty set (chapter 5), in the ordering of numbers (chapter 3) and in the zero number facts (chapter 4). The researcher did not note any of the children having difficulty writing the '0' symbol. Indeed, while a number of the 3 and 4 year old children asked the researcher to help them to write the symbols 2 and 3 no child asked for '0' to be written on their behalf. Children often have problems forming the number symbols but as '0' is symmetrical then, unlike 2 and 3, it cannot be written the 'wrong way round'. There is also the fact that children gain more practice in forming a '0' through drawing (or attempting to draw) a circle and through forming the alphabet letter 'o'.

A circle as a symbol serves a dual purpose, as a number in mathematics and as a letter in language. Children have to differentiate between a circle used as a sign in words and in numbers. This researcher wanted to discover where the emphasis would be put and how readily children picked up contextual clues to detect whether the circle is a number or an alphabet letter. The following sections consider these issues.

'0' and the young child

Sinclair and Sinclair (1984) in their study in Geneva demonstrated that pre-school children (aged 3 to 5) were aware of numbers around them and the uses to which some of them were put. These young children were able to differentiate between differing use of numerals in an environmental setting. They recognised, for example, a sequence context (such as page numbers in a book) and a convenience context (such as the numbers on the front of buses).

Young children are introduced to numbers and alphabet letters at the same time. Pound (1999) says, of children aged 3 to 5, that,

> Some show at this early stage that they can differentiate between numbers and letters, even though they may not be sure which labels to attach to them. If you watch young children busily filling in forms in banks and post offices, even quite young children often put letters in the writing spaces and numbers in the boxes intended for figures.
>
> (Pound, 1999, p.6)

This researcher noted during her experience of working with young children that there can be confusion in the recognition and reading of some symbols, for example P and 9, S and 5. It is probable that, if children are presented with **dog** and with **3045**, when asked to read the symbols the context would give them clues that the former 'o' is an alphabet letter and the latter '0' a number. But how would '0' be read when the context was removed? Where would the emphasis lie?

Two simple tasks were devised. In the first the child was asked to read a series of alphabet and number symbols, in the second only number symbols were used. It had be intended to use these two *symbol recognition tasks* only with the young children. However, (as with the Bottle and the Ribbon Activities, chapter 5, part 1) in order to gain a view of the wider picture their use was extended to include the 5 to 11 year old children. There follows a description of the design, presentation, collation and analysis of these two activities 1) *the alphanumeric recognition task* and 2) *the number symbol recognition task.*

1) The Alphanumeric Recognition Task

The aim was to see how a child would read the symbol '0', when it was presented as an individual symbol but mixed with other individual alphabet and number symbols.

a) Designing the alphanumeric recognition task

The choice of symbols was important. It was necessary to select alphabet letters and numbers which would be familiar to young children, those most likely to be taught first and those most likely to have been met by the child in and out of school. Information regarding the expected range of knowledge, in numbers and alphabet letters, of the 4 and 5 year olds, in the National Numeracy Strategy (DfEE, 1999) and National Literacy Strategy (DfEE, 1998) for Reception Children, was taken into account.

There was the expectation that the children would be more likely to know letters and numbers which appear early in the alphabet and early in the number order. The number symbols selected were **1**[2], **2,** and **3**. However, using the first three alphabet letters a, b, c, could, conceivably cause problems.

a – children would be expected to know this as the starting letter of the alphabet

(b and **d** were not selected as children confuse b and d, they also confuse b and d with p. Similarly there is confusion with the number symbols 9 and 6.)

c – a letter close to the start of the alphabet

e– an early alphabet letter, the most frequently used letter and vowel

As b and d had been discarded the alphabet letters selected were **a, c,** and **e.** Added to the number symbols was '**0**'. These seven symbols were written on individual pieces of card in a style acceptable to young children, that is, they were hand-written using the lower case format.

Alphanumeric Recognition Task Resources

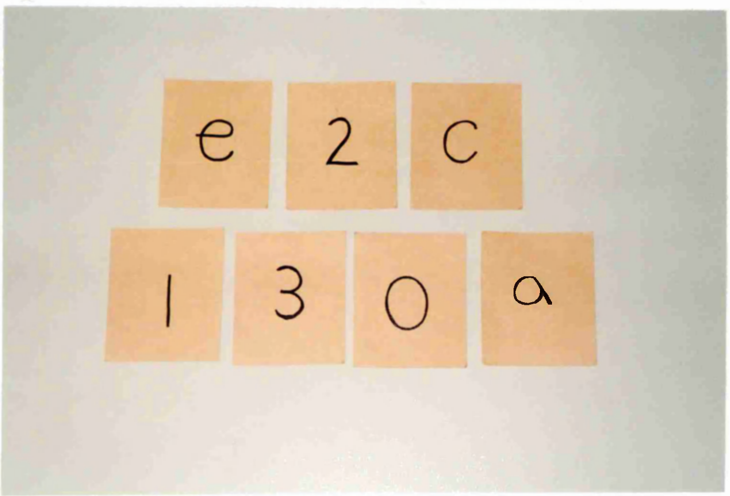


Illustration 20

[2] It had been considered that '1' (one) might be read as the letter 'l'; this would not have affected the research but in the event every child read '1' as one.

b) Presenting the alphanumeric recognition task

In order to check that the child was aware of the difference between letters and numbers the researcher asked the child to, 'Tell me a letter that you use when you are writing. Tell me another... Now tell me a number. Tell me another.' All the children appeared to know the difference between a letter and a number though, as was expected, many of the younger children referred to the alphabet letters by their phonic sound (that is C̆ as in C̆ot). The children were told that each of the cards showed a number or a letter of the alphabet. The cards were presented individually, in the correct orientation, in the order – **a, 3, c, 1, e, 0, 2.** The child was asked to read each card and the child's responses were recorded.

c) *The alphanumeric recognition task - collation and analysis of the data*

Alphanumeric Recognition Task								
Age of children	11	10	9	8	7	6	5	4
Number of children	20	12	20	16	13	13	16	13
Zero	2 10%	1 8%	4 20%	4 25%	1 8%	1 8%	7 44%	7 54%
Oh	18 90%	9 75%	15 75%	11 69%	10 77%	6 46%	3 19%	2 15%
Both (oh and zero)		2 17%	1 5%	1 6%		3 23%		
nought						1 8%		
Phonic ō					2 15%	2 15%	6 37%	4 31%

Table 51

It can be seen, in table 51, that a very high percentage of children aged 7 to 11 referred to '0' as 'oh', while in the 4 and 5 year old age range this was decidedly lower. However, it needs to be remembered that, at this age, the children would, most likely, be using phonics to learn to read. It would be more natural for them to use the phonic letter sound 'Ō', rather than the letter name 'oh'. What was surprising was that it was in this age range, 4 to 5, that the word *zero* was used most frequently. This is probably capturing a change in teacher language with young children (see appendix 16) brought about by the NNS (DfEE, 1999); this is discussed in chapter 7.

2) The Number Symbol Recognition Task

The second of the symbol recognition tasks was concerned with only the number symbols. The aim was to see how a child would read the symbol '0' when presented as an individual symbol with other single digits.

a) Designing the number symbol recognition task

Number symbols selected were **1, 2, 3** and '**0**'. These four symbols were hand-written on individual pieces of card.

b) Presenting the number symbol recognition task

The child was told that this time the cards had numbers only written on them. The cards were presented individually, in the correct orientation, in the order **3, 1, 2** and **0**. Each child was asked to read the cards and the child's responses were recorded.

c) The number symbol recognition task - collation and analysis of the data

Number Symbol Recognition Task								
Age of children	11	10	9	8	7	6	5	4
Number of children	20	12	20	16	13	13	16	13
Zero	17 85%	8 67%	19 95%	13 81%	10 77%	9 69%	14 88%	8 62%
Oh		2 17%		2 13%	1 8%	1 8%		1 8%
Both (oh and zero)		2 17%		1 6%	1 8%	1 8%	1 6%	
none			1 5%		1 8%		1 6%	
nought	3 15%					1 8%		1 8%
Phonic o						1 8%		3 23%

Table 52

In table 52, it can be seen that, when the children were presented with only number cards, a much higher percentage of children aged 4 to 11 referred to '0' as *zero.* Few children used the words *none*, *nought*, 'oh' or the phonic 'Ō'.

The language of '0' used in context

Task-Interview, reading of '0' in the equations							
	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6	Aged 5
Number of responses	20	11	20	16	12	10	9
Zero	19 95%	11 100%	18 90%	16 100%	9 75%	6 60%	9 100%
Nothing	1 5%		2 10%		1 8%	1 10%	
None					1 8%	2 20%	
Mixed					1 8%	1 10%	

Table 53

In the zero number facts Task-Interviews, each time a child was presented with one of the eight equations written on a card ($3 + 0 =$, $0 + 3 =$, $3 - 0 =$, $0 - 3 =$, $3 \times 0 =$, $0 \times 3 =$, $3 \div 0 =$, $0 \div 3 =$) the child was asked to read the card[3]. The data, in table 54, shows that most children were

[3] Not every the children read all the equations, for example the younger children tended to deal only with the addition and subtraction.

consistent in their use of zero. The two cases where there was inconsistency are recorded under the category: *Mixed.* In these two instances the child used zero with an occasionally use of the word *none* (the 7 year old) or *nothing* (the 6 year old). A very high percentage of children read the '0' as *zero*, when reading the actual zero number fact equation. Unlike the children in the Questionnaire, the Task-Interview children did not write down their answer but gave a verbal response. When zero was the answer to the equation would they change their use of language? Table 54 contains the language when the answer to an equation was zero (or the child thought the answer was zero).

Task-Interview, language used for a '0' answer						
	Aged 11	Aged 10	Aged 9	Aged 8	Aged 7	Aged 6
Number of responses	20	11	20	16	12	10
Zero	19 95%	9 82%	17 85%	16 100%	9 75%	6 60%
Nothing	1 5%		2 10%		1 8%	1 10%
None					1 8%	2 20%
Mixed		2 18%	1 5%		1 8%	1 10%

Table 54

Again most children preferred to use the word zero consistently for all the equations[4]. In the mixed category it was mainly *zero* with an occasional use of *nothing* or *none*,

- 0 – 3 = 0, *Zero take three is nothing, zero.* (Aged 9, verbal)

Summary of Part 1, The language used for the '0' symbol

When comparing the children's reading of the '0' symbol in the two tasks the majority of children in a situation where they knew only numbers were being used (as with the cards, **3, 1, 2** and **0**) then '0' was referred to as *zero.* In a situation where it was not known whether the '0' was a letter or a number (as with the cards, **a, 3, c, 1, e, 0, 2**) the majority of children referred to '0' as *'oh'*. This ability to select the neutral 'oh' for setting where '0' may be a letter or a number increased with age.

An analysis review took place of these areas:

- the reading of the '0' as a single number symbol in the Number Recognition Tasks (tables 51 and 52)
- the reading of the equations in the Zero Number Facts (table 55)
- the answers to the Zero Number Facts (table 56)

It was seen that the zero language revolved round the use of *'oh'*, *none*, *nothing*, *zero* and *nought*, with *zero* being the preferred option.

---

[4] Few of the 5 year old children attempted the zero number facts and, as the early equations (3 + 0, 0 + 3, 3 – 0) gave the answer 3 none of the 5 year old felt the need to gave a '0' answer and to use the word 'zero'.

Part 2 considers these and other alternative words for zero while the use of *'oh'*, *nought* and *zero* are discussed in detail in part 5 of this chapter.

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Chapter Six, Part Two: The Zero Words

> The writer and mathematician Lewis Carroll (1832-1898) penned this illuminating conversation in Alice's Adventures in Wonderland.
> 'Take some more tea', the March Hare said to Alice, very earnestly.
> 'I've had nothing yet,' Alice replied in an offended tone, 'so I can't take more'.
> 'You mean you can't take *less,*' said the Hatter: 'it's very easy to take *more* than nothing'. (Bloomsbury 1991, p.92)

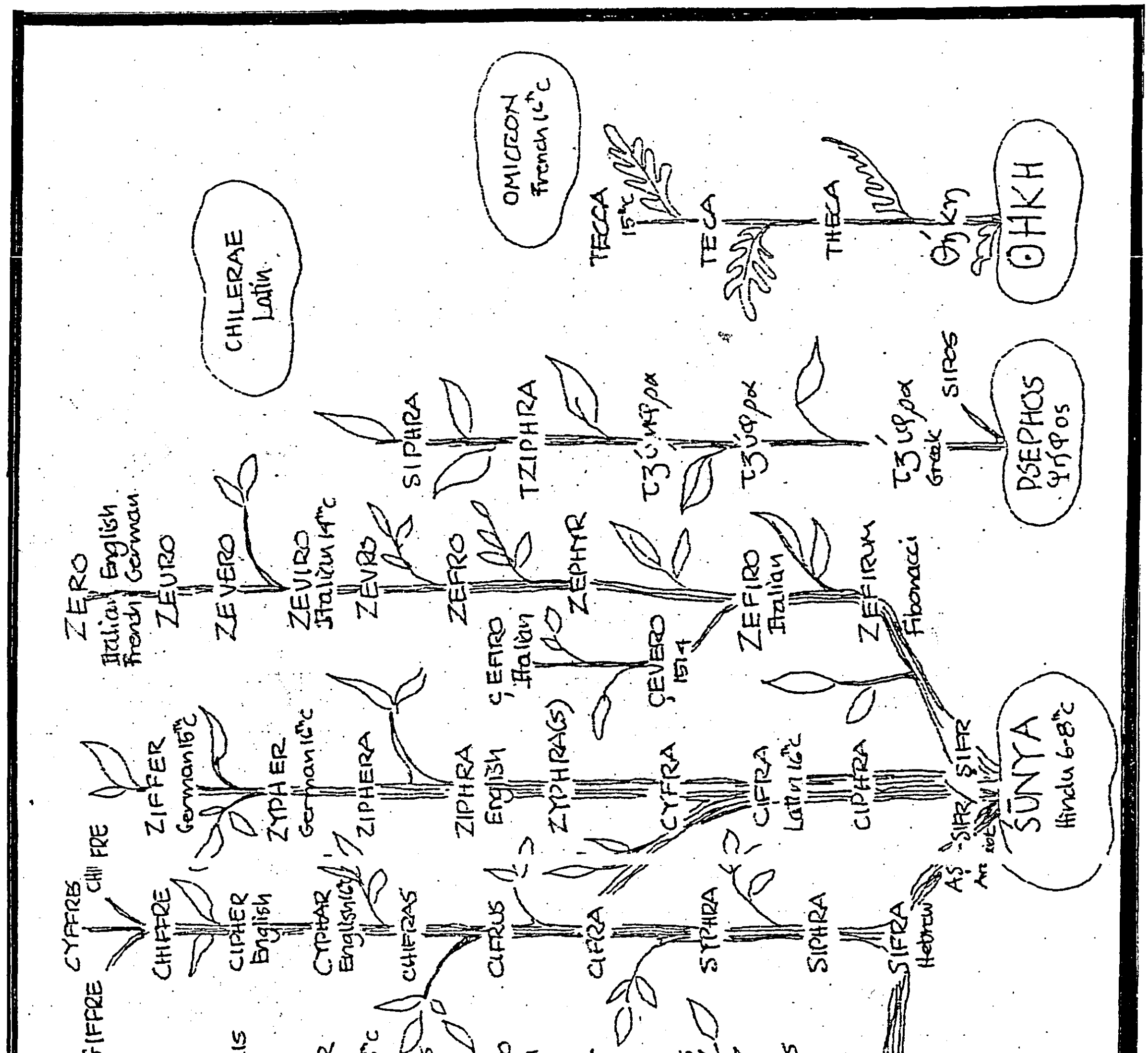
A brief history of the zero words

This historical section of the study draws on the works of Flegg (1989), Menninger (1992) and Schimmel (1993).

The Arab traders acted as intermediaries between India and the West and it was they who brought to the West the concept of zero. They also played a role in the story of the language of zero. When the Arabs became acquainted with zero, in the 9th century, they translated, literally, the Indian name *sunya* (Sanskrit for 'empty'), into the Arabic *as-sifr,* 'the empty'. The West, when it became aware of the new digit, took over its name as well as the symbol. The name was translated from the Arabic and transformed into the learned Latin form *cifra* and *cephirum.* These two Latin words then gradually worked their way into the vernacular and we see *cephirum* become *zephirum, zefiro (or zevero).* Then, in the dialect of Venice just as *libra* had become *livra* and then *lira* so *zefiro* was shortened to *zero.* The French took over both the Italian forms of *zero* and *cifra*, as can be seen in a French arithmetic textbook for merchants dated 1485.

> *Et en chiffres ne song que dix figures, des queues les neuf ant valeur et la dixieme ne vaut rien mais elle fait valloir les autres figures et se nomme zero ou chiffre,*
>
> The digits are no more than ten different figures, of which nine have value and the tenth is worth nothing [in itself] but gives [a higher] value to the others, and is called "zero" or "cipher." (Menninger, 1992, p.401)

"Digit" *(cf.* "cipher"), became manifest in the Italian *chifra* and *zero* and in the English *cipher* and *zero.* Then, in an Italian arithmetic textbook of 1484, the word *nulla* appeared as a noun. It is the 'figure of nothing' and thus no numeral, no figure at all, in Latin *nulla figura.* It can be seen how, in the English language, the zero acquired its name of 'null'. There could be no better symptom of the confusion and insecurity that *zero* produced in the minds of people in the Middle Ages than the accumulation of so many names. An interesting representation of the development of the zero words is included on the next page (Illustration 21).



The Zero Word Tree

The researcher was given this illustration many years ago.
Unfortunately she has been unable to trace its source.

Illustration 21

Indeed there were other names for the zero. These made reference to its form[5], the Latin name *circulus* being 'little circle'. The mathematician, Köbel called the zero the,

> *Ringlein 0, die Ziffer genannt, die nichts bedeut,*
> The little ring 0, that signifieth nothing. (Menninger 1992, p.401)

Recounting just some of the names for the zero gives the flavour of its movement and the change in cultural history over almost a thousand years. Are these words reflected in the zero language used in this study? The following sections look at the development of number language and at the names given to zero by the children in this research.

Communicating single digit numbers

In language development, with a few notable exceptions[6], the understanding and use of the spoken word precedes that of the written word. The use of the spoken language applies also to mathematics; hence a child will use a number word before the number symbol or the written number word.

This section begins by looking at the spoken word for a single digit number. The digits are commonly known by the number names *one, two three, four, five, six, seven, eight, nine.* An adult, with their historical and association experiences might, in certain circumstances, use alternative words for these numbers.

Alternative words for single digit numbers 1 to 9			
Number word		**Prefix**	**Examples**
one	Unit, single	uni- mono-	unicycle monorail
Two	brace, deuce, pair, twins, couple, dual	bi- di-	bicycle, binomial diabolo, diarchy
three		tri- tre-	trio, triplets, triangle, triad treble, trey
four		tetra- quad(t/r)–	tetrahedron, tetrandrous, tetrarch, quarter, quadrilateral, quadrangle, quartile, quaternity,
five		penta- quin-	pentagon, pentateuch, quintuplet, quintet,
six		sex- hex-	sextet, Hexagon, hexameter
seven		hepta	heptad, heptarchy
eight		octe/a	octet, octagon, octave
nine		none/a	nonagon, nonet

List 8

[5] *Theca* was the circular round brand that, in the Middle Ages, was burned into the cheek or forehead of convicted criminals.
[6] Such as with a deaf and dumb person

In most cases these alternative words are context specific. All nine digits have words that rely upon a prefix but only 'one' and 'two' have a complete, non-prefixed alternative word. The use of these words is more limiting than the words 'one' or 'two'. For example with the 'two' words (*twins, couple, pair, brace, deuce, dual)* there is a connection between the two objects[7]. If they are two unrelated objects, such as a dog and a door, then the general number word 'two' would be used. The teachers in the research schools agreed that their children, aged 7 to 11, would be likely to know and successfully use some of the alternative '*two*' words (*twins, couple, pair)*.

Alternative words for zero used in the Questionnaire

There does not appear to be a prefix for the word zero but there are alternative words. What word might you use instead of the word zero? This was one of the questions asked of the 11 year old children in the Questionnaire. A written question requires a written answer but this meant that the children's spelling created a problem. The answers, which included *non, nort, nout, zilch, nourght, nil, naught, note, none, nill, naut, nothing and nought,* were discussed with the class teachers who felt strongly that there were different spellings of the same word, (nort - norte, non - none, note – nout, nill - nil, naut - naught, nourght – nought). This seemed an acceptable assumption. The results were classified accordingly and can be seen in table 55.

What word might you use instead of the word zero?	
	Total of number responses - 94
nought / nourght	28 (30%)
nothing	20 (21%)
nort / norte	20 (21%)
naut / naught	13 (14%)
nil / nill	3 (3%)
none/ non	6 (6%)
zilch	2 (2%)
nout / note	2 (2%)

Table 55

Further categorisation was not as easy as was expected. The spelling of the words tended to come from the way the child spoke but these were written responses and the researcher had no way of assessing the pronunciation, so the class teachers kindly agreed to ask the children to read the words they had written. After further discussions it was decided that *nout* was the local dialect for *nothing*, that *nought, nourght, nort, norte* were a corruption of the same word *nought*, and that *nought* is a variant spelling of *naught*. This resulted in further categorisation. The final list of alternative words for *zero* being *naught, nothing, none, nil* and *zilch*. Table 56 provides the frequency of use.

[7] For example pair is a set of two that usually exist or co-ordinates as an "ordered pair". They can be used together as with a pair of gloves, shoes, eyes; or be like a pair of scissors, trousers which consists of two corresponding parts. People or animals when arranged in 'two' can be called a couple, in circumstances such as in a dance, horses harnessed together.

What word might you use instead of the word zero? Classified	
	Total of number responses from the questionnaire - 94
Nought nught, *naut nourgh, nort, norte, naut, naught*	64 (78%)
Nothing *nout, note*	22 (23%)
None *non*	6 (6%)
Nil *nill*	3 (3%)
Zilch	2 (2%)

Table 56

From the Questionnaire there was no way of knowing what the children meant by their choice of word. What was indicated was that the children saw these words as alternatives to the word *zero.* Are any of these words interchangeable and if so to what extent? Are they alternative words intended to describe the symbol '0' or the value of zero? It was the Task-Interviews that would provide a deeper insight into the use of these words.

Alternative words for zero used in the Task-Interviews

The Task-Interviews followed the same order as the Questionnaire with the question about the zero language occurring at the end. At this point in the Tasks-Interviews, because every child had used the word *zero*, the researcher remarked, 'You have done a lot of work with me today and you have used the word zero many times'. The child was then asked, 'What other word might you use instead of the word *zero*?'

Some children, particularly in the 6 and 7 year old age range, seemed to think this was a strange question,

- *Zero is zero, there's no other word for it.* (Aged 7, verbal)

A small number of children described the symbol - a 6 year old said *round,* two 6 year olds and one ten year old said *circle.* As the 10 year old girl, who really wanted to help by providing another word for zero, replied,

- *Zero is just zero. I suppose you could call it a circle.* (Aged 10, verbal)

The older the children were more inclined to use the word *nought*, this was the word most often suggested. '*Oh*' was the second suggested alternative and some children gave the answer of both *nought* and '*oh*'.

Summary of Part Two: alternative words for 'zero'

When asked a specific question as to an alternative word for zero most of the children in the Questionnaire suggested *nought,* with other notable suggestions being *nothing* or *none*. In the Task-Interview the same question was asked and the majority of the older children in the also gave the answer *nought* though the younger children tended to say '*oh*'.

Surprisingly though the children had proposed the use of *nought* when they read the '0' number symbol in the *zero number facts* the alternative word of *nought* was not used by any child, most used the word *zero,* with only a small minority using *nothing* and *none*. Similarly when answering the *zero number fact* questions most used the word *zero,* with only a small minority using *nothing* and *none*. The section of the *zero number facts* Task-Interviews, where the children were asked to explain the reasons for their answers, was particularly rich in zero language and it was possible that further contextual zero language would emerge. The data from these explanations are used in part 3 of this chapter.

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Chapter Six, Part Three: The language explaining zero's value

There were two main areas of the study which provided examples of the children explaining the value of zero, one was the direct question 'What is *zero* worth?' asked in the Task-Interviews, the other was where the children explained the reasons for their answers in the *zero number facts*. The data in these two areas were examined for further examples of zero language.

What is *zero* worth?

In answering this question the children used two zero words - *nothing* and *none.* While some children used both the words, *nothing* and *none*, some used only one. There was a tendency for the word *none* to be used by the 6 and 7 year olds and for *nothing* to be used by the older children. Overall *nothing* was used more frequently than the word *none.*

- *It means you've got none.* (Aged 7, verbal)
- *It doesn't matter what you call it. It's worth nothing.* (Aged 7, verbal)
- *It's worth nothing. Nothing there. A blank.* (Aged 8, verbal)
- *Zero, nothing, none or you can miss it out.* (Aged 9, verbal)
- *Zero is a number but it's worth nothing.* (Aged 10, verbal)

The result of asking the direct question 'What is *zero* worth?' confirmed the strong use of the zero language *nothing* or *none*. It also linked in with the language of the empty set (chapter 5) where the trend was for the children to use *none* for describing the empty ribbon box and when explaining the reasons for the emptiness to use *none* (or less commonly the word *nothing*). It appeared as if the empty set language did not add to the list of zero words or phrases. However, tables 41 and 47 show a few children who used the phrases 'not got any' [water], 'no yellow' [ribbons].

The largest piece of zero language evidence collected in the study was in the Task-Interviews where the children explained how they reached their answer to each of the *zero number facts*. Most children used a variety of zero words. Only one child continued to use only *zero* and no other alternative, this is seen in full in the following first illustrative incident. This can be compared with the varied use of language from a child of the same age, from the same school and class, reported in the second illustrative incident.

This use of a variety of zero words and expressions was typical of most children. When reading a *zero number fact* equation and giving the answer then *zero, nothing* (no-thing) and *none* (not-any) had sufficed but now, in explaining their reasoning, children were describing a practical situation; a scenario which involved objects (with many children illustrating their answer with a sweet example).

Illustrative incident

3 + 0 =, *Three add zero equals three*
Explanation – [Uses three fingers] *Add zero*
0 + 3 =, *Zero add three equals three*
Explanation - *Zero fingers* [Makes a fist] *and three fingers* [uses three fingers] *makes zero.*
3 – 0 =, *Three take zero equals three*
Explanation – [Puts out three counters] N*ow take away zero*
0 – 3 =, *Zero take three equals three, no zero*
Explanation – *Give Zero and take three, you can't so the answer is zero. The real answer is –3. I think the real answer is zero but if you were doing about minus numbers then you have to write –3. If not then it is zero.*
3 × 0 =, Three times zero equals zero
Explanation – *Because anything times zero is zero*
0 × 3 = *Zero times three equals zero*
Explanation – *because zero times by anything is zero*
3 ÷ 0 =, *Three divided by zero equals zero.*
Explanation – *You can't fit three into zero and you can't go over into the minus numbers*
0 ÷ 3 Reads - *Zero three equals three, no zero*
Explanation – *You can't fit zero into three so the answer is zero.*
(Aged 10, verbal)

Illustrative incident

3 + 0 Reads - *Three plus zero equals three*
Explanation – *I have ///* [puts out three counter] *you have* *<u>none</u>* [points to an empty space on the table.] *You add them together and you've got three.*

0 + 3 Reads - *Zero add three equals zero, I think it's three*
Explanation – *<u>Nothing</u>* [spreads hands on the table] *add three* [puts out /// counters] *and you have three counters.*

3 - 0 Reads - *Three take away zero equals three*
Explanation – *You have three* [puts out /// counters] *<u>not take any</u> away, you're left with three.*

0 - 3 Reads - *Zero take three equals three, (no zero)*
Explanation – *Give <u>Zero</u> and take three, you can't so the answer is <u>zero</u>. The real answer is –3. I think the real answer is <u>zero</u> but if you were doing about minus numbers then you have to write –3. If not then it is <u>zero</u>.*

3 × 0 Reads - *Three times zero equals zero*
Explanation – *You can't times something by <u>nothing</u>*

0 × 3 Reads - *Zero times three equals zero*
Explanation – *You can't times <u>nothing</u> by something*

3 ÷ 0 Reads - *Three divided by zero equals zero (or three).*
Explanation – *Three divided by <u>no children</u> is <u>nothing</u> because <u>no children</u> get the sweets. What about the three? Three sweets, <u>no children</u> so the sweets are still there and <u>no children</u> to eat them.*

0 ÷ 3 Reads - *Zero three equals zero*
Explanation – *<u>No sweets</u> and three children and <u>no sweets</u> to give <u>no-one.</u>*

(Aged 10, verbal)

Taking an example of 3 – 2, one could say - *I have three sweets and I give my friend two sweets so I have one sweet left.* Using a similar analogy with 3 – 0, one could say - I have three sweets and I give my friend ... How does one describe '0 sweets'? Probably as 'no sweets'. There were examples in all of the zero number facts of other words and phrases being used,

In addition 3 + 0
- *Three sweets and no more sweets.* (Aged 7, verbal)
- *I have three sweets and you haven't got any.* (Aged 8, verbal)

Subtraction 0 –3
- *You've got no sweets so I can't take any so it must be zero because you've not got any sweets so I can't take away any. Answer is zero.* (Aged 10, verbal)

Multiplication 0 × 3
- *You've got no sweets and times it by three people. If you've got none you can't times it by you can't times it by anything. Answer is zero.* (Aged 9, verbal)

Division 3 ÷ 0
- *Three divided by zero equals zero. You've got three friends and no sweets, no counters, nothing to give them.* (Aged 9, verbal)

What the researcher did not find was a pattern of use. Many of the words and phrases seemed to be interchangeable and the children seemed at ease in using any one of them,

- *you've got none*
- *you've got nothing*
- *you haven't got any sweets*
- *you have no sweets*

Only a few children used a phrase such as *zero ribbons, zero sweets* or *zero counters.*

- $0 \div 3$ *that is zero divided by three equals zero. Three cubes and share them into zero people no-one gets anything as no-one is there. That's the confusing bit as we've still got the three cubes.* (Aged 10, verbal)
- 0×3, *zero times three is three, no it's zero. Zero sweets you can't times it by three. Answer is zero.* (Aged 9, verbal)

We accept the phrase '*three sweets*' but, whether written or spoken, '*zero sweets*' is a phrase rarely used.

Illustrative incident
A 7 year old boy was asked to explain how he had got the answer $3 + 0 = 3$. He made a tower of three interlocking bricks on the table. *If I had three counters and zero counters.* [Hesitates.] *If I had three counters and zero counters.* [Hesitates for a long while and the researcher asks the child, What is the problem?] *It doesn't sound right. You can't say, zero counters.* [Hesitates.] *If I have three counters in this hand* [he picked up the tower of three bricks] *and none in the other hand, how many have I got?* [He grinned] *Three.* (Aged 7)

Summary of Part 3, The language explaining zero's value

When describing zero's value the two words used were *none* and *nothing*, these were also the words used most frequently to describe the empty set. In the context of explaining the zero number facts then the language changed, as the children tended to add a unit such as *no sweets, not got any sweets.* All the children appeared to move with confidence between *none, nothing* and the alternative zero phrases.

While it may be argued that zero does not need a unit (e.g. if you have none it matters not whether you have no eggs or no dogs) there are occasions when it is necessary to be more precise and communicate that you have 'no sweets'. So why, unlike the other digit number words used as adjectives, do we veer away from using 'zero sweets'? This question is addressed in part 5 of this chapter.

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Chapter Six, Part Four: Other modes of communicating zero

John Donne, preaching from the pulpit in the 1620s, said of zero,

> The less anything is, the less we know it: how invisible, how unintelligible a thing, then, is this Nothing! (Kaplan 1999, p.99)

So far this chapter has considered the oral and written language used to communicate zero while the previous chapter gave consideration to the ways the children used words and symbols to communicate an empty set (chapter 5). There is one other area yet to be considered as the researcher saw the children use other means of communication than language and symbols. In order to understand these other modes of communication there is the need to return to the research of Martin Hughes (1989). He found that young children responded more favourably to questions posed in the concrete than those in the abstract as can be seen in the following extracts. The first is a conversation between Martin Hughes (MH) and Ram (4 years 7 months)

MH:	What is three and one more? How many is three and one more?
Ram:	Three and what? One what? Letter? I mean number? [We had earlier been playing a game with magnetic numerals and Ram is presumably referring to them here.]
MH:	How many is three and one more?
Ram:	One more what?
MH:	Just one more, you know?
Ram:	(Disgruntled) I *don't* know. (Martin Hughes 1989, p. 45)

The second example reproduces a dialogue between Martin Hughes (MH) and Amanda (3 years 11 months). It becomes clear that the child saw no connection between the questions concerned with bricks and the more abstract questions.

MH:	How many is two and one? (Long pause. No response.) Well how many bricks is two bricks and one brick?
Amanda:	Three.
MH:	Okay. So how many is two and one?
Amanda:	(Pause.) Four (hesitantly)?
MH:	How many is one brick and one more brick?
Amanda:	Two bricks.
MH:	So how many is one and one?
Amanda:	One, maybe. (Martin Hughes 1989, p. 46)

Donaldson (1987) believes that if we offer children things to think about they can create images of unseen quantities that do not depend on the tangible. It is usual for young children to work in the concrete and the objects they use helps the formation of a mental image. The two children working with Martin Hughes were not using objects but referred to the concrete mental image. They did not appear happy to work in the next stage, that of the abstract.

Do these three stage apply to zero? As Oesterle says, 'Objectifying a null or empty set is obviously impossible' (Oesterle, 1989, p110). This researcher asks if there is no concrete stage can there be a mental image? It is possible to visualise an empty set. However, when one tries to think of an empty set in a concrete situation, by putting a unit value to the zero, this can cause problems. This is illustrated in the following incident.

Illustrative incident

An exercise, in demonstrating to students the problem children may have with imagining different numbers, was used on a number of occasions by this researcher with the same final outcome.
The students were asked to close their eyes and 'see',

- five stars. They then opened their eyes and discussed what they saw and the pattern or arrangement of the stars.

This was repeated a few times for other amounts such as 9 soldiers, 13 fish and 8 plates. They were again asked to close their eyes and 'think of no elephants'. There would be laughter as each person saw an elephant.

When thinking of 'five stars' though the number is given first (five) one cannot visualise without knowing the unit or object one is using (stars), this then takes precedence. If one thinks of '0 stars' then the object comes to mind first. So 'no elephants' meant the image of the elephant was the concrete image.

The discussion with the students would continue by looking at the problems of thinking in the concrete with regard to zero and the implications for teaching children. For example if one asks a child, 'How much have you got if you have 2 pennies and no pennies?' and the child answers 3 pennies there is a possibility that the child has visualised the 'no pennies' as a penny. The researcher has known children answer in this way and for them to go on to demonstrate their answer in the concrete with two pennies and one penny. The question then moves on to ask if it is possible to think, to visualise and/or to illustrate zero?

The researcher spoke to three artists and asked if they could illustrate zero. They said yes but as an impression, an impression which would revolve round colour. Zero made them think of black. Two of the artists, who work with children, thought it would be interesting to see how some of their pupils would illustrate *zero* or *nothing*.

One artist asked a group of 12 year olds to close their eyes and think of *nothing* and then to draw or paint what they saw. The result was that the papers were left blank. The other artist asked a group of similar aged children to close their eyes and think of *zero*. Many said you can't draw *nothing* and left the paper blank but the outcomes of a few were designs with a '0' symbol as the focus. These children followed the path of the artist whose painting is shown on the next page.

Jasper Johns, ***'The Number Zero'***, oil on canvas, 1959, private collection

(in Guedj, 1996, p.104)

Illustration 22

Throughout this research the children had been asked to explain their answers. Would it be possible for the children to illustrate zero? The opportunity came in the *zero number facts* Task-Interview. When the equations of the zero number facts were presented to the children and they were asked to explain their answer some referred to a concrete situation. This meant, in their verbal explanations, they gave *zero* a unit - 'three sweets and no sweets'. After a child had provided a rationale for a zero number fact the researcher placed a collection of resources on the table (a number line from 0 to 10, a box of interlocking cubes, a box of counters, pencil and paper). Children seem to become more expressive and articulate when given a role where they are helping someone less knowledgeable than themselves, hence a scenario was presented by the researcher. (Anna Stallard, see appendix 11, used the same strategy.) *My friend's little boy/girl doesn't know how to do that...* [the researcher would point to the specific zero fact number card]. ***How would you show him/her how to do it?***

The children were encouraged to use any of the resources. Here the intent was to see how the child would represent *zero* and for the researcher to gain further insight into the child's understanding of how he/she had arrived at their *zero number facts* answer. No child elected to use the number line, a few used a pencil and paper and the majority of children used counters to demonstrate their answer. Very few children changed their answer as a result of using counters with the exception of three children who originally said $3 - 0 = 0$ and when they used the counters they decided the answer was three. No child used the resources as an aid to finding the answer to an equation to which they did not know the answer; indeed most children used the resources to 'prove' their answer to be 'correct'.

- $3 + 0 = 0$ *You put out three counters* [for the 3] *and you need no counters* [the 3 are removed] *so you have none left.* (Aged 7, verbal)
- $3 + 0 = 0$ *You put out three counters* [for the 3] *and no more counter* [for the 0] *so you have three altogether.* (Aged 8, verbal)

Many children made no attempt to depict the numbers preferring to use a verbal explanation. How did the children who did elect to work in a 'concrete' or in a 'visual' mode depict zero? There were eight main methods,

The methods used to illustrate zero in the zero number facts

1. By showing an open hand
2. By waving the hand over the table
3. By making a clenched fist
4. By drawing a circle with a finger where the '0' sign would have been
5. By using counters to form a 0
6. By pointing to a '0' sign on one of the cards
7. By drawing a '0' on a piece of paper
8. By putting out one counter

List 9

These fall into three categories, a) *emptiness*, b) *reproducing the '0' symbol,* c) *using a representational object.*

a) Emptiness

- By showing an open hand
- By waving the hand over the table
- By making a clenched fist

These children were showing there is nothing there, they were indicating emptiness. The language used for the '0' in the equation was *nothing* or *none*. For example for the zero number bond 3 + 0 =, three counters would be put out and the child would point to the three counters and say, three and *none/nothing* (*zero* was rarely used) and for the *none/nothing* make one of the above gestures.

b) Reproducing the '0' symbol

- By drawing a circle with a finger on the table
- By using counters to form a 0
- By pointing to a '0' sign on one of the cards
- By drawing a '0' on a piece of paper

Here the children used the '0' symbol. The language used tended to be mainly *zero* though some said *nothing* or *none*. For example for 3 + 0 =, three counters would be put out and the child would point to the three counters and say, three and *zero* (*none/nothing*) and use one of the above methods. It was as if they were reproducing the equation.

c) Using a representational object

- By putting out one counter

Often this 'zero counter' had to be a different colour to the other counters being used in the demonstration. So, in 3 + 0 =, the three might be red counters and the 'zero' might be a green counter. Though an object was being used the child did not count the object in the answer. This counter might be called *nothing* or *zero* (or a ghost as was recounted in the indicative incident chapter 4, part 4).

Summary of Part 4, Other modes of communicating zero

When explaining a number children tend to describe a situation in context giving the number a unit, such as 'three apples'. This seemed to help to form a mental image. The mental image of *zero* is that of an empty set but when a unit is placed after zero, for example 'zero elephants', then it is the object which provides the mental image. Imaging *nothing* is tricky. When asked to show others how to do the zero number facts the trend was for most of the children to ignore the equipment provided and to use verbal explanations. Those who did use the resources when communicating zero used gestures to indicate emptiness while others reproduced the '0' symbol. A few used one cube to represent zero. These results show some similarity to those found in the research of Anna Stallard (1982). This will be discussed in part 5 of this chapter.

0 ~ 00 ~ 000 ~ 00000 ~ 000 ~ 00 ~ 0

Chapter Four, Part Five: discussion points

From the analysis of the language of zero there arose four main discussion areas. Each of these four aspects will be considered in turn.

1) Representing zero
2) Reading and referring to '0' as 'oh'
3) The words used for *zero*
4) The word *zero*'s development in the English language

1) Representing zero

Hughes (1986) in his chapter on 'Understanding the Written Symbol' includes some unpublished research by Anna Stallard (1982). While the focus of this study is zero and Stallard and Hughes were researching children's understanding of the symbols +, - and =, her study included three references to zero. Details of her research are to be found in appendix 11. She used sixty children aged 6 to 10. The children were shown a set of fifteen cards (see list 10) and she asked the children to show what the cards meant using the bricks to help.

Set of fifteen cards used by Anna Stallard (in Hughes 1982, p.8)				
6	2	0	-5	+6
2 + 2 = 4	3 + 1 = 4	5 + 6 = 11	3 – 1 = 2	6 – 5 = 1
5 – 5 = 0	4 + 0 = 4	7 = 5 + 2	4 = 6 - 2	3 = 3

List 10

Three of these cards involved zero; it is the two shaded cards, 0 and 4 + 0 = 4 which are pertinent to this study. The adequacy of each response was subsequently assessed, the crucial factor being that the meaning and not just the appearance of the card had to have been conveyed. The response could be done either by using the bricks, as suggested, or by a purely verbal explanation. The card showing zero on its own was adequately represented by well over three-quarters of the children. Their responses were mostly verbal but some children used the bricks to make pictorial representations of zero; the use of bricks to form a '0' symbol (illustration 23) did mirror the response of some children in this study.

Representation of zero (Stallard in Hughes 1982, p. 27)

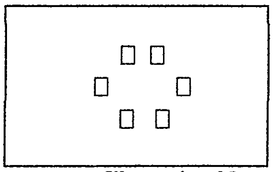

Illustration 23

The 4 + 0 = 4 card was adequately represented by less than a third of the children, the presence of zero in the second position causing particular difficulties. A number of children simply placed four bricks on the table.

There were differences in the aims of the research and in the presentation of the task. A major difference being that with an equation Stallard gave the child the answer, while in this study the child was asked to provide an answer. Even so it is interesting to compare the two findings. Looking at the three categories used to represent zero in this study - indicating emptiness, reproducing '0' and using an object to represent zero - did Stallard report findings to match any of these categories? Both Stallard and this researcher found the child's first selected method was through a verbal description. The second common method involved the indication of there being nothing there, of emptiness. The third common method was by 'reproducing the appearance of the card', in Stallard's research by forming the '0' symbol with bricks. However, in Stallard's research there was no report of a child using a counter to represent zero.

Stallard's Adequacy Score

Adequacy score (%) for each card

Card	*Score* (%)
6	95
2	93
0	85
5 – 5 = 0	62
2 + 2 = 4	62
3 – 1 = 2	60
1 + 3 = 4	58
6 - 5 = 1	57
5 + 6 = 11	55
4 = 6 - 2	47
7 = 5 + 2	45
- 5	43
+ 6	35
4 + 0 = 4	30
3 = 3	18

(Stallard, 1982, p.14)

Table 57

What is also interesting are the results seen in table 57 which show that 85% of the children 'adequately' represented '0' when used as a single number, though this was lower than for the other single digits. The equation 4 + 0 = 4 was next to the bottom of Stallard's list being 'adequately' represented by only 30% of the children.[8] One may ask what would have been 'adequate', in Stallard's view, for 4 + 0 = 4, especially with regard to the reasons given by children in this study to 3 + 0 = 3 (chapter 4, part 2).

2) Reading and referring to '0' as 'oh'

> In reciting one's telephone number, social security number, postal zip code or post office box, room number, street number or any of a variety of other numeric nominals, we carefully avoid pronouncing the digit 'zero' and instead substitute "oh."...In some parts of the world, the phrasing "naught" and "aught" are used but it is quite uncommon to hear 'zero.' (Arsham, accessed 2002)

'0' being read as 'oh' can be traced to situations where numbers and letters are mixed. By using 'oh' this covers both eventualities and possibly avoids any confusion. An example of this can be seen in the development of the telephone. In its infancy to make a call one spoke to the operator giving the town and number, such as Wimbledon 348 or Mossley 702. When the automatic dialling system came into use telephones had both numbers and letter on the dials. The telephone numbers were often formed from the first three letters of the town followed by a number, hence Mossley 702 was dialled as MOS 702. This mixing of numbers and letters could account for the '0' being treated as a letter and read as 'oh'. Today, when telephone numbers no longer contain alphabet letters and '0' must be *zero*, the custom in Britain continues with the '0' being referred to as 'oh'.

There are other 'number only' instances where '0' is read as 'oh' which cannot rest on the historical link with alphabet symbols. Three examples being bus (the 302 to Chester), the address (401, King Street) and road numbers (the A604). Reading numbers by splitting them into single digits normally applies to numbers greater than 99 but less than 1000[9], the bus number 53 would be the *fifty-three* while the 302 would be *three, oh, two*. There may be a variety of reasons for this. Reading 'larger' numbers using the place value system requires more skill and by using single digits people make fewer errors. Possibly it is for speed, *three, four, two* containing fewer syllables than *three hundred and forty two*. Why use 'oh' rather than the word zero? Again, 'oh' is quicker to say than *zero* and, also, this convention comes from an age when the word *zero* was rarely used, being considered a mathematical and an American word not an everyday word (the next section provides examples).

[8] 'Adequacy' - that the meaning and not just the appearance of the card had to be conveyed, see appendix 11.
[9] When reading numbers over 1000 they are often subdivided, for example 1070 is referred to as ten, seventy.

While telephone numbers no longer contain alphabet symbols the symbols of postal codes are alphanumeric[10]. They contain letters and number side by side, the symbol 'O' being either a letter or a number, such as in the following example of a hand-written postcode

Hand-written postcode

OL4 ODW

Illustration 24

In this particular instance the first 'O' is a letter as it stands for Oldham while the second '0' is a number. The norm is that both the 'O's are communicated as 'oh'. This conceals the fact that the reader does not know whether the '0' symbol stands for a number or a letter. Paradoxically, for years the use of 'oh' removed confusion but now it is causing confusion. In handwriting '0' (illustration 24) it is not possible to know whether the writer intended the '0' to be a number or an alphabet letter. Today, with the common use of computers it is necessary to know whether the operator should use the number or letter key, whether they should type 'o' or '0'. There is now a need for clarity, often requested, in the reading of postal codes. This may, in turn, necessitate differentiation when writing, possibly by using ø (the symbol phi) for zero.

While the reasons may be lost the custom of referring to '0' as 'oh' continues and attempts at change are not welcome by everyone.

Illustrative incident

The researcher was amused by the contents of a letter sent to the 'Home Truths' Radio 4 programme, Dec 2003.
The previous week a 'new' presenter had read out the telephone number of the programme as, zero 3, 5, zero, 7, 4, 8.
Part of a letter from an irate listener was read out the following week,

> *If you read the phone number using the word zero instead of 'oh' once more then I'll stop listening to the programme.*

(Radio 4, 2003)

Children quickly assume the customs and conventions of their home world. The code for the area where this research took place was 01457, when children were asked to recite their telephone number (or that of a friend/relative) all the children recited the number using 'oh' for any '0's. The children were then asked to read a different telephone number but one that had the same local code (01457 830450). 90% of the older children (9 to 11 year olds) used 'oh' for all the

[10] Harper Collins Dictionary - *Alphanumeric or alphameric adj. (of a character set, code, or file of data) consisting of alphabetical and numerical symbols* (Interactive, 1995)

'0s' and 34% of the 4 to 6 year olds read '0' in their local code as 'oh' but read all the other '0's' as *zero*. Possibly the convention, by the younger children, had not been fully assimilated to the point of transference to a new situation.

> **Illustrative incident**
>
> An eleven year old child was asked if *zero* was a number. Her answer was,
> *Yes, because it goes zero, one, two, three, in the number line. It can't be a letter. There isn't a zero in the alphabet in English so it must be a number. You don't mix numbers and letters and you say zero, one, two three four and so on or a, b, c, d, and so on.*
> Knowing that the local postcode contained a zero the researcher asked her to write down her telephone number which began *01457.* She was asked to read what she had written. She read the numbers quickly and for the '0' she said 'oh'. There was a pause as she realised what she'd said. She looked very surprised indeed.
> Then, quite defensively she announced, *Well, everybody says 'oh' in phone numbers.*
> (Aged 11, verbal)

3) The words used for '0'

With the exception of 'oh' being used in the convenience context, the majority of children described '0' as *zero*, a small minority using *nought*. Until the publication of the QCA National Numeracy Strategy (1999), where the use of the word *zero* took prominence, most people in England referred to the number symbol '0' as *nought*. This researcher, along with many other teachers, used the word *nought* for '0' when teaching. Indeed, during her informal discussions with teachers she found, until the implementation of the NNS, few had used *zero* in their own lives or in the classroom. They had used the words *nought* and *nothing*, unless they were counting down ... 3, 2, 1, zero. The researcher asked a group of thirty-five people aged sixty to eighty-five to read '0' (in the context of a set of single digits and in simple equations). They all read '0' as *nought* or *nothing.* When they were asked about the word *zero* the overall view was that it is a 'modern word',

- *We didn't have zero when I was young. Just plain nought or nothing.*
- *The only time we used zero was when we said – three, two, one, ZERO.*
- *It's a modern word, probably from America. We didn't have zero in our day.*
- *I think it was used in higher mathematics like algebra, not in ordinary sums.*

With the research children the majority used the word *nothing* to describe its worth, while a small minority use the word *none*. Again this may well be the influence of the National Numeracy Strategy. Throughout the NNS document the word *zero* is used exclusively with the exception of NNS Teaching Programme for Reception children which includes the word *none,*

> Begin to recognise 'none' and 'zero' in stories, rhymes and when counting.
> (QCA National Numeracy Strategy, 1999, p.2)

4) The word *zero*'s development in the English language

At some point, across the Task-Interviews, all the children used the word *zero,* 84% used the word *nothing,* 45% used *none* and 12% used the word *nought.* While the words *none* and *nought* were used very infrequently both the words *zero* and *nothing* were used regularly. Using just the four words (*zero, nothing, none, nought*) in the zero language vocabulary does not seem like overload, especially when compared with the number of words the children are likely to hear and possibly to use for 'two' (list 8). Having many words for zero is cited as a major reason for children's zero problems (Haylock and Cockburn 1989, p.25; Kaplan 1999 p.97; Arsham web site accessed 2002). Where are the 'other' words coming from? One clue may be found in what appeared to be an incongruous answer given by one child, when asked to provide another word for *zero,*

- *No.* [You can say 'no' instead of zero? I'm not sure I understand.] *No, like when you say no ribbons.* (Aged 11, verbal)

The child had envisaged a practical situation, had put the answer in context. Using phrases such as 'three sweets' or 'five eggs' are not considered problematic but, while it is to be argued that zero does not need a unit there are occasions when it is necessary to be more precise and communicate that you have 'no sweets'. So why, unlike the other digit number words, which are used as adjectives would we veer away from using the phrase 'zero sweets'?

This researcher felt that the explanation may be found in the linguistic development of the word *zero* in the English language.

> **zero,**
> **n** (pl ~*s*) Figure 0, cipher; no quantity or number, nil; starting point in scales from which positive & negative quantity is reckoned (~ in thermometers, freezing point of water or other point selected to reckon from; *absolute* ~ in temperature, point at which the particles whose motion constitutes heat would be at rest, estimated at -273.7 *C.);* (Mil.) point of time from which the start of each movement in a timed programme is at a specified interval; lowest point, bottom of scale, nullity, nadir. *fly at* ~ (under 1,000 ft). [It., contr. of *zefiro* f. Arab. As CIPHER]
> (Concise Oxford Dictionary, 1959, p.1496)

The only entry was of *zero* as a noun thus making it grammatically incorrect to have 'zero sweets'. However, in the USA, the use of zero was more flexible resulting in zero being used as a noun, an adjective, an adverb or a verb[11].

[11] Phrases such as *the result was a big, fat, zero; zero possibilities; he gave zero assistance; his speed was zero rate; we zeroed in on the cause* are cited by the American Mathematician Arsham (Arsham web site, accessed 19th Feb 2002) to illustrate this point.

There is evidence that the zero is being accepted in England in a wider context. On the Radio 4 programme (Woman's Hour, Feb 3rd, 2003) Jenny Murray discussed *Zero Tolerance,* a phrase used frequently by the media and accepted in daily life. Where one would once have said *no tolerance*, zero is replacing the 'no'.

Is not the 'zero' in zero tolerance being used as an adjective? Has the English language evolved over the past fifty years to extend zero's part of speech? Reference to various dictionaries suggested a confused picture (appendix 2). To gain clarification the editorial section of the Oxford English Dictionary (OED) was contacted with regard to the usage of zero as an adjective. The full e-mail correspondence with the Senior Editor OED, Fiona McPherson can be found in appendix 12.

> In a phrase such as 'zero tolerance', the word 'zero' is indeed an adjective; in some contexts this is completely transparent, as in the sentence 'I have zero tolerance for alcohol'. Here, 'zero' is simply qualifying the noun, without the two words combining to make a noun phrase. (Senior Editor OED, Fiona McPherson, e-mail, 09/06/05)

It is possible that, as a consequence of this researcher raising the issue with the OED, further clarification may take place. It was comforting to have the Senior Editor OED, write,

> It is likely that when the OED entry for 'zero' comes to be revised, we would exemplify the adjectival sense further. (Senior Editor OED, Fiona McPherson, e-mail, 09/06/05)

Possibly the use of zero as in 'zero sweets' may well become as commonplace as the phrase 'three sweets' if zero, as with other single digits, is recognised as both a noun and an adjective. Extending zero into an adjectival role may well simplify matters with regard to zero language. However, there are problematic shadows, for unlike other single digit numbers the *zero* is now used as a verb. Indeed the word *zero* has blossomed; from being a word rarely found outside of mathematics it is now used in all aspects of life (see appendix 13). This wider usage has brought confusion rather than clarity as the following incident shows,

Illustrative incident

This is an extract from a letter, received by this researcher, from a former colleague who worked in the Education Department in the field of Information Technology. Last year she retired to live in Spain. She wrote,
Last winter I began to feel a gap in my life. I'm beginning to miss work. I hadn't realised that you were doing some research. When I asked Kath [a mutual friend] *what the subject was and she said 'Zero', I thought that it was some modern way of saying 'I don't know!' It didn't occur to me at first that she really meant Zero!*
(05/01/05)

What has been recorded in children's use of the word zero throughout all the tasks and activities in this study has been the need to add a rider when using the word *zero*. It was common for the children to say *zero* and then add a phrase such as - *this means nothing; it means that there are none: it means it is empty.* Was this clarification for the benefit of the researcher or as reassurance for the child? It was not possible to say. The researcher did find this reminiscent of children who, when learning their numbers, read '3' as *three* and then show three fingers as if to emphasis the threeness. Chapter 8 considers this further when examining the teaching of single digits compared with the teaching of *zero.*

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CHAPTER 7

SUMMARY OF OUTCOMES AND OVERALL REVIEW OF THE RESEARCH

> … zero is exceptional. It not only took man (*sic*) centuries to invent the digit zero, it took added centuries to discover the number zero and still more centuries to accept and use it. Yet, after all this, evidence indicates that we still do not recognize its significance and importance.
>
> (Blake and Verhille 1985, p.46)

Outline of Chapter 7

Part 1 of this chapter presents a summary of the outcomes from each of the areas of the study. In part 2 there is a discussion regarding any links, and the nature of such links, between the historical development of the concept of zero and the findings in this study. In part 3 the findings are reviewed holistically.

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Chapter Seven, Part One: A summary of the outcomes from each area of study

The research aim was to explore children's conceptions of zero; the sample was from children in the age range 3 to 11. The focus areas of this exploration were,

(a) Zero as a number and its relationship to other numbers
(b) The zero number facts
(c) The empty set
(d) The language of zero.

The data came from the use of a Questionnaire, Task-Interviews and Activity-Interviews. The results were analysed and presented under the chapter headings,

Chapter 2 - Is zero a number?
Chapter 3 - Zero and the number order
Chapter 4 - Zero number facts
Chapter 5 - The empty set
Chapter 6 - The language of zero

Chapters 2, 3 and 5 looked at how these children regarded zero in the form of the empty set, as a number and in relationship to other numbers. The effect of these conceptions on the use of zero in simple equations was the focus of chapter 4. Chapter 6 considered how children communicated zero across each area of the study. While this communication was mainly in the form of the spoken word also used were written words, symbols and gestures. The following summary is under the following three headings, each section concludes with a précis of the findings.

1) The children's conceptions of zero
    a) Zero as a number
    b) Zero's relationship to other numbers
    c) The empty set
2) How these conceptions affected the zero number facts
3) The part played by the zero language

~ o ~~ o0o ~~ o ~

1) The children's conceptions of zero

a) Zero as a number

The question asked of the children (aged 7 to 11) was in two parts, 'Is zero a number? and 'Why do you think zero is/is not a number?' (see chapter 2, table 5)

<u>YES answer</u> - A high percentage said zero <u>was</u> a number, the range being from 63% to 85%, with an overall percentage of 76. The reason given by the majority was that zero was found in the company of other numbers (as in algorithms, in numbers such as 100, and in the number order 0, 1, 2, 3 ...). However, it was noted that 23% of these children said that while zero was a number it was also '*nothing*'.

<u>NO answer</u>- With children who said that zero <u>was not</u> a number (the range being from 0% to 18%, with an overall percentage of 9) the reason given by the majority of these children was that zero was not a number because it was '*nothing*'.

<u>YES and NO answer</u> - There was a small number of children (the range being from 1% to 25%, with an overall percentage of 13) who said that zero <u>was</u> and <u>was not</u> a number. Their reasoning mirrored those of the 'Yes' and 'No' groups.

- The <u>'yes'</u> reason being the explanation that zero was found in the company of other numbers.
- The <u>'no'</u> reason being that zero could not be a number because it was '*nothing*'.

The overall view was that zero could legitimately be a number as it was a number symbol amongst other number symbols. However, '*nothing*' could not be a number as,

- *Nothing cannot be anything.* [Aged 10, verbal].

In the light of the analysis of the tasks and activities it was seen that all the children in the sample (aged 7 to 11) were familiar with zero both as the symbol '0' and with zero as *nothing.* In providing an answer to the question 'Is zero a number?' the children had to choose between the two aspects. The children who gave both answers were not contradicting themselves but were not able to stress one aspect and ignore the other. Gattegno (1987) emphasised that stressing some features and ignoring others is how we generalise and abstract.

> Sameness and difference are a matter of stressing some features and ignoring others, or as Mary Bateson (1992:89) suggested 'a matter of context and point of view'.
> (Mason 2002, p.120)

The majority of the children were able to make a decision, the 'No' answers putting the emphasis upon the value of zero as '*nothing*', the 'Yes' answers putting the emphasis upon the '0' symbol. What seemed a minor point but is later seen as significant were the number of children in the 'Yes' category who wanted to express their understanding that zero was '*nothing*'.

➢ ***Zero the number, précis of findings***

- The majority of children said zero was a number.
- The reason for zero being a number was that '0' was found with other numbers.
- The reason for zero not being a number was that '0' was '*nothing*'.
- Many of the children who said zero was a number also added that it was '*nothing*'.

b) Zero's relationship to other numbers

Zero was seen as a number because it was with other numbers 'on the number line'. This response, though unknown at the start of the research, was apropos, for the second focus area was based round the number order. The children were asked to order sets of numbered cards and to explain their decisions. Depending on the numbers involved (fractions and zero, decimals and zero, negative numbers and zero or single digits including zero) the sample ages were in the range 5 to 11. Analysis of the data created three main discussion areas (chapter 3, part 3).

i The number symbol order
ii Whether zero was a whole number
iii The value of zero

i. The number symbol order - All the children were confident in placing '0' in the number order as 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 while the older children also saw zero as the pivot point, between positive and negative numbers.

When working with young children emphasis is placed on knowing the number order, orally (with the number words, zero, one, two, three, four, five, six, seven, eight, nine) and visually (using the symbols 0, 1, 2., 3, 4, 5, 6, 7, 8, 9). A number line (and the extended number line) was on display in each classroom. From the experience of the researcher, and that of her professional colleagues, it is common to observe a number line being used by teachers. With the younger children the number symbols are pointed to as the number order is recited. With the older children it is used during the mental mathematics lessons. Frequent usage provides a strong visual image of the number order.

All the children in the sample could recite the number names and order the number symbols. These skills were firmly established, so much so that many children were very reluctant to put numbers between these digits. This applied, particularly, to the 'zero-one' relationship where zero was seen as *'coming before the one'* or *'being next to the one'*. When the children were asked to place fractions and single digits in order it was the zero which caused a dilemma; often this resulted in children keeping the '0' card in their hand. Frequently the children's reasons for the positioning of the zero showed the need to preserve the number order, of 0, 1, 2, ... Preserving this order was seen as paramount.

ii Whether zero was a whole number - It was the children's discussion of zero, in relationship to fractions and decimals, which generated the fascinating debate as to whether zero was, or was not, a whole number. Individual children's thinking on this topic had an effect on where he/she placed zero in the number ordering tasks. The problem appeared to be that it was the whole number, which was seen as 'the large number in the front of a fraction', which helped to place a mixed fraction so that the 1 in 1½ placed this between 1 and 2. But ½ and ¼ were not written as 0½ to indicate that they went between the '0' and the '1'. The children compared '0' with other numbers known, by them, to be whole numbers. The discussion included,

- A quality of whole numbers, such as 1, 2, 3, is that they can be split into fractions while zero, being '*nothing*', cannot. Similarly, whole numbers can be split into parts but you cannot divide 'nothing'.
- Zero did not always follow the convention of where one would find whole numbers, particularly in fractions where 0½ would not be seen or written. Again, while £0.25 might be occasionally seen in receipt from a till (though more often without the £ sign) younger children would put 25p rather than £0.25.

The outcome of the children's debate was that zero was not a whole number. [In the selection of numbers in this task a weakness was the lack of mixed fractions which may have given more insight into the children's thinking with regard to zero and whole numbers. However, the knowledge of this shortcoming is a result of hindsight.]

iii The value of zero - The task was centred round the ordering of number symbols but in the explanations for their number orders children sometimes described the value of zero. The tension appeared to lie with the value of zero rather than whether zero was, or was not, considered a natural or counting number. Words such as '*nothing*', '*not worth anything*', or indeed '*nowt*' were used. By some, zero was considered as worthless and insignificant. A small group of children felt that a zero symbol could go anywhere in the number order, or it could be ignored and missed out, because it was '*worth nothing*'.

➢ ***Ordering numbers, précis of findings***

- The children often referred to zero as being in the number line
- The majority of the children saw '0' as the starting point of the symbol order 0, 1, 2, 3...
- The older children saw '0' as the point between the positive and negative numbers.
- Many of the children described zero as being worth *nothing*,
- A few children felt that zero was nothing and as such could be placed anywhere in the single digit number order, such as 1, 2, 3, 0, 4, 5 ... because it was of no significance, it could be ignored or missed out. There may be a linkage with the use of 'number bars' and discrete data and 'number lines' and continuous data.
- When ordering fractions and zero, decimals and zero, a number of children
    - ordered the numbers but had difficulty placing the '0'
    - wanted to know if zero was a whole number
    - tried to keep the number order intact, 0, 1, 2 ...
- In all the sets of ordering numbers there was evidence that children strove to keep the 0, 1, 2, 3 ... number order, being particularly averse to separating '0' from '1'

c) The empty set

This activity-interview was in two parts, the given empty set (in the form of an empty bottle) and the making of the empty set (in the lack of yellow ribbons to put in a yellow box).
The data analysis discussion points were,

i The reaction to the empty set
  - *Attempts to overcoming the emptiness*
  - *Re-checking for emptiness*
  - *The ignoring of the empty set*

ii The reasons given for the emptiness

iii Describing the empty set

i. The reaction to the empty set - The fact that there was no water in one of the bottles and that there were no yellow ribbons to put in the yellow box caused an emotional reaction, particularly from the 3, 4 and 5 year olds. The reaction was one of anxiety. While this anxiety was expressed verbally there were also three observable reactions, some children displayed all three, some one or two. These were,

- Attempts to overcoming the emptiness

This was particularly noticeable, across the age range 3 to 9, as children attempted to place other ribbons in the empty yellow box or to count ribbons which were not there (chapter 4, part 3). Only the younger children attempted to put water in the empty bottle.

- Re-checking for emptiness

Even though every child had been engaged in the activity, of sorting the ribbons and putting the lid on the yellow box, children opened the yellow, empty container to re-check the contents. No other container was re-opened.

- The ignoring of the empty set

When asked to order the bottles, children did not include the empty bottle in the ordering. Similarly, on being asked to describe each of the bottles and the contents of each of the containers, children (in the 3 to 9 age ranges) ignored the empty bottle and the empty yellow container.

ii. The reasons given for the emptiness - It had been expected that the majority of the children would have seen the empty sets as omission and that the reasons for the emptiness would have reflected this. There was no water in the bottle because the researcher had not put any in. However, the children did not know this and it was understandable that they saw the situation as one of removal as that was a most likely scenario in their everyday life. Indeed explanations for the bottle being empty from all the 3 and 4 year olds were of removal. In the age range 5 to 10 a notable overall percentage of 44 gave a reason which involved removal.

In the Ribbon Activity there were no yellow ribbons to be sorted because the researcher did not include any yellow ribbons. Again all the reasons given by the 3 and 4 year olds were of removal while a notable overall percentage of 37, aged 5 to 10, gave an explanation involving removal. The researcher found the responses of these older children puzzling and intriguing for she had not presented yellow ribbons so there were no yellow ribbons to be removed. The question as to why so many children give a reason involving the *removal* of ribbons they had not seen is pointed out for further research in chapter 8.

iii. Describing the empty set

When describing the empty bottle the 3, 4 and 5 year olds used mainly the word *none* (with *empty* and *nothing* less frequently), the 6 to 10 year olds used *empty* (with *none, nothing* and *zero* less frequently). In all age ranges the empty box was described using the word *none* (*zero, nothing* and *no* were also used less frequently).

When labelling, the overall trend was for the young children to label both empty containers using the '0' number symbol. The older children wrote the word 'empty' for the bottle and '0' (read as *zero*) for the ribbon box. It appeared that, from the age of 6 onwards, the children consistently described and labelled the empty container, not in isolation but with regard to its context, whether the set of containers was numerical or non-numerical in context.

➢ ***The Empty Set, précis of findings***

- The trend was that children, particularly the younger children,
  - tried to overcome the dilemma of the empty set by putting something in the set
  - felt the need to check that the empty set was empty
  - ignored the empty set
- The younger children (the 3 to 5 year olds) reacted strongly to the empty set, showing anxiety and concern.
- The reasons given for the emptiness were that the contents had been removed (removal) or that they were not there (omission). It was understandable that children saw the empty bottle as removal. However, when a set was empty, as a result of no objects being placed there (by the child), many saw this not as omission but as removal. The notable trend in both activities was for the reason to be one of removal.

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2. How these conceptions affected the zero number facts

The focus of chapter 4 was zero in use in an arithmetic setting. The inclusion of '0' in the equations of *zero number facts* resulted in some interesting and some startling explanations, for both correct and incorrect answers. The main discussion points were,

i The use of the word *'can't'*
ii Using an informed guess
iii The effect of zero as *nothing*
iv. The formation of a *personal zero rule*

<u>i The use of the word *'can't'*</u> - The word '*can't*', was used extensively by the children. They reasoned that because zero was worth *nothing* then *'you can't do it'* [the zero number fact]. It would be expected that if *'you can't do it'* then no answer would be given. Yet, invariably, an answer was given. Almost without exception the answer was one or other of numbers presented in the zero number bond equation.

<u>ii An informed guess</u> – The children realised that, in a zero number fact, the answer was generally one or other of the numbers (in this study either 3 or 0). Many children said the answer is *either zero or three.* The strong impression was that, having only two options and a 50/50

chance of being correct, a significant number of children adopted a 'lazy strategy' and took a guess. Even if they were very unsure they had a chance of being correct. The researcher felt that this willingness to guess came before trying other methods of gaining the answer.

- Response to the zero number fact 3 × 0. *It's zero, no three, no zero. I'm going to say it's zero.* [Is that a guess or do you know?] *Well I kind of know. The answer's got to be zero or three. I think that is zero.* (Aged 7, verbal)

iii The effect of zero as nothing - It was common for children to say *'because zero is nothing'*, as the explanation for their answer. However, with some children they took this to mean that as zero was *nothing* then it could be ignored, even to the point where quite a number of children covered up the zero with their hand *to make it go away.*

iv The formation of a *personal zero rule* - When analysing the explanations, patterns of *personal zero rules* emerged. The formation and use of a *personal zero rule* was a direct result of there being a zero in the equation and some *personal zero rules* were also affected by the position of the zero in the equation. One frequently implemented *personal zero rule* was that the '0' in the equation was to be covered up, ignored. This was found across all the age ranges but the younger children (unable to attempt all the four rules of number) only demonstrated this notion in addition and subtraction. Chapter 4, part 3 provided evidence of a *personal zero rule* being used consistently by children across all the zero number facts. These were extreme examples of the result of having the notion that *because zero is nothing* then it is of no significance and it can be ignored.

The wider picture, discussed in chapter 4, part 4, included Ward's (1979) research that the most common errors in subtraction calculations were to do with zero. That zero difficulties permeate whole number computation comes from the work of Suydam and Dessart (1976) who found that one of the seven most frequent whole number errors was errors with zero for each operation. Both of these pieces of research were concerned with zero in algorithms. It is possible that the way the children, in this sample, approached zero in the zero number facts will, later, be mirrored in the way they handle algorithms containing zero. This could be highly significant.

- ***The zero number facts, précis of findings***

- Because zero was *nothing* the word *'can't'* was frequently used. This was invariably followed by an answer.
- Many children thought there were only two possible answers (in the given context, '0' or '3') and there was a high tendency for the children to guess.
- The most common rationale for an answer was associated with zero being *nothing*. Some children took this to mean that zero could be ignored.

- The presence of zero in the zero number fact equations caused the children to form *personal zero rules*. There was evidence that a few children used the same *personal zero rule* across all eight of the zero number facts. Added to this, there was evidence that other children were using *personal zero rules* in just some of the zero number facts.

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3. The Language of zero

The discussion points from chapter 6 were,

i. Representing zero
ii. Reading and referring to '0' as '*oh*'
iii. The words used for *zero*
iv. The word *zero*'s development in the English language

A number of educationalists have seen zero having many names as being problematic (Haylock and Cockburn1989, p.25, Kaplan1999 p.97 and Arsham web site accessed 2002). Consequently this researcher expected to find many names for zero being used and these many names being the causation of many problems. The word 'zero' was used by almost all the children aged 3 through to 11, some using no other word. It is probable this is a direct result of the implementation of the National Numeracy Strategy (1999)[1] which, for the 5 and 6 year olds, states,

- As an outcome for year 1...
  'Recognise zero and none in stories and other contexts, including the counting sequence.' (NNS 1999, p.2)
- As an outcome for year 2,..
  'Use zero when counting and understand the function of '0' as a place holder in two-digit numbers.' (NNS 1999, p.3)

i. Representing zero –Putting numbers into visual memory is often easier if a unit is added, such as *three dogs*. However, adding a unit to zero for example *no elephants, zero cats,* is confusing as this brings to mind an image of an elephant or a cat. Most of the children used verbal means to describe '0' in an equation, which mirrored Anna Stallard's research findings. The methods used by those who tried to use representation fell into three categories. These were 1) the indicating of emptiness, 2) the reproduction of the '0' symbol and 3) the using of an object to represent zero. The children did not appear to find the representation confusing but tricky. Rather like trying to show '*air*', you know it is there, you know it does certain things but you can't see it.

ii. Reading and referring to '0' as '*oh*'- It could be seen how the children, as they got older, were assimilating everyday conventions in their use of '*oh*' for the '0' symbol (such as in the reading and in the recitation of sets of telephone numbers). Outside these conventional areas it was very

[1] See appendix 16 for NNS mathematical vocabulary for reception children.

rare for children to read the zero symbol as 'oh' unless the '0' was considered to be a letter of the alphabet.

iii. The words used for *zero* - Throughout the various task-interviews and activity interviews most of children referred to the number symbol '0' as *zero*, only a few said *'nought'*.

> The problem is that we sometimes call it zero, we sometimes call it nought and sometimes nothing. Other numbers like one have only one name. (Haylock and Cockburn 1989, p.25)

Having many names for zero suggests that this is unusual, if not unique, amongst numbers. But this is not so, as was demonstrated in chapter 6, part 2. What is unique is that while we regularly use number words as adjectives, ('five houses' and 'seven shoes') the number zero is not treated in the same way. The phrase 'no sweets' is common while 'zero sweets' sounds to be a linguistic error. In the English language until relatively recently, zero was used as a noun and a verb but not an adjective. Times are rapidly changing. In the media - newspapers, advertisements, in films, on television, on the radio - *'zero'* is used in differing ways (appendix 13). The phrase *'zero percent'* is common. *'Last year, in England, there were zero escapes from open prisons'* was a sentence used in an interview regarding the prison service, in a Radio 4 programme, 'Open Book' (12th January, 2006). Possibly phrases like *zero sweets* and *zero counters* will soon be commonplace in our schools.

When giving explanations and reasons for decisions involving zero, such as why the child felt $3 + 0 = 3$, the majority of the children used the word *'nothing'*, a few children used *'none'*. Overall the word *'zero'* was used for the symbol '0' and the word *'nothing'* to explain the value of zero. As a result of the research findings, which showed the use of zero words to be limited, the research did not find that zero having many names was problematic.

iii. The word *zero*'s development in the English language - Everyday language is notoriously ambiguous and imprecise and, when used as the basis for establishing precise notions, the results may be less than desirable. No doubt it was to establish precise language that the NNS (DfEE, 1999) strongly recommended the use of the word zero. This suggests that there is correct language one should use. As was demonstrated in this study there is ambiguity in language definitions and usage. While the NNS appears to have had some impact in the acceptance of the word *zero* it is likely that usage will have a stronger influence. Most of the children did use the word *zero* for the '0' symbol yet, where custom dictated, they changed to using 'oh'. When describing zero in the context the use of *nothing* was common. The *'zero is nothing'* analogy has evolved from the language as one of the most commonplace phrases. This analogy is deeply rooted such that it is common to virtually every conversation involving the idea of zero and it is common to virtually every publication which include this concept (Kaplan 1999, P36).

The use of the word zero, in the English language, is evolving rapidly. It is changing from being used solely as a noun to being a verb and an adjective. The Oxford English Dictionary is monitoring and noting these changes. The addition, of zero being an adjective, may open the door to phrases such as *zero apples*. Those who see *zero's* problems as being caused by it having many words (a theory disputed in this study) may feel that such terms as *zero apples* will reduce the number of *zero* words. This researcher believes it is the way the word *zero* is bring used in everyday language (examples are to be found in appendix 13), where the connection with the zero number is becoming more abstruse, which will complicate the use of the zero language in the future.

➢ *Zero Language, précis of findings*

- The majority of the children preferred to describe zero rather than attempt to represent it. A few did represent zero as an empty set, as the '0' symbol and with the use of an object.
- The majority of children referred to the symbol '0' as '*zero*', a minority used '*nought*'.
- The majority of children referred to the value of zero as '*nothing*', a minority said '*none*'.
- The use of '*oh*'
  - The older the child the more likely the child had adopted the convention of using 'oh' for '0' in situations such as telephone numbers, bus and house numbers and postal codes.
  - As the process of using typed material becomes the norm there is the need for precision, in order to differentiate between '0' and 'o', by in referring to '0' as 'zero'
- The changes in the uses of the word '*zero*'
  - In the near future, it is possible that the use of zero will be extended to included zero as an adjective (OED e-mails in appendix 12). It may be that phrases such as 'zero dogs' and 'zero pounds' will be common usage.

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Chapter Seven, Part Two: Links between the historical development and the findings of this study

Mathematics is a cumulative science with its past assimilated in its present and to understand the nature of zero it was necessary to look at its history. Various aspects of this history have been described, though briefly, in three parts of the thesis. Chapter 2, part 1, looked at the emergence of the concept of zero and its acceptance as a number. Chapter 4, part 1, considered the early problems arising from zero's interaction within the operations of addition, subtraction, multiplication and division. Chapter 6 considered the language of zero with part 1, tracing the development of the symbols used to represent zero while part 2 reviewed the zero words.

When compared with the other single digit numbers zero's history is long, complicated and turbulent. One cannot but admire the early Sumerian, Mayan and Hindu cultures where, independently, the importance of the empty space in number was recognised and where this was developed into a symbol to represent *nothing there*. Mathematics and the world at large are indebted to the Hindus for their ability to develop this notion of *nothing there* into seeing zero as a number alongside other single numbers. The stance taken by the Christian church provided understanding as to why zero took so long in arriving in and being accepted by Europe. As one would expect, great mathematicians including Brahmagupta, Bhaskara, Ibn Ezra and Fibonacci played a role in zero's advancement but it was those who appreciated the benefits, those who chose and continued to use the number system containing zero, who ensured zero's continued role alongside the other digits. Such people were the traders and accountants.

At the start of Chapter 2 it was suggested that an historical review, of the ways zero evolved, developed and was used, along with epistemological analysis, might provide an insight into why the concept of zero can be difficult to understand. Was this so? Did the research findings provide any such links? There were a few incidents, which the researcher allied with zero's history.

Compared with other single digits, one of the conceptual difficulties with zero was using a sign in order to say that *nothing is there*. If there is nothing there then why not leave the space empty? Hughes (1987), in his work on children using their own symbols to represent zero, had one 4 year old child who did not write anything when representing the empty set. In this research there was also one 4 year old child who deliberately left her paper blank because she said '*that's an empty space*' (Chapter 5, part 3, table 48). This *empty space* method to represent *nothing here* was a feature of zero's early history as was seen in the Mayan and Sumerian cultures. Hughes also noted that a number of the 3 year old children used their own symbols to denote that a box was empty; this was also the responses of some the 3 year old children in this research (Chapter 5, part 3, table 49). Again there is the historical linkage with the early representation of zero. This representation may, also, mirror the progress of a child's thinking. It seems quite logical to leave an empty space rather than use a symbol. The next step is the more sophisticated notion to put something to show there is nothing there. The use of arbitrary marks was not a reflection on the use, or lack of use, of the '0' symbol but it was an indication that these children were not at the conceptual stage of using accepted symbols for any number, including '0'.

The use of a dot in the Hindu culture, as with the Mayan and Sumerian cultures use of early symbols for zero, was intended as a kind of punctuation mark to denote an empty space. Knowing this helped this researcher to understand two similar incidents in the ordering of numbers Task-Interview (reported in Chapter 3, Part 2, section D). The first was a child who felt that zero could be used anywhere when used as a stop, as a marker between numbers.

- *If you put 9 zero, or 7 zero it could mean a stop. It* [zero] *could be a break so it could go anywhere.* (Aged 6, verbal)

The second incident was the outcome of an informal discussion with a member of staff, a teacher who had had no connection with this child. This teacher had the same understanding of zero and used the same rationale as the child. It was as a result of undertaking the background research into zero's historical development that this researcher was alerted to some possible explanations for these occurrences.

The development of Hindu mathematics, in embracing zero as a number on a par with the other number symbols 1 to 9, meant zero could be used in calculations. Between the 7th and 12th century Hindu mathematicians such as Brahmagupta, Mahavira and Bhaskara struggled with equations containing '0', particularly with 0 and $\div$. Today some adults and children are still struggling with the zero number facts. Like Brahmagupta, Mahavira and Bhaskara children and adults expect and so seek a consistent rule. Either a rule which connects all (or many) of the zero number facts or where the answer can be linked to other number facts (Chapter 4, part 3). This can be seen in the reactions of some adults to $0 \div 0$, and to the answers and reasons they provided.

- $1 \div 1 = 1$, $5 \div 5 = 1$ and therefore $0 \div 0 = 1$
- $0 \div 3 = 0$, $0 \div 5 = 0$ and therefore $0 \div 0 = 0$
- dividing by '0' cannot be done, $5 \div 0$ cannot be done and therefore $0 \div 0$ cannot be done

In our society the experience of '0' comes an early age. Very young age children meet various aspects of zero, including the '0' number symbol and the various zero language words. Situations occur such as *all gone* when the food has been eaten, filling the *empty* bucket with sand, saying *three, two one, zero* in a counting down rhyme, hearing and recalling a telephone number, seeing the number '0' on a digital time-piece and on the door. By the age of 4 most of the children in this research had accepted that there were a set of established symbols (though they did not always recognise their worth) which represented amounts and this included the '0'. Despite the possibility of alphanumerical problems no child over the age of 4 had difficulty in recognising and putting a 'zero name' to the '0' symbol. A high percentage of children aged 7 to 11 recognised zero as a number. The reasons being that the children had seen '0' with other numbers (number line), zero used to form larger numbers (102), zero being used in calculations.

> Since representations are culture-dependent, the relations between ontogeny and cultural evolution are completely changed. (Rogers, 1997, p.47)

Can the findings from this research be compared with the historical happenings in other countries hundreds of years ago? To answer this one must consider elements such as cultural and social differences as well as historical knowledge and its reporting. This researcher believes the answer is no. It is not possible to compare the experience that comes from living in a society where something is being developed and has yet to be fully accepted by society with the experience which comes from being born into a society where something is accepted, used, and is the norm. There are many modern examples where a society has slowly adapted to a new development only to witness a new generation of children who have known a world without such change. One example is the advancement of digital technology in the form of computers and mobile telephones.

Is it possible to see in children's behaviour connections between history and mathematical ideas? Piaget believed this to be so, though his connections between the history and the development of mathematical ideas were extremely general. Rogers (1997) suggests that,

> such generalisation is unjustified, and that contemporary evidence shows that the historical situation is more complex. (Rogers, 1997, p.44)

If we are to compare the findings of this research with an historical situation then we must consider the information gleaned from the history and, as Rogers (1997) did, question the continuity of historical records. He says,

> It begs questions about the continuity of records by its assumptions that our present state of knowledge about the past is complete, and it also poses problems about the biased interpretation of data. (Rogers, 1997, p.46)

As far as zero is concerned our historical knowledge is incomplete and the problems caused by using material from translated sources (often with no original material to refer back to) means historians frequently disagree on developmental details (Chapter 2, part 1). There are also the problems associated with viewing the past from a contemporary viewpoint. How the social history of mathematics is interpreted is dependent on the historian's personal point of view, the context of the time when it was written and the purpose of the writing. This researcher is in agreement with the doubts raised by Rogers (1997) about the links that have been claimed to exist between ontogeny and phylogeny. She believes that the instances reported here do not demonstrate the 'principle of parallelism', where individual development is claimed to mirror the historical development of the subject matter. As the mathematical conceptions found in this study are located in a specific age and culture no claim is made of an ontogenetic link.

However, this does not lessen the fact that this researcher was informed and this study enriched by considering the history of zero. This was particularly so when seeing past problems encountered by the use of zero. This study shows that, despite the passage of time, some of these problems are still in evidence today.

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Chapter Seven, Part Three: A Holistic View of the Research Findings

The research was divided into four focus areas,

a) Zero as a number and its relationship to other numbers
b) The zero number facts
c) The empty set
d) The language of zero

Data were collected and reported under these headings. Analysis of this data produced many interesting findings in each of these areas. Now it is time to combine this information, to try to make sense of the whole. The overall trend of the children's conceptions of zero, how these conceptions affected the outcome of the zero number facts and the part played by the zero language, all combine to form a picture of these children's conceptions of zero. This researcher can do no better than agree with and hope to achieve the sentiments in Mason's statement,

> *Patterns and distinctions may emerge during data analysis, but insight usually emerges because the researcher is able to look through the data rather than just at it ...*
> (Mason 2002, p.236)

The researcher saw the need to investigate the empty set because she believed the notion of emptiness formed part of a child's conception of zero. Kaplan also saw the strong connection between empty and zero,

> Empty ... a substantive adjective brings zero closer to numbers which lightly straddle the gap between noun and adjective ... an empty circle.
> (Kaplan 1999, p.44)

While most of the children were comfortable with the empty set many of the young children were ill at ease, perplexed, and anxious. The children's strong reactions fascinated the researcher. As there was a notable absence of relevant research and literature in mathematics education on the topic of the empty set it was felt that further insight into why emptiness should evoked negative emotions might be found in the realms of the psychology of emotions. Having no background in this area the researcher asked advice of a psychologist. She suggested that, though the links were tenuous, emptiness might stir the emotions in two areas, that of rejection psychology which could include a feeling of isolation and loneliness and that of motivational psychology such as deprivation, 'I am without, I need to have, that's mine'. Writers such as McKee (1980), Miller (1987), Langmeier and Matejcek (1975) and Laming (2004) were consulted but this researcher was not able to equate an empty set, in the form of an empty bottle or container, with the children's strong reactions. She was acutely aware that this could be the result of her own lack of expertise in the field of emotional psychology. Furthermore, to follow such lines of enquiry would make this a different thesis. Hence it was considered unwise to attempt further analysis but to leave this aspect to await further research. Interestingly when three artists were asked their reactions to *zero* (which they referred to as *empty, nothing*) each one said it provoked an emotion they described as **'black'**. Yet, when dealing with the '0' number symbol or the *zero* number word, anxiety was not evident amongst the children.

Like Kaplan (1999) this researcher expected to find a strong empty-zero connection. There was a connection but it was not strong or direct. With the exception of the activities targeted at the empty set, at no other time in the research did a child use the word *empty*. While the word zero was not associated with empty, the symbol '0' was associated with emptiness. There was a direct connection between *empty* and '0' and between '0' and *zero* but no direct language link between *empty* and *zero*.

Empty >> to >> **'0'** >> to >> **Zero**

Model 1

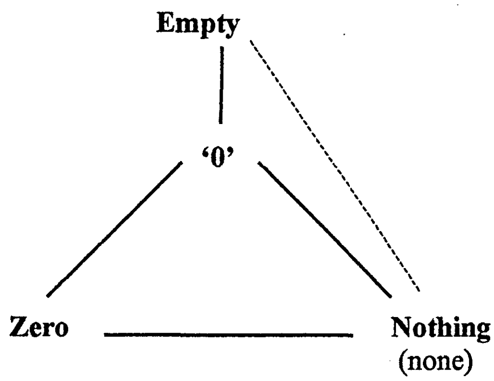
Model 1 is not a two way or cyclic model but a one-way paradigm. Understandable in that few people would refer to a container with *nothing* in it by using the word *zero*; the word *empty* would be more appropriate.

Edwards (1971) expected children to connect *zero* and *empty*,

> Nothing, nought, zero, empty, use words which present a single notion to children...
> (Edwards 1971, p.2)

Did this study find the statement of Edwards to be true? These words did not 'present a single notion' to the children in this study. This researcher sees the word *empty* as describing the state of a space. This need to enclose the empty set was seen in the research of Hughes, discussed in Chapter 5, part 3. Similarly *none* and *nothing* can also describe the state of a space, hence a few children described the empty set as *none there* or *nothing there*. The words, *empty, none, nothing,* have their roots in the concrete world of counting hence children frequently used them to describe the state when all had been removed. Yet, while empty is of a space, *none* and *nothing* need not be; they are 'of objects' none (not one), nothing (no-thing). Zero needs neither space or objects; it is abstract. The number symbol '0' can be used to denote the notion of the empty space, there being no objects and the abstract notion of zero. Thus, in model 2, it is seen that it is '0' which connects all these notions.

While there was a strong link between the words *nothing, zero* and '0', *empty* was linked, mainly, to '0'. It is these strong trends which are illustrated in model 2.



Model 2

The language of zero proved to be important when children were establishing their conceptions of zero. What was this language?

> Is it a consequence of English drawing on lots of different sources that we've ended up with all these words for '0'? (Anderson, accessed 2003)

All these words for '0' proved to be not too many words. It was seen that there were no more words used for zero than might have been commonly used for the number 'two'. Indeed there were three words used consistently by the children, *zero* to name the number symbol '0' and *nothing* and, to a much lesser extent, *none, nothing* and *none* being used to describe and explain zero (*zero is nothing, zero is worth nothing, zero means you have none*). With most of the children the use of the word *nothing* would appear to provided a working relationship sufficient for the children's current needs.

> This 'zero is nothing' analogy has evolved from the language as one of the most common speaking symbols for zero. This analogy is deeply rooted such that it is common to virtually every conversation involving the idea of zero and publications which include this concept. However, common language is notoriously ambiguous and imprecise and when used as the basis for establishing precise notions, the results may be less than desirable. ... The use of the zero-is-nothing analogy ... creates a superficial surface structure of zero that is used with some regularity for several years. It is attractive because it is easy to learn and retrieve, intuitively satisfying and it seems to work – it seems to effectively and consistently lead to correct answers. ..... The consequences of the use of such a shallow surface structure are ignored or unknown.
> (Blake and Verhille 1985, p.36)

The problem was not the many names but the understanding and use of these names, particularly that of *nothing.* The finding from this research may provide some of these 'consequences' which are 'ignored or unknown'. The trend for a great percentage of the children in this study was to see *'0' as being zero* and *zero as being nothing.* Considering the use of the analogy *'zero is nothing'* one assumes the intention is to provide insight into the meaning and understanding of *zero*. This, in turn, assumes an understanding of the word *nothing.* It is contended that the evidence in this study leads to the conclusion that there was not a common understanding of the word *nothing.*

Here we are speaking about one word, *nothing*, but as Margaret Donaldson (1987) comments, in her book 'Children's Minds', it is a common but naive assumption that one either does or does not understand a word. But, she continues this is not so, for knowledge of 'word-meaning' grows, it undergoes development and it changes. These are not consistent changes to all situations or for all people. Each person may have their own understanding of a word with nuances of meaning but with an expectation of a common foundation. What is the common foundation for the word *nothing*?

The word '*nothing*' stems from *none* which means is *not one* [derived from Old English nan, literally: not one]. From this comes the word *nothing* which is *none+thing* [derived from Old English nathing, nan thing, literally none + thing]. Over the years this has been simplified so that *nothing* is now understood to be *no-thing.*

Every child, at some point in the research activities and tasks, used the word *nothing*. It is an everyday word but it has more meanings in common usage than that of '*no-thing*',

Nothing n[2]

- *It's nothing, it doesn't matter.*
  Indicating a matter of no importance or significance:
- *It comes down to nothing, it amounts to nothing.*
  Indicating the absence of meaning, value, worth.
- *That means nothing to me.*
  Indicating disregard, not to concern or be significant to (someone).
- *Nothing doing.*
  An informal expression of dismissal.
- *Think nothing of it.*
  Indicating that one should ignore it, disregard it, forget it.
- *I think nothing of him.*
  Indicating that one has a very low opinion of someone.

[2] Summarised from Harper Collins Dictionary Interactive (1995), The Concise Oxford Dictionary (1959), Hyper-dictionary. (Accessed 2005) and xrefer.Dictionaries (Accessed 2005)

Understandably the children were bringing many of these common meanings to their dealings with '0' and *zero* [3]. Phrases used by children in this study, *forget it, ignore it, it's not important* reflect those in the list above. There was strong evidence that these meanings were taken literally and that zero was *ignored.*

Across the various tasks were seen two ways in which the language was used; these are expressed as zero being a *concrete absence* and as zero being an *abstract non-presence.* Zero as *concrete absence* may seem a contradiction in terms but this is where children are counting so three (3) is / / / while zero (0) is *no marks, not one mark, not any marks, empty, none left, not one put there.* The use of the word *nothing* may be used here in the sense that 'a thing' exists so there was a prior something and now there is 'no thing'. This research showed that the *concrete absence* seemed to hold few difficulties.

A child moves from the world of concrete objects to abstract thinking, when there is no longer need for real or imaginary objects. This is not an overnight revelation; often children will move back into the concrete for reassurance and, as was seen in this study in the zero number facts, to explain an answer. It was as if the *zero* was as real an entity as *three objects.*

Treating zero in the abstract along with the other numbers would appear to mirror the change a child moves to when he/she can function in the abstract world of number. Part of this move, into using zero in the abstract, is the use of the word *nothing* with no reference to any specific object of the kind of quantity to which it belongs; in this way *nothing* aligns itself with zero as an *abstract non-presence.*

> Being nothing is really different from being any other quantity, and this fact seems to corroborate the common sense that zero is nothing and outside all the somethings; yet it is also a number amongst numbers and so a nothing of a very determinate sort.
>
> (Brann 2001, p.58)

The *concrete absence,* where there was removal of objects, was the main explanation for the empty bottle and ribbon container. Fewer children used the reasoning of omission, *abstract non-presence.* Whether the word *nothing* was for *concrete absence* or for *abstract non-presence* for a significant number of children the outcome was the same, the *nothing* was of no-value and could be ignored.

[3] A few children felt that *nothing,* and hence zero, could not be 'anything'. From this followed, with a child's logic, that zero could not be a number – *but zero isn't anything – it's not a number, it isn't anything* (Aged 6, verbal).

It is a subtle move to go from *nothing* to *nothingness*. As Brann (2001) writes,

> Bad things are said to have badness, but Nothing itself does not have the attribute of nothingness; that is almost always said to belong to real somethings or conditions in the world, and it means worthless, a fighting word. (Brann 2001, p.169)

Once having accepted the notion that *nothing* is nothingness it is almost a logical step to take zero being *nothing* (the name of a state) to zero having *nothingness* (a property). This was a move which, in this study, was seen to have startling consequences.

Most of the children in this research understood *nothing* to be *no-thing*, whether this was or was not a specific quantity.

Zero >>>>> **nothing** >>>>> **no-thing**

Model 3

While the conception of zero as no-thing was a working definition for the majority of the children in this study there was a small but significant minority who took this a stage further. They saw *nothing* as meaning worthless, of no value and this resulted in zero being ignored. Having this *notion of nothingness* had a great effect on these children's use of zero.

Zero >>>>>**nothing** >>>>> **nothingness**

Model 4

Nothingness (n) [4]

- <u>the state or condition of being nothing; non-existence.</u>
- absence of consciousness or life.
- complete <u>insignificance or worthlessness.</u>
- <u>something that is worthless or insignificant.</u>

Strong links between emptiness and '0', and between '0', *zero*, and *nothing* had been detected. While the majority of the children used the word *nothing* this word held different meanings and that in turn produced different outcomes in the Task-Interviews. A fine line of demarcation separates the meaning of *nothing* from the meaning of *nothingness*.

The consequences of holding the notion of zero as *nothingness* were evident in the zero number facts. For,

> ... what zero is lies entangled in what zero does and what it resembles.
> (Kaplan 1999, p44)

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[4] Summarized from Harper Collins Dictionary Interactive (1995), The Concise Oxford Dictionary (1959), Hyper-dictionary. (Accessed 2005) xrefer.Dictionaries (Accessed 2005)

One frequently implemented *personal zero rule* was that, as *zero* was *nothing (nothingness)*, then it was ignored. The train of thought of a few children was reminiscent of the riddle of assumptions – a bird has wings, a bird can fly, a penguin has wings so a penguin can fly. This transferred to zero as - zero is worth nothing, nothing is worthless and of no value, if it is worthless it is of no significance, if it is of no significance it has no effect, if it has no effect then it can be ignored. This notion of insignificance and disregard was evident in each of the task-interviews from ordering empty sets to ordering numbers but was most evident in an arithmetic situation, when zero was in use in the zero number facts.

If such a rationale persists this could be the basis for future arithmetic difficulties and there could also be repercussions when zero is met in other areas of mathematics. It can only be speculation to wonder how many adults (including those who work with children) who say zero is '*just nothing*' have in their minds *the notion of nothingness.*

Conceptions of zero - general statements

What has this exploration into children's conceptions of zero discovered? What might one expect a primary child's conception of zero to be? The answer begins by stating that there should not be an expectation of one answer. If one answer is given then there should not be placed importance on that one answer. On being asked the question 'what is three?' and replying 'one more than two' can one be judged for not saying 'three is a number'? If the view that one can have a conception of zero persists then the one must allow that this conception is many faceted.

From the findings of this exploration is it possible to make some general statements with regard to the children in this research and their conception of zero? This researcher begins by stating that a child does not have a conception of zero. The children's conceptions of zero consisted of a series of notions. Each child's conceptions of zero and the change in the notions which form these conceptions are, as in all learning, dependent on the age, ability and experiences of the individual. Within each of these notions of zero the children had degrees of understanding and this understanding was being refined according to their needs. Also within these notions of zero there was diversity of thought, adding to the difficulty of stating generalised findings. However, while every child may follow a different path and be at differing stages there was some commonality. The models presented here, of children's conceptions of zero, are not of any one child but are a general overview of the trends and elements which come from the findings of this research.

The majority of the children held many of the *generally accepted notions*. These were seen as common occurrences in the research tasks. While the children generally accepted these notions as true, it does not mean that they were mathematically acceptable. The *generally accepted notions* are listed in the first column of the following charts.

Chart 1 contains the basic knowledge aspects of zero. This knowledge is similar to a child learning the letters of the alphabet. In that a child would give a 'name' (or a reading sound) to 'c', would say it is a letter because it is in words with other letters, and say it is next to 'b' or 'd' in the alphabet. So most of the children in this study called '0', *'zero'*; said that zero was a number because it was seen with other numbers (formed larger number and was in 'sums'); that zero was *said* in the number order and *seen* on the number line, before one. This is factual knowledge a child could know because he/she has been told.

Chart 1 - Conceptions of zero, factual knowledge		
Generally accepted notion	**Concerns**	**Consequences**
♦ '0' was called zero		
♦ There was a convention of using 'oh'		
♦ zero was found in the number order, next to one		That the number order, particularly '0' being next to '1', should be preserved
♦ zero was a number	As to whether zero was: odd/even, a whole number	

Model 5

Chart 1, column one contains the *generally accepted notions,* the second column is headed *concerns* and includes information a minority of children felt they needed in order to answer the Activity and Task questions. The third column, headed *consequences*, show the problems of a minority of the children which stemmed from a narrow interpretation of zero knowledge.

Chart 2 - Conceptions of zero, language based on an understanding of the word 'nothing'		
Generally accepted notion	**Distinct diverse notions**	**Consequences**
♦ zero is seen nothing, no…, not any…, empty, none left, no-thing zero > nothing > *no-thing*	zero is seen as nothing, worthless, insignificant, a non-entity zero > nothing > n*othingness*	That zero could be ignored

Model 6

Chart 2 deals with the understanding of the word *nothing*. Column one contains the *generally accepted notions.* The second column headed *distinct diverse notions,* are personal extensions of a *generally accepted notion.* These *distinct diverse notions* are held by a minority, though a substantial minority, of the children. These can be problematic for it was in the *distinct diverse notions* where most of the unsound practices were found. The third column is headed *consequences* and shows the problems which are likely to stem from the *distinct diverse notions* in column two.

Chart 3 - Conceptions of zero, in the zero number facts	
Generally accepted notion	**Consequences**
♦ zero number facts	
– use of the word *'can't'*	
– the answer was '0' or 'the other number'	A 'lazy strategy' using a 50/50 guess was employed
– the presence of '0' indicated these equations were different	The formation and use of *a personal zero rule(s)*

Model 7

Chart 3 is concerned with zero in the number facts. Column one contains the *generally accepted notions,* the second column is headed *consequences,* and these may arise from the *generally accepted notions.*

The order of the charts is developmental in that the younger children tended to hold the *generally accepted notions* in chart 1. There is not a linear development from charts 2 and 3 but a relationship development, each being dependent upon the other. With the children in this research some, or all, of these generally accepted notions formed the basis of their conception of zero. No model was found whereby one could predict whether or not a child would adopt any one of these diverse thoughts or problematic consequences. This study found four possible consequences,

- The number order, particularly '0' being next to '1', should be preserved
- Zero could be ignored
- A 'lazy strategy' using a 50/50 guess was employed
- The formation and use of *a personal zero rule(s).*

It is suggested that, in this study, some of these consequences formed the basis of individual children's problems with zero. It is possible that some of these may be of short duration but others may be the basis of long-term future zero problems, such as those reported in Chapter 4, part 4.

What needs to be asked is how does a child acquire the *generally accepted notions* and *diversity of thought* and how, in turn, might these develop into such problematic consequences? In the light of the research findings the next section considers the teaching and learning of these notions of zero.

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CHAPTER 8

IN CONCLUSION

Outline of Chapter 8

Part 1 considers possible changes in the educational world as a result of the study's findings. Part 2 begins with a discussion as to whether the study has achieved its aims and draws to a close by considering the researcher's own path through the research process.

0 ~ 00 ~ 000 ~ 00 ~ 0

Part One: The Educational World

Teaching zero is easy

Gelman and Gallistel (1986) state the counting concepts, which children need to acquire to become proficient at counting, are

- the one to one principle
  (assigning a distinct counting word to each of the items to be counted)
- the stable order principle
  (knowing the list of counting words must be consistent in name and order)
- the cardinal principle
  (counting is not merely a process in which one engages but that it can actually yield a product)
- the abstraction principle
  (that the count principles can be applied to any array or collection of entities whether these are physical or non-physical, heterogeneous or homogeneous)
- the order irrelevance principle
  (that the order in which items are counted is irrelevant)

Ensuring that children understand and can apply these five concepts often requires a considerable amount of time and patience. As zero is not a counting number then these five counting concepts do not apply to zero. This leads teachers to believe that, in comparison to the counting numbers, zero is easy to learn and easy to teach; children are expected to have no problems. Over the years this researcher has asked numerous teachers, working in primary schools, about their teaching of zero. All were surprised at the question. Some made a reference to putting zero on the number line; many made a reference to zero being *nothing*. As one teacher said, '*Nothing isn't a problem, it's the numbers you have to count which cause problems*'. While it may be generally accepted that zero is easy to teach, in comparison to other single digits, this study challenges this view. As quoted before,

> The uniqueness of zero is more general and deeper than that of any other number and yet pedagogically zero is treated superficially as a trivial and obvious notion.
> (Blake and Verhille 1985, p.46)

Gelman and Gallistel state that zero needs formal teaching,

> We think that formal instruction is necessary for the development of a true understanding of zero as a number. (Gelman and Gallistel 1986, p.240)

An explanation of 'formal instruction' is not given so it is difficult to agree fully with this statement. What this researcher feels is necessary is that all teaching staff understand the long term effects of the approach where zero is treated as 'a trivial and obvious notion'. This section of the study considers possible changes in the educational world as a result of the research findings. Classroom practice with regard to the empty set, zero as a number, zero and the number order and the language of zero are discussed, together with the contentious subject of when children should be introduced to zero in arithmetic. Some of the expressed opinions are those of mathematicians, of the researcher, of teachers and of educationalists. To these are added the perspective of this researcher, acquired over many years of working with children, teachers and students and, now, augmented as a result of this study.

Pre-school children and number

Children come into contact with number in informal situations prior to formal schooling but what they learn and how they learn is open to discussion. Sue Gifford (2003), in her article 'How should we teach mathematics to 3 and 4 year olds?' expressed the essence of current thinking about pre-school mathematics. She reported the opinions of various educationalists, that,

- children learn about number mainly from adults in the home context, in accidental conversation and as the result of direct teaching (Durkin, Shire et al. 1986)
- children's mathematical learning contexts are shopping and cooking, cars, calculators, clocks, calendars, money and games (Young-Lovage, 1989)
- children have their own social purpose for numbers, such as indicating their age, but may also be interested in counting for its own sake as a 'culturally significant' skill (Carr, 1992)
- young children do not share adult purposes for number and so they may not learn as much from the integration of mathematics in everyday activities as has been understood (Munn, 1994)

(Summarized from Gifford 2003, p35)

There are various, differing, schools of thought as to how zero fits into this pattern, if indeed it does. Over the next few pages these differing views will be discussed. These include those who believe that zero, being an abstract concept, should not be taught until a child is reasonably confident with single digit numbers in the abstract. Others argue that, as zero is meaningless in everyday problems, it is unnecessary until later in schooling.

> The point about zero is that we do not need to use it in the operations of daily life. No one goes out to buy zero fish. It is in a way the most civilized of all the cardinals, and its use is only forced on us by the needs of cultivated modes of thought.
> (Whitehead 1911, cited in Blake and Verhille 1985, p.43)

While we may still not go out to buy *zero fish*, or zero anything, the number zero is seen in our daily lives more than it was in 1911. There are the digital displays of timepieces seen on videos, watches, microwaves; we have decimal notation in our money and in other measurements; the use of keyboards and keypads are commonplace, such as those used for computers and calculators, all of these contain zero. What cannot be denied is that children will have heard the word *zero* and will have seen the '0' number symbol at an early age. This needs to be taken into account when considering teaching zero to young children. Today it is not a matter of whether to introduce zero but of building on, and possibly correcting, the understanding the child brings to the nursery and school.

The empty set – pre-number

Evidence was given, in chapter 5, that the empty set was a cause of anxiety, a cause for concern to the point where children wanted to overcome the emptiness by placing objects in the containers. On the whole, emptiness was seen as removal not as omission. Removal of the contents resulting in emptiness assumes that objects were, initially, there. Yet, prior to the objects being placed there the original state must have been one of emptiness. When cooking, eggs are removed from an egg box until the box is empty; this exemplifies one of many situations with which a child would be familiar. But the egg box must originally have been empty before the eggs were added.

> Zero arises either as a consequence of subtraction, when, for example, there are two marbles in the pot, and both are taken out, and nothing remains; or of finding nothing such as opening a box and finding that it is empty. (Montague-Smith 1997, p.26)

Children meet emptiness in pre number situations such as filling and emptying activities in capacity and volume; this emptiness needs to be made apparent. Children need to be engaged in tasks where it is accepted that an empty set can be the result of removal and of omission. When ordering containers when appropriate an empty set should be included. Young child are less likely to feel apprehensive about emptiness if it is met in an enjoyable situation such as a game where 'emptiness', having no objects left, is the aim. Whatever the context discussion about 'the emptiness' should be a priority.

The empty set – counting

The teaching of arithmetic has a structure that proceeds from the simple to the complex, from the concrete to the abstract.

> For example, contemporary school children are introduced to the natural numbers before they encounter the concept of integers ... It appears that authors of mathematical texts have always followed such a scheme. (Swete in Katz 2000, p.1)

If zero is not a natural or a concrete number then this suggests that the teaching of zero would come later in the teaching structure. Anne Montague-Smith (1997) refers to the understanding of number as evolving gradually through the counting experiences where the development of counting skills

> ... runs alongside the development of number concepts and these are inextricably entwined. (Montague-Smith 1997, p.6)

As zero is not a counting number, then, by implication, zero would not be included in such early developmental experiences.

Zero has cardinal value. It is the cardinal value, the empty set, which needs to be included when dealing with the counting numbers. This researcher advocates that children should be made comfortable with emptiness and especially with the original state of emptiness, as omission and that an important element in a child's skill of counting is the empty set. Two counters are placed in a container. How many counters are in the container? Two, but only if the container was empty prior to the two counters being placed inside. The following illustrative incident provides an amusing episode and acts as an example of the importance, in counting, that the original state is one of emptiness.

Illustrative incident

'In a school the Principal is assigning duties to the staff on the occasion of entrance examination by incoming students. The professor of English is to check spelling, the professor of history is to look into the background of students, the professor of geography is to make notes of where they come from, etc. The professor of mathematics offered his abilities, but being perceived as 'not practical', the Principal declined his help. After seeing the unhappy face of the professor, the Principal yielded and suggested that the mathematics professor keep the count of students entering the room and leaving the room 'so that we can know at any time how many students are in the room (taking the test)'. The mathematics professor was happy and with pencil and paper started marking students entering and leaving the examination room. After a while, not because of curiosity but courtesy, the Principal asked the mathematics professor how many students were in the room. The mathematics professor quickly added and subtracted his counts and came up with an answer: 'Minus two.' What do you mean by 'minus two' asked the Principal. 'Well', answered the professor, 'two must enter the room and the room will be empty'.

While most people laugh hearing this anecdote, a mathematician to whom this anecdote was once told did not laugh at all. His answer was, 'Well, there is nothing funny about this, he was right, the room was not empty when he started the count'.

(Pogliani et al 1998, p.730)

As Pogliani and his co-writers suggest,

> The anecdote provides yet another illustration of the overlapping use of zero, 0, and empty, Ø. It also perhaps illustrates the limitations of the current use of labels 'natural numbers' or 'counting numbers' that could easily embrace zero, and even possibly negative numbers! (Pogliani et al 1998, p.730)

So, this researcher strongly believes, it is important that the empty set is recognised before objects are placed in the set and conversely that the empty set is recognised when all the objects have been removed. For example, children count in 2s by reciting 2, 4, 6, 8 ... (Similarly they count in 5s as 5, 10, 15, 20, 25... and in 10s as 10, 20, 30, 40...).[1] However, from the experience of this researcher, and that of her professional colleagues, when asked what number comes at the start of each sequence most children will reply, 'one', giving 1, 2, 4, 6, 8 ... (1, 5, 10, 15, 20, 25... and 1, 10, 20, 30...). There is the need to show that the original sequence is true only if one begins at zero, with an empty set. This can be illustrated with counters or with 2p coins. One has an empty purse (0) in which is placed a 2p coin, one continues to add 2p coins showing the sequence as 0, 2, 4, 6, 8 ... similarly in their removal 8, 6, 4, 2, 0. However, if there is 1p coin already in the purse then the sequence is 1, 3, 5, 7.... The use of 5p and 10p coins help to illustrate the 0, 5, 10, 15 ... and the 0, 10, 20, 30, 40 ... number sequences.

When counting, children should be made aware that the empty set is the starting point - first the box is empty (zero), then we have one, then we have two, etc. This will help children to form a visual image. It will strengthen the links between *empty* and *zero* and will give more meaning to '0' being at the start of the 0, 1, 2, 3, 4 ... number order. This may, also, help to overcome the difficulty some children had in adding to zero, when the zero number fact 0 + 3 proved to be more problematic than 3 + 0 (chapter 4, part 2).

[1] This was illustrated in the Questionnaire for, when asked to write the first five lines of the five times table, 25% of the children wrote 5, 10, 15, 20, 25. No child wrote 0, 5, 10, 15, 20 (see appendix 15).

The number order

Whereas young children used to be taught the counting numbers, now, with the more widespread use of zero, this seems to have changed to their being taught the number order beginning with zero, as in 0, 1, 2, 3... At the prospect of introducing children to zero through the reciting of the number order one teacher voiced an understandable concern, that the children might use zero as the first counting word, thus a group of four objects would be counted as five. Though this is possible the researcher has seen no evidence of this happening. However, as suggested in the previous section, by noting the empty set and naming it *zero* this may pre-empt such an occurrence.

All the children sampled knew the ordinal numbers; they could recite them and place the number symbols in the correct order. The visual image of the number order was in each classroom. The importance as to which number order presentation was used (as discussed at the beginning of chapter 3) was seen in some of the outcomes of this study.

The number line is a useful tool and provides a child with a powerful visual image, which then forms the basis of his/her conceptual image. The teacher needs to be aware of the strengths and weaknesses of these discrete and continuous images. It is the continuous number line that can be refined to incorporate other numbers. This researcher suggests that teachers place simple fractions and decimals on the classroom number line (when these numbers are being taught). A further suggestion, which may prevent errors in measurement, is to add '0' to a measuring scale such as that on a ruler or a measuring jug. This could act as a reminder to children that '0', not '1', is the starting point and help to prevent the errors referred to in chapter 3, part 1.

Zero as a number, zero as nothing

Rather than learning a new piece of information in isolation children are taught to make connections or associations between other learned knowledge and small elements of experience. This is an economical way of learning. This means that the information a child has learned about the number three can, in the main, be applied to the number four. The problem with zero is that it does not conform to all the associations and connections which apply to other numbers. Children do have not the width of experience to know which associations and which connections do and do not apply, as was seen in the discussion as to whether zero was a whole number (chapter 3, part 3). This makes the teaching of zero difficult. While zero should be given its rightful place and be treated as a number, it needs to be treated with caution as it does not always follow the conventions of other numbers. A child's mathematical needs, his/her mathematical ability and understanding will determine when these zero idiosyncrasies are addressed. However, there are times when children ask questions the answers to which may go beyond their present understanding, such as the thorny issue of dividing by zero. As Oesterle writes, it is important that the child's,

> ... definition should be one that may be enlarged and refined and of such a nature that it need not be denied or dis-allowed at a later stage. (Oesterle 1959, p.111)

As was seen in chapter 4, with some of the zero number facts a correct answer may be achieved by '*covering up the '0'*, by taking '*the biggest number as the answer*'. These are 'bad habits' used by adults, who know when they can be used and when they will work. Teaching these bad habits to children may produce an immediate 'tick for a right answer' but they may need to 'be denied or dis-allowed at a later stage'. It is possible that these bad habits form the foundation of the *personal zero rules* discussed in chapter 4, part 3.

In this study the majority of children defined zero as *nothing*. Providing that the definition of '*nothing*' was that of no-thing then this was a working definition which this researcher found adequate for the needs of this age range of children and one which could be 'enlarged and refined'. This research found strong evidence of children who said zero was '*nothing*' but whose understanding of '*nothing*' was that of '*nothingness*'. When this *nothingness – no value, worthless, can be ignored* – was applied to *zero* then this was the underlying cause of many problems. It is crucial that teachers are aware that their choice of language, used to explain the value of zero, is of great importance. As a result of this study the researcher concluded that with primary children, the use of zero meaning *nothing* was not detrimental. In the words of Blake and Verhille the 'zero is nothing' analogy,

> ... is attractive because it is easy to learn and retrieve, intuitively satisfying and it seems to work – it seems to effectively and consistently lead to correct answers.
> (Blake and Verhille 1985, p.36)

But, the researcher strongly recommends that teachers listen to children, that they ask the children for explanations in situations where a child is providing correct as well as incorrect answers, that they look for signs of *nothing* becomes *nothingness*. A fine line of demarcation separates the meaning of *nothing* from the meaning of *nothingness* and this implies a fine line of judgement on the part of the teacher. Positive statements 'zero might be nothing but zero is an important number and we mustn't ignore it' should replace the offhand 'zero is just nothing, forget it'. Blake and Verhille say that 'the consequences of the use of such a shallow surface structure are ignored or unknown' (Blake and Verhille 1985, p.36). They are no longer 'unknown' in the sense that this research shows some of the consequences of treating *nothing* and hence *zero* with this particular 'shallow surface structure'. This researcher feels that if teachers were aware of the consequences likely to emanate from a child's understanding that zero is *nothingness*, which in turn originates from the use of *nothing*, then they would be more vigilant.

The zero number facts

> The knowledge of what is and what is not a number is the mental bedrock upon which arithmetic is built. The laws of arithmetic are dictated by this knowledge of what a number is. The concept of number is primary ... (Gelman and Gallistel 1986, p.180)

Most primary schools practise an eclectic model of teaching, where children need to develop an understanding by using apparatus, engaging in activities and making sense of the world (the cognitive view), but where there is a firm place for practice and even rote learning (behaviourist theoretical view). This mixed model approach is seen in the *Framework for Teaching Mathematics* of the National Numeracy Strategy (1999). Teachers find it difficult to follow this mixed model approach in the teaching of zero as the teaching and learning of zero differs from all other numbers. Demonstrating and illustrating zero is not easy and the use of apparatus in equations involving zero can create misconceptions.

> Passing from the concrete, to the picture, to the semi-concrete stage in order until the abstract symbolism is presented, the teacher helps the child to build a fundamental concept of arithmetic. But what happens to the concrete to the abstract developmental sequence when the zero facts are included? Objectifying a null or empty set is obviously impossible. Perhaps in exasperation the teacher defines zero as 'nothing'.
> (Oesterle 1959, p.110)

The core of the debate throughout most of the 20th century was not how but when the teaching of the zero number facts should occur. Often the rationale for not using the zero facts was that to a child these zero number facts had no meaning (Hickerson 1952; Brueckner & Grossnickle, 1953). Herbert Spitzer (1954) advocated delaying the use of zero facts until the pupil had a need for them in a two number computation. Some researchers suggested that notions of zero were not adequately developed until the stage of formal operations had been obtained (Inhelder and Piaget, 1967). Oesterle (1959) epitomised these views in the following passage,

> It appears that the consensus of opinion supports the view that at the primary level to operate in any meaningful way with zero is somewhat nonsensical. One may have occasion to add 3 pence and 5 pence but not to add 0 pence and 3 pence. The empty set in concrete situations is not attended to; there is no necessity to incorporate it into one's thinking. However when the need to operate with two or multi-digit numbers arises then it is necessary to operate with zero as in
>
> $$\begin{array}{r} 67 \\ \underline{+20} \end{array}$$
>
> Until this time there seems to be no real reason for introducing the zero facts in any of the basic operations. (Oesterle 1959, p.109)

What Oesterle does not appear to consider is that children who do not know their zero number facts are likely to have difficulties with algorithms ('formal algorithms'[2]). In such 'formal algorithms' it is often the practice of children to negotiate the different parts of an algorithm as if they were single number facts. His example of 67 + 20 would be seen, by most children doing the algorithm, as two separate equations 7 + 0 = and 6 + 2 =. Chapter 4, part 4 considered the wider picture and the effect the zero number facts have on these 'formal algorithms'. The conclusion was that the problems children have with zero's presence in the number bonds will probably be mirrored in the algorithms. Detecting and correcting problems at the zero number facts stage of learning should prevent, at least, some of the zero problems in the algorithms.

In the quotation above Oesterle made a second point, that the zero number facts are meaningless, that one would have no occasion 'to add 0 pence and 3 pence'. If a child has no pence in his/her pocket and he/she puts in three pence is this not a contextual, concrete situation? Oesterle does not appear to take into account the possibility of using apparatus and the empty set to demonstrate some of the zero number facts.

It was noted, at the beginning of this chapter, that teaching staff do not see the teaching of zero as a problem. It would appear that the findings of both Wilson (1951) and Baroody (1984) support this view. Wilson (1951), when summarising the research from 1911 to 1927, concluded that,

> When attention is given to the zero facts, they are very quickly mastered. While it is no doubt better to teach all the zero combinations, they cause little or no trouble to pupils if the initial teaching of zero in combination is careful and thorough and at the right stage of mental level. (Wilson 1951, p.130-1)

The research of Baroody (1984) suggested that number size does not account entirely for which number bond children find easiest to learn.

> Double facts and their related subtraction, such as 2 + 2, 4 + 4, 6 – 3, 10 – 5, are easiest to learn. 'Zero' facts, such as 3 + 0, 5 – 0 are also very easy to learn.
> (Baroody 1984, in Aubrey 1994, p.63)

These views were written before the implementation of the National Curriculum, the National Numeracy Strategy and the Foundation Stage. When and how the zero topic is covered, in nursery and primary schools, has been greatly influenced by these documents (chapter 4, part 1). In the classroom today emphasis is placed upon knowing the number facts, including the zero number facts. As was discussed in the section on 'Memory', at the beginning of chapter 4, part 3, there is the need for repetition. But, while the implementation of 'the law of practice ... to

[2] While the NNS (DfEE, 1999) encourages the use of 'multiple algorithms', 'formal algorithms' are still taught but at a later stage.

secure over-learning' (Hughes et al, 2000, p.11) occurs with the number bonds, it appears to be lacking with the zero number facts.

As knowledge of the zero number facts, along with the other number facts, permeates algorithms it is essential that children know these facts. This study has shown that a notable percentage of children did have problems with the zero number facts. There would appear to be varying reasons. Teachers need to be aware of and to address these possible areas of difficulty.

- The children's understanding of zero as *nothingness* which resulted in zero being ignored
- The formation of a *personal zero rule* caused by the presence of zero in an equation
- The temptation to guess. Children realised that, in a zero number fact, the answer was one or other of the numbers (in this study either 3 or 0). Having a 50/50 chance of being correct, many guessed the answer
- Children do not gain sufficient practice. The zero number facts are not practised as frequently as the other single digit number bonds, including the teaching of the multiplication tables
- That there were children who acquired a conception of zero which went unnoticed (this was especially true when a correct answer was given) and thus was not corrected. Inaccurate information was put into memory

Teacher knowledge

There was strong evidence, in this study, that children obtained some unsound practices, which were revealed during the explanations of the zero number facts.

- *you cover up the '0'*
- *the answer is the first number,*
- *the answer is the last number,*
- *the answer is the number other than zero*
- *zero is like a ghost*
- *saying 'you can't' when presented with a zero number fact equation*

Such beliefs, which together form a child's conception of zero, could have been acquired from home, school or the wider world. However, because of the nature of the zero number facts it was more likely that the source of this knowledge was home (adults, other children such as siblings) or school (adults and peers). It was probable that the origins of some of these practices come from the staff in schools.

Do teachers have misconceptions and are they being passed on to the children? Arsham (2002) believes that problems are often initiated by our educators, world wide, both in the content of textbooks and in the teaching and that,

> We should not be surprised to see errors persist among students just as teachers learned the practice from their own teachers. Research of children's understanding of zero has been sporadic and meagre, as has the teachers' understanding of zero.
>
> (Arsham accessed 2002)

Wheeler and Feghali (1983) investigated student-teacher knowledge of zero. They found it to be inadequate and concluded that they were not sufficiently prepared to teach concepts of zero. Ball (1990) interviewed practising teachers and found similar results.

How would adults have reacted to the questions asked in this study? What information as to their conceptions of zero would emerge? While this research is centred round children some data were collected from adults, who work with, or intend to work with children. The adults were asked to complete the same Questionnaire as that given to the children (appendix 1). These adults were from three sample groups,

a) 18 members of staff from the five questionnaire schools
b) 16 students in their third, final year of the BA Education (training to teach children in the age range 5 to 11)
c) 21 post-graduate students[3] starting a research degree course (all were teachers in varying subject disciplines and of various age groups)

The tables of results, showing their answers to the zero number facts, are to be found in appendix 14 while some of the written comments are given in list 11. (The letter after the comment refers to the grouping above.)

A section of adult comments

- $3 \div 0 =$ *Too philosophical for me.* (a)
  $0 \div 3 =$ *I don't work for NASA.* (a)
- $3 + 0 = 3$, *I looked at the biggest number only as one number was a zero.* (a)
- $3 + 0 = 3$ and $0 + 3 = 3$ *The number stays the same if you add zero.* (a)
- $3 - 0 = 3$, *Taking zero away from a number keeps the number the same.* (a)
- $0 + 3 = 3$, *3 is the only number, then it's the answer.* (a)
- $3 \div 0 = 3$, *I would say you can't divide 3 by 0 so the answer is 3.* (a)
- $3 - 0 = 3$, *I instantly know that when you add 0 or take 0 the answer always stays the same.* (b)
- $3 - 0 = 3$, *As you are taking away nothing you don't change the first number.* (b)
- $3 \times 0 = 0$, *No sets of 3 leaves nothing.* (c)
- Ordering ¾, ¼, 0, ½. *I don't know where to put '0'. Is '0' nothing or less than one?* (a)
- $3 \times 0 = 0$, *I just seem to remember from learning times tables that multiplying by 0 you will be left with 0.* (c)
- *I don't see that zero can be a number because it doesn't have a quantity.* (b)

List 11

[3] The post-graduate students were peers of this researcher attending the same research course.

The answers to and the explanations for the zero number facts were reflective of those found in the study, or rather the children's explanations reflected those of the adults. Where did these adults acquire their knowledge and their habits? One should be aware of the power of one's own background in shaping one's beliefs and one's actions as these may have a marked effect on the teaching-learning situation. Teachers are products of a primary and secondary education. They, too, acquired knowledge and skills in out-of school settings, everyday experiences, activities in a social context. They, too, were taught and use *unsound practices*. When asking a student, young teacher or a teacher of many years standing, why they have taught in a certain way or used a certain phrase, this researcher has found the answer has, frequently, been that that was the way they were taught.

Unsound practices are perpetuated. This is a difficult cycle to break. Getting teachers to review and question their own practice and providing them with information as to the consequences of continuing with some of these customs would be beneficial. The findings in this study would be of benefit to students and to practising teachers in order to change their expectation that *children will have no problems with zero, because it is only nothing* (student teacher, verbal). It needs to be recognised that the teaching of zero is more difficult than the teaching of any of the other single digit and that the consequences of teaching *unsound practices* can be far reaching.

Illustrative incidents

This is an incident recounted by Oesterle (1959, p.111).
'The consequences of thinking of zero as 'nothing', is epitomised by the elementary school teacher in one of the writer's methods classes who insisted that 1/0 implies one over 'nothing'.
Since 'zero is nothing', this teacher casually erased the zero from the board, 'proving' to her satisfaction that the answer is *one*!' [4]

A friend and colleague of this researcher provided another incident. This person was acting as group leader for the KS2 SATs marking, one of her tasks was to address the markers' problems.
One of the questions on the 2001 SATs paper was, (□ ÷ □) + 90 = 100. The children were expected to fill in numbers to make the equation true. One of the markers in the group contacted her group leader asking,

'Can I accept 10 ÷ 0 as the missing numbers?'

> ***Possible changes in the approach to the teaching of zero, précis of suggestions***

- Place the emphasis on the empty set
- Make use of number lines showing zero and its relationship with other single digits and with simple fractions and decimals

[4] The adult groups, referred to on the previous page, were asked the answer to 3 ÷ 0. 36% gave the answer as 3. (See *Table Division D1 adults* in appendix 14.)

- Maintain a positive attitude to zero and to its importance as a number
- Be aware that referring to zero as *nothing* can develop into the *notion of nothingness*
- Put in place measures which ensure that the zero number facts are frequently included with the number facts in the children's written and mental work
- Appreciate that the teaching of zero is more difficult than the teaching of any of the other single digits and that the consequences of children adopting *unsound practices* can be far reaching.

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Chapter Eight, Part Two: Reflecting on the research process

Achievement of the research aims

A question, which all researchers must ask and answer, is whether their study has achieved its aim. In this instance the main aim was to **explore children's conceptions of zero.** Has this been an exploration? The researcher has worked in known and unknown areas and has systematically collected and analysed data. This constitutes 'exploration' according to a summary of dictionary definitions, [5]

explore vb.

- to examine or investigate, esp. systematically
- to travel to or into (unfamiliar or unknown regions), esp. for organised purposes
- to examine (an organ or part) for diagnostic purposes

The researcher wanted to know what ideas the children held about zero, within the confines of the study area. Information about the children's conceptions of zero was collected in the form of a Questionnaire, Tasks-Interviews and Activities-Interviews. What is a conception? According to a summary of dictionary definitions,

conception n.

- something conceived; notion, idea, design, or plan
- the description under which someone considers something
- the act or power of forming notions; invention

conceive vb.

- to have an idea (of); imagine; think
- to hold as an opinion; believe
- to develop or form, esp. in the mind

The researcher took her understanding of *conceptions* to be a collection of children's thoughts, views and beliefs about a specific topic given under certain circumstances. These conceptions

[5] All the following words have been summarised from Harper Collins Dictionary Interactive (1995), The Concise Oxford Dictionary (1959), Hyper-dictionary (Accessed 2005), xrefer.Dictionaries (Accessed 2005)

were gathered by listening to what a child said in his/her answers, opinions, explanations and rationale and by observing what a child did. Some of the answers and reasoning were incorrect, some of the answers were correct but the reasoning was incorrect. If these conceptions included incorrect understanding then were these misconceptions, mistakes or misunderstandings rather than conceptions? An interesting question as, initially, the researcher had considered the research title would be 'Children's Misconceptions of Zero'.

While she agreed with Smith and Roschelle (1993/4) that to cast misconceptions as mistakes was too narrow a view of their role in learning, the researcher found limiting their belief that these misconceptions might be conceived as faulty extensions of productive prior knowledge. She did not feel that sufficient allowance had been made for *unsound* initial 'prior knowledge'. A further consideration was the children's lack of experience, which produced a limited view of zero and so any misconceptions could be related to their developing understanding. Together with an individual's limited prior knowledge there are the complexities of the zero number to be taken into consideration. These were the factors the researcher considered before deciding that the study was primarily concerned with children's conceptions of zero. A wide view of conceptions was taken; this included misconceptions and mistakes together with correct answers and notions of zero, which, for the age of the child may be viewed as sound notions. It was not part of the study to diagnose misconceptions; the aim was to show trends, to provide an overall view of the conceptions of children in the age range 3 to 11.

The researcher feels that this study has achieved the stated aim, 'To explore children's conceptions of zero'. Yet, while acknowledging the success of the study, it is important to acknowledge its weaknesses and its limitations.

Weaknesses and limitations of the study

While it is difficult to illustrate omissions, the literature search did not reveal any significant research on children's conceptions of zero. This meant the researcher had to search wider sources of theoretical literature in order to obtain small pertinent pieces of information. This limitation was unavoidable and understandable as this was an exploratory investigation of an, as-yet, uncharted area of investigation.

With the exception of the material from the questionnaire all data were collated and analysed at the end of the data collection process. In the methodology section of chapter 1, part 7, the researcher stressed the importance of remaining uncoloured by early data analysis. It was felt such knowledge could have affected the researcher's perceptions, affected the running of the tasks and activities and distorted the results. The researcher was aware of the limits of this open-ended approach and of leaving analysis until the end of the data collection process. This prevented questions provoked by early analysis being highlighted and followed up. With hindsight she still feels that the correct decision was made as it aided the discovery of what were often masked lines of enquiry necessary for a successful exploratory study.

When analysing the data, issues arose which opened up many routes down which the researcher could have travelled. Each route was enticing for a researcher to explore, but as is the nature of research some could only be given a superficial examination in order to concentrate on those aspects which had relevance to this research focus.

At one point, particularly when categorising the data in the zero number facts (chapter 4), the researcher felt that the necessity to simplify was becoming a weakness. However, it was this process of carefully considered categorisation which highlighted areas allowing for a clearer pattern and paradox search to be undertaken in the analysis. This in turn strengthened the process of generalising within the sample, leading to some interesting findings.

Generalisations

This study did not frame a hypothesis and so the intent was not to seek evidence measured against criteria of validity and reliability as is needed when hypothesis testing. The data collected during this research were used to look at the nature of attitudes and trends; there was no call for any external criterion of adequacy, even if one had been available. Nor was there any intention, within this study, to assume generalisations beyond the area of the data collection contained in the research.

> Researchers want to generalise to populations that have not been sampled, whether based on qualitative or quantitative data such generalisations can never be fully justified.
> (Cook and Reichardt 1979, p.11)

This researcher cannot generalise only particularize, open generalisations would be dangerous, but from the findings local generalization might be possible. These local generalizations may then find resonance within a larger field. It is felt that this study has contributed rich, complex, new information that it will add to the body of knowledge with regard to mathematics education. As Caleb Gattegno said in 1988 when he addressed the ATM Conference at Winchester,

> There is only one instrument in research to find answers. One instrument. And that is to raise questions. (Williams 1996, p.8)

One of the results of this research was to raise questions but as a result of the research there are now different questions, the debate has moved on and has highlighted new areas of study.

Directions for future research

Study findings of this nature could provide valuable starting points for future research. Analysis of the data, at each stage in the exploration, raised further questions, some of which were clearly beyond the scope of the study. Some study findings were tentative, some findings were both tentative and complex. This research is seen as an awareness raiser, a prelude to further work in the same field. Hypotheses have first to be generated then later these hypotheses are tested. Possibly an aspect of this study may provide some appealing hypotheses for a future researcher to state and test. The main areas seen as providing a wealth of enquiry for future research are itemised below.

i. Longitudinal case studies

While there were occasions when comments were made as to whether a group of children were displaying certain tendencies in other related areas on the whole this study concentrated on general trends and not specific children. It would be of value to see how, over time, a child's conception of zero developed. This could be achieved by longitudinal case studies on children's conceptions of zero. They might also provide data on possible development stages or test those provisionally tabled in chapter 7, part 2.

ii. Young children and the empty set

One startling outcome was the anxiety displayed, particularly by the younger children, of the empty set. Further work is needed around the children's concern, of being presented with an empty set and of being involved in an activity leading to an empty set. Are the findings indicative of a widespread happening? Is there is a developmental age when this occurs and is resolved; what might be the reasoning behind the concern?

iii. Zero in simple equations and in algorithms

As was demonstrated in chapter 4, part 4, zero is known to cause problems in algorithms and it is highly likely that some of these problems stem from the children's conceptions of zero. The teaching of arithmetic would benefit from the outcome of further work as to how the children's conceptions of zero in the zero number facts affect the outcome of two digit algorithms which contain zero.

iv. Children's conceptions of nothing and nothingness

Most of the problems, particularly evident in the zero number facts, stemmed not from the choice of language but from a child's understanding of the word *nothing*. The root cause of many difficulties appeared to be *the notion of nothingness*. This is an important aspect, which needs further investigation.

A personal journey

The researcher embarked on this study in order to investigate a subject of long-term professional interest. She followed Monly's (1978) five empirical steps, those of experience, classification, quantification, discovery of relationships, approximation of truth (in Cohen & Manion 1994, p.13).

The research has also been a personal journey for the researcher. A journey which started with the making of important initial decisions, on the data to collect and the methods of data collection; decisions which could make or mar the project research. Working with the children when collecting the data was a joy and a privilege. Writing the thesis was an expected trial but this was where the greatest personal and professional learning took place. Admittedly there were times of despondency, especially when the ability to communicate a complicated set of data to a reader, in a clear succinct form but without losing its intricacy, eluded the writer.

> The skill ... is to provide sufficient 'thick description' to illuminate the contexts and participants yet maintain a clear analytical story. (Aubrey 1994, p.27)

Reflective times were of great value in the process of clarification. Low points were far outweighed by pleasure when problems were resolved. There were many 'small achievement' moments such as when pieces of data were classified and patterns emerged. There were the feelings of disbelief when an idea surfaced in the analysis process. There was the sense of amazement when stubborn suspicions turned into a Eureka moment. Much to the surprise of the writer the writing of the thesis provided the greatest personal sense of fulfilment.

Adding to the body of knowledge

This research was a concentrated, informed and systematic effort to provide accurate data, justifiable analysis and well balanced discussions of the main findings. It is felt that the contribution, as a result of this research, is rich, complex, new knowledge.

The researcher's intent was to produce a sound piece of research in a significant educational area. At the same time it is part of that 'sound piece of research' to admit weaknesses and to show where perplexity remains. To acknowledge that it was the work of a fallible person.

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Appendix 1

Questionnaire

Today's date …………… Your full name……………………………………………
Your age ……………….. Your date of birth …………
Name of School ………………………………………… Class ……………….

Don't worry whether or not you are writing the same answer as other students. It is very important that you give only your own answers or the survey data will not be of any value. You might think some questions are very easy, others harder but this is because the same questions need to be used with people older and younger than you. Thank you for your help.

>>>>>>>>>>>>>>>>>>><<<<<<<<<<<<<<<<<<<

[1] Please write down the first few lines of your 5 times table.

………………………………
………………………………
………………………………
………………………………

>>>>>>>>>>>>>>>>>>><<<<<<<<<<<<<<<<<<<

[2] Put the following in order beginning with the smallest amount:

(a) ½ ¼ 2 1 0

(b) .3 .4 0 .5 .1

(c) -9 0 -7 -1 -4

(d) ¾ ¼ 0 ½

(e) 0.4 5 1.2 8 0

(f) 3 0 5 4 7

(g) 8 5 7 1 0 4 3 2 9 6

>>>>>>>>>>>>>>>>>>><<<<<<<<<<<<<<<<<<<

[3]Write in your answers to the following sums and explain how you got your answer:

3 + 0 =

……………………………………………………………………………………………

……………………………………………………………………………………………

……………………………………………………………………………………………

0 + 3 =

……………………………………………………………………………………………

……………………………………………………………………………………………

……………………………………………………………………………………………

3 - 0 =

……………………………………………………………………………………………

……………………………………………………………………………………………

……………………………………………………………………………………………

0 - 3 =

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……………………………………………………………………………………………

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3 × 0 =

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……………………………………………………………………………………………

0 × 3 =

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……………………………………………………………………………………………

……………………………………………………………………………………………

3 ÷ 0 =

……………………………………………………………………………………………

……………………………………………………………………………………………

……………………………………………………………………………………………

0 ÷ 3 =

……………………………………………………………………………………………

……………………………………………………………………………………………

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>>>>>>>>>>>>>>>>>>>><<<<<<<<<<<<<<<<<<<

[4] Is ZERO a number? Yes/No What makes you think zero is or is not a number?

……………………………………………………………………………………………

……………………………………………………………………………………………

……………………………………………………………………………………………

>>>>>>>>>>>>>>>>>>>><<<<<<<<<<<<<<<<<<<

[5} What words might you use instead of the word zero?

……………………………………………………………………………………………

……………………………………………………………………………………………

……………………………………………………………………………………………

>>>>>>>>>>>>>>>>>>>><<<<<<<<<<<<<<<<<<<

[6] Write down anything else you feel is important about zero

……………………………………………………………………………………………

……………………………………………………………………………………………

……………………………………………………………………………………………

>>>>>>>>>>>>>>>>>>>><<<<<<<<<<<<<<<<<<<

Thank you very much indeed for completing this questionnaire.

Appendix 2

Dictionary definitions of Zero
1959 – 1995

The Concise Oxford Dictionary
Fowler, H. W. and Fowler, F. G. (Eds.) (1959) *The Concise Oxford Dictionary,* Oxford University Press, Oxford.

zero,
n (pl *~s*) Figure 0, cipher; no quantity or number, nil; starting point in scales from which positive & negative quantity is reckoned (-in thermometers, freezing point of water or other point selected to reckon from; *absolute* - in temperature, point at which the particles whose motion constitutes heat would be at rest, estimated at - 273.7 *C.);* (Mil.) point of time from which the start of each movement in a timed programme is at a specified interval; lowest point, bottom of scale, nullity, nadir. *fly at ~-* (under 1,000 ft).
[It., contr. Of *zefiro* f. Arab. As CIPHER]

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Webster's Dictionary
Read, A. W. (Ed.) (1998) *The New International Webster's Comprehensive Dictionary of the English Language, Deluxe Encyclopedic Edition,* Trident International, Florida.

Ze-ro (zir'o, zd'ro)
n. pl. Ze-ros or ze-roes
I The numeral or symbol **0;** a cipher. • In nontechnical speech, this symbol is often pronounced *(ā).*
2 *Math.* The element of a number system that leaves any element unchanged under addition, in particular, a real number *0* such that $a + 0 = 0+a=a$ for any real number *a.*
3 The point o a scale, as of a thermometer, from which measures are counted.
4 Mil A setting for a gunsight which adjusts both for elevation and wind.
5. The lowest point in any standard of comparison; nullity.
vt. Ze-roed, ze-ro-ing
To adjust (instruments) to an arbitrary zero point for synchronized readings.
-to zero in
1 To bring aircraft into a desired position, as for bombing or landing.
2 To adjust the sight of (a gun) calibrated results of firings.
-to zero in on
1 direct gunfire, bombs, etc., toward (a spec target).
2 To concentrate or focus one's enei attention, etc., on.
adj. Without value or appreciable change.
[<F *zero* <Ital. *zero* <Arabic *sifi.* Doublet of CIPHER.]

~~~~~~~~~~ ✵ ~~~~~~~~~~

There are expressions, which in some areas are considered slang -an example of this being the word **zilch** which is not found in the Oxford Dictionary but in Websters Dictionary:

zilch slang for nothing, naught, while Collins sees *zilch* as - Chiefly U.S. and Canadian slang. nothing.[of uncertain origin].

~~~~~~~~~~ ✵ ~~~~~~~~~~

Collins Dictionary:
Interactive Harper Collins English Dictionary (1995)

zero (ˋziarau)
n., pl. -ros or -roes.
1. the symbol 0, indicating an absence of quantity or magnitude; nought. Former name:cipher.
2. the integer denoted by the symbol 0; nought.
3. the ordinal number between +1 and -1.
4. nothing; nil.
5. a person or thing of no significance; nonentity.
6. the lowest point or degree: his prospects were put at zero.
7. the line or point on a scale of measurement from which the graduations commence.
8. a. the temperature, pressure, etc., that registers a reading of zero on a scale.
   b. the value of a variable, such as temperature, obtained under specified conditions.
9. a gunsight setting in which accurate allowance has been made for both windage and elevation for a specified range.
10. Maths.
   a. the cardinal number of a set with no members.
   b. the identity element of addition.
11. Linguistics.
   a. an allomorph with no phonetic realization, as the plural marker of English sheep.
   b. (as modifier): a zero form.
12. Finance.Also called zero-coupon bond.a bond that pays no interest, the equivalent being paid in its redemption value. Compare Zebra.
adj.
13. having no measurable quantity, magnitude, etc.
14. Meteorol.
   a. (of a cloud ceiling) limiting visibility to 15 metres (50 feet) or less.
   b. (of horizontal visibility) limited to 50 metres (165 feet) or less.
vb. -roes, -roing, -roed.
15. (tr.) to adjust (an instrument, apparatus, etc.) so as to read zero or a position taken as zero.determiner.16. Informal, chiefly U.S.no (thing) at all:this job has zero interest.
[C17: from Italian, from Medieval Latin zephirum, from Arabic sifr empty, CIPHER]

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Appendix 3

A chronology of inventions and progress in the history of mathematics

Chronology

The dating of inventions and progress in the history of mathematics is not always precise. Many early discoveries occurred in several cultures at different times. Principal sources for this time line are Motz and Boyer and Merzbach

BC

c. 30,000: Paleolithic tally bones with numerical notches
4th millennium: appearance of *calculi*, clay counting stones, in Mesopotamia and other regions of the Middle East
c. 3300: writing, and the first digits, invented in Sumer and Elam
3d millennium: use of hieroglyphic numerals and additive, decimal notation in Egypt
c. 2700: Sumerian cuneiform digits in use
c. 2000: appearance of the decimal base system
c. 1800: positional notation appears in Babylon
c. 1700: the Rhind Papyrus, in Egypt, in which the scribe Ahmes explores the measurement of the area of a circle
c. 1450: earliest Chinese notation system
c. 1300: appearance of digits in China
6th century: Pythagoras (c. 580–c. 500), Greek mathematical philosopher, founds the Pythagorean school in southern Italy; Pythagoreans distinguish between odd and even numbers; develop concept of connectivity and other theories of number
5th century: Greek philosophers who advance concepts of mathematics include Philolaus of Croton (470–390), Anaxagoras of Clazomenae (c. 500–c. 428), Zeno of Elea (c. 495–c. 430), Democritus of Abdera (c. 460–370), Plato (c. 428–348), in Athens; in Persia, knotted cords used for calculation
c. 425: the discovery of incommensurable magnitudes and irrational values disrupts Pythagorean mathematics; the discovery is sometimes credited to Hippasus of Metapontum, a former Pythagorean
4th century: in Greece, alphabetic numeration system in use; Aristotle (384–322) develops first significant mathematical concept of infinity
c. 300: Euclid, in Alexandria, writes the *Elements*, establishing mathematical method of postulate and proof; foundation of Alexandria as a center of mathematics study
3d century: appearance of the first zero, in Babylon; numerals partially resembling modern Indo-Arabic numerals appear in inscriptions in India; Archimedes (c. 287–212), Greek inventor and mathematician, writes a treatise on π
2d century: use in China of positional numeration, without the zero
c. 140: Hipparchus, Greek astronomer, invents an early form of trigonometry
1st century: Chinese place-value system

AD

c. 1st century: first use of negative numbers
2d century: paper invented in China
120: in China, Chang Hing calculates π as 3.1555…
4th–5th century: Indian positional notation, with zero
415: the death of Hypatia (c. 370–415), Alexandrian Greek philosopher and author of texts on mathematics, marks the close of the golden age of Alexandrian mathematics
458: the *Lokavibhaga (The Parts of the Universe)*, a Sanskrit treatise, uses the place-value system
5th–9th century: the Maya place-value system, with zero, in use in Mesoamerica
6th century: Boethius (c. 480–524), Roman philosopher and mathematician, writes treatises on logic and arithmetic; in India, Āryabhata I (476–c. 550) refines the calculation of π to 3.1416
8th century: in England, Bede (c. 673–735) writes a treatise on digital calculation; Indian calculation method reaches Baghdad: beginning of the golden age of Arabic mathematics
9th century: Thābit ibn Qurrah (c. 836–901), Arabic mathematician, describes amicable numbers
c. 825: in Baghdad, Muhammad ibn Mūsā al-Khwārizmī (c. 780–c. 850), called the "father of algebra," writes the *Kitab al-jabr wa al-muqa balah (Treatise on Restoration or Completion and of Reduction or Balancing)*, on algebra
10th century: the mathematician and poet Omar Khayyám (1048?–1131), in Persia, develops a general number theory and publishes an influential treatise on algebra; Gobar numerals in use in North Africa and the Iberian Peninsula; these differ from the Indo-Arabic numerals in use in the Arab Middle East and are the direct ancestors of the numerals used today in the West
12th century: in India, the *Bakshali* manuscript on arithmetic is written, although parts of it may be much older; in Europe, Arabic numerals (without zero) begin to appear; they are definitively in use there by the 14th century
13th century: Sacrobosco (John of Halifax, c. 1200–1256), English mathematician, writes *General Algorism;* in Peru, Inca *quipu*, knotted calculating cord, in use; in Mongolia, Nasir Eddin al-Tusi (1201–74), writes on trigonometry and astronomy and offers a proof of the parallel postulate; decline of the Arab golden age of mathematics
c. 1200: in France Alexandre de Villedieu writes the *Carmen de algorismo (Poem on Algorism)*, describing the operations of integers
1202: Fibonacci (Leonardo of Pisa, c. 1180–1250) writes the *Liber abaci (Book of the Abacus)*, on algebra
14th century: the zero appears in Europe
15th century: with the development of the printing

press, Indo-Arabic numerals acquire a definitive form in Europe; negative numbers appear

1427: al-Kashi (died c. 1436), in Samarkand, defines decimal fractions and writes the *Miftah al-hisab (The Key to Arithmetic)*

c. 1450–92: Piero della Francesca (1410?–92), Italian painter and mathematician, writes the treatises *Trattato d'abaco (Abacus Treatise)* and *De prospectiva pingendi (On Perspective for Painting)*, and an essay on Euclid, uniting mathematics and aesthetic theory

1484: Nicolas Chuquet (c. 1445–c. 1500), in France, writes the *Triparty en la science des nombres (Three Parts in the Science of Numbers)*, on fractions, decimals, and irrational numbers

1489: Johannes Widman (c. 1462–98?), in Germany, first uses the plus and minus signs (+ and –) in print

1492: in Italy, Francesco Pellos (c. 1450–c. 1500) first uses the decimal point, in his *Compendio de lo abaco (Compendium of the Abacus)*

1494: Luca Pacioli (1445?–1514?), in Italy, writes the *Summa de arithmetica, geometrica, proportioni et proportionalità (Summary of Arithmetic, Geometry, Proportions, and Proportionality)*

16th century: galley division in use in Europe; in China a form of digital calculation permits computations involving numbers in the billions

1545: Geronimo Cardano (1501–76), in Italy, publishes the *Ars magna (Great Art)*, giving the solutions of cubic and quartic equations, and using a negative square root; with Rafael Bombelli (c. 1526–73) he describes complex numbers for the first time

1556–60: Niccolò Tartaglia (1499–1557), in Italy, publishes his *Trattato di numeri et misure (Treatise on Numbers and Measures)*

1557: in England, Robert Recorde (1501–58) writes a treatise on algebra, *The Whetstone of Witte*, in which the sign for equality (=) first appears

1585: Simon Stevin (1548–1620), in Holland, writes *De thiende (The Tenth)*, on decimals

1614: John Napier (1550–1617), a Scot, invents logarithms

1632: Galileo Galilei (1564–1642) Italian mathematician and astronomer, writes *The Two Chief Systems* and, in 1638, *The Two New Sciences*, texts that explore the infinite and the infinitesimal

1635: Francesco Cavalieri (1598–1647), in Italy, originates the method of indivisibles, precursor of integral calculus; he also writes on trigonometry

1637: René Descartes (1596–1650), in France and Holland, publishes the *Discours de la méthode (Discourse on Method)*, on establishing mathematical certainty

1640: Pierre de Fermat (1601–65), in France, formulates his famous "last conjecture," stating that there is no cube divisible into two cubes; he contributes to the foundation of modern number theory and differential calculus

c. 1642: Blaise Pascal (1632–62), in France, invents a Calculating Machine, forerunner of the computer, and contributes to number theory

1644: Marin Mersenne (1588–1648), in France, advances the study of prime numbers

c. 1650: John Wallis (1616–1703), in England, researches the problem of π

1665: Isaac Newton (1642–1727), English physicist and mathematician, writes on the calculus; in 1687, publishes *Philosophiae naturalis principia mathematica* (known as the *Principia*)

1684: Gottfried Wilhelm Leibniz (1646–1716), in Germany, publishes his first paper on the differential calculus; in 1686 he describes the integral calculus; in 1703 he first uses the binary system

c. 1700: the ratio of a circle's diameter to its circumference is named *pi*, written π

1742: Christian Goldbach (1690–1764), in Germany and Russia, writes on prime numbers, contributes to the study of number theory, and formulates the "Goldbach conjecture," stating that every even natural number is the sum of two prime numbers

1754: Jean-Etienne Montucla (1725–99), in France, writes the *Histoire des recherches sur la quadrature du cercle (History of Inquiries on Squaring the Circle)* and, in 1758, *Histoire des mathématiques (History of Mathematics)*

1761: Johann Heinrich Lambert (1728–77), in Germany, offers the first proof that π is an irrational number

c. 1777: Leonhard Euler (1707–83), Swiss mathematician in Russia and Germany, devises the modern notation system for calculus, with first use of *i* to signify the square root of –1, an imaginary number; he pursues research on integral calculus and develops theories of complex numbers

1797: Caspar Wessel (1745–1818), in Norway, develops the idea of the complex plane

1801: Carl Friedrich Gauss (1777–1855), in Germany, proposes the method of least squares and a solution for binomial equations

1806: Jean Robert Argand (1768–1822), in Switzerland, devises the Argand diagram for complex numbers

1830: Evariste Galois (1811–32), in France, researches algebraic numbers, equations, and equation theory

1833: William Rowan Hamilton (1805–65), in Ireland, introduces a formal algebra of real number

1834: Nicolai Ivanovich Lobachevsky (1792–1856), a Russian, discovers a method of approximating the roots of algebraic equations

1844: Joseph Liouville (1809–82), in France, identifies a class of nonalgebraic real numbers, or transcendental numbers, now called Liouville numbers

1872: Richard Dedekind (1831–1916), in Germany, develops (with Georg Cantor) an arithmetic theory of irrational numbers, publishing *Stetigkeit und irrationale Zahlen (Continuity and Irrational Numbers)*

1874: Georg Cantor (1845–1918), in Germany, publishes *Mengenlehre (Set Theory)*, establishing modern set theory; he develops (with Richard Dedekind) theories of irrational numbers and of the arithmetic of actual infinity, introduces transfinite numbers, and explores new aspects of infinite numbers

1882: Ferdinand von Lindemann (1852–1939), in Germany, establishes the impossibility of squaring the circle, proving that π is a transcendental number
1893: David Hilbert (1862–1943), in Germany, publishes the *Zahlbericht (The Theory of Algebraic Number Fields*, literally, *Report on Numbers)*, on number theory; he develops innovative theories of invariants and integral equations. Gottlob Frege (1848–1925), in Germany, writes *Grundgesetze der Arithmetik (The Basic Laws of Arithmetic)* and works on mathematical logic
1895: Henri Poincaré (1854–1912), in France, publishes *Analysis situs (Topology)*, introducing the new mathematical field of topology
1896: proof of the prime number theorem, by Jacques Hadamard (1865–1963), a Frenchman, and C. J. de la Vallée-Poussin (1866–1962), a Belgian, working independently
1903: Bertrand Russell (1872–1970), English mathematician and philosopher, publishes *Principles of Mathematics* and works on mathematical logic
1910–13: Russell and Alfred North Whitehead publish their *Principia mathematica (Principles of Mathematics)* in England
1931: Kurt Gödel (1906–78) an Austrian, proposes a theorem of undecidable propositions, demonstrating that within a logical arithmetical system some axioms can neither be proved nor disproved
1995: Andrew Wiles completes his proof of Fermat's conjecture (now theorem), in England

In Guedj 1996, p.169 – 171
[who states the principle source to be Motz, Boyer and Merzbach in Boyer, C.B. and Merzbach, U.C., A History of Mathematics, 2d ed., 1991]

Appendix 4

Classification categories for the 'Is zero a number?' responses

The category definitions and rationale for the answers 'Yes', 'No' and 'Yes and No'.

The categories where a child made reference to:

- Number line order:
  Zero in the number line, the order of the numbers (spoken or written), the extended number line with reference to positive and negative numbers.
- Place value:
  Zero seen, used in numbers greater than 9.
- Algorithms:
  Zero found, used in sums, number facts, algorithms, calculations.
- Context:
  Zero found, used in everyday life such as in temperature, on a thermometer.
- No value:
  Zero was referred to as being no thing.
- Non-numerical:
  Zero as a word, as a circle.
- Meaning unclear:
  Where there was a mismatch between the question and the answer

Zero IS a number :	**Zero IS NOT a number :**
The number order	**The number order**
because when you count down from say ten you will eventually reach zero (7)	*Numbers are 1,2,3, that is just a zero, just a blank like oh (8)*
because it goes zero, one, two, three in the number line, to show it's the start. (10)	
because the numbers below zero like –3 are called numbers and so it has to be a number (11)	
because zero is in the number line (8)	
Because without zero it would go from positive to –1, there has to be a number in-between to take one off to get –1 (11)	
Its halfway between negative and positive numbers, got to be a number because it's there between negative and positive (10)	
Place value	**Place value**
if zero wasn't a number there wouldent be 10, 20, 30 ect. (11)	*Not a number it's a zero, never a number unless you put 1 in front of it (7)*
If you look in 100 and other numbers like 120 then it's in that number. (11)	
Yes, I think, because it's got zero at the end of 10, 20, 30 (9)	
because all the numbers like 110 if theres no zero it would be 11(8)	

Zero IS a number :	Zero IS NOT a number :
Algorithms	**Algorithms**
because you see it in sums(9)	*If added to itself it doesn't make another number (10)*
it is not a representable amount but it can mean something like 3 + 0 = 3, 0 + 3 = 3 (11)	*When you add 3 + 0 it = 3 so I think it is not a number, it doesn't do anything (11)*
it is used in sums like all other numbers (10)	*It is nothing because it does not (times) or (divide) to anything (11)*
when you say something like 1 times 1 equals zero then it must be a number (11)	*if you add it to 3 you ad nothing and so on because you can't add it up can you (11)*
Like 10-10 is zero (9)	*because when you add any other number to it it will leave you with the number you added to it (11)*
Contextual	**Contextual**
because the number zero is used in lots of things including temperature and decimals	*because if you had 1 or 2 babies and then had none zero would not be a number*
No value	**No value**
	No It don't mean anything like seven of them(8)
	it is not a representable amount (11)
	it is not [a number] because nothing there (10)
	because it has no [child waves hand over pile of counter] – like 1/2/3 – it's just nothing (11)
No mathematical reasoning	
I think zero is a number because I've heard	*I was told*
it is the oposite of another Opposite of another	*I now*
Yes Because no one would say it wasn't a number	
Non numerical	
Yes it is another word for nout and nout is a number	*Zero is a word not a number*
It's in mathematics, there isn't a zero when you speak and write English.	
It's a circle, a ring, a letter, a number	
Zero isn't a number it is a word. 0 is a number.	

Appendix 5

The number order display - critique of recommendations in the NNS

The NNS, states that children should appreciate the relationship between single digits. The NNS advocates, as a display resource, each classroom should have a *number track or number line*. What is of particular interest to this study is how the NNS include the position *of* zero in this number order relationship.

Reception children (aged 4 -5)
The NNS Supplement of Examples states that children should be taught to:

> *Recognise and use numerals 1 to 9, extending to 0 and 10, then beyond 10*
> (DfEE, 1999, NNS Reception p.9)

The NNS end of year outcome should be to:

> *Begin to recognise 0 as the numeral associated with 'none', or the space before 1 on the number track.*
> (DfEE, 1999, NNS Reception p.9)

Children in Y123
[Y1 -Year One aged 5 to 6; Y2 - Year Two aged 6 to 7; Y3 - Year Two aged 7 to 8.]
Pupils should be taught to:

> *Order a set of numbers and position them on a number line....*
> (DfEE, 1999, NNS Y123 p.14-15)

There are three modes of displaying the numbers found in the NNS. In this study, for clarity, they are called Diagram One, Diagram Two and Diagram Three.

Diagram One

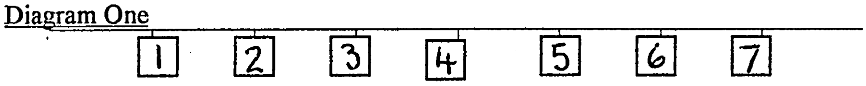


Diagram One is a series of rectangles with a number in each, these numbers are attached to a marked line. Interestingly this researcher has not seen Diagram One format displayed in a classroom but remove the line and this becomes the *number card* format

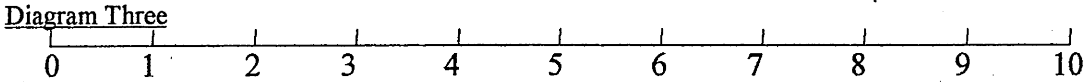
In the NNS Reception section this is called a **'number track'** (Reception p.12). This format appears for children in Y2 and Y3 (Y123 page 15) and for Y4 children (Y456 p.8) but it is now called a **'number line'**. There is no example shown which includes '0'.

Diagram Two

1	2	3	4	5	6	7	8	9	10

Diagram Two is the same as the *number bar* seen previously in this study (refer back to previous section). This format is seen in examples for Y1, Y2 and Y3 children (Y123 p.2, 3, 10 and 12) and is called **a number track.** There is no example shown which includes '0'.

Diagram Three



This format is seen in examples for Y2 and Y3 children (Y123 p.11, 15, 23) and for Y4 and Y5 children (Y456 p.8, 14 and 15) and is called **a number line.** There are examples shown which include '0'.

So Diagram One is called a number track and a number line, Diagram Two is called a number track, Diagram Three is called a number line. The NNS terminology used for these formats is confusing and inconsistent. In essence Diagram One is the *number card* format, Diagram Two is the *number bar,* Diagram Three is the *number line* described in the main text of this study. It would appear that the three forms and the age of the children who use them show little change from those in use prior to the NNS. What is noteworthy is that while the NNS states that children should appreciate that zero is in the space before 1 on the number track; zero is not included in the number track illustrations. Possibly this is connected with the problems muted earlier with the use of '0' on *a number bar.*

Appendix 6

Classification categories for the zero number facts

Some Examples of the Categorisation Criteria development

Key to first categorisation

Rationale for addition, 3 + 0 =	
X	No answer given to the sum
NR	No Reason
R w	Repeats the question in words
Rs	Repeats the question in symbols
IK	I know
SS	Stays the same,
NE	Adding 0 has no effect (3 or 0),
Hn	Higher number is the answer
Ln	Lower number is the answer
Nl	Refers to number line order
A	Refers to an amount, objects
Df	Derived fact
Ig	Ignore the zero
n	0 is not anything, nothing
CT	the word CAN'T is used, then an answer is given
^	Conservation, same as the paired zero number fact (3+0 same as 0+3)
?	Don't understand the explanation

ARS (1)

A combined categorisation:

Rationale for addition, 3 + 0 =	
NR	No Reason
R	Repeats the question in words or symbols
IK	I know
SS	Stays the same, adding 0 has no effect (3 or 0),
P	Pattern higher number is the answer, lower number is the answer
NL	Refers to number line order
A	Refers to an amount, objects
DR	Derived Fact
IG	Ignore the zero
n	0 is not anything, nothing
CT	The word CAN'T is used, then an answer is given
^	Conservation, same as the paired zero number fact (3+0 same as 0+3)
?	Don't understand the explanation

ARS (2)

Appendix 7

Standard Assessment Tasks (SATs) papers and Staffordshire Arithmetic Test

Standard Assessment Tasks (SATs)

Ma 3

$2 + 3 = 5$

$4 + 4 =$

$5 + 1 =$

$7 + 2 =$

$3 + 5 =$

$6 + 4 =$

$6 - 1 = 5$

$7 - 5 =$

$5 - 2 =$

$8 - 4 =$

$9 - 6 =$

$10 - 3 =$

Ma 3

Ma 3 Pupil Sheet 1

92/Ma3/PS1

THE STAFFORDSHIRE ARITHMETIC TEST

(Formerly Revised Southend Attainment Test in Mechanical Arithmetic)

SHEET I (for pupils)

(Work across the page)

(1)

(2) Add:

5	3	4	6	7	4
+ 5	+ 4	+ 6	+ 2	+ 3	+ 5

(3) Subtract:

8	10	7	9	10	9
− 4	− 2	− 6	− 5	− 7	− 8

(4) Add:

8	9	11	4	9	7
+ 4	+ 8	+ 5	+ 13	+ 11	+ 7

Add:

(5)	(6)	(7)	(8)
64	37	38	28
17	45	16	27
+ 89	+ 3	+ 47	+ 35

[P.T.O.

Sheet 1 (contd.)

Subtract:

(9) $\begin{array}{r} 37 \\ -\ 23 \\ \hline \end{array}$

(10) $\begin{array}{r} 82 \\ -\ 30 \\ \hline \end{array}$

(11) $\begin{array}{r} 45 \\ -\ 26 \\ \hline \end{array}$

(12) $\begin{array}{r} 80 \\ -\ 43 \\ \hline \end{array}$

Subtract:

(13) $\begin{array}{r} 73 \\ -\ 68 \\ \hline \end{array}$

(14) $\begin{array}{r} 879 \\ -\ 273 \\ \hline \end{array}$

(15) $\begin{array}{r} 657 \\ -\ 349 \\ \hline \end{array}$

Subtract:

(16) $\begin{array}{r} 762 \\ -\ 487 \\ \hline \end{array}$

(17) $\begin{array}{r} 543 \\ -\ 465 \\ \hline \end{array}$

(18) $\begin{array}{r} 991 \\ -\ 198 \\ \hline \end{array}$

Multiply:

(19) $\begin{array}{r} 486 \\ \times\ 4 \\ \hline \end{array}$

(20) $\begin{array}{r} 498 \\ \times\ 5 \\ \hline \end{array}$

(21) $\begin{array}{r} 237 \\ \times\ 6 \\ \hline \end{array}$

(22) $\begin{array}{r} 596 \\ \times\ 11 \\ \hline \end{array}$

(23) $\begin{array}{r} 256 \\ \times\ 12 \\ \hline \end{array}$

Divide:

(24) $4\overline{)804}$

(25) $5\overline{)768}$

(26) $6\overline{)905}$

(27) $11\overline{)870}$

(28) Add:

$\begin{array}{rrr} & \text{s.} & \text{d.} \\ & 2 & 7\frac{1}{2} \\ & 1 & 2 \\ + & 1 & 11\frac{1}{2} \\ \hline \end{array}$

(29) Add:

$\begin{array}{rrr} & \text{s.} & \text{d.} \\ & 1 & 4\frac{1}{2} \\ & & 7\frac{1}{2} \\ + & 4 & 5\frac{1}{2} \\ \hline \end{array}$

(30) Subtract:

$\begin{array}{rrr} & \text{s.} & \text{d.} \\ & 4 & 3 \\ - & 1 & 8 \\ \hline \end{array}$

(31) Subtract:

$\begin{array}{rrr} & \text{s.} & \text{d.} \\ & 5 & 9 \\ - & 2 & 4\frac{1}{2} \\ \hline \end{array}$

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Appendix 8

Incorrect answers – Patterns
Zero Number Facts

	Number of responses *	Questionnaire		Number of responses	Interview Task
First Number					
0 + 3 = 0	8 (89)	**9%**		2 (20)	**10%**
0 – 3 = 0	18 (86)	**21%**		6 (20)	**30%**
3 × 0 = 3	25 (86)	**29%**		4 (20)	**25%**
3 ÷ 0 = 3	38 (82)	**46%**		10 (20)	**50%**
	38 (343)	**26%**		22 (20)	**28%**
Second Number					
3 + 0 = 0	4 (93)	**4%**		4 (20)	**5%**
3 – 0 = 0	2 (88)	**2%**		5 (20)	**25%**
0 – 3 = 3	24 (86)	**28%**		5 (20)	**25%**
0 × 3 = 3	21 (84)	**25%**		4 (20)	**20%**
3 ÷ 0 = 0	44 (82)	**54%**		10 (20)	**50%**
0 ÷ 3 = 3	21 (80)	**26%**		5 (17)	**25%**
	116 (513)	**23%**		33 (117)	**28%**
Zero as the answer					
0 + 3 = 0	8 (89)	**9%**		2 (20)	**10%**
3 + 0 = 0	4 (93)	**4%**		4 (20)	**5%**
0 – 3 = 0	18 (86)	**21%**		6 (20)	**30%**
3 – 0 = 0	2 (88)	**2%**		5 (20)	**25%**
3 ÷ 0 = 0	44 (82)	**54%**		10 (20)	**50%**
	76 (438)	**17%**		27 (100)	**27%**
Three as the answer					
0 × 3 = 3	21 (84)	**25%**		4 (20)	**20%**
0 – 3 = 3	24 (86)	**28%**		5 (20)	**25%**
3 × 0 = 3	25 (86)	**29%**		4 (20)	**25%**
3 ÷ 0 = 3	38 (82)	**46%**		10 (20)	**50%**
0 ÷ 3 = 3	21 (80)	**26%**		5 (17)	**25%**
	129 (418)	**31%**		28 (97)	**29%**

Zero number facts - Pattern Table

* the first number is the number of children who gave that particular answer, the number in parenthesis is the overall number of responses to that equation

Appendix 9

The Tins Game, Martin Hughes'

The following is from Martin Hughes' book *Children and Number: Difficulties in Learning Mathematics* (1986, p.64). In this passage he describes the Tins Game.

The Tins Game

I decided to devise a game in which the children's written representations would serve a clear communicative purpose. The idea for this game arose fairly naturally from my earlier work with boxes and bricks. Young children seemed to be attracted by a closed box containing a number of bricks, and I thought they might be intrigued by the idea of putting a written message on the lid of a box to show how many bricks were inside.

The game centred on four identical tobacco tins, containing different numbers of bricks: usually there were three, two, one and no bricks inside each tin. After letting the child see inside the tins, I shuffled them around, and asked the child to pick out 'the tin with two bricks in', 'the tin with no bricks in' and so on. At this stage the child had no alternative but to guess. After a few guesses, I interrupted the game with 'an idea which might help'. I attached a piece of paper to the lid of each tin, gave the child a pen, and suggested that they 'put something on the paper' so that they would know how many bricks were inside. The children dealt with each tin in turn, its lid being removed so that they could see inside. When they had finished, the tins were shuffled around again, and the children were asked once more to identify particular tins and see whether their representations had 'helped them play the game'. The Tins game thus provided not only a clear rationale for making written representations, but also an opportunity to discover what children understood about what they had done.

(Hughes, 1986, p.64)

Appendix 10

Zero Division Dilemma

Page 43 to 44 from:
Blake, R. and Verhille, C. (1985) The story of zero. *For the Learning of Mathematics,* **5,** 35-47.

Division involving zero requires the mathematical analysis of three cases as follows:

CASE I: **$0 \div a, a \neq 0$**

$$0/a = \square \rightarrow 0 = a \times \square$$

There is one number, zero, which makes this statement true so, $0/a = 0$
Mathematically we say that $0 \div a$ is uniquely defined and is meaningful.

CASE 2: **$a \div 0, \ a \neq 0$**

$$a/0 = \square \rightarrow a = 0 \times \square$$

There is no number which makes this statement true. Thus, $a \div 0$ where $a \neq 0$ is not uniquely defined and, therefore, meaningless.

CASE3: **$a \div 0, a = 0$**

$$0/0 = \square \rightarrow 0 = 0 \times \square$$

All numbers make this statement true but there is no way of determining which number to choose as the answer. Thus, 0/0 is indeterminate.

There are no instances prior to this in the mathematics curriculum where consideration of the question of uniqueness is necessary. As demonstrated above, the mathematical analysis of division involving zero is otherwise incomplete. Mathematically, the behaviour of zero is division is exceptional because:

1. A different, conceptually deeper approach is necessary to analyze this behaviour, compared to other numbers.
2. Three cases must be analyzed.
3. Each case results in different outcomes, two of which are startling contrasts with previous computational exercises.
4. For the first time it is necessary to use the notion of uniqueness.
5. The new language needed because of (I) to (4).

....

Web site: http://ubmail.ubalt.ed/~harsham/zero/ZERO.HTM (Accessed 19th Feb, 2002)
Some extracts giving the opinions of visitors to the site on the subject of 'The Zero Saga'.

- Another author wrote that perhaps a / 0 = a because if he should divide a units of apple pies among 0 people, he would be left with the entire pie. Unfortunately his analysis also leaves him with pie in his face since his analysis references the results of a division is a distribution transaction, but as he is not making any transaction, therefore the result is meaningless.

For example, if one distributes a pie to 2 people, each would get 1/2. In other words, the distribution takes place across an equal sign among the number specified in the denominator on the left hand side. Since he specifies zero people in the distribution, the transaction is not taken place. The result of interest is on the right-hand side.
It is certainly true that if you do not distribute the pie, you retain it; but when you use an equal sign, you imply the result of some real transaction to the other side. Similarly, a / 0 is also

meaningless as being the amount of pie the zero people received. This is certainly true for apple pies. The results may differ with other varieties of pies, not all of which have been reviewed! Once you understand what division **is,** then there should be little difficulty in understanding why division by zero is not allowed.

- A visitor of this site wrote:

  "... but I think it's simply because the division operation is defined in that way. There could be many interpretations for this definition, but there is no reason. As you demonstrated many times, allowing division by zero causes contradiction, thereby making that mathematical system useless. On the other hand, prohibiting division by zero has not yet known to cause any contradiction. If any, that's the only reason why we define division as it currently is."

Division by zero does cause contradiction. That's why we can not divide 2 apples among zero people. It's meaningless7~and has no other "interpretations". The act of "distributing" apples cannot be performed. However, adding zero apples to 2 apples we get 2 apples. Notice that, in addition and subtraction operations we must have the same "dimensions", i.e., not adding up apples with oranges. However, in division operation, this is not a necessary condition, such as in speed, expressed, e.g., as kilometers/hour. It might even have a hybrid dimensionality, like momentum in physics.

- A Software Engineer kindly wrote to me that:

  for $2/0 = \propto$, I think you get all fired up about not much. Of course if there is any implied assertion that ~ is a number then one gets into all sorts of contradictions that you describe. But I don't think anyone in his right mind considers this the textbook author simply uses the infinity symbol as a conventional way to denote something that doesn't exist -- something that's impossible (impossible for exactly the reasons you state)."

My concern here is not whether $2/0 = \propto$ is true or not. It could have been a hundred times worse than this and I will not lift a finger against it. But, what I'm combating is the act of dividing by zero in the first place, the carelessness of our educators, and not willing to know that they are misleading students. Adding to these, by the way of doubling our difficulties, now it is claimed that is "a conventional way to denote something that doesn't exist." There is no such conventional usage for ~. Do you know of any? Whenever, mathematics is distorted and sensationalized, or even pseudo-mathematics is used uncritically, a disservice is done to public understanding of mathematical fact. What I am attempting to signify here is nothing more than this: in applied mathematics dividing by zero is a meaningless operation. Remember that, the *sign is not for any number it is only for a concept, and "infinite quantity" is* unmeasurable by any numerical scale. Therefore, one should never do any kinds of arithmetic operations with it, such as $+ 100 = \propto$, which gives the silly result, $100 = 0$.
Remember also that, a good Logic is a strong container where we put our Ideas to delivered it to someone else. Therefore, empty logic is useless. Also, having useful ideas but not using strong logic to make it common is dead. One must look for both the container and what it contains. Both are needed good ideas communicated by good logic.

- One of my British colleagues kindly wrote to me that:

  "...Personally I like the numeral zero as it provokes people to think about their preconceptions. it is possible that people find the notion that you can NOT divide a number by zero unnerving because they like their life organized. Again this could be that when they "do" maths they only consider a "tick" for a correct answer their own reward! Math is far more fun and interesting, don't you think? As a non-specialist lecturer I am pleased to point students to consider your web page. Please keep such ideas in the public domain."

Appendix 11

The research of Anna Stallard (1982)
Unpublished, in Hughes (1986 pp.103-109)

Children's explanations of arithmetical symbolism

These questions provided the focus of a much larger study carried out by Anna Stallard (1982). She used sixty children aged 6 years 3 months to 10 years 6 months, from a middle-class school in Edinburgh. The children were chosen at random from five separate classes, but their teachers were afterwards asked to rate each child as 'good', 'average', or 'poor' at mathematics.
Anna Stallard improved on my procedure by presenting the children with a much clearer rationale for explaining the meaning of the symbols. She used the large toy panda called'Chu-Chu' (mentioned briefly in chapter 5) who had featured in several earlier experiments carried out by Margaret Donaldson. These experiments had shown that young children often gave explanations to the panda which they would not do to an adult. This was particularly likely to happen if it was put to the children that the panda did not understand things and needed their help: children seem to be more expressive and articulate in a role whereby they imagine that they are helping someone smaller and less knowledgeable than themselves.

Anna Stallard introduced Chu-Chu to the children and asked them to pretend that the panda did not understand what was written on the cards. She said, 'I want you to try and show the panda what the cards mean, so that he will understand them, using these bricks to help you, okay?' After they had attempted to explain a particular card to the panda, they were asked whether or not the panda could understand the card now, and were generally encouraged to think about whether or not their response conveyed the meaning of the card. The adequacy of each response was subsequently assessed, the crucial factor being that the meaning and not just the appearance of the card had to have been conveyed. This could be done either by using the bricks, as suggested, or by a purely verbal explanation.

The children were shown a much wider range of cards than I had used in my pilot work. For example, one of the cards just showed the zero sign ('0'), while two cards included zero as part of complete addition and subtraction sums ('5 — 5 = 0' and '4 + 0 = 4'). Stallard also used a number of cards which showed numerals and operator signs being used in an unusual and unfamiliar form: two showed a single numeral and operator sign ('— 5' and '+ 6'), similar to those I had used earlier; two showed complete addition and subtraction sums, but presented in a 'reversed' form ('7 = 5 + 2' and '4 = 6 — 2'); and one card showed the equality '3 = 3'. The complete set of cards used is shown in figure 7.4.

Anna Stallard found, as I had, that some cards were much easier to explain than others. As before, the cards bearing just the numerals '6' and '2' were the easiest, with most children simply showing the appropriate number of bricks. There were, however, a few examples of children who, like Jamie, produced pictorial representations of the numerals. In contrast, cards showing both numerals and operator signs were much harder to explain (see table 7.1).

6	2	0	-5	+6
2 + 2 = 4	3 + 1 = 4	5 + 6 = 11	3 – 1 = 2	6 – 5 = 1
5 – 5 = 0	4 + 0 = 4	7 = 5 + 2	4 = 6 - 2	3 = 3

Figure 1, Set of fifteen cards used by Anna Stallard (1982)

TABLE 7.1 Adequacy score (%) for each card (from Stallard, 1982)

Card	Score (%)
6	95
2	93
0	85
$5-5=0$	62
$2+2=4$	62
$3-1=2$	60
$1+3=4$	58
$6-5=1$	57
$5+6=11$	55
$4=6-2$	47
$7=5+2$	45
-5	43
$+6$	35
$4+0=4$	30
$3=3$	18

Of the cards showing operator signs as well as numerals, those showing conventional 'sums' were the easiest, with between half and two-thirds of the children producing adequate representations. Most of these involved the dynamic movement of groups of bricks to demonstrate the additions and subtractions. Some of the children, however, gave inventive verbal descriptions. There was one 9-year-old girl who, for '3 — 1 = 2', explained: 'There are three buses at the bus stop and you get on one and it goes away and there are two left.' For '1 + 3 = 4' she said: 'You go to the shop and buy an apple and your mum says, "Your dad and me and Tracy want one too", so you go back and buy three more and you've got four.' As before, some children produced static demonstrations of the superficial appearance of tl_um, as in figure 7.3. There were also some responses which mixed a static display with a verbal description. For example, one child laid out groups of one, three and four bricks and said 'Take one brick and three more bricks, count them up and you've got four': this child was credited with an 'adequate' explanation.

The cards bearing or including zero produced an interesting range of responses. The card showing zero on its own was adequately represented by well over three-quarters of the children. Their responses were mostly verbal, such as 'None there', 'Put no bricks in front', 'Haven't got any bricks at all', in each case accompanied by either the removal of all the bricks from the table, or a refusal to add any bricks to the empty space in front of them. Some children, on the other hand, used the bricks to make pictorial representations of zero (see figure 7.5).

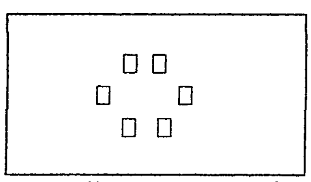

Figure 7.5 Representation of zero
(from Stallard, 1982, *p.* 27)

The two addition and subtraction problems involving zero differed quite strikingly in the adequacy of the responses which they elicited. The card with '5 — 5 = 0' caused no particular difficulties due to the presence of the zero, and this sum was adequately represented by two-thirds of the children. In contrast, '4 + 0 = 4' was adequately represented by less than a third of the children, with the presence of zero in the second position causing particular difficulties. A number of children simply placed four bricks on the table and said nothing. Other children did

this but at the same time made various verbal attempts to convey the meaning of the sum. Thus one child put four bricks on the table and said, 'Add nothing'; she then moved the group over to the right-hand side of the table, said 'Makes four', and moved the group back. Another child put four bricks on the table, took them off to show zero, and finally placed them back again.
The cards which showed a numeral and operator sign ('-5' and + 6') were more difficult than those showing a complete sum. Some of the children suggested that these cards were in some sense not fair:

'We don't do them kind of sums', or 'We haven't done those yet'. Another child said, 'You wouldn't show the sign as it's not got a number at the start.' As before, the most common response was simply to count out the number of bricks represented by the numeral and ignore the operator sign. Curiously enough, this type of response was produced over three times as often to the '+ 6' card as to the '-5' card.

Other children made more successful attempts to convey what was going on. One child responded to the '+ 6' card by saying, 'There'd be nothing there and then put six', while several children converted these messages into a complete sum which they could represent. This response was judged to be adequate if the child gave a full explanation which retained the meaning of what was on the card. The card showing '— 5' was usually converted into the sum '5 — 5 = 0'. One child, however, said 'Nothing take away five leaves...' and started to produce a static display of this sum (figure 7.6). Leaving an empty space at the left-hand side of the table, he positioned four bricks in a row to make the '—'; he added a group of five to the right of that, and then positioned six bricks (in two rows) for the '='. At this point he refused to continue, saying: 'You can't do that. You've got to have a bigger number on that side to take away from that side.'

Appendix 12

E-mail correspondence with the Oxford English Dictionary Editorial Department

Sent to Oxford English Dictionary 26/05/05
Dear Sir/Madam,
I would be grateful if you could help me with some queries.
I am in the process of writing up my research for a PhD on *Children's Conceptions of Zero.* One of the chapters is on the language of zero. I am trying to trace the changes in meaning and in grammatical usage of the word zero.
I appreciate that American English differs from British English. It is the latter which interests me. I have two main questions but would appreciate any other information regarding the changes in use and meaning of zero.

- In the British English dictionaries I have had access to there seems to have been changes over the past 40 years. I have the impression, which I would appreciate if you could confirm or not, that zero was originally only used as a noun but that it can now be used as other parts of speech. Could you give me details of the changes in use, such as the years these occurred?
- I was surprised to see that phrases such *as zero option, zero gravity, zero tolerance* are considered as nouns. Could you explain why the zero is not an adjective in such cases? My thinking was going along the lines of instances such as *little option* where I would see *little* as being an adjective then why is the *zero* in *zero option* not an adjective?

I thank you, in anticipation, for your help and look forward to your reply.
I am,
Yours Sincerely,
Rona Catterall.

Reply 09/06/05
Dear Ms Catterall,
Thank you very much for your e-mail to the Oxford English Dictionary which has been passed on to me for reply.
I have e-mailed you a separate link to the entries for 'zero' noun and 'zero' verb, so that you can see it for yourself. These contain the sense progressions through time, with the changes in meaning, and should give you the kind of information you require.

In a phrase such as 'zero tolerance', the word 'zero' is indeed an adjective; in some contexts this is completely transparent, as in the sentence 'I have zero tolerance for alcohol'. Here, 'zero' is simply qualifying the noun, without the two words combining to make a noun phrase. In some contexts though, the phrase as a whole is a noun, as in 'The council introduced a policy of zero tolerance'. Here, 'zero' is still an adjective, but it is the combination of the two words which make the noun. As far as I am aware, 'little option' is never used in the second of these two ways, ie it is always an adjective 'little' plus a noun 'option'.
I hope this is of interest. If you have any further questions, please do not hesitate to get in touch.

Yours sincerely,
Fiona McPherson
Senior Editor OED.

08/09/05
To Fiona McPherson, Senior Editor OED.
Thank you very much for allowing me the 3 day free access to the OED entry for zero, n. and zero, v.. I can't tell you how valuable this material will be. May I ask if there is an entry for zero as an adjective? It is zero's 'apparent' use as an adjective which is of particular relevance to my PhD and I was wondering if you have anything specific on zero's use as an adjective. Thank you again for your considerable help. It really is appreciated.
Rona Catterall.

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09/06/05
Dear Ms Catterall,
The OED does not have any separate adjectival entry, but in the entry for 'zero' noun sense 7c, it does mention the adjectival use. The Shorter OED treats it like this: "B attrib. or as adjective. Marked by temperatures of zero or below. m19. That amounts to zero; colloq. no, not any. L19. Linguistics. Marked by the absence of a feature (e.g. an inflection) which is sometimes present. e20."
It also includes a few examples, and it would probably be easiest to look at these in the book itself, which I assume you should be able to gain access to.
It is likely that when the OED entry for 'zero' comes to be revised, we would exemplify the adjectival sense further.
I hope this is helpful.
Best wishes,
Fiona

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Some examples, collected by the researcher (2002 - 2006), of the contextual use of the word 'zero'.

Radio

BBC Radio 4
'Open Book' 16.00 to 16.20, 12th January, 2006

Discussion on the prison service: 'Last year there were zero escapes recorded from open prisons in England.'

In measurement zero is seen as the start, the beginning, the first mark or spot that does not carry the zero symbol or word. So this researcher felt that it was unusual to see a stone, in the Peace Park in Hiroshima, Japan, on which was written.
'Zero Milestone of Hiroshima City. The distance from Hiroshima was always measured from this point... a wooden pillar was set here as the zero milestone of the Hiroshima Prefecture. After a municipality system was established in 1889 the spot was marked with a stone pillar as the zero milestone of Hiroshima City'. (9th May, 2005)

Radio

BBC Radio 4
'The Money Programme'
10th April, 2003.

Discussion on the Budget: 'Zero change in the tax high rate band.'

Radio

BBC Radio 4
'The Today Programme'
1st Dec, 2003.

Two different commentators reading of the telephone number 0 900 01:

- Zero, nine hundred, 'oh' one
- 'Oh', nine hundred, 'oh' one

Radio

'The News' BBC Radio 4, 6th Dec, 2003.

'Students paying zero fees ...'

Notice

Notice in a doctor's surgery 11th April, 2003.

'This practice supports the Governments NHS zero tolerance zone campaign. www.nhs.uk/zerotolerance'

Book

Jonathan Cash, 'The Possessions of a Lady' (1996) Century UK, London

Lovejoy, an antique dealer, reflects on how Joseph Briggs rescued Art Nouveau 1890s Louis Comfort Tiffany glass pieces, which were being thrown out. Joseph Briggs stood against the tide of fashion and virtually saved an art form.

Lovejoy says, 'See? Fashions as near to zero performance as mankind gets. It's almost always wrong.' (p.7)

Book

Dan Brown, 'De Vinci Code', 2003, Bantam Press, London

The police are searching for signs of Vernet's criminal record. Vernet's credit card record shows nothing untoward. ' *Zero,* Callet sighed.' (p.478)

Advertisement

An advert sent by the Halifax Building Society, Jan 2003.

The envelope stated

'We're giving you Nothing ...

Zilch, Rien, Cero, Dim, Null, Nought, Zero, Nada.

0% for balance transfers and purchases fixed for five months.'

Television

5th Jan, 2004, Channel 5, 7.30 pm

'The Big Question'

Stephen Hawking, 'When the universe had zero size and was infinitely hot.'

Newspaper

The Oldhamer
14th March 2003,
Sub headline, page 7,
Zero tolerance on dumped cars continues.

Television programme

Saturday, 21st Dec, 2002
ITV transmitted 23.55
Bethlehem Year Zero

Martyn Lewis presents a news bulletin chronicling the events of the first Christmas Week as if it were happening now.

Radio

BBC Radio 4 Programme : Woman's Hour
23rd February, 2002.

The programme presenter, Jenni Murray, interviews Members of Parliament, members of the public and representatives of the police force on the strategy of Zero Tolerance.

Lecture

An Egyptian Art lecture by Victor Blunden
Mottram WEA.
Thursday, 23rd January, 2003,

Between Predydastic Period c5000-3100 BC and the Archain Period C3100-2890BC was the *dynasty and reign of Narmer who was recognised as the first King of Egypt.*

The Predydastic Period is split into

4,500-4,000 BC	*Badarium Period*
4,000-3,500 BC	*Naqada One*
3,500-3,100 BC	*Naqada Two*
3,100-300 BC	*Naqada Three*

Naquada Three is normally referred to as Dynasty Zero. The term was coined by an American archaeologist as Dynasty Zero to represent the time between not having a king and having a king.

Film

Television screening of the Warner Brothers Film
"Outbreaker",
ITV, Monday 9th Dec, 2003 at 9pm.

The principal character is a scientist, played by Dustin Hoffman, who is trying to trace the source of a virus. He refers to a patient infected with a deadly virus as

"First Patient – Patient Zero."

Adult responses to the Questionnaire

Results of Ordering Numbers	Questionnaire	
Question B .3, .4, 0, .5, .1	Staff	Student
Number of responses	16	16
0 .1 .3 .4 .5	10 63%	15 94%
5 .4 .3 .1 0	6 37%	1 6%

Results of Ordering Numbers	Questionnaire	
Question D ¾, ¼, 0, ½	Staff	Student
Number of responses	16	16
0, ¼, ½, ¾,	12 75%	15 94%
¼, ½, ¾, 0	4 25%	1 6%

Results of Ordering Numbers	Questionnaire	
Question A ½, ¼, 2, 1, 0	Staff	Student
Number of responses	16	16
0, ¼, ½, 1, 2,	13 81%	14 87%
¼, ½, 0, 1, 2, **012 in order**	3 19%	2 13%

Multiplication tables	Questionnaire		
	Staff	Student	Post-grad
Number of responses	16	16	21
Beginning with 0 e.g. 0 × 5 or 5 × 0	5 31%	2 13%	2 10%
Beginning with 1 e.g. 1 × 5 or 1 × 5	11 69%	14 87%	19 90%

83% began tables with 1, 17% began tables with 0

Table Multiplication adults

3 + 0 Answers	Questionnaire		
	Staff	Student	Post-grad
Number of responses	16	16	21
3 + 0 = 3	16 100%	16 100%	21 100%
3 + 0 = 0			

Table Addition adults

0 + 3 Answers	Questionnaire		
	Staff	Student	Post-grad
Number of responses	16	16	21
0 + 3 = 3	16 100%	16 100%	21 100%
0 + 3 = 0			

Table Addition adults

3 – 0 Answers	Questionnaire		
	Staff	Student	Post-grad
Number of responses	16	16	21
3 – 0 = 3	16 100%	16 100%	19 90%
3 – 0 = 0			1 5%
Can't be done			1 5%

Table Subtraction adults

0 – 3 Answers	Questionnaire		
	Staff	Student	Post-grad
Number of responses	16	16	21
0 - 3 = -3	13 81%	16 * 100%	18 86%
0 - 3 = 0	1 6%		1 5%
0 - 3 = 3			2 10%
Put ? or -3	1 6%		
This can't be done	1 6%		

* two altered the answer from 0 to -3

Table Subtraction adults

3 × 0	Questionnaire		
	Staff	Student	Post-grad
Number of responses	16	16	21
3 × 0 = 0	13 81%	15 94%	21 100%
3 × 0 = 3	2 13%	1 6%	
Put ?	1 6%		

Table Multiplication adults

0 × 3	Questionnaire		
	Staff	Student	Post-grad
Number of responses	16	16	21
0 × 3 = 0	14 88%	16 100%	21 100%
Put ?	1 6%		
0 or -3	1 6%		

Table Multiplication adult

3 ÷ 0	Questionnaire		
	Staff	Student	Post-grad
Number of responses	16	16	21
3 ÷ 0 = 0	5 31%	10 63%	5 24%
3 ÷ 0 = 3	5 31%	4 25%	10 48%
This can't be done	1 6%	1 6%	1 5%
∞	2 13%		5 24%
Can't be done, puts 0	1 6%		
Puts ?	2 13%		
1		1 6%	

Table Division D1 adults

0 ÷ 3	Questionnaire		
	Staff	Student	Post-grad
Number of responses	16	16	21
0 ÷ 3 = 0	12 75%	13 81%	17 81%
0 ÷ 3 = 3		1 6%	
This can't be done		2 13%	
∞			2 10%
0 or -3	1 6%		
Puts ?	3 19%		2 10%

Table Division D2 adults

Children's Questionnaire responses to:
'Write down the first few line of your five times table'

The five times table questionnaire responses	
Age of children	11
Total number of responses	95
Counted in 5s, beginning at 5 (5, 20, 15, 20, 25)	24 25%
Counted in 5s, beginning at 0 (0, 5, 10, 15, 20)	
1×5	37 39%
5×1	7 7%
0×5	19 20%
5×0	8 8%

T5TimesT

The National Numeracy Strategy

Mathematical Vocabulary

RECEPTION

Counting and Recognising Numbers

number
zero, one, two, three... to twenty and beyond
zero, ten, twenty... one hundred
none
how many...?
count, count (up) to
count on (from, to)
count back (from, to)
count in ones, twos... tens...
more, less
odd, even
every other
how many times?
pattern, pair
guess, estimate
nearly, close to, about the same as
just over, just under
too many, too few, enough, not enough

Capacity

full
half full
empty
holds
container