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DYNAMICAL INVARIANTS FOR GROUP AUTOMORPHISMS

RICHARD MILES, MATTHEW STAINES, AND THOMAS WARD

Abstract. We discuss some of the issues that arise in attempts to classify automorphisms of compact abelian groups from a dynamical point of view. In the particular case of automorphisms of one-dimensional solenoids, a complete description is given and the problem of determining the range of certain invariants of topological conjugacy is discussed. Several new results and old and new open problems are described.

1. Introduction

Automorphisms and rotations of compact abelian metric groups provide the simplest examples of dynamical systems, and their special structure makes them amenable to detailed analysis. In particular, their strong homogeneity properties allow global invariants (notably the topological entropy) to be calculated locally.

Here we wish to discuss the problem of classifying compact group automorphisms viewed as dynamical systems in the most naive sense. Denote by $\mathcal{G}$ the space of all pairs (G, T), where T is an automorphism of a compact metric abelian group G, and let $\sim$ denote a dynamically meaningful notion of equivalence between two such systems. Then the problem we wish to discuss has three facets:

- *Classification*: can the quotient space $\mathcal{G}/\!\sim$ be described?
- *Range*: for some invariant π of such systems, what is
$$\{\pi(G, T) \mid (G, T) \in \mathcal{G}\}?$$
- *Fibre*: for a given value f of some invariant or collection of invariants π, what can be said about
$$\{(G, T) \in \mathcal{G} \mid \pi(G, T) = f\}?$$

We will describe some of the issues that arise in formulating these questions, recalling in particular some of the well-known difficulties in the classification problem both from an ergodic theory and a set theory point of view. For the range problem we will describe some recent work and present some new arguments concerned with orbit growth in particular. Many but not all of the problems identified here are well-known.

The same questions make sense for group actions. Given a countable group Γ, there is an associated space $\mathcal{G}_\Gamma$ comprising all pairs (G, T), where $T : \Gamma \to \mathrm{Aut}(G)$ is a representation of Γ by automorphisms of the compact metric abelian group G. While our emphasis is not on any setting beyond $G = \mathbb{Z}$, one or two examples will be given to indicate where the theory has new features for other group actions.

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An element $(G,T)\in\mathcal{G}$ (by assumption) carries a rotation-invariant metric giving the topology, a σ-algebra $\mathcal{B}_G$ of Borel sets, and a probability measure (Haar measure) m_G preserved by T. The most natural notion of equivalence between (G_1,T_1) and (G_2,T_2) in $\mathcal{G}$ is a commutative diagram

$$\begin{tikzcd} G_1 \arrow[r, "T_1"] \arrow[d, "\phi"'] & G_1 \arrow[d, "\phi"] \\ G_2 \arrow[r, "T_2"'] & G_2 \end{tikzcd}$$

where the equivariant map ϕ is required to be:

- a continuous isomorphism of groups, giving the equivalence of *algebraic isomorphism*;
- a homeomorphism, giving the equivalence of *topological conjugacy*; or
- an almost-everywhere defined isomorphism between the measure spaces
$$(G_1,\mathcal{B}_{G_1},m_{G_1})$$
and
$$(G_2,\mathcal{B}_{G_2},m_{G_2}),$$
giving the equivalence of *measurable isomorphism.*

Clearly algebraic isomorphism implies topological conjugacy. Entropy arguments show that topological conjugacy implies measurable isomorphism. An easy consequence of the type of constructions discussed later — and an instance of the type of question one might ask about the structure of $\mathcal{G}/{\sim}$ — is that each measurable isomorphism equivalence class splits into uncountably many topological conjugacy classes. We have not included the equivalence relation that is in some ways the most natural in dynamics (*finitary isomorphism*, essentially measurable isomorphism by a map that is continuous off an invariant null set) because there is little that can be said about it beyond those examples where it is almost self-evident how to construct such a map.

We will also be interested in various invariants for $(G,T)\in\mathcal{G}$. The most significant of these is the entropy,

$$h(T)=\lim_{\varepsilon\searrow 0}\lim_{n\to\infty}-\frac{1}{n}\log m_G\left(\bigcap_{j=0}^{n-1}T^{-j}B_\varepsilon(0)\right) \tag{1.1}$$

where $B_\varepsilon(0)$ denotes an open metric ball around the identity of G. A manifestation of the homogeneity of the dynamics of group automorphisms is that this quantity coincides with the topological entropy and with the measure-theoretic entropy with respect to Haar measure (see Bowen [12] or [17]), and is therefore an invariant of any of the notions of equivalence above.

Topological conjugacy also preserves closed orbits and periodic points. This combinatorial data is all contained in the dynamical zeta function

$$\zeta_T(z)=\exp\sum_{n\geqslant 1}\frac{\mathsf{F}_T(n)}{n}z^n, \tag{1.2}$$

where $\mathsf{F}_T(n)=|\{g\in G\mid T^ng=g\}|$, with radius of convergence

$$1/\limsup_{n\to\infty}\mathsf{F}_T(n)^{1/n}.$$

Definition 1. For elements $(G,T),(G',T')\in\mathcal{G}$ write

$$(G,T)\sim_h(G',T')$$

if $h(T)=h(T')$, and

$$(G,T)\sim_\zeta(G',T')$$

if $\zeta_T(z)=\zeta_{T'}(z)$ as formal power series.

The primary growth rate measures are

$$p(T)=\lim_{n\to\infty}\frac{1}{n}\log\mathsf{F}_T(n)$$

if this exists, and

$$\pi_T(X)=\sum_{n\leqslant X}\mathsf{O}_T(n),$$

where $\mathsf{O}_T(n)$ denotes the number of closed orbits of length n. As we will see, for many (in several precise senses, most) examples the limit defining p will not exist, and a more averaged measure of orbit growth comes from the dynamical Mertens' theorem, giving asymptotics for the quantity

$$\mathsf{M}_T(N)=\sum_{n\leqslant N}\frac{\mathsf{O}_T(n)}{\mathrm{e}^{h(T)n}}=\sum_{n\leqslant N}\sum_{d|n}\frac{\mu(n/d)\,\mathsf{F}_T(d)}{n\mathrm{e}^{h(T)n}},$$

where the expression on the right is simply obtained by Möbius inversion, as

$$\mathsf{F}_T(n)=\sum_{d|n}d\,\mathsf{O}_T(n).$$

In general, for an action T of a countable group Γ, the periodic point data for the action is a map from the space of finite-index subgroups of Γ to $\mathbb{N}\cup\{\infty\}$, associating to each subgroup $L<\Gamma$ the cardinality of the set of L-periodic points

$$\mathsf{F}_T(L)=\{x\in X\mid T_\ell x=x\text{ for all }\ell\in L\}.\tag{1.3}$$

There is an extension of the dynamical zeta function to the setting of $\mathbb{Z}^d$-actions due to Lind [40], but it does not carry all the periodic point data unless $d=1$. Some of the questions considered here have been studied for full-shift actions of nilpotent groups in [46], and for specific examples of $\mathbb{Z}^2$-actions by automorphisms of compact connected groups in [47].

Several of the questions we will discuss involve adopting a point of view on what is a natural family of dynamical systems, and what is a dynamically meaningful notion of equivalence. In this context it is reasonable to restrict attention to ergodic group automorphisms throughout, and in some cases the more interesting questions arise if we also insist that the group be connected. From this point of view the natural starting point is to study automorphisms of one-dimensional solenoids, and much of what we will discuss will be concerned with these. A remarkable consequence of the work of Markus and Meyer is that all of these seemingly exotic one-dimensional solenoids appear generically in Hamiltonian flows on compact symplectic manifolds of sufficiently high dimension [43] (we thank Alex Clark for pointing this out). Thus, despite the seemingly strange phenomena and arithmetic questions that arise here, the dynamical systems discussed here arise naturally in smooth dynamical situations. Another reason to restrict much of our attention to one-dimensional solenoids is cowardice. In one natural direction of extension, the S-integer construction builds families of compact group automorphisms starting with algebraic number fields or function fields of positive characteristic (see [15] for the details), and the arithmetic questions that arise become more intricate but are in principle amenable to the same methods in those settings. Another natural direction is to simply replace 'one' by $d \geqslant 2$, and study automorphisms of d-dimensional solenoids. In this complete generality the difficulties are more formidable, and this is discussed further at the end of Section 5.1.

The assumption of ergodicity is a mild requirement and avoids degeneracies – automorphisms of compact abelian groups cannot be non-ergodic in dynamically interesting ways. Ergodicity guarantees (completely) positive entropy, and the structure of compact group automorphisms with zero entropy has been described by Seethoff [56].

2. Algebraic isomorphism

Deciding if (G_1, T_1) and (G_2, T_2) are algebraically isomorphic is not a dynamical question at all. The equivalence implies that $\phi : G_1 \to G_2$ is an isomorphism of groups, so we are asking if the groups are isomorphic to a single group G, and then if the two elements T_1 and $\phi^{-1}T_2\phi$ are conjugate as elements of the group $\mathrm{Aut}(G)$ of continuous automorphisms of G. If the systems are expansive (or satisfy the descending chain condition) and connected, then the problem of determining conjugacy in the automorphism group is algorithmically decidable by a result of Grunewald and Segal [22] (see Kitchens and Schmidt [28, Sec. 6]). In general this cannot be expected: if the group is infinite-dimensional then elements of the automorphism group in general correspond to matrices of infinite rank.

Example 2. Toral automorphisms are conjugate if the corresponding matrices have the same characteristic polynomial and share an additional number-theoretic invariant related to ideal classes in the splitting field of the characteristic polynomial (see Latimer and Macduffee [32] for the details). This additional invariant is already visible even on the 2-torus. For example, if

$$A = \begin{pmatrix} 3 & 10 \\ 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$$

then it is easy to check that if $Q \in \mathrm{M}_2(\mathbb{Z})$ has $QA = BQ$ we have $5 \big| \det Q$.

A flavour of the sort of examples that will arise in this setting is given by the next problem.

Problem 1. Given a monic polynomial $\chi \in \mathbb{Z}[x]$ of degree d, describe the combinatorial properties of the following poset. At the bottom level $n = 0$ there is a vertex for each of the finitely many algebraic conjugacy classes of the set

$$M_\chi = \{A \in \mathrm{GL}_d(\mathbb{Z}) \mid \det(tI_d - A) = \chi(t)\},$$

which may be enumerated using data from the ideal class groups of the splitting fields of the various irreducible factors of χ using [32]. At level $n \geqslant 1$ there is a vertex for each of the conjugacy classes of the elements of M_χ over the ring $\mathbb{Z}[\frac{1}{p_j} : j \leqslant n]$ where $p_1, p_2, \dots$ are the rational primes in their natural order. The edges reflect conjugacy classes merging in the larger group at each level, and the top of the poset has a vertex for each conjugacy class of the set M_χ in $\mathrm{GL}_d(\mathbb{Q})$, which are described by the rational canonical forms of matrices with characteristic polynomial χ.

For example, Problem 1 is readily solved for Example 2 as follows. The polynomial $\chi(x) = x^2 - 6x - 1$ is irreducible over $\mathbb{Q}$ with splitting field $\mathbb{Q}(\sqrt{10})$, with ideal class group isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The representative matrices in Example 2 are not conjugate over $\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{Z}[\frac{1}{6}]$, but become conjugate over $\mathbb{Z}[\frac{1}{30}]$, resulting in the diagram in Figure 1.

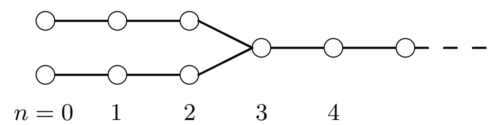

FIGURE 1. Conjugacy for the characteristic polynomial $x^2 - 6x - 1$.

3. Topological conjugacy

As we will see, for connected groups this notion of equivalence collapses to algebraic isomorphism. In fact there is a rigidity phenomenon at work, which means that in many situations not only does the existence of a topological conjugacy force there to be an algebraic isomorphism, it is required to be algebraic itself.

Theorem 3 (Adler and Palais [2]). *If $T : \mathbb{T}^d \to \mathbb{T}^d$ and $S : G \to G$ are topologically conjugate ergodic automorphisms via a homeomorphism ϕ from $\mathbb{T}^d$ to G, then $G \cong \mathbb{T}^d$ and the map ϕ is itself an automorphism of $\mathbb{T}^d$ composed with rotation by a fixed point of S.*

This extends in several ways to other connected finite-dimensional groups, and to other group and semi-group actions (see Clark and Fokkink [16] and Bhattacharya and the third author [10]). Notice that the phenomena is not universal, and in particular requires the topological entropy to be finite. The next example shows that in infinite entropy situations there is no topological rigidity.

Example 4. If $\theta : \mathbb{T} \to \mathbb{T}$ is any homeomorphism, then the map $\Theta : \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}$ defined by $(\Theta((x_n)_{n\in\mathbb{Z}}))_k = \theta(x_k)$ for all $k \in \mathbb{Z}$ is a topological conjugacy between the left shift on $\mathbb{T}^{\mathbb{Z}}$ and itself.

Example 5. In contrast to the topological rigidity in Theorem 3 (as pointed out by Kitchens and Schmidt [28]) topological conjugacy on zero-dimensional groups preserves very little of the algebraic structure.

(1) A finite abelian group G has an associated shift automorphism $(G^{\mathbb{Z}}, \sigma_G)$ where $(\sigma((x_n)))_k = x_{k+1}$ for all $k \in \mathbb{Z}$. Clearly $(G^{\mathbb{Z}}, \sigma_G)$ will be topologically conjugate to $(H^{\mathbb{Z}}, \sigma_H)$ if and only if $|G| = |H|$. Thus no part of the internal structure of the group alphabet is preserved by topological conjugacy.

(2) Kitchens [29] proves that if G is zero-dimensional and T is expansive, meaning that there is an open neighbourhood U of the identity with
$$\bigcap_{n\in\mathbb{Z}} T^{-n}U = \{e\},$$
and has a dense orbit, then (G, T) is topologically conjugate to a full shift on $\mathrm{e}^{h(T)}$ symbols. In the same paper the problem of classifying such automorphisms up to simultaneous algebraic isomorphism and topological

conjugacy is also addressed but not completely solved. It is shown that for any given entropy there are only finitely many equivalence classes for simultaneous algebraic isomorphism and algebraic conjugacy. If the entropy is $\log p$ for some prime p then it is shown that there is only one equivalence class, namely that containing $(\mathbb{F}_p^{\mathbb{Z}}, \text{shift})$.

(3) For other group actions entirely new rigidity phenomena appear. Given a finite abelian group G, we may define an action σ_G of $\mathbb{Z}^2$ on the compact group

$$X_G = \{x = (x_{n,m}) \in G^{\mathbb{Z}^2} \mid x_{n,m} + x_{n+1,m} = x_{n,m+1} \text{ for all } n, m \in \mathbb{Z}\}$$

by shifting. Röttger [53], extending a partial result in [67], shows that the periodic point data in the sense of (1.3) for the system determines the structure of G, and so in particular if (X_G, σ_G) is topologically conjugate to (X_H, σ_H) then $G \cong H$.

(4) For the case $G = \mathbb{Z}/2\mathbb{Z}$ in the setting of X_G as in (3), the only homeomorphisms $X_G \to X_G$ commuting with the action of σ_G are elements of the action itself by [64].

Problem 2 (Kitchens [29]). Classify expansive ergodic automorphisms of compact abelian zero-dimensional groups up to simultaneous algebraic isomorphism and topological conjugacy.

Example 5(3) is associated in a natural way to the polynomial

$$1 + u_1 - u_2 \in \mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]$$

since the condition defining the elements of X_G may be thought of as requiring that the element of $G^{\mathbb{Z}^2}$ is annihilated by $1 + u_1 - u_2$, with u_1 corresponding to a shift in the first coordinate and u_2 to a shift in the second coordinate.

Problem 3. Associate to a polynomial $f \in \mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]$ and a finite abelian group G the shift action $\sigma_{G,f}$ on the space $X_{G,f}$ of elements of $G^{\mathbb{Z}^2}$ annihilated by f. Does the periodic point data for $\sigma_{G,f}$ determine the structure of G?

4. Measurable isomorphism

This is an opaque equivalence relation, because as measure spaces any two infinite compact metric abelian groups are isomorphic. A consequence of the homogeneity mentioned above is that the entropy $h(T)$ is an invariant of measurable isomorphism. A much deeper fact is that there is an abstract model as an independent identically distributed stochastic process for any ergodic group automorphism. These independent identically distributed processes – Bernoulli shifts – in turn are classified up to measurable isomorphism by their entropy (see Theorem 8 below).

Example 6. Example 5 gave some instances of topological rigidity for actions of larger groups. There is a remarkable theory of rigidity for measurable isomorphism of algebraic $\mathbb{Z}^d$ actions, and we refer to the survey of Schmidt [55] and the references therein for this.

Theorem 7. *Let $(G, T) \in \mathcal{G}$ be an ergodic compact group automorphism. Then there is a probability vector $(p_1, p_2, \dots)$ defining a measure μ_p on $\mathbb{N}$ with the property that (G, T) is measurably isomorphic to the shift map σ on the space $\mathbb{N}^{\mathbb{Z}}$ preserving the product measure $\prod_{n \in \mathbb{Z}} \mu_p$. Moreover, $h(T)$ is equal to the measure-theoretic* *entropy* $\sum_{n\in\mathbb{N}} p_n \log p_n$ *of* σ *with respect to this measure. If* $h(T) < \infty$ *then we may assume that* $p_k = 0$ *for all but finitely many* k*, or equivalently we may replace the alphabet* $\mathbb{N}$ *with a finite set.*

This was proved by Katznelson [27] in the case $G = \mathbb{T}^d$, and shown in general by Lind [38] and independently by Miles and Thomas [44] and Aoki [4], using abstract characterizations of the property of being measurably isomorphic to a Bernoulli shift developed by Ornstein [50]. It means that an ergodic compact group automorphism is measurably indistinguishable from an abstract independent identically distributed stochastic process. Notice that the space $A^{\mathbb{Z}}$ for a finite alphabet A is a compact metric space in a natural way, so the question of whether or not the measurable isomorphism in Theorem 7 may be chosen to be finitary – continuous off a null set – in the case of finite entropy arises. Little is known in general, but Smorodinsky (unpublished) pointed out that a necessary condition is exponential recurrence – for a given non-empty open set O, the Haar measure of the set of points first returning to O after n iterations of the automorphism decays exponentially in n. Lind [39] showed that any ergodic group automorphism is exponentially recurrent, so all that is known is that this particular property does not preclude the possibility that ergodic group automorphisms are finitarily isomorphic to Bernoulli shifts. Rudolph has given a characterization of the property of being finitarily isomorphic to a Bernoulli shift [54].

Problem 4. Determine when an ergodic group automorphism is finitarily isomorphic to a Bernoulli shift.

Theorem 8 (Ornstein [49])**.** *Bernoulli shifts with the same entropy are measurably isomorphic.*

Theorems 7 and 8 together mean that the space $\mathcal{G}/{\sim}$ for the equivalence of measurable isomorphism embeds into $\mathbb{R}_{>0}$. There is a representative Bernoulli shift for each measurable isomorphism class of compact group automorphisms – but is there a compact group automorphism for each Bernoulli shift? Equivalently, is there a compact group automorphism for each possible entropy?

In order to describe what is known about this problem, let $f \in \mathbb{Z}[x]$ be a polynomial with integer coefficients, with factorization $f(z) = \prod_{i=1}^{d}(z - \lambda_i)$ over $\mathbb{C}$. Then the *logarithmic Mahler measure of* f is the quantity

$$m(f) = \int_0^1 \log|f(\mathrm{e}^{2\pi\mathrm{i}s})|\mathrm{d}s = \sum_{|\lambda_i|>1} \log|\lambda_i| = \sum_{i=1}^{d} \log^+|\lambda_i|.$$

Thus Kronecker's lemma is the statement that $m(f) = 0$ if and only if f is a product of cyclotomic polynomials. Lehmer [35] raised the question of whether there could be small positive values of the Mahler measure. He found the smallest value known to date, corresponding to the polynomial

$$f(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

with Mahler measure approximately $\log 1.176$. Write

$$L = \inf\{m(f) \mid m(f) > 0\}.$$

Yuzvinskiĭ [72] computed the entropy of endomorphisms of solenoids (see Lind and the third author [41] for the history of earlier results and a simpler approach), and this led to a complete description of the range of possible values. These relate to the dynamics of group automorphisms in several ways.

Theorem 9 (Lind [37]). *The infinite torus $\mathbb{T}^\infty$ has ergodic automorphisms with finite entropy if and only if $L = 0$.*

An expansive automorphism is guaranteed to have finite entropy, but unfortunately Hastings [24] shows that no automorphisms of the infinite torus can be expansive.

Theorem 10 (Lind [38]). *The set of possible entropies of ergodic automorphisms of compact groups is $(0,\infty]$ if $L = 0$, and is the countable set*

$$\{m(f) \mid f \in \mathbb{Z}[x], m(f) > 0\}$$

if $L > 0$.

This means that the cardinality of the quotient space $\mathcal{G}/\sim$ is uncountable or countable depending on the solution to Lehmer's problem. This is now a problem of some antiquity, and means that we do not know if group automorphisms are special or not when viewed as measure-preserving transformations among the systems measurably isomorphic to Bernoulli shifts.

Problem 5 (Lehmer [35]). Determine if $L = \inf\{m(f) \mid m(f) > 0\} > 0$.

We refer to Boyd [13] for an overview of Lehmer's problem. Expressed in dynamical systems, Theorem 10 may be used to express the same question as follows.

Problem 6 (Lind [38]). Determine if it is possible to construct a compact group automorphism whose entropy is not of the form $m(f)$ for some $f \in \mathbb{Z}[x]$.

Attempting to describe the fibres (that is, the equivalence classes) for measurable isomorphism requires answering an already difficult problem in number theory, the *inverse Mahler measure problem.*

Problem 7. Given $f \in \mathbb{Z}[x]$, characterize the set $\{g \in \mathbb{Z}[x] \mid m(g) = m(f)\}$.

We refer to Boyd [14] for an overview of this problem, and to Staines [58] for recent work and references.

To conclude this section, it is worth noting that for other group actions the same entropy range problem, that is, the problem of determining the set

$$\{h(T) \mid (G,T) \in \mathcal{G}_\Gamma\} \subset \mathbb{R}, \tag{4.1}$$

can turn out to be considerably simpler than providing an answer to Lehmer's problem corresponding to the case $\Gamma = \mathbb{Z}$. For example, Björklund and Miles [11, Th. 4.1] show that if Γ is any countably infinite amenable group with arbitrarily large finite normal subgroups, then the entropy range (4.1) is $[0,\infty]$. In contrast, for $\Gamma = \mathbb{Z}^d$, the entropy range problem again reduces to Lehmer's problem by [36]. Further discussion of the entropy range problem for group actions may be found in [11].

5. One-solenoids and subgroups of the rationals

Some of the classification and range problems we wish to discuss are already interesting in the special case of one-dimensional solenoids. A compact metric abelian group is called a one-solenoid if it is connected and has topological dimension one. Equivalently, a one-solenoid is a group whose dual or character group is a subgroup of $\mathbb{Q}$. In this section we briefly recall the well-known description of these subgroups.

5.1. **Subgroups of the rationals.** Subgroups of $\mathbb{Q}$ are readily classified (see Baer [6], or Beaumont and Zuckerman [8] for a modern account). Let $H \leqslant \mathbb{Q}$ be an additive subgroup, and $x \in H \setminus \{0\}$. Write $\mathbb{P} = \{2, 3, 5, 7, \dots\}$ for the set of rational primes. For each prime $p \in \mathbb{P}$, the *p-height* of x is

$$k_p(x) = \sup\{n \in \mathbb{N} \mid p^n y = x \text{ has a solution } y \in H\} \in \mathbb{N} \cup \{\infty\}$$

and the *characteristic* of x is the sequence

$$k(x) = (k_p(x))_{p \in \mathbb{P}} \in (\mathbb{N} \cup \{\infty\})^{\mathbb{N}}.$$

Two sequences $(a_p), (b_p) \in (\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$ *belong to the same type* if $a(p) = b(p)$ for all but finitely many p, and for any p with $a(p) \neq b(p)$, both $a(p)$ and $b(p)$ are finite. Notice that if $x, y \in H \setminus \{0\}$ then $k(x)$ and $k(y)$ belong to the same type, allowing us to define the *type of* H, $k(H)$, to be the type of any non-zero element of H.

On the other hand, given any sequence $(k_p)_{p \in \mathbb{P}}$ with $k_p \in \mathbb{N} \cup \{\infty\}$, we may define

$$H\left((k_p)_{p \in \mathbb{P}}\right) = \{\tfrac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z}, \gcd(a, b) = 1, \operatorname{ord}_p(b) \leqslant k_p \text{ for all } p \in \mathbb{P}\},$$

an additive subgroup of $\mathbb{Q}$.

Theorem 11 (Baer [6])**.** *Any subgroup of $\mathbb{Q}$ is of the form $H\left((k_p)_{p \in \mathbb{P}}\right)$ for some sequence $(k_p) \in (\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$. Two subgroups of $\mathbb{Q}$ are isomorphic if and only if they are of the same type.*

Clearly the only self-homomorphisms of $H = H\left((k_p)\right)$ are the maps $x \mapsto \frac{a}{b}x$ with $\gcd(a, b) = 1$ and $k_p(H) = \infty$ for any prime p dividing b. This gives a description of all continuous automorphisms of one-solenoids: $(G, T) \in \mathcal{G}$ has G connected with topological dimension 1 if and only if G is dual to a group $H\left((k_p)_{p \in \mathbb{P}}\right)$ and T is dual to a map $x \mapsto \frac{a}{b}x$ with $\gcd(a, b) = 1$ and with $k_p = \infty$ for any prime p dividing a or b. Writing $\mathcal{S}(H)$ for the set of primes p with $k_p(H) = \infty$, Theorem 11 shows in particular that $G \cong G'$ implies that $\mathcal{S}(\widehat{G}) = \mathcal{S}(\widehat{G}')$.

Example 12. Some of the diversity of subgroups may be seen in the following examples.

(1) $H\left((\infty)_{p \in \mathbb{P}}\right) = \mathbb{Q}$.
(2) $H\left((0)_{p \in \mathbb{P}}\right) = \mathbb{Z}$.
(3) $H\left((\infty, \infty, 0, 0, \dots)\right) = \mathbb{Z}[\frac{1}{6}]$.
(4) $H\left((0, 1, 1, 1, 1, \dots)\right)$ is the subgroup of rationals with odd square-free denominator.
(5) For a set $\mathcal{S} \subset \mathbb{P}$ of primes, there is an associated subgroup of $\mathcal{S}$-integers

$$R_{\mathcal{S}} = \{x \in \mathbb{Q} \mid |x|_p \leqslant 1 \text{ for all } p \notin \mathcal{S}\} = \mathbb{Z}[\tfrac{1}{p} : p \in \mathcal{S}] = H\left((k_p)_{p \in \mathbb{P}}\right)$$

where $k_p = \infty$ if $p \in S$ and 0 otherwise.
(6) Let $\omega = (\omega_j) \in \{H, T\}^{\mathbb{N}}$ be the outcome of an infinitely repeated fair coin toss, and let

$$k_{p_j} = \begin{cases} \infty & \text{if } j = H; \\ 0 & \text{if } j = T. \end{cases}$$

Then $H\left((k_p)_{p \in \mathbb{P}}\right)$ is a 'random' $\mathcal{S}$-integer subgroup.

Solenoids of higher topological dimension, dual to subgroups of $\mathbb{Q}^d$ with d greater than 1, present peculiar difficulties unless there are additional assumptions. A solenoid that carries an expansive automorphism has a prescribed structure (Lawton [33], [34]; see also Kitchens and Schmidt [28, Sec. 5] for this result in the context of a more general treatment of finiteness conditions on group automorphisms) rendering it amenable to analysis; solenoids dual to rings of $\mathcal{S}$-integers in number fields also have a prescribed structure, and this is used by Chothi *et al.* to study orbit growth in these systems [15], [19] and [20]. Around the same time as Baer's classification of subgroups of $\mathbb{Q}$, Kurosh [31] and Malcev [42] described systems of complete invariants for subgroups of $\mathbb{Q}^d$, but determining when the invariants correspond to isomorphic groups is intractable for $d > 1$. The difficulties encountered here now have a precise formulation in terms of descriptive set theory, and Thomas [61] proves the remarkable result that the classification problem for subgroups of $\mathbb{Q}^{n+1}$ is strictly more difficult than the same problem for $\mathbb{Q}^n$ for $n \geqslant 1$ (we refer to the survey by Thomas [62] for an overview of these results).

5.2. **Fixed points and entropy on one-solenoids.** There are several routes to a formula for $\mathsf{F}_T(n)$ if T is an automorphism of a one-solenoid: a group-theoretic argument due to England and Smith [18], a general valuation-theoretic approach due to Miles [45] which is independent of the characteristic and the topological dimension, and in the $\mathcal{S}$-integer case an adelic argument due to Chothi, Everest *et al.* [15]. Here we adapt the adelic approach of Weil [70] to cover all subgroups, because this geometrical approach also evaluates the entropy (the harmonic analysis used here is not strictly adelic, but closer to the constructions in Tate's thesis [60], reproduced as [59]). For completeness we explain in Lemma 14 the basic idea from [15, Lem. 5.1], and refer to Hewitt and Ross [26] for background on the duality theory between discrete abelian groups and compact abelian groups.

The dual of a one-solenoid may be identified with a subgroup of $\mathbb{Q}$, and we now describe such a group using a direct limit of fractional ideals over a ring of $\mathcal{S}$-integers (see Example 12(5)) based on the concept of types introduced in the previous section. Let $R_{\mathcal{S}}$ be a ring of $\mathcal{S}$-integers. A set of rational primes $\mathcal{P}$ disjoint from $\mathcal{S}$ corresponds to a set of non-zero prime ideals of $R_{\mathcal{S}}$. Let $k = (k_p)_{p\in\mathcal{P}}$ be a sequence of positive integers and define a submodule of the $R_{\mathcal{S}}$-module $\mathbb{Q}$ by $H = \sum_Q c_Q R_{\mathcal{S}}$, where $c_Q = \prod_{p\in Q} p^{-k_p}$ and Q runs over all finite subsets of $\mathcal{P}$. The group H is isomorphic to $R_{\mathcal{S}}$ if and only if $\mathcal{P}$ is finite, otherwise H may be expressed using a direct limit $\operatorname{inj\,lim}_{i\to\infty} H_i$ of $R_{\mathcal{S}}$-modules, where

$$H_i = p(1)^{-k_{p(1)}} \cdot \dots \cdot p(i)^{-k_{p(i)}} R_{\mathcal{S}},$$

and $p(1), p(2), \dots$ is an enumeration of $\mathcal{P}$. By Theorem 11, for any given type we may choose a representative subgroup $H = H((k_p)_{p\in\mathbb{P}})$ and note that the set

$$\mathcal{S}(H) = \{p \in \mathbb{P} \mid k_p = \infty\}$$

is independent of the choice of representative. We set

$$\mathcal{P}(H) = \{p \in \mathbb{P} \mid 0 < k_p < \infty\}$$

and

$$k(H) = (k_p)_{p\in\mathcal{P}(H)},$$

and use this data as above to obtain an $R_{\mathcal{S}(H)}$-module having the same type as H. Any other choice of representative subgroup produces an isomorphic $R_{\mathcal{S}(H)}$-module. This description now gives us a canonical way to view a one-solenoid.

Since there is a short exact sequence of discrete groups

$$\{0\} \longrightarrow \mathbb{Z} \longrightarrow R_{\mathcal{S}(H)} \longrightarrow \sum_{p\in\mathcal{S}(H)} \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z} \longrightarrow \{0\},$$

and the dual group of $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ is isomorphic to the ring of p-adic integers $\mathbb{Z}_p$, via duality, there is a short exact sequence of compact groups

$$\{0\} \longrightarrow \prod_{p\in\mathcal{S}(H)} \mathbb{Z}_p \longrightarrow \widehat{R_{\mathcal{S}(H)}} \longrightarrow \mathbb{T} \longrightarrow \{0\}. \tag{5.1}$$

Therefore, $\widehat{R_{\mathcal{S}(H)}}$ is a central extension of $\mathbb{T}$ by a cocycle

$$w : \mathbb{T}\times\mathbb{T} \longrightarrow \prod_{p\in\mathcal{S}(H)} \mathbb{Z}_p, \tag{5.2}$$

and a simple explicit calculation along the lines of [63] shows that we may take

$$w(s,t) = -\lfloor s+t\rfloor\,((1,1,\dots)).$$

Furthermore, there is an explicit geometrical description of the solenoid $\widehat{R_{\mathcal{S}(H)}}$ as follows.

Lemma 13. *The diagonal map $\delta : x \mapsto (x,x,\dots)$ embeds $R_{\mathcal{S}(H)}$ as a discrete and co-compact subgroup of the restricted directed product*

$$\mathbb{A}_{\mathcal{S}(H)} = \left\{ (x_\infty,(x_p)) \in \mathbb{R}\times \prod_{p\in\mathcal{S}(H)} \mathbb{Q}_p \mid x_p\in\mathbb{Z}_p \textit{ for all but finitely many } p \right\},$$

which is a locally compact topological group. The set

$$F = [0,1)\times \prod_{p\in\mathcal{S}(H)} \mathbb{Z}_p$$

is a fundamental domain for $\delta(R_{\mathcal{S}(H)})$ in $\mathbb{A}_{\mathcal{S}(H)}$, and

$$\widehat{R_{\mathcal{S}(H)}} \cong \mathbb{A}_{\mathcal{S}(H)}/\delta(R_{\mathcal{S}(H)}).$$

Proof. This is a special case of the construction introduced in [60]. See also [52] for an accessible account. □

Recall that a subgroup $\Gamma < X$ in a locally compact topological group is called a *uniform lattice* if Γ is discrete and the quotient space X/Γ is compact in the quotient topology, and a measurable set $F \subset X$ is a *fundamental domain* for Γ if it contains exactly one representative of each coset of Γ. The *module* of a homomorphism

$$A : X \to X$$

is the quantity $m(AU)/m(U)$ for an open set U of finite measure, where m is a choice of Haar measure on X.

Lemma 14. *Let Γ be a uniform lattice in a locally compact abelian group X, let F be a fundamental domain for Γ, and let m denote the Haar measure on X normalized to have $m(F) = 1$. Let $\widetilde{A} : X \to X$ be a continuous surjective homomorphism with $\widetilde{A}(\Gamma) \subset \Gamma$, and let $A : X/\Gamma \to X/\Gamma$ be the induced map. If* $\ker A$ *is discrete, then*

$$\operatorname{mod}_X(\widetilde{A}) = |\ker A| = m(\widetilde{A}F).$$

Proof. Choose F so that it contains an open neighborhood U of the identity in X (this is possible since Γ is discrete in X). Since X/Γ is compact, $\ker A$ is finite and so there is a disjoint union $A^{-1}V = V_1 \sqcup \cdots \sqcup V_{|\ker A|}$ if V is a sufficiently small neighbourhood of the identity in X/Γ, with each V_i an open neighbourhood of a point in the fibre $A^{-1}(0_{X/\Gamma})$. Since A is measure–preserving,

$$m\left(A^{-1}V\right) = m\left(V\right).$$

If U and V are sufficiently small, then the quotient map $x \mapsto x + \Gamma$ is a homeomorphism between U and V, and so

$$m\left(\widetilde{A}U\right) = m\left(AV\right) = |\ker A| m\left(V\right) = |\ker A| m\left(U\right);$$

furthermore, since $U(0_X) \subset F$, $m(\widetilde{A}F) = |\ker A|$. □

Example 15. A simple situation to which Lemma 14 may be applied is endomorphisms of the torus.

(1) Taking $\Gamma = \mathbb{Z} < X = \mathbb{R}$, $\widetilde{A}(x) = bx$ for some $b \in \mathbb{Z} \setminus \{0\}$, and m to be Lebesgue measure with $m\left(F = [0,1)\right) = 1$, we see that

$$|\ker(x \mapsto bx \pmod 1)| = |b|.$$

In particular, it follows that if $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the map $x \mapsto ax$ for some a in $\mathbb{Z} \setminus \{0, \pm 1\}$ then $\mathsf{F}_T(n) = |a^n - 1|$ and so

$$\zeta_T(z) = \frac{1-z}{1-az}$$

if $a > 0$, and

$$\zeta_T(z) = \exp\left(\sum_{n\geqslant 1} \frac{|a|^n}{n} z^n + \sum_{n\geqslant 1} \frac{1}{n} z^n - 2\sum_{n\geqslant 1} \frac{1}{2n} z^{2n}\right) = \frac{1+z}{1-|a|z}$$

if $a < 0$.

(2) As seen above, some care is needed in dealing with the distinction between the expressions $|a^n - 1|$ and $|a|^n - 1$: for an automorphism T of the d-torus $G = \mathbb{T}^d$ defined by an integer matrix $A_T \in \mathrm{GL}_d(\mathbb{Z})$ the same argument gives the formula

$$\mathsf{F}_T(n) = |\det(A_T^n - I)|,$$

and *a priori* an argument is needed to show that $\zeta_T(z)$ is a rational function of z because of the absolute value. This is discussed in Smale [57, Prop. 4.15], and an elementary algorithmic way to compute the zeta function in integer arithmetic is given by Baake, Lau and Paskunas [5].

We can also use Lemma 14 to find a formula for the periodic points of an automorphism of a one-solenoid, recovering in different notation the result of [18] and the one-dimensional case of [45].

Proposition 16. *If $T : \widehat{H} \to \widehat{H}$ is an automorphism of a one-solenoid dual to the map $x \mapsto rx$ on $H \leqslant \mathbb{Q}$, then*

$$\mathsf{F}_T(n) = |r^n - 1| \times \prod_{p \in \mathcal{S}(H)} |r^n - 1|_p.$$

Proof. Since $x \mapsto rx$ is an automorphism of H, $p \in \mathcal{S}(H)$ whenever $|r|_p \neq 1$. Lemmas 13 and 14 show that if T' is the automorphism dual to $x \mapsto rx$ on $\mathbb{A}_{\mathcal{S}(H)}$, then

$$\mathsf{F}_{T'}(n) = \operatorname{mod}_{\mathbb{A}_{\mathcal{S}(H)}}(x \mapsto rx) = |r^n - 1| \times \prod_{p \in \mathcal{S}(H)} |r^n - 1|_p.$$

The set of points of period n is dual to the group $H/(r^n - 1)H$, and multiplication by p^{k_p} for any $p \in \mathcal{P}(H)$ remains non-invertible in the direct limit H, so

$$H/(r^n - 1)H \cong R_{\mathcal{S}(H)}/(r^n - 1)R_{\mathcal{S}(H)}$$

for each $n \geqslant 1$, showing the proposition. □

The geometric viewpoint using adeles also allows the entropy to be computed easily using (1.1); this calculation is originally due to Abramov [1]. Write

$$\log^+(t) = \max\{0, \log t\}$$

for $t > 0$.

Proposition 17 (Abramov's formula)**.** *If $T : \widehat{H} \to \widehat{H}$ is an automorphism of a one-solenoid dual to the map $x \mapsto rx$ on $H \leqslant \mathbb{Q}$, then*

$$h(T) = \sum_{p \in \mathbb{P} \cup \{\infty\}} \log^+ |r|_p = \log \max\{|a|, |b|\}$$

where $r = \frac{a}{b}$ with $\gcd(a, b) = 1$.

Proof. This formula is explained under the assumption that $H = R_{\mathcal{S}(H)}$ by the following argument. First, the calculation (1.1) may be performed in the covering space $\mathbb{A}_{\mathcal{S}(H)}$ because the quotient map

$$\mathbb{A}_{\mathcal{S}(H)} \to \widehat{R_{\mathcal{S}(H)}}$$

is a local isometry (we refer to [17, Prop. 4.7] for the details of this general lifting principle). Secondly, a small metric ball around 0 in the covering space $\mathbb{A}_{\mathcal{S}(H)}$ takes the form

$$(-\varepsilon, \varepsilon) \times \prod_{p \in \mathcal{S}(H)} p^{n_p} \mathbb{Z}_p,$$

and the action $\widetilde{T_r}$ of multiplication by r^{-1} on each of the coordinates $\mathbb{R}$ or $\mathbb{Q}_p$ gives

$$\bigcap_{j=0}^{n-1} T_r^{-j}(-\varepsilon, \varepsilon) = \begin{cases} (-\varepsilon, \varepsilon) & \text{if } |r| \leqslant 1; \\ (-\varepsilon |r|^{-(n-1)}, \varepsilon |r|^{-(n-1)}) & \text{if } |r| > 1, \end{cases}$$

on the quasi-factor $\mathbb{R}$, and

$$\bigcap_{j=0}^{n-1} T_r^{-j} p^{n_p} \mathbb{Z}_p = \begin{cases} p^{n_p} \mathbb{Z}_p & \text{if } |r|_p \leqslant 1; \\ |r|_p^{-(n-1)} p^{n_p} \mathbb{Z}_p & \text{if } |r| > 1, \end{cases}$$

showing the claimed formula.

In the general case an argument is needed to ensure that the entropy is not changed by passing from $R_{\mathcal{S}(H)}$ to the direct limit H, and we refer to [41, Prop. 3.1] for the details. $\square$

5.3. **Equivalence relations for one-solenoids.** We now consider how the equivalence relations from Definition 1 behave for the subspace $\mathcal{G}_1 \subset \mathcal{G}$ consisting of pairs (G,T) where $T: G \to G$ is an automorphism of a one-solenoid. Thus we may assume that G is dual to a group $H = H((k_p))$, and $T = T_{a/b}$ is dual to the map $x \mapsto \frac{a}{b}x$ with $\gcd(a,b) = 1$ on H, with the property that any prime p dividing ab has $k_p = \infty$.

Example 18. It is clear that the entropy and the zeta function do not together determine an element of $\mathcal{G}_1$.

(1) We have $(\widehat{\mathbb{Z}[\frac{1}{2}]}, T_2) \sim_\zeta (\widehat{\mathbb{Z}[\frac{1}{2}]}, T_{1/2})$ and $(\widehat{\mathbb{Z}[\frac{1}{2}]}, T_2) \sim_h (\widehat{\mathbb{Z}[\frac{1}{2}]}, T_{1/2})$.
(2) By varying the group instead of the map, much larger joint equivalence classes may be found. As pointed out by Miles [45], a consequence of Proposition 16 and Proposition 17 is that the set
$$\left\{(G,T) \in \mathcal{G}_1 \mid \zeta_T(z) = \tfrac{1-z}{1-2z}, h(T) = \log 2\right\}$$
is uncountable. To see this, let $\{S_\lambda\}_{\lambda\in\Lambda}$ be an uncountable set of infinite subsets of $\mathbb{P}$, all containing 2, with the property that $|S_\lambda \triangle S_\nu| = |\mathbb{N}|$ for all $\lambda \neq \nu$. Associate to each S_λ the subgroup $H_\lambda = H((k_p^{(\lambda)}))$ where
$$k_p^{(\lambda)} = \begin{cases} 0 & \text{if } p \notin S_\lambda, \\ 1 & \text{if } p \in S_\lambda \setminus \{2\}, \\ \infty & \text{if } p = 2. \end{cases}$$
Then the automorphism dual to $x \mapsto 2x$ on H_λ has zeta function $\frac{1-z}{1-2z}$, and by Theorem 11 these are all algebraically, and hence topologically, distinct systems.

Lemma 19. *If $(G,T) \sim_\zeta (G',T')$ then $\mathcal{S}(G) = \mathcal{S}(G')$.*

Proof. Assume that $(G,T) \in \mathcal{G}_1$ with $T = T_{a/b}$ and G dual to $H((k_p))$. We claim that
$$\{p \in \mathbb{P} \mid p\big| \mathsf{F}_T(n) \text{ for some } n \geqslant 1\} = \{p \in \mathbb{P} \mid p \notin \mathcal{S}(G) \text{ and } p\not| ab\}. \tag{5.3}$$
Since by hypothesis T is an automorphism, we have
$$p\big|ab \implies p \in \mathcal{S}(G)$$
so (5.3) gives the lemma. To see (5.3), notice that by Proposition 16 no prime in $\mathcal{S}(G)$ can divide any $\mathsf{F}_T(n)$, and any prime $p \notin \mathcal{S}(G)$ will divide $a^{p-1} - b^{p-1}$ and so will divide some $\mathsf{F}_T(n)$. $\square$

Problem 8. The case of endomorphisms is slightly different, because for example $(\widehat{\mathbb{Z}}, T_2) \sim_\zeta (\widehat{\mathbb{Z}[\frac{1}{2}]}, T_2) \sim_\zeta (\widehat{\mathbb{Z}[\frac{1}{2}]}, T_{1/2})$. Formulate a version of Lemma 19 for endomorphisms of one-solenoids.

The case of subrings, or equivalently of $\mathcal{S}$-integer subgroups of $\mathbb{Q}$, has distinctive features. Let $\overline{\mathcal{G}_1}$ denote the collection of pairs $(G,T) \in \mathcal{G}_1$ with the property that $\widehat{G}$ is a subring of $\mathbb{Q}$. This means that $k_p(G)$ is 0 or ∞ for any prime p.

Proposition 20. *On the space $\overline{\mathcal{G}_1}$,*

(1) *$\sim_h$ has uncountable equivalence classes;*
(2) *$\sim_\zeta$ has countable equivalence classes;*
(3) *the joint relation $\sim_h$ and $\sim_\zeta$ has finite equivalence classes.*

Proof. The pair $(\widehat{\mathbb{Z}[\frac{1}{ab}]}, T_{a/b})$ has entropy $\log\max\{|a|,|b|\}$, and by Abramov's formula for any subset $\mathcal{S} \subset \mathbb{P} \setminus \{p \mid p\big|ab\}$ of primes the pair $(\widehat{R_{\mathcal{S}}[\frac{1}{ab}]}, T_{a/b})$ has the same entropy, showing (1).

By Lemma 19, the $\sim_\zeta$ equivalence class determines the set of primes of infinite height, and so the only parameter that can change is the rational $\frac{a}{b}$ defining the map $T_{a/b}$, and there are only countably many of these.

For (3), notice that if $(G_2, T_{a/b})$ has entropy h then

$$\max\{|a|,|b|\} \leqslant \exp h,$$

so there are only finitely many choices for the rational r defining the map T_r with a given entropy. Lemma 19 shows that there is no further choice for a fixed zeta function. □

Example 21. Some of the diversity possible in Proposition 20 is illustrated via simple examples.

(1) For any $r \in \mathbb{Q} \setminus \{0, \pm 1\}$ the system $(\widehat{\mathbb{Q}}, T_r)$ has zeta function $\frac{1}{1-z}$, showing that the $\sim_\zeta$ equivalence class may be infinite.
(2) The $\sim_\zeta$ equivalence class may be infinite in a less degenerate way. Let G be the group dual to $\mathbb{Z}_{(3)}$, the subgroup obtained by inverting all the primes except 3. Then Proposition 16 shows that
$$\mathsf{F}_{T_r}(n) = |r^n - 1|_3^{-1}$$
by the product formula
$$\prod_{p \in \mathbb{P} \cup \{\infty\}} |t|_p = |t| \times \prod_{p \in \mathbb{P} : |t|_p \neq 1} |t|_p = 1 \tag{5.4}$$
for all $t \in \mathbb{Q} \setminus \{0\}$. It follows that $(G, T_2) \sim_\zeta (G, T_r)$ if $r = \frac{a}{b}$ where (a, b) is of the form
$$\left(\tfrac{9k+3m}{2} + 2, \tfrac{9k-3m}{2} + 1\right)$$
or
$$\left(\tfrac{9k+3m}{2} + 4, \tfrac{9k-3m}{2} + 2\right)$$
for integers m, k of the same parity chosen with a, b coprime and positive, or of the form
$$\left(\tfrac{9k+3m+5}{2}, \tfrac{9k-3m+1}{2}\right)$$
or
$$\left(\tfrac{9k+3m+7}{2}, \tfrac{9k-3m+5}{2}\right)$$
for integers m, k of opposite parity chosen with a, b coprime and positive.
(3) The $\sim_\zeta$ equivalence class may be finite. For example, if $(G, T) \in \overline{\mathcal{G}_1}$ has zeta function $\frac{1-z}{1-2z}$, then we claim that (G, T) can only be $(\widehat{\mathbb{Z}[\frac{1}{2}]}, T_2)$ or $(\widehat{\mathbb{Z}[\frac{1}{2}]}, T_{1/2})$. To see this, notice first that these both have the claimed zeta function, and apply Lemma 19 to deduce that any element of $\overline{\mathcal{G}_1}$ with the same zeta function has the form $(\widehat{\mathbb{Z}[\frac{1}{2}]}, T_{a/b})$. Since the map $T_{a/b}$ is an

automorphism, the only prime dividing ab is 2, so $\frac{a}{b}$ is $\pm 2^k$ for some $k \in \mathbb{Z}$ and we may use Proposition 16 to calculate

$$\zeta_T(z) = \begin{cases} \frac{1-z}{1-2^k z} & \text{if } T = T_{2^k}; \\ \frac{1+z}{1-2^k z} & \text{if } T = T_{-2^k}. \end{cases}$$

The constraints seen in Example 21(3) hold more generally for systems with the property that $|\mathcal{S}(\widehat{G})|$ is finite.

Proposition 22. *Let* $(G_1, T_r), (G_2, T_s) \in \mathcal{G}_1$ *have* $|\mathcal{S}(\widehat{G_1})| < \infty$ *and*

$$(G_1, T_r) \sim_\zeta (G_2, T_s).$$

Then $r = s$ *or* $r = s^{-1}$.

Proof. First notice that $\mathcal{S}(\widehat{G_2}) = \mathcal{S}(\widehat{G_1})$ by Lemma 19. Denote this common set of primes by $\mathcal{S}$. Write $r = \frac{a_1}{b_1}$, $s = \frac{a_2}{b_2}$ with $\gcd(a_i, b_i) = 1$ and $a_i > 0$ for $i = 1, 2$. Since $\zeta_{T_r} = \zeta_{T_r^{-1}}$ and $\zeta_{T_s} = \zeta_{T_s^{-1}}$, without loss of generality we may assume that $a_i > |b_i|$ for $i = 1, 2$. Then, using Proposition 16 (see [15, Th 6.1] for the details), it follows that

$$\log a_1 = \lim_{n\to\infty} \frac{1}{n} \log \mathsf{F}_{T_r}(n) \text{ and } \log a_2 = \lim_{n\to\infty} \frac{1}{n} \log \mathsf{F}_{T_s}(n),$$

so that $a_1 = a_2$. Let $a = a_1 = a_2$,

$$\mathcal{T}_i = \{p \in \mathcal{S} \mid |a|_p < |b_i|_p\} \text{ and } \mathcal{T}_i' = \{p \in \mathcal{S} \mid |a|_p > |b_i|_p\},$$

for $i = 1, 2$. Notice that since $\gcd(a, b_i) = 1$, $|b_i|_p = 1$ for all $p \in \mathcal{T}_i$, and $|a|_p = 1$ for all $p \in \mathcal{T}_i'$, $i = 1, 2$. Hence

$$\prod_{p \in \mathcal{T}_i \cup \mathcal{T}_i'} \left| \left(\frac{a}{b_i}\right)^n - 1 \right|_p = \prod_{p \in \mathcal{T}_i'} \left| \left(\frac{a}{b_i}\right)^n - 1 \right|_p = \prod_{p \in \mathcal{T}_i'} |b_i|_p^{-n} = |b_i|^n, \tag{5.5}$$

where the last equality follows by the Artin product formula, as $|b_i|_p = 1$ for all $p \in \mathbb{P} \setminus \mathcal{T}_i'$.

For each $p \in \mathcal{S} \setminus (\mathcal{T}_i \cup \mathcal{T}_i')$, $|\frac{a}{b_i}|_p = 1$, that is, $\frac{a}{b_i}$ is a unit in the valuation ring $\mathbb{Z}_{(p)}$. Let $m_i(p)$ denote the multiplicative order of $\frac{a}{b_i}$ in the residue field $\mathbb{Z}_{(p)}/(p)$, and note that if $m_i(p) \nmid n$, then $|(\frac{a}{b_i})^n - 1|_p = 1$. Therefore, whenever n is coprime to

$$m = \prod_{i=1,2} \prod_{p \in \mathcal{S} \setminus (\mathcal{T}_i \cup \mathcal{T}_i')} m_i(p),$$

for both $i = 1$ and $i = 2$ we have

$$\prod_{p \in \mathcal{S} \setminus (\mathcal{T}_i \cup \mathcal{T}_i')} \left| \left(\frac{a}{b_i}\right)^n - 1 \right|_p = 1. \tag{5.6}$$

Substituting (5.5) and (5.6) into the formula given by Proposition 16 yields

$$\mathsf{F}_{T_r}(n) = |a^n - b_1^n| \text{ and } \mathsf{F}_{T_s}(n) = |a^n - b_2^n|,$$

whenever $\gcd(n, m) = 1$. Since $a > 0$ and $a > |b_1|$, it follows that $b_1 = \pm b_2$. □

Corollary 23. *The* $\sim_\zeta$ *equivalence class of* $(G, T_{a/b})$ *in the subset of* $\mathcal{G}_1$ *with* $\mathcal{S}(\widehat{G})$ *finite has cardinality* $2^{\omega(a)} + 2^{\omega(b)}$, *where as usual* $\omega(k)$ *denotes the number of distinct prime divisors of* k.

6. Counting closed orbits

The most transparent combinatorial invariant of topological conjugacy is the count of periodic orbits, which may be captured using generating functions like the dynamical zeta function. For a class of ergodic group automorphisms with polynomially bounded growth in the number of closed orbits, a more natural generating function is given by the orbit Dirichlet series [20]. A consequence of the structure theorem for expansive automorphisms of connected groups due to Lawton [34] is the following.

Theorem 24 (Lawton [34])**.** *If $T : G \to G$ is an ergodic expansive automorphism of a compact connected group, then $\zeta_T(z)$ is a rational function of z.*

It is clear that not every function can be a zeta function of a map – in particular the coefficients in (1.2) must be non-negative. In fact more is true, and it is shown in [51] that a function $\zeta(z) = \exp\sum_{n\geqslant 1}\frac{a_n}{n}z^n$ is the dynamical zeta function of some map if and only if

$$0 \leqslant \sum_{d|n}\mu\left(\tfrac{n}{d}\right)a_d \equiv 0 \pmod{n} \tag{6.1}$$

for all $n \geqslant 1$. A beautiful remark of Windsor is that the same condition is equivalent to being the dynamical zeta function of a C^∞ diffeomorphism of the 2-torus [71]. Thus, for example,

$$\frac{\mathrm{e}^{-z^2}}{(1-2z)} = \exp\left(2z + z^2 + \frac{2^3}{3}z^3 + \frac{2^4}{4}z^4 + \frac{2^5}{5}z^5 + \cdots\right)$$

has non-negative coefficients but cannot be a dynamical zeta function of any map. England and Smyth [18] characterized in combinatorial terms the property of being the dynamical zeta function of an automorphism of a one-dimensional solenoid, and Moss [48] considered the more general question of when a function could be the dynamical zeta function of a group automorphism. Clearly the non-negativity and congruence condition in (6.1) is not sufficient, since the sequence $\mathsf{F}_T = (\mathsf{F}_T(n))_{n\geqslant 1}$ for a group automorphism must be a divisibility sequence. Moss showed that adding this further condition is also not sufficient, even for linear recurrence sequences.

Lemma 25 (Moss [48])**.** *The function $f(z) = \frac{1}{(1-z)(1-z^5)}$ is the dynamical zeta function of the permutation $\tau = (1)(23456)$ on the set $\{1,2,3,4,5,6\}$, which has the property that F_τ is a divisibility sequence, but f is not the zeta function of any group automorphism.*

Proof. The given permutation τ has

$$(\mathsf{F}_\tau) = (1,1,1,1,6,1,1,1,1,6,\ldots),$$

which is a divisibility sequence satisfying the linear recurrence relation

$$u_{n+5} = u_n$$

for $n \geqslant 1$ with the initial conditions

$$u_1 = u_2 = u_3 = u_4 = 1, u_5 = 6.$$

To see that f cannot be the dynamical zeta function of an abelian group automorphism, notice that if there is a group automorphism $T : G \to G$ with $\zeta_T(z) = f(z)$ then there is such a realization with $|G| = 6$. In this case T is an automorphism of G with $T^5 = I$, the identity. If some $g \in G \setminus \{0\}$ has orbit under T of cardinality less than 5, then there are integers $0 \leqslant m < n \leqslant 4$ with $T^m x = T^n x$, which implies that $T^{n-m}x = x$ and so $x = 0$ since $|n-m| \leqslant 4$. It follows that any non-identity element of G has an orbit of length 5 under the automorphism T, and so in particular every non-identity element of G has the same order, which is impossible. □

England and Smyth [18] showed that

$$f(z) = \exp \sum_{n \geqslant 1} \frac{a_n}{n} z^n$$

is the dynamical zeta function of a group automorphism dual to $x \mapsto \frac{m}{n}x$ on a one-solenoid if and only if

$$a_k \big| n^k - m^k \text{ for all } k \geqslant 1 \tag{6.2}$$

and

$$\gcd\left(a_k, \tfrac{n^\ell - m^\ell}{a_\ell}\right) = 1 \text{ for all } k \neq \ell. \tag{6.3}$$

They also gave an example (which does not seem to be correct) of a group automorphism with irrational zeta function for an automorphism of a one-solenoid; there are now several ways to see that these must exist. In [65] it is shown that there are uncountably many distinct zeta functions of automorphisms dual to $x \mapsto 2x$ on a one-solenoid (and hence most in cardinality are irrational functions); Everest, Stangoe and the third author [21, Lem. 4.1] showed that the map dual to $x \mapsto 2x$ on $\mathbb{Z}[\frac{1}{6}]$ has a natural boundary at $|z| = \frac{1}{2}$. In order to relate the characterization from [18] to the S-integer constructions of Chothi, Everest *et al.* [15] outlined in Proposition 16, we show that they are equivalent.

Lemma 26. *Given coprime integers m, n, a sequence (a_n) of positive integers satisfies* (6.2) *and* (6.3) *if and only if there is a subset of the primes $\mathcal{S} \subset \mathbb{P}$ for which*

$$a_k = |n^k - m^k| \prod_{p \in \mathcal{S}} |n^k - m^k|_p \tag{6.4}$$

for all $k \geqslant 1$.

Proof. Clearly the existence of a set $\mathcal{S} \subset \mathbb{P}$ with (6.4) implies (6.2) and (6.3).

Assume (6.2) and (6.3), let

$$\mathcal{S} = \{p \in \mathbb{P} \mid p \big| \tfrac{n^\ell - m^\ell}{a_\ell} \text{ for some } \ell \geqslant 1\},$$

and write $b_k = |n^k - m^k| \prod_{p \in \mathcal{S}} |n^k - m^k|_p$. If some prime power p^e divides some a_k, then $p^e | n^k - m^k$ so by (6.3) $p^e \big| b_k$, and so $a_k \big| b_k$ for all $k \geqslant 1$. If some prime power p^e divides some b_k, then $p^e \big| n^k - m^k$ and $p \notin \mathcal{S}$ and hence $p^e \big| a_k$. Thus $b_k \big| a_k$ for all $k \geqslant 1$. □

Miles gave a general characterization for finite entropy automorphisms on finite-dimensional groups. In order to state this, recall from Weil [70] that a global field (or $\mathbb{A}$-field) is an algebraic number field or a function field of transcendence degree one over a finite field. For a global field K we write $\mathbb{P}(K)$ for the set of places of K, and $\mathbb{P}_\infty(K)$ for the set of infinite places of K

Theorem 27 (Miles [45]). *If $T : G \to G$ is an ergodic automorphism of a finite-dimensional compact abelian group with finite entropy, then there exist global fields $K_1, \dots, K_n$, sets of finite places $P_i \subset \mathbb{P}(K_i)$ and elements $\xi_i \in K_i$, no one of which is a root of unity for $i = 1, \dots, n$, such that*

$$\mathsf{F}_T(k) = \prod_{i=1}^{n} \prod_{v \in P_i} |\xi_i^k - 1|_v^{-1}.$$

We now turn to another range problem, concerned with the growth in orbits. This may be thought of as a combinatorial analogue of Lehmer's problem.

Theorem 28 (Ward [69]). *For any $C \in [0, \infty]$ there is a compact group automorphism $T : G \to G$ with*

$$\frac{1}{n} \log \mathsf{F}_T(n) \longrightarrow C$$

as $n \to \infty$.

Theorem 28 exploits known bounds for the appearance of primes in arithmetic progressions, and the proof works because the growth rate targeted is exponential (the case $C = 0$ is trivial). The faster the growth rate sought, the easier the proof becomes (as may be seen in the proof of Theorem 29). Constructing compact group automorphisms with prescribed polynomial growth rates of periodic points is a far more delicate problem, and we refer to work of Haynes and White [25] for the current state of the problem.

6.1. **Sets of primes and constructions in one-solenoids.** The examples constructed in Theorem 28 are not really satisfactory, since the groups used are zero-dimensional and (worse) the automorphisms are not ergodic. As a result it is natural to ask about the range of various invariants for ergodic automorphisms of connected groups. For rapid growth and a crude approximation, it is possible to exhibit growth in periodic orbits in a controlled way on connected groups, as illustrated in the next result (the proof of which we postpone briefly).

Theorem 29. *Let $\theta = (\theta_n)$ be a sequence of positive integers with*

$$\frac{n}{\log \theta_n} \longrightarrow 0$$

as $n \to \infty$. Then there is an automorphism $T : X \to X$ of a connected compact group with

$$\frac{\log \mathsf{F}_T(n)}{\log \theta_n} \longrightarrow 1$$

as $n \to \infty$.

With more effort, taking advantage of known estimates for the size of the primitive part of linear recurrence sequence, it is likely that this result can be improved, but it is unlikely that any such approach will lead to an (unexpected) positive answer to the following.

Problem 9. Does Theorem 28 still hold if the compact groups are required to be connected?

It is expected that this reduces directly to Lehmer's problem, and is as a result equally intractable. A refined version (in light of the expected negative answer) is the following; background and partial results in this direction may be found in some work of the third author [66], [65], exposing in particular some of the Diophantine problems that arise.

Problem 10. Characterize the set

$$\left\{\limsup_{n\to\infty}\frac{1}{n}\log \mathsf{F}_T(n) \mid (G,T)\in\mathcal{G} \text{ and } G \text{ is connected}\right\}$$

of real numbers.

As we have seen in Proposition 16, orbit-growth questions on the space $\mathcal{G}_1$ essentially amount to questions about rational numbers and sets of primes. In this section we will give an overview of what is known in this setting specifically for the map dual to $x\mapsto 2x$ on subrings of $\mathbb{Q}$. Thus the systems are parameterised by subsets of $\mathbb{P}$, and Figure 2 gives a guide to the results assembled here on the Hasse diagram of subsets of $\mathbb{P}$, or equivalently of subrings of $\mathbb{Q}$. In the diagram the standard notation for localization of rings is used for convenience, so for example $\mathbb{Z}[\frac{1}{p}\mid p\neq 2,3]$ is written $\mathbb{Z}_{(2,3)}$ and so on.

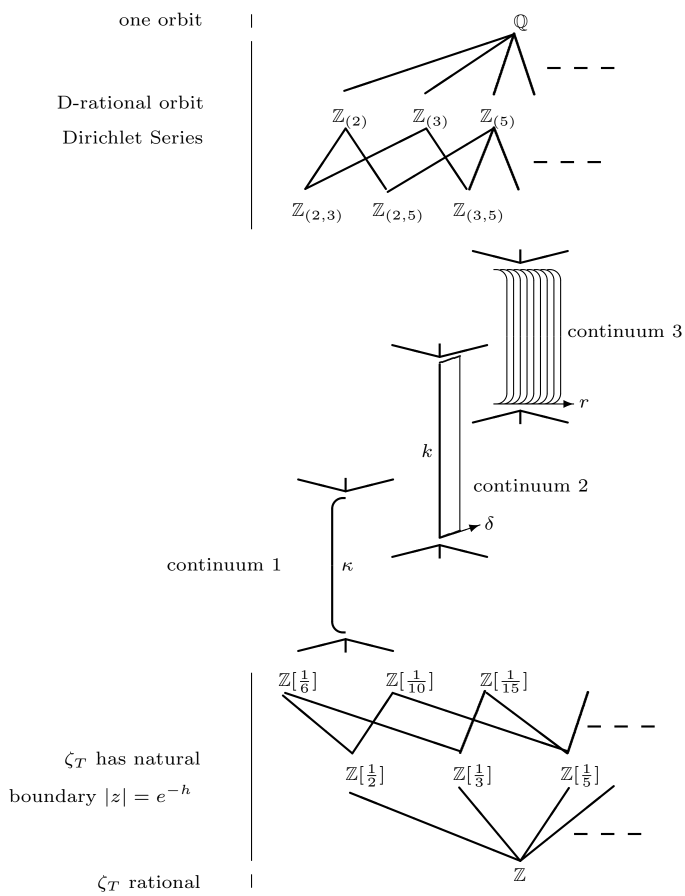

FIGURE 2. The Hasse diagram of subrings of $\mathbb{Q}$.

At the bottom of Figure 2 the compact group is the circle, and strictly speaking this has no ergodic automorphisms. However, as seen in Example 15, if we permit the endomorphisms $x \mapsto ax$ (or invert only those primes appearing in a) then the zeta functions arising are rational.

The first observation one can make about the lower part of Figure 2 (subrings of $\mathbb{Q}$ in which $k_p = \infty$ for only finitely many primes p) concerns the exponential growth rate of periodic points. If $S \subset \mathbb{P}$ has $|S| < \infty$, then there are constants $A, B > 0$ with

$$\frac{A}{n^B} \leqslant \prod_{p \in S} |a^n - b^n|_p \leqslant 1 \tag{6.5}$$

for all $n \geqslant 1$ (see [15, Th 6.1] for the details). A simple consequence of Proposition (16) and the estimate (6.5) gives the construction needed for Theorem 29.

Proof of Theorem 29. Define a sequence (S_k) of sets of primes by

$$S_k = \{p \in \mathbb{P} \mid p \big| 2^n - 1 \text{ for some } n < k\} \cup \{2\}.$$

Thus $S_1 = \{2\}$, $S_2 = \{2, 3\}$, $S_3 = \{2, 3, 7\}$, and so on, and let $T_k : X_k \to X_k$ be the automorphism dual to $x \mapsto 2x$ on $\mathbb{Z}[\frac{1}{p} : p \in S_k]$. By Proposition 16 and (6.5) it follows that $\mathsf{F}_{T_k}(n) = 1$ for $n < k$ and $\lim_{n\to\infty}(1/n)\log \mathsf{F}_{T_k}(n) = \log 2$. It may be helpful to have in mind that the sequence of fixed point counts for these systems begins as follows:

$$\begin{aligned}
\mathsf{F}_{T_1} &= (1, 3, 7, 15, 31, \dots) \\
\mathsf{F}_{T_2} &= (1, 1, 7, 5, 31, \dots) \\
\mathsf{F}_{T_3} &= (1, 1, 1, 5, 31, \dots) \\
\mathsf{F}_{T_4} &= (1, 1, 1, 1, 31, \dots),
\end{aligned}$$

and so on. We now inductively construct an infinite-dimensional solenoid X together with an automorphism T as follows, starting at 2 for convenience.

At the first stage, let $n_2 = \lceil \log_3 \theta_2 \rceil$ and let $(X^{(2)}, T^{(2)})$ be the n_2-fold Cartesian product of copies of (X_1, T_1). By construction,

$$1 \leqslant \frac{\mathsf{F}_{T^{(2)}}(2)}{\theta_2} \leqslant 3 = \mathsf{F}_{T_1}(2). \tag{6.6}$$

At the next stage, let $n_3 = \lceil \log_7 \theta_3 \rceil - m_2$ (and assume, as we may, that this is non-negative by amending the early terms of the sequence θ), and let $(X^{(3)}, T^{(3)})$ be the product of $(X^{(2)}, T^{(2)})$ and the n_3-fold Cartesian product of copies of (X_2, T_2), where m_2 is chosen to have

$$\mathsf{F}_{T^{(3)}}(3) \geqslant \theta_3$$

and to minimize

$$\mathsf{F}_{T^{(3)}}(3) - \theta_3.$$

By construction, we retain (6.6) but also have

$$1 \leqslant \frac{\mathsf{F}_{T^{(3)}}(3)}{\theta_3} \leqslant 7 = \mathsf{F}_{T_2}(3).$$

We continue in this way, multiplying at the nth stage by a number of copies of (X_k, T_k) to approximate as well as possible the number of points of period n sought, eventually producing an infinite product of one-dimensional solenoids T with an automorphism T so that

$$1 \leqslant \frac{\mathsf{F}_T(n)}{\theta_n} \leqslant \mathsf{F}_{T_{n-1}}(n) \leqslant 2^n.$$

Thus
$$1 \leqslant \frac{\log \mathsf{F}_T(n)}{\log \theta_n} \leqslant 1 + \frac{n \log 2}{\log \theta_n},$$
giving the result by the growth-rate assumption on θ. $\square$

Example 30. The map T dual to $x \mapsto 2x$ on $\mathbb{Z}[\frac{1}{6}]$ has
$$\zeta_T(z) = \exp \sum_{n \geqslant 1} \frac{|2^n - 1||2^n - 1|_3}{n} z^n.$$

The radius of convergence of ζ_T is $\frac{1}{2}$ by (6.5), and a simple argument from [21, Lem. 4.1] shows that the series defining $\zeta_T(z)$ has a zero at all points of the form $\frac{1}{2}\mathrm{e}^{2\pi \mathrm{i} j/3^r}$ for $j \in \mathbb{Z}$ and $r \geqslant 1$, so $|z| = \frac{1}{2}$ is a natural boundary.

The appearance of a natural boundary for the zeta function at $\frac{1}{2} = \exp(-h(T))$ prevents an analysis of π_T and M_T using analytic methods. On the other hand, the dense set of singular points on the circle $|z| = \frac{1}{2}$ form a discrete set in $\mathbb{C}_3$ (the completion of the algebraic closure of $\mathbb{Q}_3$), raising the following question.

Problem 11. For the map in Example 30 develop a notion of a dynamical zeta function $\zeta_T : \mathbb{C}_3 \to \mathbb{C}_3$ in such a way that it has a meromorphic extension beyond its radius of convergence, and exploit this to give analytic or Tauberian proofs of statements like
$$\mathsf{M}_T(X) = \tfrac{5}{8} \log N + C_T + \mathrm{O}(1/N).$$

There are several difficulties involved here, starting with the fact that the formal power series exp on $\mathbb{Q}_3$ has radius of convergence $3^{-1/2} < 1$; we refer to Koblitz [30] for the background needed to address this problem.

Problem 12. For a system in $\mathcal{G}_1$ for which the set of primes of infinite height is finite, show that if the zeta function is irrational, then it has natural boundary at e^{-h}.

More generally, there is some reason to suspect a stronger result (see [9] for more on the methods involved here, and some partial results). In the disconnected case it is known that the zeta function is typically non-algebraic [68].

Problem 13 (Bell, Miles and Ward [9]). Is there a Pólya–Carlson dichotomy for zeta functions of automorphisms of compact groups? That is, is it true that such a zeta function is either rational or has a natural boundary at its circle of convergence?

Results of [19] give precise information about the systems at the lower end of Figure 2, and in the special case of one-solenoids they take the following form.

Theorem 31 (Everest, Miles, Stevens and Ward [19]). *If T is an automorphism in $\mathcal{G}_1$ with a finite set of primes of infinite height, then there are constants k_T in $\mathbb{Q}$ and $C_T > 0$ with*
$$\mathsf{M}_T(n) = k_T \log N + C_T + \mathrm{O}(1/N).$$

Example 32. If T is the automorphism dual to $x \mapsto 2x$ on $\mathbb{Z}[\frac{1}{21}]$ then
$$\mathsf{M}_T(N) = \tfrac{269}{576} \log N + C + \mathrm{O}(1/N).$$

Problem 11 also arises in this setting, taking the following form.

Problem 14. For systems in $\mathcal{G}_1$ for which the set of primes of infinite height is finite, develop a notion of dynamical zeta function on the space $\prod_{p:k_p=\infty} \mathbb{C}_p$ so that it has a meromorphic extension beyond its radius of convergence, and exploit this to give an analytic proof of Theorem 31.

At the opposite extreme, the system corresponding to the subring $\mathbb{Q}$ has one closed orbit of length 1 by the product formula. If the set of primes of infinite height is co-finite, then the same basic p-adic estimate (6.5) shows that

$$\mathsf{F}_T(n) \leqslant n^A \tag{6.7}$$

for some $A > 0$. This polynomial bound makes it natural to study the orbit growth using the orbit Dirichlet series

$$\mathsf{d}_T(s) = \sum_{n \geqslant 1} \frac{\mathsf{O}_T(n)}{n^s},$$

and in [20] this is done for a large class of automorphisms of connected groups where (6.7) is guaranteed. At the level of generality adopted in [20], estimates for the growth of $\pi_T(N)$ depend on the abscissa of convergence of $\mathsf{d}_T(s)$. However, if T is assumed to be an automorphism in $\mathcal{G}_1$, then $\mathsf{d}_T(s)$ is found to have abscissa of convergence $\sigma = 0$ (see [20, Rmk. 3.7(1)]), and the following applies.

Theorem 33 (Everest, Miles, Stevens and Ward [20])**.** *If T is an automorphism in $\mathcal{G}_1$ with a co-finite set of primes of infinite height, then there is a finite set $M \subset \mathbb{N}$ for which $\mathsf{d}_T(s)$ is a rational function of the variables $\{s^{-m} \mid m \in M\}$, and there is a constant $C > 0$ such that*

$$\pi_T(N) = C (\log N)^K + \mathrm{O}\left((\log N)^{K-1}\right),$$

where K is the order of the pole at $s = 0$.

The special structure of the orbit Dirichlet series arising in the co-finite case, which might be called Dirichlet rationality, raises the possibility of proving some or all of Theorem 33 using Tauberian methods. Unfortunately, as the next example shows, Theorem 33 does not seem to be provable using Tauberian methods for a reason related to the natural boundary phenomenon in Example 30: in all but the simplest of settings, the analytic behavior on the critical line is poor. The following example, which is an easy calculation using Proposition 16, is taken from Everest, Miles, Stevens *et al.* [20, Ex. 4.2].

Example 34. Let $T : G \to G$ be the automorphism dual to the map $x \mapsto 2x$ on the subring $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)} \subset \mathbb{Q}$. Then

$$\begin{aligned}\mathsf{d}_T(z) = 1 - \frac{1}{2^{z+1}} + \frac{3}{2^{z+1}} \left(1 - \frac{1}{3^{z+1}} - \frac{1}{2^{z+1}} + \frac{1}{6^{z+1}}\right) \frac{1}{1 - 3^{-z}} \\ + \frac{15}{4^{z+1}} \left(1 - \frac{1}{3^{z+1}} - \frac{1}{5^{z+1}} + \frac{1}{15^{z+1}}\right) \frac{1}{(1 - 3^{-z})(1 - 5^{-z})}.\end{aligned}$$

The obstacle to using Agmon's Tauberian theorem [3] or even Perron's Theorem [23, Th. 13] is illustrated in Example 34: not only does d_T have infinitely many singularities on the critical line, it has singularities that are arbitrarily close together. On the other hand, they are located at arithmetically special points, raising the analogue of Problem 14.

Problem 15. Develop a notion of orbit Dirichlet series with variables in the space $\prod_{p:k_p=0}\mathbb{C}_p$, and develop p-adic versions of Tauberian theorems to find a meromorphic extension of the orbit Dirichlet series arising in the setting of Theorem 33, and use this to give analytic proofs of Theorem 33(1) and (3).

Theorems 31 and 33 give some insight into the range of orbit-growth invariants across those groups with finite or co-finite set of primes of infinite height. Of course most elements of $\mathcal{G}_1$ fall into neither the co-finite nor the finite camp, and new phenomena appear in this large middle ground. Using delicate constructions of thin sets of primes, Baier, Jaidee, Stevens *et al.* [7] find several different continua of different orbit-growth invariants inside $\mathcal{G}_1$.

Theorem 35 (Baier, Jaidee, Stevens and Ward [7]). *As illustrated in Figure 2, three different continua of orbit-growth rates exist in $\mathcal{G}_1$ as follows.*

(1) *For any $\kappa\in(0,1)$ there is a system $(G,T)\in\mathcal{G}_1$ with*

$$\mathsf{M}_T(N)\sim\kappa\log N$$

(see Continuum 1*).*

(2) *For any $\delta\in(0,1)$ and $k>0$ there is a system $(G,T)\in\mathcal{G}_1$ with*

$$\mathsf{M}_T(N)\sim k(\log N)^\delta$$

(see Continuum 2*).*

(3) *For any $r\in\mathbb{N}$ and $k>0$ there is a system $(G,T)\in\mathcal{G}_1$ with*

$$\mathsf{M}_T(N)\sim k(\log\log N)^r$$

(see Continuum 3*).*

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