A natural boundary for the dynamical zeta function for commuting group automorphisms

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A NATURAL BOUNDARY FOR THE DYNAMICAL ZETA FUNCTION FOR COMMUTING GROUP AUTOMORPHISMS

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Abstract. For an action α of $\mathbb{Z}^d$ by homeomorphisms of a compact metric space, D. Lind introduced a dynamical zeta function and conjectured that this function has a natural boundary when $d \geqslant 2$. In this note, under the assumption that α is a mixing action by continuous automorphisms of a compact connected abelian group of finite topological dimension, it is shown that the upper growth rate of periodic points is zero and that the unit circle is a natural boundary for the dynamical zeta function.

1. Introduction

Since the definitive work of Artin and Mazur [1], the dynamical zeta function ζ_T of a single transformation T has been studied extensively in various contexts. For example, if T is a smooth map with sufficiently uniform hyperbolic behaviour, Manning [10] shows that ζ_T is rational. In contrast, Everest, Stangoe and Ward [7] give a simple example of a compact group automorphism T for which ζ_T has a natural boundary at the circle of convergence. For an ergodic automorphism T of a compact connected abelian group of finite topological dimension, there is strong evidence [2] to suggest that ζ_T is either rational or admits a natural boundary.

For a $\mathbb{Z}^d$-action α generated by d commuting homeomorphisms of a compact metric space X, a dynamical zeta function was introduced by Lind [9]. For $\mathbf{n} \in \mathbb{Z}^d$, let $\alpha^{\mathbf{n}}$ denote the element of the action corresponding to $\mathbf{n}$. Denote the set of finite index subgroups of $\mathbb{Z}^d$ by $\mathcal{L}$. For any $L \in \mathcal{L}$, let $[L] = |\mathbb{Z}^d/L|$ and let $\mathsf{F}(L)$ denote the cardinality of the set of points $x \in X$ for which $\alpha^{\mathbf{n}}(x) = x$ for all $\mathbf{n} \in L$. The *dynamical zeta function* of α is defined formally as

$$\zeta_\alpha(z) = \exp\left(\sum_{L \in \mathcal{L}} \frac{\mathsf{F}(L)}{[L]} z^{[L]}\right).$$

If $d = 1$, $\mathcal{L} = \{n\mathbb{Z} : n \geqslant 1\}$ and the definition of ζ_α agrees with the one given by Artin and Mazur for a single transformation. Lind studies several key

Date: January 9, 2014.
2010 *Mathematics Subject Classification.* 37A45,37B05,37C25,37C30,37C85,22D40.
I would like to thank Pietro Corvaja and Umberto Zannier for their useful comments.

examples and, assuming that $\mathsf{F}(L)$ is finite for all $L \in \mathcal{L}$, shows [9, Th. 5.3] that ζ_α has radius of convergence $e^{-\mathsf{g}(\alpha)}$, where $\mathsf{g}(\alpha)$ is the *upper growth rate of periodic points*, given by

$$\mathsf{g}(\alpha) = \limsup_{[L]\to\infty} \frac{1}{[L]} \log \mathsf{F}(L).$$

It is important to note that the upper growth rate of periodic points $\mathsf{g}(\alpha)$ need not coincide with the growth rate of periodic points obtained by replacing $[L] \to \infty$ with $\min\{||\mathbf{n}|| : \mathbf{n} \in L \setminus \{\mathbf{0}\}\} \to \infty$ in the definition above. In many natural situations, the latter limit $\mathsf{h}(\alpha)$ coincides with the topological entropy of the system [8, Sec. 7]. In general, $0 \leqslant \mathsf{h}(\alpha) \leqslant \mathsf{g}(\alpha)$, and Lind gives an example [9, Ex. 6.2(a)] with $\mathsf{h}(\alpha) = \log 3$ and $\mathsf{g}(\alpha) = \log 4$. Furthermore, Lind conjectures [9, Sec. 7] that for a $\mathbb{Z}^d$-action with $d \geqslant 2$, the circle $|z| = e^{-\mathsf{h}(\alpha)}$ is a natural boundary for ζ_α and that ζ_α is meromorphic inside this circle. The main result of this note is the following.

Theorem 1.1. *Suppose X is a compact connected abelian group of finite topological dimension and α is a mixing $\mathbb{Z}^d$-action by continuous automorphisms of X. If $d \geqslant 2$, then $\mathsf{g}(\alpha) = 0$ and the unit circle is a natural boundary for ζ_α.*

This confirms Lind's conjecture in a setting which includes, for example, the well-known $\mathbb{Z}^2$-action generated by multiplication by 2 and by 3 on the solenoid dual to $\mathbb{Z}[\frac{1}{6}]$, any mixing action generated by finitely many commuting toral automorphisms, and all of the mixing entropy rank one actions on connected groups considered in [6]. In addition, and somewhat surprisingly, Theorem 1.1 also shows that $\mathsf{g}(\alpha) = 0$ in our setting. This result is obtained via an upper estimate for $\mathsf{F}(L)$ obtained using techniques from [12] and a theorem of Corvaja and Zannier [5] concerning bounds on quantities related to greatest common divisors for rings of S-integers. Hence, we obtain the stronger result that the circle of convergence for ζ_α is actually a natural boundary for the function. The other main ingredient in the proof of Theorem 1.1 is the fundamental theorem of Pólya and Carlson which states that a power series with integer coefficients and radius of convergence 1 is either rational or has the unit circle as a natural boundary (see [3], [14] and [16]). Before turning to the proof of the main result, we consider a familiar example.

2. Periodic point counts for the $\times 2 \times 3$ example

Since any abelian group is a $\mathbb{Z}$-module, the maps $x \mapsto 2x$ and $x \mapsto 3x$ are homomorphisms of the torus $\mathbb{T}$. The most natural compact abelian group X for which both these maps are automorphisms, and for which there is a projection $X \twoheadrightarrow \mathbb{T}$ that commutes with both the maps, is the Pontryagin dual of $\mathbb{Z}[\frac{1}{6}]$. Hence, the two maps generate a $\mathbb{Z}^2$-action on this one-dimensional solenoid.

a	b	c	$\mathsf{F}_{a,b,c}$	a	b	c	$\mathsf{F}_{a,b,c}$	a	b	c	$\mathsf{F}_{a,b,c}$	a	b	c	$\mathsf{F}_{a,b,c}$
1	0	1	1	6	5	1	1	9	4	1	1	11	8	1	1
1	0	2	1	1	0	7	1	9	5	1	1	11	9	1	1
2	0	1	1	7	0	1	1	9	6	1	1	11	10	1	1
2	1	1	1	7	1	1	1	9	7	1	1	1	0	12	1
1	0	3	1	7	2	1	1	9	8	1	1	2	0	6	1
3	0	1	1	7	3	1	1	1	0	10	1	2	1	6	1
3	1	1	1	7	4	1	1	2	0	5	1	3	0	4	1
3	2	1	1	7	5	1	1	2	1	5	1	3	1	4	7
1	0	4	1	7	6	1	1	5	0	2	1	3	2	4	1
2	0	2	1	1	0	8	1	5	1	2	1	4	0	3	1
2	1	2	1	2	0	4	1	5	2	2	1	4	1	3	1
4	0	1	1	2	1	4	1	5	3	2	1	4	2	3	1
4	1	1	5	4	0	2	1	5	4	2	1	4	3	3	5
4	2	1	1	4	1	2	1	10	0	1	1	6	0	2	1
4	3	1	1	4	2	2	5	10	1	1	1	6	1	2	1
1	0	5	1	4	3	2	1	10	2	1	11	6	2	2	7
5	0	1	1	8	0	1	1	10	3	1	1	6	3	2	1
5	1	1	1	8	1	1	5	10	4	1	1	6	4	2	1
5	2	1	1	8	2	1	1	10	5	1	1	6	5	2	7
5	3	1	1	8	3	1	1	10	6	1	1	12	0	1	1
5	4	1	1	8	4	1	1	10	7	1	1	12	1	1	5
1	0	6	1	8	5	1	5	10	8	1	1	12	2	1	1
2	0	3	1	8	6	1	1	10	9	1	1	12	3	1	1
2	1	3	1	8	7	1	1	1	0	11	1	12	4	1	1
3	0	2	1	1	0	9	1	11	0	1	1	12	5	1	5
3	1	2	1	3	0	3	1	11	1	1	1	12	6	1	1
3	2	2	7	3	1	3	1	11	2	1	1	12	7	1	1
6	0	1	1	3	2	3	1	11	3	1	23	12	8	1	13
6	1	1	1	9	0	1	1	11	4	1	1	12	9	1	5
6	2	1	1	9	1	1	1	11	5	1	1	12	10	1	1
6	3	1	1	9	2	1	1	11	6	1	1	12	11	1	1
6	4	1	1	9	3	1	1	11	7	1	1				

TABLE 1. Periodic point counts $\mathsf{F}_{a,b,c}$ for subgroups parameterized by integers a, b and c for the $\times 2 \times 3$ example.

The subgroups of index n in $\mathbb{Z}^2$ may be parameterized using non-negative integers a, b, c, where $ac = n$, $0 \leqslant b \leqslant a-1$, and

$$L = L(a,b,c) = \langle (a,0),(b,c) \rangle$$

is the corresponding subgroup of index n (for a more thorough explanation of this form originally due to Hermite, see the next section). For this example, the number of periodic points for rectangular subgroups (that is, those with $b=0$) was considered by Ward [17], whose focus was periodic point distribution. In general, we have

$$\begin{aligned} \mathsf{F}_{a,b,c} &= \mathsf{F}(L(a,b,c)) \\ &= |\{x \in X : 2^a x = x \text{ and } 2^b 3^c x = x\}| \\ &= |\ker(x \mapsto (2^a - 1)x) \cap \ker(x \mapsto (2^b 3^c - 1)x)|. \end{aligned}$$

Since the intersection of the kernels in the expression on the right is a finite closed subgroup of X, by Pontryagin duality (see [8, Sec. 7]), it follows that

$$\mathsf{F}_{a,b,c} = \left| \frac{\mathbb{Z}[1/6]}{(2^a - 1, 2^b 3^c - 1)} \right| = \gcd(2^a - 1, 2^b 3^c - 1).$$

Using the resulting greatest common divisor, $\mathsf{F}_{a,b,c}$ has been calculated for all subgroups of index at most 12 in $\mathbb{Z}^2$ in Table 1. Heuristically, potential growth in $\mathsf{F}_{a,b,c}$ appears to be slow in relation to $[L] = [L(a,b,c)] = ac$. In particular, in the next section, we will show that

$$\limsup_{[L]\to\infty} \frac{1}{[L]} \log \mathsf{F}(L) = 0.$$

Thus, the dynamical zeta function for this action has radius of convergence 1. This also means that the unit circle is a natural boundary for the function (see Lemma 3.3).

3. Proof of Theorem 1.1

The standard tools from commutative algebra used to study an action α of $\mathbb{Z}^d$ by continuous automorphisms of a compact abelian group X are described in Schmidt's monograph [15] (see also Einsiedler and Lind's paper [6] which is useful here). A key observation is that the Pontryagin dual of X, denoted $M = \widehat{X}$, becomes a module over the Laurent polynomial ring $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ by identifying application of the dual automorphism $\widehat{\alpha}^{\mathbf{n}}$ with multiplication by $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$, and extending this in a natural way to polynomials.

When X is connected and has finite topological dimension (that is, X is a solenoid), M is a subgroup of a finite-dimensional vector space over $\mathbb{Q}$ and we may use the method of canonical filtration described in Method 4.4 of [12]. Doing so, we obtain a chain of submodules

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

and natural module embeddings

$$R_d/\mathfrak{p}_i \hookrightarrow M_i/M_{i-1} \hookrightarrow \mathbb{K}_i,$$

where $\mathfrak{p}_i \subset R_d$ is a prime ideal, and $\mathbb{K}_i$ is an algebraic number field which is the field of fractions of $R_d/\mathfrak{p}_i$, $1 \leqslant i \leqslant r$. The list $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ comprises all associated primes of M (note that repetitions are permitted in the list). Let $\mathcal{P}(\mathbb{K})$ denote the set of all places (both finite and infinite) of an algebraic number field $\mathbb{K}$, and let $\mathcal{P}_0(\mathbb{K})$ be the set of finite places. We always assume that the places of an algebraic number field are normalized so that the Artin product formula holds. For each i, let

$$S_i = \{v \in \mathcal{P}_0(\mathbb{K}_i) : |\cdot|_v \text{ is unbounded on } R_d/\mathfrak{p}_i\},$$

and note that S_i is finite as $R_d/\mathfrak{p}_i$ is finitely generated as a ring over $\mathbb{Z}$. Define a homomorphism $\chi_i : \mathbb{Z}^d \to \mathbb{K}_i^\times$ by the composition of $\mathbf{n} \mapsto u^{\mathbf{n}}$ and the natural quotient map $R_d \to R_d/\mathfrak{p}_i$. Then $\chi_i(\mathbf{n})$ is an S_i-unit for all $\mathbf{n} \in \mathbb{Z}^d$ and the mixing hypothesis implies that χ_i is injective (see [15, Th. 6.5]). To proceed, we shall make use of the following result due to Corvaja and Zannier [5].

Theorem 3.1 (Corvaja and Zannier [5])**.** *Let S be a finite set of finite places of an algebraic number field $\mathbb{K}$. Then given any $\varepsilon > 0$, there are only finitely many pairs of multiplicatively independent S-units ξ, η that do not satisfy the inequality*

$$\prod_{v\in\mathcal{P}(\mathbb{K})} \max\{1, \min\{|\xi-1|_v^{-1}, |\eta-1|_v^{-1}\}\} < \max\{H(\xi), H(\eta)\}^\varepsilon,$$

where $H(\cdot) = \prod_{v\in\mathcal{P}(\mathbb{K})} \max\{1, |\cdot|_v\}$ denotes the absolute Weil height.

Let $\varepsilon > 0$ be given and let $||\cdot||$ denote the supremum norm. Since each χ_i is injective, applying the theorem above, there exists a constant $A = A(\varepsilon)$ such that for any linearly independent $\mathbf{j}, \mathbf{k} \in \mathbb{Z}^d$ with $\max\{||\mathbf{j}||, ||\mathbf{k}||\} > A$, and all $1 \leqslant i \leqslant r$, we have

$$\prod_{v\in\mathcal{P}(\mathbb{K}_i)} \max\{1, \min\{|\chi_i(\mathbf{j})-1|_v^{-1}, |\chi_i(\mathbf{k})-1|_v^{-1}\}\} < \max\{H(\chi_i(\mathbf{j}))H(\chi_i(\mathbf{k}))\}^\varepsilon$$

and hence also

$$\prod_{v\in P_i} \min\{|\chi_i(\mathbf{j})-1|_v^{-1}, |\chi_i(\mathbf{k})-1|_v^{-1}\} < \max\{H(\chi_i(\mathbf{j})), H(\chi_i(\mathbf{k}))\}^\varepsilon, \quad (1)$$

for any $P_i \subset \mathcal{P}_0(\mathbb{K}_i)$ with $|\chi_i(\mathbf{j})|_v = |\chi_i(\mathbf{k})|_v = 1$ for all $v \in P_i$. For each i, define

$$P_i = \{v \in \mathcal{P}_0(\mathbb{K}_i) : |\cdot|_v \text{ is bounded on } M_i/M_{i-1}\}$$

and note that since $\{\chi_i(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^d\} \hookrightarrow M_i/M_{i-1}$, we have $|\chi_i(\mathbf{n})|_v = 1$ for all $v \in P_i$ and all $\mathbf{n} \in \mathbb{Z}^d$ (so (1) holds with P_i defined in this way). A straightforward adaption of [12, Lem. 4.5] also gives the following.

Lemma 3.2. *Let $L \in \mathcal{L}$. Then for any non-zero $\mathbf{j}, \mathbf{k} \in L$*

$$\mathsf{F}(L) \leqslant \mathsf{F}(\langle \mathbf{j}, \mathbf{k}\rangle) \leqslant C \prod_{i=1}^{r} \prod_{v\in P_i} \min\{|\chi_i(\mathbf{j})-1|_v^{-1}, |\chi_i(\mathbf{k})-1|_v^{-1}\},$$

where C is a constant depending only on M.

Before attempting to combine the lemma above with (1), we need a more concrete description of the elements of $\mathcal{L}$. This is provided by the Hermite normal form of an integer matrix [9, Th. 4.1]. Any $L \in \mathcal{L}$ has a unique representation as the image of $\mathbb{Z}^d$ under a matrix of the form

$$T_L = \begin{pmatrix} a_1 & b_{12} & b_{13} & \dots & b_{1d} \\ 0 & a_2 & b_{22} & \dots & b_{2d} \\ 0 & 0 & a_3 & \dots & b_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_d \end{pmatrix}$$

where $a_m \geqslant 1$ for $1 \leqslant m \leqslant d$, $0 \leqslant b_{mn} \leqslant a_m - 1$ for $m+1 \leqslant n \leqslant d$ and $a_1 a_2 \cdots a_d = [L]$. Whenever $[L] > A^d$, we have $\max\{a_1, \dots, a_d\} > A$, and by taking two appropriate basis vectors in $\mathbb{Z}^d$ and multiplying these by T_L, it follows that there exist linearly independent $\mathbf{j}, \mathbf{k} \in L$ such that

$$A < \max\{||\mathbf{j}||, ||\mathbf{k}||\} \leqslant [L].$$

Furthermore, since the Weil height is submultiplicative, Lemma 3.2 and (1) can be combined to show that there is a constant $B > 1$, depending only on the homomorphisms χ_i, $1 \leqslant i \leqslant r$, such that

$$\mathsf{F}(L) < CB^{\varepsilon \max\{||\mathbf{j}||, ||\mathbf{k}||\}} \leqslant CB^{\varepsilon [L]},$$

whenever $[L] > A^d$. Therefore,

$$\mathsf{g}(\alpha) = \limsup_{[L] \to \infty} \frac{1}{[L]} \log \mathsf{F}(L) < \varepsilon \log B,$$

and since ε was arbitrary, it follows that $\mathsf{g}(\alpha) = 0$. Hence, ζ_α has radius of convergence 1. To show that the circle of convergence is a natural boundary for ζ_α, we use the following application of the Pólya–Carlson theorem.

Lemma 3.3. *Suppose ζ_α has radius of convergence 1 and $d \geqslant 2$. Then ζ_α admits the unit circle as a natural boundary.*

Proof. We proceed similarly to the conclusion of [9, Ex. 3.4], with some slight modifications, as follows. Suppose for a contradiction that ζ_α is rational, that is

$$\zeta_\alpha(z) = \frac{\omega \prod_{i=1}^k (1 - \lambda_i z)}{\prod_{j=1}^l (1 - \mu_j z)}$$

for some $\omega, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_l \in \mathbb{C}$. Some straightforward calculus then shows $\omega = 1$ and

$$\sum_{L \in \mathcal{L} : [L] = n} \mathsf{F}(L) = \sum_{j=1}^l \mu_j^n - \sum_{i=1}^k \lambda_i^n. \tag{2}$$

Since ζ_α is analytic inside the unit disk, we must have $|\mu_j| \leqslant 1$ for $1 \leqslant j \leqslant l$. Furthermore, as ζ_α is the exponential of a convergent power series inside the unit disk, we must also have $|\lambda_i| \leqslant 1$ for $1 \leqslant i \leqslant k$. Hence, (2) gives $\sum_{L \in \mathcal{L} : [L] = n} \mathsf{F}(L) \leqslant k + l$. Since $\mathsf{F}(L) \geqslant 1$ for all $L \in \mathcal{L}$, this implies that the number of subgroups $L \in \mathcal{L}$ with $[L] = n$ is at most $k + l$ for all $n \geqslant 1$.

Since this is obviously not the case when $d \geqslant 2$, this gives the required contradiction.

Hence, ζ_α is irrational with radius of convergence 1 and since the Taylor series for ζ_α has integer coefficients [9, Cor. 5.5], the result now follows from the theorem of Pólya and Carlson. □

4. Concluding remarks

Clearly, the generality of Theorem 1.1 is far from that of Lind's Conjecture described in the introduction. In particular, the algebraic $\mathbb{Z}^d$-actions considered here all have zero entropy. Without classical results such as the Polya–Carlson theorem, which requires $\mathsf{g}(\alpha) = 0$, or gap theorems, establishing the existence of a natural boundary can be difficult. For a single automorphism T with ζ_T irrational, this problem is explored in detail in [2].

It is also interesting to see that, together with Rudnick, Corvaja and Zannier themselves applied their result [4, Th. 2] in a way related to our calculations here, motivated by problems in quantum dynamics.

Finally, note that for algebraic $\mathbb{Z}^d$-actions on solenoids, other perspectives involving dynamical zeta functions can be fruitful. For example, the zeta functions of individual elements of the algebraic $\mathbb{Z}^d$-actions considered here are shown to be intimately linked with expansive subdynamics in [11] and [13].

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