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A NATURAL BOUNDARY FOR THE DYNAMICAL ZETA FUNCTION FOR COMMUTING GROUP AUTOMORPHISMS

RICHARD MILES

ABSTRACT. For an action α of \mathbb{Z}^d by homeomorphisms of a compact metric space, D. Lind introduced a dynamical zeta function and conjectured that this function has a natural boundary when $d \geq 2$. In this note, under the assumption that α is a mixing action by continuous automorphisms of a compact connected abelian group of finite topological dimension, it is shown that the upper growth rate of periodic points is zero and that the unit circle is a natural boundary for the dynamical zeta function.

1. INTRODUCTION

Since the definitive work of Artin and Mazur [1], the dynamical zeta function ζ_T of a single transformation T has been studied extensively in various contexts. For example, if T is a smooth map with sufficiently uniform hyperbolic behaviour, Manning [10] shows that ζ_T is rational. In contrast, Everest, Stangoe and Ward [7] give a simple example of a compact group automorphism T for which ζ_T has a natural boundary at the circle of convergence. For an ergodic automorphism T of a compact connected abelian group of finite topological dimension, there is strong evidence [2] to suggest that ζ_T is either rational or admits a natural boundary.

For a \mathbb{Z}^d -action α generated by d commuting homeomorphisms of a compact metric space X , a dynamical zeta function was introduced by Lind [9]. For $\mathbf{n} \in \mathbb{Z}^d$, let $\alpha^{\mathbf{n}}$ denote the element of the action corresponding to \mathbf{n} . Denote the set of finite index subgroups of \mathbb{Z}^d by \mathcal{L} . For any $L \in \mathcal{L}$, let $[L] = |\mathbb{Z}^d/L|$ and let $F(L)$ denote the cardinality of the set of points $x \in X$ for which $\alpha^{\mathbf{n}}(x) = x$ for all $\mathbf{n} \in L$. The *dynamical zeta function* of α is defined formally as

$$\zeta_{\alpha}(z) = \exp \left(\sum_{L \in \mathcal{L}} \frac{F(L)}{[L]} z^{[L]} \right).$$

If $d = 1$, $\mathcal{L} = \{n\mathbb{Z} : n \geq 1\}$ and the definition of ζ_{α} agrees with the one given by Artin and Mazur for a single transformation. Lind studies several key

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examples and, assuming that $F(L)$ is finite for all $L \in \mathcal{L}$, shows [9, Th. 5.3] that ζ_α has radius of convergence $e^{-g(\alpha)}$, where $g(\alpha)$ is the *upper growth rate of periodic points*, given by

$$g(\alpha) = \limsup_{[L] \rightarrow \infty} \frac{1}{[L]} \log F(L).$$

It is important to note that the upper growth rate of periodic points $g(\alpha)$ need not coincide with the growth rate of periodic points obtained by replacing $[L] \rightarrow \infty$ with $\min\{|\mathbf{n}| : \mathbf{n} \in L \setminus \{\mathbf{0}\}\} \rightarrow \infty$ in the definition above. In many natural situations, the latter limit $h(\alpha)$ coincides with the topological entropy of the system [8, Sec. 7]. In general, $0 \leq h(\alpha) \leq g(\alpha)$, and Lind gives an example [9, Ex. 6.2(a)] with $h(\alpha) = \log 3$ and $g(\alpha) = \log 4$. Furthermore, Lind conjectures [9, Sec. 7] that for a \mathbb{Z}^d -action with $d \geq 2$, the circle $|z| = e^{-h(\alpha)}$ is a natural boundary for ζ_α and that ζ_α is meromorphic inside this circle. The main result of this note is the following.

Theorem 1.1. *Suppose X is a compact connected abelian group of finite topological dimension and α is a mixing \mathbb{Z}^d -action by continuous automorphisms of X . If $d \geq 2$, then $g(\alpha) = 0$ and the unit circle is a natural boundary for ζ_α .*

This confirms Lind's conjecture in a setting which includes, for example, the well-known \mathbb{Z}^2 -action generated by multiplication by 2 and by 3 on the solenoid dual to $\mathbb{Z}[\frac{1}{6}]$, any mixing action generated by finitely many commuting toral automorphisms, and all of the mixing entropy rank one actions on connected groups considered in [6]. In addition, and somewhat surprisingly, Theorem 1.1 also shows that $g(\alpha) = 0$ in our setting. This result is obtained via an upper estimate for $F(L)$ obtained using techniques from [12] and a theorem of Corvaja and Zannier [5] concerning bounds on quantities related to greatest common divisors for rings of S -integers. Hence, we obtain the stronger result that the circle of convergence for ζ_α is actually a natural boundary for the function. The other main ingredient in the proof of Theorem 1.1 is the fundamental theorem of Pólya and Carlson which states that a power series with integer coefficients and radius of convergence 1 is either rational or has the unit circle as a natural boundary (see [3], [14] and [16]). Before turning to the proof of the main result, we consider a familiar example.

2. PERIODIC POINT COUNTS FOR THE $\times 2 \times 3$ EXAMPLE

Since any abelian group is a \mathbb{Z} -module, the maps $x \mapsto 2x$ and $x \mapsto 3x$ are homomorphisms of the torus \mathbb{T} . The most natural compact abelian group X for which both these maps are automorphisms, and for which there is a projection $X \rightarrow \mathbb{T}$ that commutes with both the maps, is the Pontryagin dual of $\mathbb{Z}[\frac{1}{6}]$. Hence, the two maps generate a \mathbb{Z}^2 -action on this one-dimensional solenoid.

a	b	c	$F_{a,b,c}$	a	b	c	$F_{a,b,c}$	a	b	c	$F_{a,b,c}$	a	b	c	$F_{a,b,c}$
1	0	1	1	6	5	1	1	9	4	1	1	11	8	1	1
1	0	2	1	1	0	7	1	9	5	1	1	11	9	1	1
2	0	1	1	7	0	1	1	9	6	1	1	11	10	1	1
2	1	1	1	7	1	1	1	9	7	1	1	1	0	12	1
1	0	3	1	7	2	1	1	9	8	1	1	2	0	6	1
3	0	1	1	7	3	1	1	1	0	10	1	2	1	6	1
3	1	1	1	7	4	1	1	2	0	5	1	3	0	4	1
3	2	1	1	7	5	1	1	2	1	5	1	3	1	4	7
1	0	4	1	7	6	1	1	5	0	2	1	3	2	4	1
2	0	2	1	1	0	8	1	5	1	2	1	4	0	3	1
2	1	2	1	2	0	4	1	5	2	2	1	4	1	3	1
4	0	1	1	2	1	4	1	5	3	2	1	4	2	3	1
4	1	1	5	4	0	2	1	5	4	2	1	4	3	3	5
4	2	1	1	4	1	2	1	10	0	1	1	6	0	2	1
4	3	1	1	4	2	2	5	10	1	1	1	6	1	2	1
1	0	5	1	4	3	2	1	10	2	1	11	6	2	2	7
5	0	1	1	8	0	1	1	10	3	1	1	6	3	2	1
5	1	1	1	8	1	1	5	10	4	1	1	6	4	2	1
5	2	1	1	8	2	1	1	10	5	1	1	6	5	2	7
5	3	1	1	8	3	1	1	10	6	1	1	12	0	1	1
5	4	1	1	8	4	1	1	10	7	1	1	12	1	1	5
1	0	6	1	8	5	1	5	10	8	1	1	12	2	1	1
2	0	3	1	8	6	1	1	10	9	1	1	12	3	1	1
2	1	3	1	8	7	1	1	1	0	11	1	12	4	1	1
3	0	2	1	1	0	9	1	11	0	1	1	12	5	1	5
3	1	2	1	3	0	3	1	11	1	1	1	12	6	1	1
3	2	2	7	3	1	3	1	11	2	1	1	12	7	1	1
6	0	1	1	3	2	3	1	11	3	1	23	12	8	1	13
6	1	1	1	9	0	1	1	11	4	1	1	12	9	1	5
6	2	1	1	9	1	1	1	11	5	1	1	12	10	1	1
6	3	1	1	9	2	1	1	11	6	1	1	12	11	1	1
6	4	1	1	9	3	1	1	11	7	1	1				

TABLE 1. Periodic point counts $F_{a,b,c}$ for subgroups parameterized by integers a , b and c for the $\times 2 \times 3$ example.

The subgroups of index n in \mathbb{Z}^2 may be parameterized using non-negative integers a , b , c , where $ac = n$, $0 \leq b \leq a - 1$, and

$$L = L(a, b, c) = \langle (a, 0), (b, c) \rangle$$

is the corresponding subgroup of index n (for a more thorough explanation of this form originally due to Hermite, see the next section). For this example, the number of periodic points for rectangular subgroups (that is, those

with $b = 0$) was considered by Ward [17], whose focus was periodic point distribution. In general, we have

$$\begin{aligned} F_{a,b,c} &= F(L(a, b, c)) \\ &= |\{x \in X : 2^a x = x \text{ and } 2^b 3^c x = x\}| \\ &= |\ker(x \mapsto (2^a - 1)x) \cap \ker(x \mapsto (2^b 3^c - 1)x)|. \end{aligned}$$

Since the intersection of the kernels in the expression on the right is a finite closed subgroup of X , by Pontryagin duality (see [8, Sec. 7]), it follows that

$$F_{a,b,c} = \left| \frac{\mathbb{Z}[1/6]}{(2^a - 1, 2^b 3^c - 1)} \right| = \gcd(2^a - 1, 2^b 3^c - 1).$$

Using the resulting greatest common divisor, $F_{a,b,c}$ has been calculated for all subgroups of index at most 12 in \mathbb{Z}^2 in Table 1. Heuristically, potential growth in $F_{a,b,c}$ appears to be slow in relation to $[L] = [L(a, b, c)] = ac$. In particular, in the next section, we will show that

$$\limsup_{[L] \rightarrow \infty} \frac{1}{[L]} \log F(L) = 0.$$

Thus, the dynamical zeta function for this action has radius of convergence 1. This also means that the unit circle is a natural boundary for the function (see Lemma 3.3).

3. PROOF OF THEOREM 1.1

The standard tools from commutative algebra used to study an action α of \mathbb{Z}^d by continuous automorphisms of a compact abelian group X are described in Schmidt's monograph [15] (see also Einsiedler and Lind's paper [6] which is useful here). A key observation is that the Pontryagin dual of X , denoted $M = \widehat{X}$, becomes a module over the Laurent polynomial ring $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ by identifying application of the dual automorphism $\widehat{\alpha}^{\mathbf{n}}$ with multiplication by $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$, and extending this in a natural way to polynomials.

When X is connected and has finite topological dimension (that is, X is a solenoid), M is a subgroup of a finite-dimensional vector space over \mathbb{Q} and we may use the method of canonical filtration described in Method 4.4 of [12]. Doing so, we obtain a chain of submodules

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

and natural module embeddings

$$R_d/\mathfrak{p}_i \hookrightarrow M_i/M_{i-1} \hookrightarrow \mathbb{K}_i,$$

where $\mathfrak{p}_i \subset R_d$ is a prime ideal, and \mathbb{K}_i is an algebraic number field which is the field of fractions of R_d/\mathfrak{p}_i , $1 \leq i \leq r$. The list $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ comprises all associated primes of M (note that repetitions are permitted in the list). Let $\mathcal{P}(\mathbb{K})$ denote the set of all places (both finite and infinite) of an algebraic number field \mathbb{K} , and let $\mathcal{P}_0(\mathbb{K})$ be the set of finite places. We always assume

that the places of an algebraic number field are normalized so that the Artin product formula holds. For each i , let

$$S_i = \{v \in \mathcal{P}_0(\mathbb{K}_i) : |\cdot|_v \text{ is unbounded on } R_d/\mathfrak{p}_i\},$$

and note that S_i is finite as R_d/\mathfrak{p}_i is finitely generated as a ring over \mathbb{Z} . Define a homomorphism $\chi_i : \mathbb{Z}^d \rightarrow \mathbb{K}_i^\times$ by the composition of $\mathbf{n} \mapsto u^{\mathbf{n}}$ and the natural quotient map $R_d \rightarrow R_d/\mathfrak{p}_i$. Then $\chi_i(\mathbf{n})$ is an S_i -unit for all $\mathbf{n} \in \mathbb{Z}^d$ and the mixing hypothesis implies that χ_i is injective (see [15, Th. 6.5]). To proceed, we shall make use of the following result due to Corvaja and Zannier [5].

Theorem 3.1 (Corvaja and Zannier [5]). *Let S be a finite set of finite places of an algebraic number field \mathbb{K} . Then given any $\varepsilon > 0$, there are only finitely many pairs of multiplicatively independent S -units ξ, η that do not satisfy the inequality*

$$\prod_{v \in \mathcal{P}(\mathbb{K})} \max\{1, \min\{|\xi - 1|_v^{-1}, |\eta - 1|_v^{-1}\}\} < \max\{H(\xi), H(\eta)\}^\varepsilon,$$

where $H(\cdot) = \prod_{v \in \mathcal{P}(\mathbb{K})} \max\{1, |\cdot|_v\}$ denotes the absolute Weil height.

Let $\varepsilon > 0$ be given and let $\|\cdot\|$ denote the supremum norm. Since each χ_i is injective, applying the theorem above, there exists a constant $A = A(\varepsilon)$ such that for any linearly independent $\mathbf{j}, \mathbf{k} \in \mathbb{Z}^d$ with $\max\{\|\mathbf{j}\|, \|\mathbf{k}\|\} > A$, and all $1 \leq i \leq r$, we have

$$\prod_{v \in \mathcal{P}(\mathbb{K}_i)} \max\{1, \min\{|\chi_i(\mathbf{j}) - 1|_v^{-1}, |\chi_i(\mathbf{k}) - 1|_v^{-1}\}\} < \max\{H(\chi_i(\mathbf{j}))H(\chi_i(\mathbf{k}))\}^\varepsilon$$

and hence also

$$\prod_{v \in P_i} \min\{|\chi_i(\mathbf{j}) - 1|_v^{-1}, |\chi_i(\mathbf{k}) - 1|_v^{-1}\} < \max\{H(\chi_i(\mathbf{j})), H(\chi_i(\mathbf{k}))\}^\varepsilon, \quad (1)$$

for any $P_i \subset \mathcal{P}_0(\mathbb{K}_i)$ with $|\chi_i(\mathbf{j})|_v = |\chi_i(\mathbf{k})|_v = 1$ for all $v \in P_i$. For each i , define

$$P_i = \{v \in \mathcal{P}_0(\mathbb{K}_i) : |\cdot|_v \text{ is bounded on } M_i/M_{i-1}\}$$

and note that since $\{\chi_i(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^d\} \hookrightarrow M_i/M_{i-1}$, we have $|\chi_i(\mathbf{n})|_v = 1$ for all $v \in P_i$ and all $\mathbf{n} \in \mathbb{Z}^d$ (so (1) holds with P_i defined in this way). A straightforward adaption of [12, Lem. 4.5] also gives the following.

Lemma 3.2. *Let $L \in \mathcal{L}$. Then for any non-zero $\mathbf{j}, \mathbf{k} \in L$*

$$F(L) \leq F(\langle \mathbf{j}, \mathbf{k} \rangle) \leq C \prod_{i=1}^r \prod_{v \in P_i} \min\{|\chi_i(\mathbf{j}) - 1|_v^{-1}, |\chi_i(\mathbf{k}) - 1|_v^{-1}\},$$

where C is a constant depending only on M .

Before attempting to combine the lemma above with (1), we need a more concrete description of the elements of \mathcal{L} . This is provided by the Hermite

normal form of an integer matrix [9, Th. 4.1]. Any $L \in \mathcal{L}$ has a unique representation as the image of \mathbb{Z}^d under a matrix of the form

$$T_L = \begin{pmatrix} a_1 & b_{12} & b_{13} & \dots & b_{1d} \\ 0 & a_2 & b_{22} & \dots & b_{2d} \\ 0 & 0 & a_3 & \dots & b_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_d \end{pmatrix}$$

where $a_m \geq 1$ for $1 \leq m \leq d$, $0 \leq b_{mn} \leq a_m - 1$ for $m + 1 \leq n \leq d$ and $a_1 a_2 \cdots a_d = [L]$. Whenever $[L] > A^d$, we have $\max\{a_1, \dots, a_d\} > A$, and by taking two appropriate basis vectors in \mathbb{Z}^d and multiplying these by T_L , it follows that there exist linearly independent $\mathbf{j}, \mathbf{k} \in L$ such that

$$A < \max\{\|\mathbf{j}\|, \|\mathbf{k}\|\} \leq [L].$$

Furthermore, since the Weil height is submultiplicative, Lemma 3.2 and (1) can be combined to show that there is a constant $B > 1$, depending only on the homomorphisms χ_i , $1 \leq i \leq r$, such that

$$F(L) < CB^{\varepsilon \max\{\|\mathbf{j}\|, \|\mathbf{k}\|\}} \leq CB^{\varepsilon [L]},$$

whenever $[L] > A^d$. Therefore,

$$g(\alpha) = \limsup_{[L] \rightarrow \infty} \frac{1}{[L]} \log F(L) < \varepsilon \log B,$$

and since ε was arbitrary, it follows that $g(\alpha) = 0$. Hence, ζ_α has radius of convergence 1. To show that the circle of convergence is a natural boundary for ζ_α , we use the following application of the Pólya–Carlson theorem.

Lemma 3.3. *Suppose ζ_α has radius of convergence 1 and $d \geq 2$. Then ζ_α admits the unit circle as a natural boundary.*

Proof. We proceed similarly to the conclusion of [9, Ex. 3.4], with some slight modifications, as follows. Suppose for a contradiction that ζ_α is rational, that is

$$\zeta_\alpha(z) = \frac{\omega \prod_{i=1}^k (1 - \lambda_i z)}{\prod_{j=1}^l (1 - \mu_j z)}$$

for some $\omega, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_l \in \mathbb{C}$. Some straightforward calculus then shows $\omega = 1$ and

$$\sum_{L \in \mathcal{L}: [L]=n} F(L) = \sum_{j=1}^l \mu_j^n - \sum_{i=1}^k \lambda_i^n. \quad (2)$$

Since ζ_α is analytic inside the unit disk, we must have $|\mu_j| \leq 1$ for $1 \leq j \leq l$. Furthermore, as ζ_α is the exponential of a convergent power series inside the unit disk, we must also have $|\lambda_i| \leq 1$ for $1 \leq i \leq k$. Hence, (2) gives $\sum_{L \in \mathcal{L}: [L]=n} F(L) \leq k + l$. Since $F(L) \geq 1$ for all $L \in \mathcal{L}$, this implies that the number of subgroups $L \in \mathcal{L}$ with $[L] = n$ is at most $k + l$ for all $n \geq 1$.

Since this is obviously not the case when $d \geq 2$, this gives the required contradiction.

Hence, ζ_α is irrational with radius of convergence 1 and since the Taylor series for ζ_α has integer coefficients [9, Cor. 5.5], the result now follows from the theorem of Pólya and Carlson. \square

4. CONCLUDING REMARKS

Clearly, the generality of Theorem 1.1 is far from that of Lind's Conjecture described in the introduction. In particular, the algebraic \mathbb{Z}^d -actions considered here all have zero entropy. Without classical results such as the Pólya–Carlson theorem, which requires $g(\alpha) = 0$, or gap theorems, establishing the existence of a natural boundary can be difficult. For a single automorphism T with ζ_T irrational, this problem is explored in detail in [2].

It is also interesting to see that, together with Rudnick, Corvaja and Zannier themselves applied their result [4, Th. 2] in a way related to our calculations here, motivated by problems in quantum dynamics.

Finally, note that for algebraic \mathbb{Z}^d -actions on solenoids, other perspectives involving dynamical zeta functions can be fruitful. For example, the zeta functions of individual elements of the algebraic \mathbb{Z}^d -actions considered here are shown to be intimately linked with expansive subdynamics in [11] and [13].

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